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Anti-Gaussian Quadrature Rule for Trigonometric Polynomials

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Abstract. In this paper, anti-Gaussian quadrature rules for trigonometric polynomials are introduced. Special attention is paid to an even weight function on $[-\pi, \pi)$. The main properties of such quadrature rules are proved and a numerical method for their construction is presented. That method is based on relations between nodes and weights of the quadrature rule for trigonometric polynomials and the quadrature rule for algebraic polynomials. Some numerical examples are included. Also, we compare our method with other available methods.

1. Introduction

Let u be a given weight function over an interval $[a, b]$. By $\mathcal{P}_n$ and $\mathcal{T}_n$, $n \in \mathbb{N}_0$, we denote the linear space of all algebraic and trigonometric polynomials of degree less than or equal to n, respectively. Let G_n be the corresponding n-point Gaussian quadrature rule:

$$G_n(f) = \sum_{k=1}^{n} \omega_k f(x_k)$$

of degree $2n - 1$ for the integral

$$I(f) = \int_a^b f(x)u(x)\,\mathrm{d}x.$$

The formula G_n has property $G_n(p) = I(p)$, for all $p \in \mathcal{P}_{2n-1}$. This famous method for numerical integration was derived by C.F. Gauss in 1814 in [7]. During the period of more than 200 years, such rules were considered by large number of mathematicians, and were generalized in different ways.

For an arbitrary function f, it may be very difficult to determine an accurate estimate of the error $I(f) - G_n(f)$. By using some quadrature rule A with at least $n + 1$ additional points and degree greater than $2n - 1$, this error can be estimated by difference $A(f) - G_n(f)$. In order to increase the accuracy of the approximation value of a desired integral, Laurie [14] introduced an anti-Gaussian quadrature rule, with

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algebraic degree of exactness $2n+1$, that gives an error equal in magnitude but of opposite sign to that of the corresponding Gaussian rule. A similar idea has been used in the numerical solution of initial value problems in ordinary differential equations [5, 18, 20].

The anti-Gaussian quadrature rule

$$H_{n+1}(f) = \sum_{k=1}^{n+1} \lambda_k f(\xi_k)$$

is an $(n+1)$-point formula such that $I(p) - H_{n+1}(p) = -(I(p) - G_n(p))$, for all $p \in \mathcal{P}_{2n+1}$. Here, we see that $H_{n+1}(p) = (2I - G_n)(p)$, for each $p \in \mathcal{P}_{2n+1}$. Because of that, we introduce the following inner product:

$$(f, g)_u = (2I - G_n)(fg). \tag{1}$$

Let us denote by (p_k) the sequence of monic polynomials orthogonal with respect to the inner product $\langle f, g \rangle_u = I(fg)$. It is well known that such polynomials satisfy the following three-term recurrence relation of the following form (see, e.g., [8]):

$$p_{k+1}(x) = (x - a_k)p_k(x) - b_k p_{k-1}(x), \quad k = 0, 1, \ldots, \qquad p_0(x) = 1, \quad p_{-1}(x) = 0.$$

Let (π_k) be the sequence of monic polynomials orthogonal with respect to the inner product (1). They also satisfy the three-term recurrence relation (see [14]):

$$\pi_{k+1}(x) = (x - \alpha_k)\pi_k(x) - \beta_k \pi_{k-1}(x), \quad k = 0, 1, \ldots, n, \qquad \pi_0(x) = 1, \quad \pi_{-1}(x) = 0.$$

Coefficients $\{a_k \mid k = 0, 1, \ldots\}$, $\{b_k \mid k = 1, 2, \ldots\}$, $\{\alpha_k \mid k = 0, 1, \ldots, n\}$ and $\{\beta_k \mid k = 1, \ldots, n\}$ are given by:

$$a_k = \frac{I(xp_k^2)}{I(p_k^2)}, \quad b_k = \frac{I(p_k^2)}{I(p_{k-1}^2)}, \qquad \alpha_k = \frac{(2I - G_n)(x\pi_k^2)}{(2I - G_n)(\pi_k^2)}, \quad \beta_k = \frac{(2I - G_n)(\pi_k^2)}{(2I - G_n)(\pi_{k-1}^2)}.$$

It is easy to see that these coefficients satisfy:

$$\alpha_k = a_k, \quad k = 0, \ldots, n; \quad \beta_k = b_k, \quad k = 0, \ldots, n-1; \quad \beta_n = 2b_n,$$

and, therefore $\pi_k = p_k$, for $k = 0, 1, \ldots, n$ (see [14]).

Modified and generalized anti-Gaussian quadrature rules for real-valued measures were investigated in [2] and [17]. Other generalizations to matrix-valued measures were given in [6] and [1]. Applications of anti-Gaussian quadrature rules for the estimates of the error $I(f) - G_n(f)$ can be found in [23]. Based on anti-Gaussian quadrature rule, Laurie in [14] introduced an averaged Gaussian rule. Different averaged Gaussian quadrature rules relative to those introduced by Laurie in [14] can be found in [19, 22, 24, 25].

Sun-mi Kim and Reichel in [12] introduced anti-Szegő quadrature rules which were characterized by the property that the quadrature error for Laurent polynomials of order at most n is a specified negative multiple of the quadrature error obtained with the n-node Szegő rule. By using Szegő and anti-Szegő quadrature rules, Jages, Reichel, and Tang in [13] defined generalized averaged Szegő quadrature rules which are exact for all Laurent polynomial in $\Lambda_{-n+1,n-1}$.

In this paper, we combine two different types of generalizations of Gaussian rules: generalizations to the rules which are exact on spaces of functions different from the space of algebraic polynomials as well as generalizations to an anti-Gaussian quadrature rule. Our attention is restricted to the anti-Gaussian quadrature rules with trigonometric degree of exactness for an even weight function on $[-\pi, \pi)$. Also, we introduce the averaged Gaussian quadrature formula with trigonometric degree of exactness.

The paper is organized as follows. In Section 2 we introduce and consider an anti-Gaussian quadrature rule for trigonometric polynomials. In Section 3, we present method for numerical construction of the anti-Gaussian quadrature rules with respect to symmetric weight function. Some numerical examples and comparisons with other available methods are presented in Section 4.

2. Anti-Gaussian quadrature rule for trigonometric polynomials

Let w be a weight function, integrable and nonnegative on the interval $[-\pi,\pi)$. For every nonnegative integer n and $\gamma\in\{0,1/2\}$ by $\mathcal{T}_{n,\gamma}$ we denote the linear span of the set $\{\cos(k+\gamma)x, \sin(k+\gamma)x \mid k=0,1,\dots,n\}$. Thus, $\mathcal{T}_{n,0}=\mathcal{T}_n$ is the linear space of all trigonometric polynomials of degree less than or equal to n, and $\mathcal{T}_{n,1/2}$ is the linear space of all trigonometric polynomials of semi-integer degree less than or equal to $n+1/2$.

Let us consider the integral:

$$\widehat{I}(f)=\int_{-\pi}^{\pi} f(x)w(x)\,\mathrm{d}x.$$

By $(A_{k,\gamma})$, $A_{k,\gamma}\in\mathcal{T}_{n,\gamma}$, $k=0,1,\dots$, we denote the sequence of trigonometric polynomials orthogonal on $[-\pi,\pi)$ with respect to the inner product

$$\langle f,g\rangle_w=\widehat{I}(fg). \tag{2}$$

The corresponding Gaussian quadrature rule

$$\widehat{G_{\widetilde{n}+1}}(f)=\sum_{k=0}^{\widetilde{n}}\omega_k f(x_k),\quad \widetilde{n}=2n-1+2\gamma,\quad \gamma\in\{0,1/2\}, \tag{3}$$

is exact for every $t\in\mathcal{T}_{2n-1+2\gamma}$ if and only if the nodes x_k, $k=0,1,\dots,\widetilde{n}$ are the zeros of the trigonometric polynomial $A_{n,\gamma}\in\mathcal{T}_{n,\gamma}$ (see [3, 4, 16, 26–28]).

