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On the Stability of Multicubic-Quartic and Multimixed Cubic-Quartic Mappings

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Abstract. In this paper, we define the multicubic-quartic and the multimixed cubic-quartic mappings and characterize them. In other words, we unify the system of functional equations defining a multimixed cubic-quartic (resp., multicubic-quartic) mapping to a single equation, namely, the multimixed cubic-quartic (resp., multicubic-quartic) functional equation. We also show that under what conditions a multimixed cubic-quartic mapping can be multicubic, multiquartic and multicubic-quartic. Moreover, by using a fixed point theorem, we study the generalized Hyers-Ulam stability of multimixed cubic-quartic functional equations in non-Archimedean normed spaces. As a corollary, we show that every multimixed cubic-quartic mapping under some mild conditions can be hyperstable. Lastly, we present a non-stable example for the multiquartic mappings.

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [31] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [17]. Later on, various generalizations and extension of Hyers' result were ascertained by Th. M. Rassias [27], Aoki [1] and J. M. Rassias [26] in different versions. Next, several stability problems of various functional equations have been investigated by many mathematicians (see the mentioned papers as the above for the comprehensive accounts of the subject), but mainly in classical spaces; for instance refer to [10], [11], [29] and [30].

In two last decades, the stability problem has been studied by authors for several variables mappings such as multiadditive, multiquadratic, multicubic and multiquartic mappings. In what follows, we review them as given in the lecturers. Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall that a mapping $f : V^n \longrightarrow W$ is called *multiadditive* if it is additive (satisfies Cauchy's functional equation $A(x + y) = A(x) + A(y)$) in each variable. Some facts on such mappings can be found in [22] and

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many other sources. Ciepliński in [9] showed that f is multiadditive if and only if it satisfies the equation

$$f(x_1+x_2)=\sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{1j_1},x_{2j_2},\ldots,x_{nj_n}),$$

where $x_j=(x_{1j},x_{2j},\ldots,x_{nj})\in V^n$ with $j\in\{1,2\}$. In addition, f is said to be *multiquadratic* if it is quadratic (satisfies quadratic functional equation $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$) in each variable [23]. Zhao et al., [32] showed that the mentioned mapping f is multiquadratic if and only if the equation

$$\sum_{t\in\{-1,1\}^n} f(x_1+tx_2)=2^n\sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{1j_1},x_{2j_2},\ldots,x_{nj_n}),$$

holds for all $x_j=(x_{1j},x_{2j},\ldots,x_{nj})\in V^n$, where $j\in\{1,2\}$. For the generalization of multiquadratic mappings which was recently studied, we refer to [3].

Recall from [6] that a mapping $f:V^n\longrightarrow W$ is also called *multicubic* if it is cubic in each of variable [19], namely, it satisfies the equation

$$C(2x+y)+C(2x-y)=2C(x+y)+2C(x-y)+12C(x), \tag{1}$$

in all variables. A general system of cubic functional equations is available in [14]. In [6], the authors unified the system of functional equations defining a multicubic mapping to a single equation, as multicubic functional equation. We mention that some different forms of the cubic functional equations can be found in [18] and [25].

The quartic functional equation

$$\mathfrak{Q}(x+2y)+\mathfrak{Q}(x-2y)=4\mathfrak{Q}(x+y)+4\mathfrak{Q}(x-y)-6\mathfrak{Q}(x)+24\mathfrak{Q}(y) \tag{2}$$

was introduced for the first time by Rassias [24]. The functional equation (2) was generalized by Bodaghi and Kang in [4] and [20], respectively. Motivated by equation (2), Bodaghi et al. [5] defined multiquartic mappings and provided a characterization of such mappings. In other words, they showed that every multiquartic mapping can be described as a single functional equation and vice versa.

In [13], Eshaghi Gordji et al., introduced and obtained the general solution of the mixed type cubic and quartic functional equation

$$f(x+2y)+f(x-2y)-3f(2y)=4f(x+y)+4f(x-y)-6f(x)-24f(y). \tag{3}$$

They also established the Hyers-Ulam-Rassias stability of the above functional equation in the setting quasi-Banach spaces. The stability of (3) in non-Archimedean normed spaces was studied in [12]. It is easily seen that the function $f(x)=\alpha x^4+\beta x^3$ is a solution of equation (3). Here, we remember that the characterization and the stability of multimixed additive-quadratic mappings in Banach spaces was studied recently in [28].

In this article, we first define the multicubic-quartic mappings and include a characterization of such mappings. In fact, we prove that every multicubic-quartic mapping can be shown as a single functional equation and vice versa (under some extra conditions). Moreover, motivated by equation (3), we introduce the multimixed cubic-quartic mappings and reduce the system of n equations defining the multimixed cubicquartic mappings to a single functional equation. We also investigate the generalized Hyers-Ulam stability for multimixed cubic-quartic mappings by applying the fixed point method in non-Archimedean normed spaces which has been presented in [8]; for more applications of this approach for the stability of multi-Cauchy-Jensen and multiadditive-quadratic mappings see [2]. Eventually, an appropriate counterexample is supplied to invalidate the results in the case of singularity for multiquartic functions.