For the fixed positive integer n, we introduce the inner product $(*,*)_\omega$ as follows:

$$(f,g)_w=(2\widehat{I}-\widehat{G_{\widetilde{n}+1}})(fg). \tag{4}$$

Remark 2.1. *It is important to emphasize that the inner product* (4) *depends on the number n. Because of that, in what follows when we write that positive integer n is fixed in advance, we assume that we consider the inner product* (4) *with respect to that fixed number n.*

Let positive integer n be fixed in advance. By $(B_{k,\gamma})$ we denote the sequence of trigonometric polynomials orthogonal with respect to the inner product (4). The corresponding Gaussian rule has $\widetilde{n}+1$ nodes, while the corresponding anti-Gaussian rule (where $n\mapsto n+1$) has $\widetilde{n}+3$ nodes (so we denote it by $\widehat{H_{\widetilde{n}+3}}$, and it is exact for all $t\in\mathcal{T}_{2n+1+2\gamma}$), which are the zeros of the trigonometric polynomial

$$B_{n+1,\gamma}(x)=\sum_{k=0}^{n+1}\Big(p_{k,\gamma}\cos(k+\gamma)x+q_{k,\gamma}\sin(k+\gamma)x\Big),\quad |p_{n+1,\gamma}|+|q_{n+1,\gamma}|\neq 0.$$

Specially, for $(p_{n+1,\gamma},q_{n+1,\gamma})=(1,0)$ and $(p_{n+1,\gamma},q_{n+1,\gamma})=(0,1)$ we get

$$B^C_{n+1,\gamma}(x)=\cos(n+1+\gamma)x+\sum_{k=0}^{n}\Big(p^{(n+1)}_{k,\gamma}\cos(k+\gamma)x+q^{(n+1)}_{k,\gamma}\sin(k+\gamma)x\Big)$$

and

$$B^S_{n+1,\gamma}(x)=\sin(n+1+\gamma)x+\sum_{k=0}^{n}\Big(r^{(n+1)}_{k,\gamma}\cos(k+\gamma)x+s^{(n+1)}_{k,\gamma}\sin(k+\gamma)x\Big),$$

respectively. For $k,\ell\in\mathbb{N}_0$ we define:

$$\widehat{I}^{C,\gamma}_k=\left(B^C_{k,\gamma},B^C_{k,\gamma}\right)_w,\quad \widehat{I}^{S,\gamma}_k=\left(B^S_{k,\gamma},B^S_{k,\gamma}\right)_w,\quad \widehat{I}^{\gamma}_k=\left(B^C_{k,\gamma},B^S_{k,\gamma}\right)_w,$$
$$\widehat{J}^{C,\gamma}_{k,\ell}=\left(2\cos x B^C_{k,\gamma},B^C_{\ell,\gamma}\right)_w,\quad \widehat{J}^{S,\gamma}_{k,\ell}=\left(2\cos x B^S_{k,\gamma},B^S_{\ell,\gamma}\right)_w,\quad \widehat{J}^{\gamma}_{k,\ell}=\left(2\cos x B^C_{k,\gamma},B^S_{\ell,\gamma}\right)_w.$$

By $I_k^{C,\gamma}$, $I_k^{S,\gamma}$, I_k^{γ}, $J_{k,\ell}^{C,\gamma}$, $J_{k,\ell}^{S,\gamma}$, $J_{k,\ell}^{\gamma}$, $k,\ell \in \mathbb{N}_0$, we denote the corresponding values related to the inner product (2) (e.g., $I_k^{C,\gamma} = \left\langle A_{k,\gamma}^C, A_{k,\gamma}^C \right\rangle_w, \ldots$).

Theorem 2.2. *Let positive integer n be fixed in advance. Trigonometric polynomials $B_{k,\gamma}^C(x)$ and $B_{k,\gamma}^S(x)$, $k \geqslant 1$, orthogonal with respect to the inner product* (4)*, satisfy the following recurrence relations:*

$$B_{k,\gamma}^C(x) = \left(2\cos x - a_{k,\gamma}^{(1)}\right) B_{k-1,\gamma}^C(x) - b_{k,\gamma}^{(1)} B_{k-1,\gamma}^S(x) - a_{k,\gamma}^{(2)} B_{k-2,\gamma}^C(x) - b_{k,\gamma}^{(2)} B_{k-2,\gamma}^S(x), \tag{5}$$

$$B_{k,\gamma}^S(x) = \left(2\cos x - d_{k,\gamma}^{(1)}\right) B_{k-1,\gamma}^S(x) - c_{k,\gamma}^{(1)} B_{k-1,\gamma}^C(x) - d_{k,\gamma}^{(2)} B_{k-2,\gamma}^S(x) - c_{k,\gamma}^{(2)} B_{k-2,\gamma}^C(x), \tag{6}$$

where the coefficients are the solutions of the following systems for $j = 1, 2$:

$$\begin{aligned} &\widehat{J}_{k-1,k-j}^{C,\gamma} = a_{k,\gamma}^{(j)} \widehat{I}_{k-j}^{C,\gamma} + b_{k,\gamma}^{(j)} \widehat{I}_{k-j}^{\gamma}, \quad \widehat{J}_{k-1,k-j}^{\gamma} = a_{k,\gamma}^{(j)} \widehat{I}_{k-j}^{\gamma} + b_{k,\gamma}^{(j)} \widehat{I}_{k-j}^{S,\gamma}, \\ &\widehat{J}_{k-1,k-j}^{\gamma} = c_{k,\gamma}^{(j)} \widehat{I}_{k-j}^{C,\gamma} + d_{k,\gamma}^{(j)} \widehat{I}_{k-j}^{\gamma}, \quad \widehat{J}_{k-1,k-j}^{S,\gamma} = c_{k,\gamma}^{(j)} \widehat{I}_{k-j}^{\gamma} + d_{k,\gamma}^{(j)} \widehat{I}_{k-j}^{S,\gamma}, \end{aligned} \tag{7}$$

with $a_{1,\gamma}^{(2)} = b_{1,\gamma}^{(2)} = c_{1,\gamma}^{(2)} = d_{1,\gamma}^{(2)} = 0$.

Proof. The polynomials $B_{j,\gamma}^C(x)$ and $B_{j,\gamma}^S(x)$, for $j = 0, \ldots, k$, are linearly independent, so we have

$$2\cos x B_{k-1,\gamma}^C(x) = B_{k,\gamma}^C(x) + \sum_{j=0}^{k-1} \left[a_{k,\gamma}^{(k-j)} B_{j,\gamma}^C(x) + b_{k,\gamma}^{(k-j)} B_{j,\gamma}^S(x) \right]. \tag{8}$$

Applying the inner product (4) for $i = 0, 1, \ldots, k-1$ we get:

$$\left(2\cos x B_{k-1,\gamma}^C(x), B_{i,\gamma}^C(x)\right)_w = \left(B_{k,\gamma}^C(x), B_{i,\gamma}^C(x)\right)_w + \sum_{j=0}^{k-1} a_{k,\gamma}^{(k-j)} \left(B_{j,\gamma}^C(x), B_{i,\gamma}^C(x)\right)_w + \sum_{j=0}^{k-1} b_{k,\gamma}^{(k-j)} \left(B_{j,\gamma}^S(x), B_{i,\gamma}^C(x)\right)_w.$$

Now, for $i = 0, \ldots, k-3$, one has:

$$a_{k,\gamma}^{(k-i)} \widehat{I}_i^{C,\gamma} + b_{k,\gamma}^{(k-i)} \widehat{I}_i^{\gamma} = 0. \tag{9}$$

Similarly, for $i = 0, 1, \ldots, k-1$ we get:

$$\left(2\cos x B_{k-1,\gamma}^C(x), B_{i,\gamma}^S(x)\right)_w = \left(B_{k,\gamma}^C(x), B_{i,\gamma}^S(x)\right)_w + \sum_{j=0}^{k-1} a_{k,\gamma}^{(k-j)} \left(B_{j,\gamma}^C(x), B_{i,\gamma}^S(x)\right)_w + \sum_{j=0}^{k-1} b_{k,\gamma}^{(k-j)} \left(B_{j,\gamma}^S(x), B_{i,\gamma}^S(x)\right)_w.$$

Again, for $i = 0, \ldots, k-3$, one has

$$a_{k,\gamma}^{(k-i)} \widehat{I}_i^{\gamma} + b_{k,\gamma}^{(k-i)} \widehat{I}_i^{S,\gamma} = 0. \tag{10}$$

Therefore, we get homogeneous systems of linear equations:

$$\begin{aligned} &\widehat{I}_i^{C,\gamma} \cdot a_{k,\gamma}^{(k-i)} + \widehat{I}_i^{\gamma} \cdot b_{k,\gamma}^{(k-i)} = 0, \\ &\widehat{I}_i^{\gamma} \cdot a_{k,\gamma}^{(k-i)} + \widehat{I}_i^{S,\gamma} \cdot b_{k,\gamma}^{(k-i)} = 0, \end{aligned} \qquad i = 0, \ldots, k-3, \tag{11}$$

and the corresponding determinants of these systems are given by

$$\widehat{D}_i^{\gamma} = \begin{vmatrix} \widehat{I}_i^{C,\gamma} & \widehat{I}_i^{\gamma} \\ \widehat{I}_i^{\gamma} & \widehat{I}_i^{S,\gamma} \end{vmatrix}.$$

Since $\widehat{I}(f) = \widehat{G}_{\widetilde{n}+1}(f)$ for all $f \in \mathcal{T}_{\widetilde{n}}$, we have

$$\widehat{H}_{\widetilde{n}+3}(f) = (2\widehat{I} - \widehat{G}_{\widetilde{n}+1})(f) = \widehat{I}(f) + (\widehat{I} - \widehat{G}_{\widetilde{n}+1})(f) = \widehat{I}(f) \quad \text{for each } f \in \mathcal{T}_{\widetilde{n}}.$$

Further, $B^C_{k,\gamma}(x) = A^C_{k,\gamma}(x)$ and $B^S_{k,\gamma}(x) = A^S_{k,\gamma}(x)$, for every $k = 0, \ldots, n$. For $2k+1 < 2n$ it follows:

$$\widehat{I}^{C,\gamma}_k = \left(B^C_{k,\gamma}, B^C_{k,\gamma}\right)_w = (2\widehat{I} - \widehat{G}_{\widetilde{n}+1})\left(B^C_{k,\gamma} \cdot B^C_{k,\gamma}\right) = \widehat{I}\left(B^C_{k,\gamma} \cdot B^C_{k,\gamma}\right) = I^{C,\gamma}_k.$$

Similarly, $\widehat{I}^{S,\gamma}_k = I^{S,\gamma}_k$ and $\widehat{I}^{\gamma}_k = I^{\gamma}_k$. Now, using Cauchy-Schwarz-Bunyakovsky integral inequality, for $i = 0, \ldots, k-3$, we have:

$$\begin{aligned}
\widehat{D}^{\gamma}_i =& \widehat{I}^{C,\gamma}_i \widehat{I}^{S,\gamma}_i - (\widehat{I}^{\gamma}_i)^2 = I^{C,\gamma}_i I^{S,\gamma}_i - (I^{\gamma}_i)^2 \\
=& \left(\int_{-\pi}^{\pi} \left(A^C_{i,\gamma}(x)\right)^2 w(x)\,\mathrm{d}x\right) \cdot \left(\int_{-\pi}^{\pi} \left(A^S_{i,\gamma}(x)\right)^2 w(x)\,\mathrm{d}x\right) - \left(\int_{-\pi}^{\pi} A^C_{i,\gamma}(x) A^S_{i,\gamma}(x) w(x)\,\mathrm{d}x\right)^2 \\
>& \left(\int_{-\pi}^{\pi} \left(A^C_{i,\gamma}(x)\right)^2 w(x)\,\mathrm{d}x\right) \cdot \left(\int_{-\pi}^{\pi} \left(A^S_{i,\gamma}(x)\right)^2 w(x)\,\mathrm{d}x\right) - \left(\int_{-\pi}^{\pi} \left(A^C_{i,\gamma}(x)\right)^2 w(x)\,\mathrm{d}x\right) \cdot \left(\int_{-\pi}^{\pi} \left(A^S_{i,\gamma}(x)\right)^2 w(x)\,\mathrm{d}x\right) \\
=& 0.
\end{aligned}$$

Therefore, for all $i = 0, \ldots, k-3$ the system (11) has only the trivial solutions $a^{(k-i)}_{k,\gamma} = b^{(k-i)}_{k,\gamma} = 0$. Thus, equality (8) gets the following form

$$2\cos x B^C_{k-1,\gamma}(x) = B^C_{k,\gamma}(x) + a^{(1)}_{k,\gamma} B^C_{k-1,\gamma}(x) + b^{(1)}_{k,\gamma} B^S_{k-1,\gamma}(x) + a^{(2)}_{k,\gamma} B^C_{k-2,\gamma}(x) + b^{(2)}_{k,\gamma} B^S_{k-2,\gamma}(x), \tag{12}$$

i.e., form (5). Analogously, one can obtain the recurrence relation (6) for $B^S_{k,\gamma}(x)$.

Using recurrence relation (12), for $j = 1, 2$, we have:

$$\begin{aligned}
\left(2\cos x B^C_{k-1,\gamma}(x), B^C_{k-j,\gamma}(x)\right)_w = & a^{(1)}_{k,\gamma}\left(B^C_{k-1,\gamma}(x), B^C_{k-j,\gamma}(x)\right)_w + b^{(1)}_{k,\gamma}\left(B^S_{k-1,\gamma}(x), B^C_{k-j,\gamma}(x)\right)_w \\
& + a^{(2)}_{k,\gamma}\left(B^C_{k-2,\gamma}(x), B^C_{k-j,\gamma}(x)\right)_w + b^{(2)}_{k,\gamma}\left(B^S_{k-2,\gamma}(x), B^C_{k-j,\gamma}(x)\right)_w,
\end{aligned}$$

i.e., $\widehat{J}^{C,\gamma}_{k-1,k-j} = a^{(j)}_{k,\gamma} \widehat{I}^{C,\gamma}_{k-j} + b^{(j)}_{k,\gamma} \widehat{I}^{\gamma}_{k-j}$, $j = 1, 2$.

Analogously, one gets $\widehat{J}^{\gamma}_{k-1,k-j} = a^{(j)}_{k,\gamma} \widehat{I}^{\gamma}_{k-j} + b^{(j)}_{k,\gamma} \widehat{I}^{S,\gamma}_{k-j}$, $j = 1, 2$.

Similarly, starting from (6) one can obtain $\widehat{J}^{\gamma}_{k-1,k-j} = c^{(j)}_{k,\gamma} \widehat{I}^{C,\gamma}_{k-j} + d^{(j)}_{k,\gamma} \widehat{I}^{\gamma}_{k-j}$, $\widehat{J}^{S,\gamma}_{k-1,k-j} = c^{(j)}_{k,\gamma} \widehat{I}^{\gamma}_{k-j} + d^{(j)}_{k,\gamma} \widehat{I}^{S,\gamma}_{k-j}$, $j = 1, 2$. □

Using the recurrence relations (5), (6) and orthogonality conditions, the following Lemma could be easily proved.

Lemma 2.3. *For $k \geqslant 1$ the following equations hold:*

$$\widehat{I}^{C,\gamma}_k = \widehat{J}^{C,\gamma}_{k,k-1}, \quad \widehat{I}^{S,\gamma}_k = \widehat{J}^{S,\gamma}_{k,k-1}, \quad \widehat{I}^{\gamma}_k = \widehat{J}^{\gamma}_{k,k-1}.$$

Let us denote $\widehat{J}^{C,\gamma}_k = \widehat{J}^{C,\gamma}_{k,k}$, $\widehat{J}^{S,\gamma}_k = \widehat{J}^{S,\gamma}_{k,k}$, $\widehat{J}^{\gamma}_k = \widehat{J}^{\gamma}_{k,k}$.

Corollary 2.4. *The recurrence coefficients in (5) and (6) are given as follows:*

$$a^{(1)}_{k,\gamma} = \frac{\widehat{I}^{S,\gamma}_{k-1}\widehat{J}^{C,\gamma}_{k-1} - \widehat{I}^{\gamma}_{k-1}\widehat{J}^{\gamma}_{k-1}}{\widehat{D}^{\gamma}_{k-1}}, \quad a^{(2)}_{k,\gamma} = \frac{\widehat{I}^{C,\gamma}_{k-1}\widehat{I}^{S,\gamma}_{k-2} - \widehat{I}^{\gamma}_{k-1}\widehat{I}^{\gamma}_{k-2}}{\widehat{D}^{\gamma}_{k-2}}, \quad b^{(1)}_{k,\gamma} = \frac{\widehat{I}^{C,\gamma}_{k-1}\widehat{J}^{\gamma}_{k-1} - \widehat{I}^{\gamma}_{k-1}\widehat{J}^{C,\gamma}_{k-1}}{\widehat{D}^{\gamma}_{k-1}}, \quad b^{(2)}_{k,\gamma} = \frac{\widehat{I}^{\gamma}_{k-1}\widehat{I}^{C,\gamma}_{k-2} - \widehat{I}^{C,\gamma}_{k-1}\widehat{I}^{\gamma}_{k-2}}{\widehat{D}^{\gamma}_{k-2}},$$

$$c^{(1)}_{k,\gamma} = \frac{\widehat{I}^{S,\gamma}_{k-1}\widehat{J}^{\gamma}_{k-1} - \widehat{I}^{\gamma}_{k-1}\widehat{J}^{S,\gamma}_{k-1}}{\widehat{D}^{\gamma}_{k-1}}, \quad c^{(2)}_{k,\gamma} = \frac{\widehat{I}^{\gamma}_{k-1}\widehat{I}^{S,\gamma}_{k-2} - \widehat{I}^{S,\gamma}_{k-1}\widehat{I}^{\gamma}_{k-2}}{\widehat{D}^{\gamma}_{k-2}}, \quad d^{(1)}_{k,\gamma} = \frac{\widehat{I}^{C,\gamma}_{k-1}\widehat{J}^{S,\gamma}_{k-1} - \widehat{I}^{\gamma}_{k-1}\widehat{J}^{\gamma}_{k-1}}{\widehat{D}^{\gamma}_{k-1}}, \quad d^{(2)}_{k,\gamma} = \frac{\widehat{I}^{S,\gamma}_{k-1}\widehat{I}^{C,\gamma}_{k-2} - \widehat{I}^{\gamma}_{k-1}\widehat{I}^{\gamma}_{k-2}}{\widehat{D}^{\gamma}_{k-2}},$$

where $\widehat{D}^{\gamma}_{k-j} = \widehat{I}^{C,\gamma}_{k-j}\widehat{I}^{S,\gamma}_{k-j} - (\widehat{I}^{\gamma}_{k-j})^2$, $j = 1, 2$, *for* $k \geqslant 1$.

Proof. Formulas can be obtained solving systems (7). □

Let positive integer n be fixed in advance. Since $B^C_{k,\gamma}(x) = A^C_{k,\gamma}(x)$ and $B^S_{k,\gamma}(x) = A^S_{k,\gamma}(x)$, for every $k = 0, \dots, n$, we have:

$$a^{(j)}_{k,\gamma} = \alpha^{(j)}_{k,\gamma}, \quad b^{(j)}_{k,\gamma} = \beta^{(j)}_{k,\gamma}, \quad c^{(j)}_{k,\gamma} = \gamma^{(j)}_{k,\gamma}, \quad d^{(j)}_{k,\gamma} = \delta^{(j)}_{k,\gamma}, \quad j = 1, 2, \tag{13}$$

where $\alpha^{(j)}_{k,\gamma}$, $\beta^{(j)}_{k,\gamma}$, $\gamma^{(j)}_{k,\gamma}$ and $\delta^{(j)}_{k,\gamma}$ are the coefficients in the corresponding recurrence relations for polynomials $A^C_{k,\gamma}$ and $A^S_{k,\gamma}$ (see [16, 28]).

Lemma 2.5. *Let positive integer n be fixed in advance. For $k = 0, \dots, n-1$ the following equations*

$$\widehat{I}^{S,\gamma}_k = I^{S,\gamma}_k, \ \widehat{J}^{S,\gamma}_k = J^{S,\gamma}_k, \ \widehat{I}^{C,\gamma}_k = I^{C,\gamma}_k, \ \widehat{J}^{C,\gamma}_k = J^{C,\gamma}_k, \ \widehat{I}^{\gamma}_k = I^{\gamma}_k, \ \widehat{J}^{\gamma}_k = J^{\gamma}_k,$$

hold, while

$$\widehat{I}^{S,\gamma}_n = 2I^{S,\gamma}_n, \ \widehat{J}^{S,\gamma}_n = 2J^{S,\gamma}_n, \ \widehat{I}^{C,\gamma}_n = 2I^{C,\gamma}_n, \ \widehat{J}^{C,\gamma}_n = 2J^{C,\gamma}_n, \ \widehat{I}^{\gamma}_n = 2I^{\gamma}_n, \ \widehat{J}^{\gamma}_n = 2J^{\gamma}_n,$$

and for $k = n+1$ coefficients in the five-term recurrence relations (5) *and* (6) *satisfy*

$$\begin{array}{llll} a^{(1)}_{n+1,\gamma} = \alpha^{(1)}_{n+1,\gamma}, & a^{(2)}_{n+1,\gamma} = 2\alpha^{(2)}_{n+1,\gamma}, & b^{(1)}_{n+1,\gamma} = \beta^{(1)}_{n+1,\gamma}, & b^{(2)}_{n+1,\gamma} = 2\beta^{(2)}_{n+1,\gamma}, \\ c^{(1)}_{n+1,\gamma} = \gamma^{(1)}_{n+1,\gamma}, & c^{(2)}_{n+1,\gamma} = 2\gamma^{(2)}_{n+1,\gamma}, & d^{(1)}_{n+1,\gamma} = \delta^{(1)}_{n+1,\gamma}, & d^{(2)}_{n+1,\gamma} = 2\delta^{(2)}_{n+1,\gamma}. \end{array} \tag{14}$$

Proof. For $k = 0, \dots, n-1$, we have

$$\begin{aligned} \widehat{I}^{\gamma}_k &= \left(B^C_{k,\gamma}, B^S_{k,\gamma}\right)_w = \left(2\widehat{I} - \widehat{G}_{\widetilde{n}+1}\right)\left(B^C_{k,\gamma} B^S_{k,\gamma}\right) = \left(2\widehat{I} - \widehat{G}_{\widetilde{n}+1}\right)\left(A^C_{k,\gamma} A^S_{k,\gamma}\right) \\ &= \widehat{I}\left(A^C_{k,\gamma} A^S_{k,\gamma}\right) + \left(\widehat{I} - \widehat{G}_{\widetilde{n}+1}\right)\left(A^C_{k,\gamma} A^S_{k,\gamma}\right) = \left\langle A^C_{k,\gamma}, A^S_{k,\gamma}\right\rangle_w = I^{\gamma}_k. \end{aligned}$$

In a similar way, one can prove the other equalities for $k \leqslant n-1$.

Due to the fact that the nodes of the Gaussian quadrature rule are the zeros of trigonometric polynomial $A_{n,\gamma}(x)$, for $k = n$, we have

$$\widehat{I}^{\gamma}_n = \left(B^C_{n,\gamma}, B^S_{n,\gamma}\right)_w = (2\widehat{I} - \widehat{G}_{\widetilde{n}+1})\left(B^C_{n,\gamma} B^S_{n,\gamma}\right) = (2\widehat{I} - \widehat{G}_{\widetilde{n}+1})\left(A^C_{n,\gamma} A^S_{n,\gamma}\right) = 2I^{\gamma}_n.$$

Analogously, the other equalities can be proved.

Using the previous equalities, we deduce:

$$\begin{aligned} \widehat{D}^{\gamma}_k &= \widehat{I}^{C,\gamma}_k \widehat{I}^{S,\gamma}_k - (\widehat{I}^{\gamma}_k)^2 = I^{C,\gamma}_k I^{S,\gamma}_k - (I^{\gamma}_k)^2 = D^{\gamma}_k, \quad k = 0, \dots, n-1, \\ \widehat{D}^{\gamma}_n &= \widehat{I}^{C,\gamma}_n \widehat{I}^{S,\gamma}_n - (\widehat{I}^{\gamma}_n)^2 = 2I^{C,\gamma}_n \cdot 2I^{S,\gamma}_n - (2I^{\gamma}_n)^2 = 4D^{\gamma}_n, \end{aligned}$$

where $D^{\gamma}_k = I^{C,\gamma}_k I^{S,\gamma}_k - (I^{\gamma}_k)^2$, $k = 0, 1, \dots, n$.

Finally, for the coefficients $a^{(1)}_{n+1,\gamma}$ and $a^{(2)}_{n+1,\gamma}$ we have

$$a^{(1)}_{n+1,\gamma} = \frac{\widehat{I}^{S,\gamma}_n \widehat{J}^{C,\gamma}_n - \widehat{I}^{\gamma}_n \widehat{J}^{\gamma}_n}{\widehat{D}^{\gamma}_n} = \frac{2I^{S,\gamma}_n \cdot 2J^{C,\gamma}_n - 2I^{\gamma}_n \cdot 2J^{\gamma}_n}{4 \cdot D^{\gamma}_n} = \alpha^{(1)}_{n+1,\gamma}$$

and

$$a^{(2)}_{n+1,\gamma} = \frac{\widehat{I}^{C,\gamma}_n \widehat{I}^{S,\gamma}_{n-1} - \widehat{I}^{\gamma}_n \widehat{I}^{\gamma}_{n-1}}{\widehat{D}^{\gamma}_{n-1}} = \frac{2I^{C,\gamma}_n I^{S,\gamma}_{n-1} - 2I^{\gamma}_n I^{\gamma}_{n-1}}{D^{\gamma}_{n-1}} = 2\alpha^{(2)}_{n+1,\gamma}.$$

The equalities for other coefficients can be proved in a similar way. □

3. Numerical construction of anti-Gaussian quadrature rule

In this section we consider only even weight functions w on $(-\pi,\pi)$. Let positive integer n be fixed in advance. Using equalities $B^C_{k,\gamma}(x)=A^C_{k,\gamma}(x)$, $B^S_{k,\gamma}(x)=A^S_{k,\gamma}(x)$, for $k=0,\dots,n$, and (14), we get

$$\begin{aligned}B^C_{n+1,\gamma}(x)&=\left(2\cos x-a^{(1)}_{n+1,\gamma}\right)B^C_{n,\gamma}(x)-b^{(1)}_{n+1,\gamma}B^S_{n,\gamma}(x)-a^{(2)}_{n+1,\gamma}B^C_{n-1,\gamma}(x)-b^{(2)}_{n+1,\gamma}B^S_{n-1,\gamma}(x)\\&=\left(2\cos x-\alpha^{(1)}_{n+1,\gamma}\right)A^C_{n,\gamma}(x)-\beta^{(1)}_{n+1,\gamma}A^S_{n,\gamma}(x)-2\alpha^{(2)}_{n+1,\gamma}A^C_{n-1,\gamma}(x)-2\beta^{(2)}_{n+1,\gamma}A^S_{n-1,\gamma}(x)\\&=A^C_{n+1,\gamma}(x)-\alpha^{(2)}_{n+1,\gamma}A^C_{n-1,\gamma}(x)-\beta^{(2)}_{n+1,\gamma}A^S_{n-1,\gamma}(x).\end{aligned}$$

Due to [16, Lemma 4.1.] and [28, Lemma 2.5.] we know that $\beta^{(j)}_{k,\gamma}=0$, $j=1,2$, for each $k\in\mathbb{N}$, so $\beta^{(2)}_{n+1,\gamma}$ is equal to zero. Now, because of the form $A^C_{k,\gamma}(x)=\sum_{j=0}^{k}u^{(k)}_{j,\gamma}\cos(j+\gamma)x$, with $u^{(k)}_{k,\gamma}=1$, we have:

$$B^C_{n+1,\gamma}(x)=A^C_{n+1,\gamma}(x)-\alpha^{(2)}_{n+1,\gamma}A^C_{n-1,\gamma}(x)=\sum_{k=0}^{n+1}p^{(n+1)}_{k,\gamma}\cos(k+\gamma)x,$$

with $p^{(n+1)}_{n+1,\gamma}=1$. Similarly, using (14), [16, Lemma 4.1] and [28, Lemma 2.5.], we get:

$$B^S_{n+1,\gamma}(x)=A^S_{n+1,\gamma}(x)-\delta^{(2)}_{n+1,\gamma}A^S_{n-1,\gamma}(x)=\sum_{k=0}^{n+1}s^{(n+1)}_{k,\gamma}\sin(k+\gamma)x,$$

with $s^{(n+1)}_{n+1,\gamma}=1$.

Also, polynomials $B^C_{n+1,\gamma}$ and $B^S_{n+1,\gamma}$ satisfy the following three-term recurrence relations:

$$B^C_{n+1,\gamma}(x)=\left(2\cos x-a^{(1)}_{n+1,\gamma}\right)B^C_{n,\gamma}(x)-a^{(2)}_{n+1,\gamma}B^C_{n-1,\gamma}(x),\tag{15}$$

$$B^S_{n+1,\gamma}(x)=\left(2\cos x-d^{(1)}_{n+1,\gamma}\right)B^S_{n,\gamma}(x)-d^{(2)}_{n+1,\gamma}B^S_{n-1,\gamma}(x).\tag{16}$$

The following result is obvious.

Lemma 3.1. *For every $k\in\mathbb{N}_0$ we have*

$$B^C_{k+1,\gamma}(\pi)=0,\quad B^S_{k+1,\gamma}(0)=0.$$

3.1. Quadrature rule with an even number of nodes

Let positive integer n be fixed in advance. For $\gamma=0$ we get a quadrature formula with an even number of nodes.

Let us introduce the following notation for $k=0,1,\dots$:

$$C_{k,0}(x)=\sum_{j=0}^{k}p^{(k)}_{j,0}T_j(x),\quad S_{k,0}(x)=\sum_{j=0}^{k}s^{(k)}_{j,0}U_{j-1}(x),$$

$$u_1(x)=\frac{w(\arccos x)}{\sqrt{1-x^2}},\qquad u_2(x)=\sqrt{1-x^2}\,w(\arccos x),$$

where $T_k(x)=\cos(k\cdot\arccos x)$ and $U_k(x)=\sin((k+1)\arccos x)/\sqrt{1-x^2}$, $x\in(-1,1)$, are Chebyshev polynomials of the first and second kind, respectively. Also, by $\tau^{(i)}_k$ and $\sigma^{(i)}_k$, $k=1,\dots,n$, we denote the nodes and weights of the Gaussian quadrature rule constructed for the algebraic polynomials with respect to the weight function $u_i(x)$, $i=1,2$.

The connections of quadrature rule (3) with certain Gaussian quadrature rule for algebraic polynomials in the case $\gamma=0$ were considered in [27]. Using these connections we get the following results.

Theorem 3.2. *Let positive integer n be fixed in advance. For an even weight function $w(x)$ on $(-\pi,\pi)$, the following equalities*

$$\left(C_{n,0}(x), C_{k,0}(x)\right)_{u_1} = 0 \quad \text{and} \quad \left(S_{n,0}(x), S_{k,0}(x)\right)_{u_2} = 0 \tag{17}$$

hold for all $k = 0,1,\dots,n-1$.

Proof. Using [27, Lemma 2.4.], orthogonality of the polynomials $B^C_{k,0}(x)$ and substitution $x = \arccos t$, for each $k = 0,\dots,n-1$, we have

$$\begin{aligned}
0 = &\left(B^C_{n,0}(x), B^C_{k,0}(x)\right)_w = \left(2\widehat{I} - \widehat{G}_{2n}\right)\left(B^C_{n,0}(x)B^C_{k,0}(x)\right) = 4\int_0^\pi B^C_{n,0}(x)B^C_{k,0}(x)w(x)\,\mathrm{d}x - \sum_{j=0}^{2n-1}\omega_j B^C_{n,0}(x_j)B^C_{k,0}(x_j)\\
=&4\int_{-1}^1 B^C_{n,0}(\arccos t)B^C_{k,0}(\arccos t)\frac{w(\arccos t)}{\sqrt{1-t^2}}\,\mathrm{d}t - 2\sum_{j=1}^n \sigma_j^{(1)} B^C_{n,0}\left(\arccos\tau_j^{(1)}\right)B^C_{k,0}\left(\arccos\tau_j^{(1)}\right)\\
=&2\left(2I - G_n\right)\left(B^C_{n,0}(\arccos x)B^C_{k,0}(\arccos x)\right) = 2\left(B^C_{n,0}(\arccos x), B^C_{k,0}(\arccos x)\right)_{u_1}.
\end{aligned}$$

Now, using $B^C_{k,0}(\arccos x) = \sum\limits_{j=0}^k p^{(k)}_{j,0}\cos(j\arccos x) = \sum\limits_{j=0}^k p^{(k)}_{j,0}T_j(x)$, for all $k = 0,\dots,n-1$ we get:

$$\left(B^C_{n,0}(\arccos x), B^C_{k,0}(\arccos x)\right)_{u_1} = \left(\sum_{j=0}^n p^{(n)}_{j,0}T_j(x), \sum_{j=0}^k p^{(k)}_{j,0}T_j(x)\right)_{u_1} = \left(C_{n,0}(x), C_{k,0}(x)\right)_{u_1},$$

i.e., $\left(C_n(x), C_k(x)\right)_{u_1} = 0$.
The second equality can be proved in the similar way, using [27, Lemma 2.5.]:

$$\begin{aligned}
0 = &\left(B^S_{n,0}(x), B^S_{k,0}(x)\right)_w = \left(2\widehat{I} - \widehat{G}_{2n}\right)\left(B^S_{n,0}(x)B^S_{k,0}(x)\right) = 4\int_0^\pi B^S_{n,0}(x)B^S_{k,0}(x)w(x)\,\mathrm{d}x - \sum_{j=0}^{2n-1}\omega_j B^S_{n,0}(x_j)B^S_{k,0}(x_j)\\
=&2\cdot\left[2\int_{-1}^1 \frac{B^S_{n,0}(\arccos t)}{\sqrt{1-t^2}}\cdot\frac{B^S_{k,0}(\arccos t)}{\sqrt{1-t^2}}\,u_2(t)\,\mathrm{d}t - \sum_{j=1}^n \sigma_j^{(2)}\frac{B^S_{n,0}\left(\arccos\tau_j^{(2)}\right)}{\sqrt{1-\left(\tau_j^{(2)}\right)^2}}\cdot\frac{B^S_{k,0}\left(\arccos\tau_j^{(2)}\right)}{\sqrt{1-\left(\tau_j^{(2)}\right)^2}}\right]\\
=&2\cdot\left(2I - G_n\right)\left(\frac{B^S_{n,0}(\arccos x)}{\sqrt{1-x^2}}\cdot\frac{B^S_{k,0}(\arccos x)}{\sqrt{1-x^2}}\right) = 2\left(\frac{B^S_{n,0}(\arccos x)}{\sqrt{1-x^2}}, \frac{B^S_{k,0}(\arccos x)}{\sqrt{1-x^2}}\right)_{u_2}.
\end{aligned}$$

Now, using

$$B^S_{k,0}(\arccos x) = \sum_{j=0}^k s^{(k)}_{j,0}\sin(j\arccos x) = \sum_{j=0}^k s^{(k)}_{j,0}\sqrt{1-x^2}\,U_{k-1}(x),$$

i.e.,

$$\frac{B^S_{k,0}(\arccos x)}{\sqrt{1-x^2}} = \sum_{j=0}^k s^{(k)}_{j,0}U_{j-1}(x),$$

for all $k = 0,\dots,n-1$ we get:

$$\left(\frac{B^S_{n,0}(\arccos x)}{\sqrt{1-x^2}}, \frac{B^S_{k,0}(\arccos x)}{\sqrt{1-x^2}}\right)_{u_2} = \left(\sum_{j=0}^n s^{(n)}_{j,0}U_{j-1}(x), \sum_{j=0}^k s^{(k)}_{j,0}U_{j-1}(x)\right)_{u_2} = \left(S_{n,0}(x), S_{k,0}(x)\right)_{u_2},$$

i.e., $\left(S_n(x), S_k(x)\right)_{u_2} = 0$. □

If we put $x := \arccos x$ in (15), using equality $B^C_{n,0}(\arccos x) = C_{n,0}(x)$, we get:

$$C_{n,0}(x) = \left(2x - a^{(1)}_{n,0}\right) C_{n-1,0}(x) - a^{(2)}_{n,0} C_{n-2,0}(x), \quad a^{(2)}_{1,0} = 0, \quad C_{0,0}(x) = 1.$$

Therefore, the zeros of the polynomial $C_{n+1,0}(x)$ (and hence the zeros of the trigonometric polynomial $B^C_{n+1,0}(x)$) can be calculated using QR-algorithm.

Lemma 3.3. *Let w be the even weight function on $(-\pi, \pi)$ and let τ_k and σ_k, $k = 1, \ldots, n+1$, be the nodes and weights of the $(n + 1)$-point anti-Gaussian quadrature rule constructed for algebraic polynomials with respect to the weight function $u_1(x) = w(\arccos x)/\sqrt{1 - x^2}$ on $(-1, 1)$. Then the weights ω_k and the nodes x_k, $k = 0, 1, \ldots, 2n + 1$, of the $(2n + 2)$-point anti-Gaussian quadrature rule with respect to w are given as follows:*

$$\omega_k = \omega_{2n+1-k} = \sigma_{k+1}, \quad k = 0, \ldots, n,$$
$$x_k = -x_{2n+1-k} = -\arccos \tau_{k+1}, \quad k = 0, \ldots, n.$$

Proof. Due to the equality $B^C_{n+1,0}(\arccos x) = C_{n+1,0}(x)$ it is easy to see that the nodes of the anti-Gaussian quadrature rule for trigonometric polynomials are given by

$$x_k = -x_{2n+1-k} = -\arccos \tau_{k+1}, \quad k = 0, \ldots, n.$$

Weights can be constructed using Shohat formula (see [16, 21]):

$$\sigma_k = \mu_0 \left[\sum_{j=0}^{n} \left(\frac{C_{j,0}(\tau_k)}{\prod_{i=2}^{j} a^{(2)}_{i,0}} \right)^2 \right]^{-1} = \frac{\mu_0}{\sum_{j=0}^{n} \left(\frac{B^C_{j,0}(\arccos \tau_k)}{\prod_{i=2}^{j} a^{(2)}_{i,0}} \right)^2}, \quad k = 1, \ldots, n+1,$$

where

$$\mu_0 = \int_{-1}^{1} \frac{w(\arccos x)}{\sqrt{1 - x^2}} \, \mathrm{d}x.$$

Due to the fact that the function $w(x)$ is even on $(-\pi, \pi)$, we have

$$\omega_{2n+1-k} = \frac{\widehat{\mu_0}}{2 \cdot \sum_{j=0}^{n} \left(\frac{B^C_{j,0}(x_{2n+1-k})}{\prod_{i=2}^{j} a^{(2)}_{i,0}} \right)^2}, \quad k = 0, \ldots, n,$$

where

$$\widehat{\mu_0} = \int_{-\pi}^{\pi} w(x) \, \mathrm{d}x = 2 \int_{-1}^{1} w(\arccos t) \frac{\mathrm{d}t}{\sqrt{1 - t^2}} = 2\mu_0.$$

Therefore we get $\omega_{2n+1-k} = \omega_k = \sigma_{k+1}$, for $k = 0, \ldots, n$. □

The following result could be proved by using the similar arguments.

Lemma 3.4. *Let w be the even weight function on $(-\pi, \pi)$ and let τ_k and σ_k, $k = 1, \ldots, n+1$, be the nodes and weights of the $(n + 1)$-point anti-Gaussian quadrature rule constructed for algebraic polynomials with respect to the weight function $u_2(x) = \sqrt{1 - x^2}\, w(\arccos x)$ on $(-1, 1)$. Then the weights ω_k and the nodes x_k, $k = 0, 1, \ldots, 2n + 1$, of the $(2n + 2)$-point anti-Gaussian quadrature rule with respect to w are given as follows:*

$$\omega_k = \omega_{2n+1-k} = \frac{\sigma_{k+1}}{1 - \tau^2_{k+1}}, \quad k = 0, \ldots, n,$$
$$x_k = -x_{2n+1-k} = -\arccos \tau_{k+1}, \quad k = 0, \ldots, n.$$

3.2. Quadrature rule with an odd number of nodes

In the case $\gamma = 1/2$, we get quadrature rule with an odd number of nodes.

Let us introduce the following notation for $k = 0, 1, \ldots$:

$$C_{k,1/2}(x) = \sum_{j=0}^{k} p_{j,1/2}^{(k)}\Big(T_j(x) - (1-x)U_{j-1}(x)\Big), \quad u_3(x) = \sqrt{\frac{1+x}{1-x}}\, w(\arccos x),$$
$$S_{k,1/2}(x) = \sum_{j=0}^{k} s_{j,1/2}^{(k)}\Big(T_j(x) + (1+x)U_{j-1}(x)\Big), \quad u_4(x) = \sqrt{\frac{1-x}{1+x}}\, w(\arccos x).$$

Also, by $\tau_k^{(i)}$ and $\sigma_k^{(i)}$, $k = 1, \ldots, n$, we denote the nodes and weights of the Gaussian quadrature rule constructed for algebraic polynomials with respect to the weight function $u_i(x)$, $i = 3, 4$.

Theorem 3.5. *Let positive integer n be fixed in advance. For an even weight function $w(x)$ on $(-\pi, \pi)$, the following equalities*

$$(C_{n,1/2}(x), C_{k,1/2}(x))_{u_3} = 0 \quad \text{and} \quad (S_{n,1/2}(x), S_{k,1/2}(x))_{u_4} = 0$$

hold for all $k = 0, 1, \ldots, n-1$.

Proof. Using the following simple trigonometric equality:

$$\cos\left(\left(k + \frac{1}{2}\right)\arccos x\right) = T_k(x)\sqrt{\frac{1+x}{2}} - \sqrt{1-x^2}\cdot U_{k-1}(x)\sqrt{\frac{1-x}{2}},$$

we have

$$B_{k,1/2}^{C}(\arccos x) = \sum_{j=0}^{k} p_{j,1/2}^{(k)} \cos\left(j + \frac{1}{2}\right)\arccos x = \sqrt{\frac{1+x}{2}}\, C_{k,1/2}(x), \tag{18}$$

i.e., $B_{k,1/2}^{C}(\arccos x)/\sqrt{1+x} = C_{k,1/2}(x)/\sqrt{2}$. Using these equalities, orthogonality of the polynomials $B_{k,1/2}^{C}(x)$, Lemma 3.1, and [16, Lemma 5.2], for every $k = 0, \ldots, n-1$, we get:

$$\begin{aligned}
0 =& \left(B_{n,1/2}^{C}(x), B_{k,1/2}^{C}(x)\right)_w = \left(\widehat{2I} - \widehat{G}_{2n+1}\right)\left(B_{n,1/2}^{C}(x)\cdot B_{k,1/2}^{C}(x)\right)\\
=&4\int_0^{\pi} B_{n,1/2}^{C}(x) B_{k,1/2}^{C}(x) w(x)\,\mathrm{d}x - \sum_{j=0}^{2n} \omega_j B_{n,1/2}^{C}(x_j) B_{k,1/2}^{C}(x_j)\\
=&4\int_{-1}^{1} \frac{B_{n,1/2}^{C}(\arccos t)}{\sqrt{1+t}}\cdot\frac{B_{k,1/2}^{C}(\arccos t)}{\sqrt{1+t}}\sqrt{\frac{1+t}{1-t}}\, w(\arccos t)\,\mathrm{d}t\\
&- 2\sum_{j=1}^{n} \sigma_j^{(3)} \frac{B_{n,1/2}^{C}\left(\arccos \tau_j^{(3)}\right)}{\sqrt{1+\tau_j^{(3)}}}\cdot\frac{B_{k,1/2}^{C}\left(\arccos \tau_j^{(3)}\right)}{\sqrt{1+\tau_j^{(3)}}}\\
=&2\int_{-1}^{1} C_{n,1/2}(t) C_{k,1/2}(t) u_3(t)\,\mathrm{d}t - \sum_{j=1}^{n} \sigma_j^{(3)} C_{n,1/2}(\tau_j^{(3)}) C_{k,1/2}(\tau_j^{(3)})\\
=& (2I - G_n)\left(C_{n,1/2}(x) C_{k,1/2}(x)\right) = \left(C_{n,1/2}(x), C_{k,1/2}(x)\right)_{u_3},
\end{aligned}$$

i.e., $(C_{n,1/2}(x), C_{k,1/2}(x))_{u_3} = 0$.

For the second part, we use the following trigonometric equality

$$\sin\left(\left(k + \frac{1}{2}\right)\arccos x\right) = \sqrt{1-x^2}\, U_{k-1}(x)\sqrt{\frac{1+x}{2}} + T_k(x)\sqrt{\frac{1-x}{2}},$$

and get

$$B^S_{k,1/2}(\arccos x) = \sum_{j=0}^{k} s^{(k)}_{j,1/2} \sin\left(\left(j+\frac{1}{2}\right)\arccos x\right) = \sqrt{\frac{1-x}{2}}\, S_{n,1/2}(x), \tag{19}$$

i.e., $B^S_{k,1/2}(\arccos x)/\sqrt{1-x} = S_{k,1/2}(x)/\sqrt{2}$. Now, using these equalities, the orthogonality of the polynomials $B^S_{k,1/2}(x)$, Lemma 3.1, and [16, Lemma 5.3], for every $k = 0,\dots,n-1$, we get:

$$0 = \left(B^S_{n,1/2}(x), B^S_{k,1/2}(x)\right)_w = (S_{n,1/2}(x), S_{k,1/2}(x))_{u_4}.$$

□

If we put $x := \arccos x$ in (15), using equality (18), and dividing both sides by $\sqrt{(1+x)/2}$, we get

$$C_{n+1,1/2}(x) = \left(2x - a^{(1)}_{n+1,1/2}\right) C_{n,1/2}(x) - a^{(2)}_{n+1,1/2} C_{n-1,1/2}(x), \quad a^{(2)}_{1,1/2} = 0,\ C_{0,1/2}(x) = 1. \tag{20}$$

Lemma 3.6. *Let w be the even weight function on $(-\pi,\pi)$ and let τ_k and σ_k, $k = 1,\dots,n+1$, be the nodes and weights of the $(n+1)$-point anti-Gaussian quadrature rule constructed for algebraic polynomials with respect to the weight function $u_3(x) = w(\arccos x)\sqrt{1+x}/\sqrt{1-x}$ on $(-1,1)$. Then the weights ω_k and the nodes x_k, $k = 0,1,\dots,2n+2$, of the $(2n+3)$-point anti-Gaussian quadrature rule with respect to w are given as follows:*

$$\omega_k = \omega_{2n+2-k-1} = \frac{\sigma_{k+1}}{1+\tau_{k+1}}, \quad k = 0,\dots,n, \quad \omega_{2n+2} = \int_{-\pi}^{\pi} w(x)\,\mathrm{d}x - \sum_{k=0}^{2n+1}\omega_k,$$
$$x_k = -x_{2n+2-k-1} = -\arccos\tau_{k+1}, \quad k = 0,\dots,n, \quad x_{2n+2} = \pi.$$

Proof. Using the three-term recurrence relation (20), we can construct the sequence of the polynomials $(C_{k,1/2}(x))$, and using QR-algorithm we can obtain the zeros of the polynomial $C_{n+1,1/2}(x)$, i.e., nodes τ_k, $k = 1,\dots,n+1$. According to (18) it is easy to see that $x_k = -x_{2n+2-k-1} = -\arccos\tau_{k+1}$, $k = 0,\dots,n$. Due to Lemma 3.1 we have $x_{2n+2} = \pi$.

The weights can be constructed using Shohat formula (see [16, 21]):

$$\sigma_k = \mu_0\left[\sum_{j=0}^{n}\left(\frac{C_{j,1/2}(\tau_k)}{\prod_{i=2}^{j} a^{(2)}_{i,1/2}}\right)^2\right]^{-1} = \frac{\mu_0(1+\tau_k)}{2\sum_{j=0}^{n}\left(\frac{B^C_{j,1/2}(x_{2n+2-k})}{\prod_{i=2}^{j} a^{(2)}_{i,1/2}}\right)^2}, \quad k = 1,\dots,n+1,$$

where

$$\mu_0 = \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\, w(\arccos x)\,\mathrm{d}x.$$

Since the function $w(x)$ is even on $(-\pi,\pi)$, we have

$$\omega_{2n+2-k-1} = \frac{\widehat{\mu}_0}{2\sum_{j=0}^{n}\left(\frac{B^C_{j,1/2}(x_{2n+2-k-1})}{\prod_{i=2}^{j} a^{(2)}_{i,1/2}}\right)^2}, \quad k = 0,\dots,n,$$

where

$$\widehat{\mu}_0 = \int_{-\pi}^{\pi}\cos^2\frac{x}{2}\, w(x)\,\mathrm{d}x = 2\int_{-1}^{1}\frac{1+t}{2}\, w(\arccos t)\frac{\mathrm{d}t}{\sqrt{1-t^2}} = \mu_0,$$

so we obtain:

$$\omega_{2n+2-k-1} = \omega_k = \frac{\sigma_{k+1}}{1+\tau_{k+1}}, \quad k = 0, \ldots, n.$$

Finally, using equality $\sum\limits_{k=0}^{2n+2} \omega_k = \int\limits_{-\pi}^{\pi} w(x)\,\mathrm{d}x$, we get the last weight

$$\omega_{2n+2} = \int_{-\pi}^{\pi} w(x)\,\mathrm{d}x - \sum_{k=0}^{2n+1} \omega_k.$$

□

Lemma 3.7. *Let w be the even weight function on $(-\pi,\pi)$ and let τ_k and σ_k, $k = 1,\ldots,n+1$, be the nodes and weights of the $(n+1)$-point anti-Gaussian quadrature rule constructed for algebraic polynomials with respect to the weight function $u_4(x) = w(\arccos x)\sqrt{1-x}/\sqrt{1+x}$ on $(-1,1)$. Then the weights ω_k and the nodes x_k, $k = 0,1,\ldots,2n+2$, of the $(2n+3)$-point anti-Gaussian quadrature rule with respect to w are given as follows:*

$$\omega_k = \omega_{2n+2-k} = \frac{\sigma_{k+1}}{1-\tau_{k+1}}, \quad k = 0,\ldots,n, \quad \omega_{n+1} = \int_{-\pi}^{\pi} w(x)\,\mathrm{d}x - \sum_{\substack{k=0\\k\neq n+1}}^{2n+2} \omega_k,$$

$$x_k = -x_{2n+2-k} = -\arccos\tau_{k+1}, \quad k = 0,\ldots,n, \quad x_{n+1} = 0.$$

Proof. This lemma can be proved in the same way as the previous one, using polynomials $B^S_{j,1/2}(x)$, relation (19) and $\widehat{\mu}_0 = \int\limits_{-\pi}^{\pi} \sin^2\frac{x}{2}\, w(x)\,\mathrm{d}x$. □

Finally, using Gaussian ($\widehat{G}_{\widetilde{n}+1}(f)$) and anti-Gaussian ($\widehat{H}_{\widetilde{n}+3}(f)$) quadrature rule, we can introduce the so-called *averaged Gaussian rule* for trigonometric polynomials:

$$\widehat{A}_{2\widetilde{n}+4}(f) = \frac{1}{2}(\widehat{G}_{\widetilde{n}+1} + \widehat{H}_{\widetilde{n}+3})(f).$$

4. Numerical examples

4.1. Quadrature formulas with an even number of nodes

The errors in the Gaussian ($\widehat{G}_{\widetilde{n}+1}$), anti-Gaussian ($\widehat{H}_{\widetilde{n}+3}$) and averaged ($\widehat{A}_{2\widetilde{n}+4}$) quadrature rules in the case

$$w(x) = 1 - \cos^2 x, \text{ for } x \in (-\pi,\pi) \text{ and}$$
$$u_1 = \frac{w(\arccos x)}{\sqrt{1-x^2}} = \sqrt{1-x^2}, \text{ for } x \in (-1,1)$$

for the integrand $f(x) = (1+\cos x)(\mathrm{e}^{-x} + 4/3)$ are given in Table 1.

Table 1: Errors in the Gaussian ($\widehat{G_{\widetilde{n}+1}}$), anti-Gaussian ($\widehat{H_{\widetilde{n}+3}}$) and averaged ($\widehat{A_{2\widetilde{n}+4}}$) formula for $w(x) = 1 - \cos^2 x$, $x \in (-\pi, \pi)$ and $f(x) = (1 + \cos x)(e^{-x} + 4/3)$

$\widetilde{n}+1$	$\widehat{I(f)} - \widehat{G_{\widetilde{n}+1}(f)}$	$\widehat{I(f)} - \widehat{H_{\widetilde{n}+3}(f)}$	$\widehat{I(f)} - \widehat{A_{2\widetilde{n}+4}(f)}$
80	$-9.30463 \cdot 10^{-9}$	$8.99386 \cdot 10^{-9}$	$-1.55389 \cdot 10^{-10}$
60	$-4.97942 \cdot 10^{-8}$	$4.82213 \cdot 10^{-8}$	$-7.86464 \cdot 10^{-10}$
40	$-5.16734 \cdot 10^{-7}$	$5.00653 \cdot 10^{-7}$	$-8.04024 \cdot 10^{-9}$
20	-0.0000254069	0.0000246255	$-3.90685 \cdot 10^{-7}$

4.2. Quadrature formulas with odd number of nodes

Errors in the Gaussian ($\widehat{G_{\widetilde{n}+1}}$), anti-Gaussian ($\widehat{H_{\widetilde{n}+3}}$) and averaged ($\widehat{A_{2\widetilde{n}+4}}$) quadrature rules in the case

$$w(x) = 1 + \cos x, \text{ for } x \in (-\pi, \pi) \text{ and}$$

$$u_4(x) = w(\arccos x)\sqrt{\frac{1-x}{1+x}} = \sqrt{1-x^2}, \text{ for } x \in (-1, 1)$$

for the integrand $f(x) = (1 + \cos x)(e^{-x} + 4/3)$ are given in Table 2.

Table 2: Errors in the Gaussian ($\widehat{G_{\widetilde{n}+1}}$), anti-Gaussian ($\widehat{H_{\widetilde{n}+3}}$) and averaged ($\widehat{A_{2\widetilde{n}+4}}$) formula for $w(x) = 1 + \cos x$, $x \in (-\pi, \pi)$ and $f(x) = (1 + \cos x)(e^{-x} + 4/3)$

$\widetilde{n}+1$	$\widehat{I(f)} - \widehat{G_{\widetilde{n}+1}(f)}$	$\widehat{I(f)} - \widehat{H_{\widetilde{n}+3}(f)}$	$\widehat{I(f)} - \widehat{A_{2\widetilde{n}+4}(f)}$
81	$-4.63804 \cdot 10^{-9}$	$4.49229 \cdot 10^{-9}$	$-7.28786 \cdot 10^{-11}$
61	$-2.48222 \cdot 10^{-8}$	$2.40457 \cdot 10^{-8}$	$-3.88281 \cdot 10^{-10}$
41	$-2.56852 \cdot 10^{-7}$	$2.48826 \cdot 10^{-7}$	$-4.01318 \cdot 10^{-9}$
21	-0.0000124339	0.0000120453	$-1.94297 \cdot 10^{-7}$

Finally, we compare our method with the other methods.

The nodes of the generalized averaged Szegő quadrature rule are the eigenvalues of $(2n-2) \times (2n-2)$ unitary upper Hessenberg matrix $\check{H}_{2n-2}(\tau)$, determined by parameter τ from unit circle and the so-called Schur parameters $\gamma_1, \ldots, \gamma_{n-1}$. Weights are determined by the square of the first component of associated unit eigenvectors (see [12, 13]). Matrix $\check{H}_{2n-2}(\tau)$ is given by $\check{H}_{2n-2}(\tau) = \widehat{D}_{2n-2}^{-1/2}\widehat{H}_{2n-2}(\tau)\widehat{D}_{2n-2}^{1/2}$, where

$$\widehat{H}_{2n-2}(\tau) = \begin{bmatrix} -\bar{\gamma}_0\gamma_1 & -\bar{\gamma}_0\gamma_2 & \cdots & -\bar{\gamma}_0\gamma_{n-1} & -\bar{\gamma}_0\gamma_{n-2} & \cdots & -\bar{\gamma}_0\gamma_1 & -\bar{\gamma}_0\tau \\ 1-|\gamma_1|^2 & -\bar{\gamma}_1\gamma_2 & \cdots & -\bar{\gamma}_1\gamma_{n-1} & -\bar{\gamma}_1\gamma_{n-2} & \cdots & -\bar{\gamma}_1\gamma_1 & -\bar{\gamma}_1\tau \\ 0 & 1-|\gamma_2|^2 & \cdots & -\bar{\gamma}_2\gamma_{n-1} & -\bar{\gamma}_2\gamma_{n-2} & \cdots & -\bar{\gamma}_2\gamma_1 & -\bar{\gamma}_2\tau \\ \vdots & & & & & & & \\ 0 & 0 & & \cdots & & 0 & 1-|\gamma_1|^2 & -\bar{\gamma}_1\tau \end{bmatrix},$$

$\gamma_0 = 1$, $\widehat{D}_{2n-2} = \mathrm{diag}[\widehat{\delta_0}, \widehat{\delta_1}, \ldots, \widehat{\delta_{2n-3}}]$, $\widehat{\delta_0} = 1$, $\widehat{\delta_j} = \widehat{\delta_{j-1}}\left(1 - |\widehat{\gamma_j}|^2\right)$, $j = 1, \ldots, 2n-3$, $\widehat{\gamma_j} = \gamma_j$, $j = 1, \ldots, n-1$ and $\widehat{\gamma_j} = \gamma_{2n-2-j}$, $j = n, \ldots, 2n-3$. There are several algorithms for the eigen decomposition of such kind of matrices (see [9–11]). Computation of eigensystem can be performed very efficiently by using compact representation of Hessenberg matrices (see e.g., [9, 11]).

In our method we use recurrence relations to obtain wanted orthogonal systems in order to escape numerical non-stability which is characteristic for Gram-Schmidt method. Also, recurrence relations provide a stable way for computation of values of trigonometric polynomials in contrast to using expanded forms. We demonstrated how in the case of symmetric weight function the anti-Gaussian quadrature rules can be constructed using orthogonal polynomials on the real line.

Using averaged Gaussian quadrature rules for trigonometric polynomials introduced in this article, we were able to achieve much greater accuracy in comparison with averaged Szegő quadrature rules on the class of symmetric weight functions, which we will demonstrate with the following example. Also, introducing the anti-Gaussian quadrature rules with an odd number of nodes (the case $\gamma = 1/2$) we achieved accuracy for the trigonometric polynomials of higher degree, for all $t \in \mathcal{T}_{2n+2}$. We give one example from [12] supplemented by our results.

Consider the weight function $w(x) = 2\sin^2(x/2)$ on the interval $(-\pi, \pi)$ and the integrand $f(x) = \frac{1}{2}\log(5 + 4\cos x)$. In Table 3 we give errors in Szegő ($S_1^n(f)$), anti-Szegő ($A_1^n(f)$), generalized averaged Szegő ($\widehat{S}_1^{(2n-2)}(f)$); Gaussian ($\widehat{G}_{\widetilde{n}+1}(f)$), anti-Gaussian ($\widehat{H}_{\widetilde{n}+3}(f)$) and averaged Gaussian ($\widehat{A}_{2\widetilde{n}+4}$) formula, with $\widetilde{n} + 1 = n$. One can see that errors obtained by using Gaussian and Szegő are similar with those obtained by using anti-Gaussian and anti-Szegő quadrature rules, respectively. However, it is obvious that averaged Gaussian quadrature rules for trigonometric polynomials give a significant improvement over the generalized averaged Szegő quadrature rules (which gave the best results in [12]). That improvement is more significant with increasing number of nodes. Also, we can see that in this example quadrature rules with an even number of nodes lead to a greater improvement of results than rules with an odd number of nodes.

Table 3: Errors for several quadrature rules for $w(x) = 2\sin^2(x/2)$, $x \in (-\pi, \pi)$ and $f(x) = \frac{1}{2}\log(5 + 4\cos x)$

Rule	$n = 12$	$n = 15$	$n = 18$
$S_1^n(f)$	$-2.2 \cdot 10^{-5}$	$2.2 \cdot 10^{-6}$	$-2.3 \cdot 10^{-7}$
$A_1^n(f)$	$2.3 \cdot 10^{-5}$	$-2.3 \cdot 10^{-6}$	$2.4 \cdot 10^{-7}$
$\widehat{S}_1^{(2n-2)}(f)$	$-1.5 \cdot 10^{-7}$	$9.2 \cdot 10^{-9}$	$-6.7 \cdot 10^{-10}$
$\widehat{G}_{\widetilde{n}+1}(f)$	$-1.98 \cdot 10^{-5}$	$1.38 \cdot 10^{-5}$	$-2 \cdot 10^{-7}$
$\widehat{H}_{\widetilde{n}+3}(f)$	$1.98 \cdot 10^{-5}$	$-1.38 \cdot 10^{-5}$	$2 \cdot 10^{-7}$
$\widehat{A}_{2\widetilde{n}+4}$	$-5.31 \cdot 10^{-10}$	$1.04 \cdot 10^{-10}$	$-8.75 \cdot 10^{-14}$

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