2. Characterization of multicubic-quartic mappings

Throughout this paper, $\mathbb{N}$ and $\mathbb{Q}$ stand for the set of all positive integers and the rational numbers, respectively, $\mathbb{N}_0:=\mathbb{N}\cup\{0\},\mathbb{R}_+:=[0,\infty)$. For any $l\in\mathbb{N}_0, m\in\mathbb{N}$, $t=(t_1,\ldots,t_m)\in\{-2,-1,1,2\}^m$ and

$x = (x_1, \ldots, x_m) \in V^m$ we write $lx := (lx_1, \ldots, lx_m)$ and $tx := (t_1x_1, \ldots, t_mx_m)$, where ra stands, as usual, for the rth power of an element a of the commutative group V.

Let V and W be linear spaces, $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$. A mapping $f : V^n \longrightarrow W$ is called k-cubic and $n-k$-quartic (briefly, multicubic-quartic) if f satisfies (1) in each of some k variables and is quartic in each of the other variables (see equation (2)). In this note, we suppose for simplicity that f is cubic in each of the first k variables, but one can obtain analogous results without this assumption. Let us note that for $k = n$ (resp., $k = 0$), the above definition leads to the so-called multicubic (resp., multiquartic) mappings; some basic facts on such mappings can be found for instance in [5] and [6].

Here and subsequently, we assume that V and W are vector spaces over $\mathbb{Q}$. Moreover, we identify $x = (x_1, \ldots, x_n) \in V^n$ with $(x^k, x^{n-k}) \in V^k \times V^{n-k}$, where $x^k := (x_1, \ldots, x_k)$ and $x^{n-k} := (x_{k+1}, \ldots, x_n)$, and we adopt the convention that $(x^n, x^0) := x^n := (x^0, x^n)$.

Let $n \in \mathbb{N}$ with $n \geq 2$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. Throughout, we will write x_i^n simply x_i when no confusion can arise. Put also $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1} \ldots, x_{in}) \in V^{n-k}$. For $x_1, x_2 \in V^n$ and $p_i, T \in \mathbb{N}_0$ with $0 \leq p_i \leq n-k$, $0 \leq T \leq k$ and $0 \leq k \leq n-1$. Set

$$\mathbb{M}^k = \left\{\mathfrak{N}_k = (N_1, \ldots, N_k) |\ N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\right\},$$

where $j \in \{1, \ldots, k\}$ and

$$\mathcal{N}^{n-k} = \left\{\mathfrak{N}_{n-k} = (N_{k+1}, \ldots, N_n) |\ N_j \in \{x_{1j} \pm x_{2j}, x_{1j}, x_{2j}\}\right\},$$

for all $j \in \{k+1, \ldots, n\}$. Consider the subsets $\mathbb{M}_T^k$ and $\mathcal{N}_{(p_1,p_2)}^{n-k}$ of $\mathbb{M}^k$ and $\mathcal{N}^{n-k}$, respectively as follows:

$$\mathbb{M}_T^k := \left\{\mathfrak{N}_k \in \mathbb{M}^k |\ \mathrm{Card}\{N_j :\ N_j = x_{1j}\} = T\right\},$$

$$\mathcal{N}_{(p_1,p_2)}^{n-k} := \left\{\mathfrak{N}_{n-k} \in \mathcal{N}^{n-k} |\ \mathrm{Card}\{N_j :\ N_j = x_{ij}\} = p_i \ \ (i \in \{1, 2\})\right\}.$$

In addition, we use the following notations for the multicubic-quartic mappings.

$$f\left(\mathbb{M}_T^k, x_i^{n-k}\right) := \sum_{\mathfrak{N}_k \in \mathbb{M}_T^k} f\left(\mathfrak{N}_k, x_i^{n-k}\right),$$

$$f\left(x_i^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right) := \sum_{\mathfrak{N}_{n-k} \in \mathcal{N}_{(p_1,p_2)}^{n-k}} f\left(x_i^k, \mathfrak{N}_{n-k}\right),$$

$$f\left(\mathbb{M}_T^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right) := \sum_{\mathfrak{N}_k \in \mathbb{M}_T^k} \sum_{\mathfrak{N}_{n-k} \in \mathcal{N}_{(p_1,p_2)}^{n-k}} f\left(\mathfrak{N}_k, \mathfrak{N}_{n-k}\right).$$

Moreover, for a multiquartic mapping f (in the case $k = 0$), we use the notation

$$f\left(\mathcal{N}_{(p_1,p_2)}^{n}\right) := \sum_{\mathfrak{N}_n \in \mathcal{N}_{(p_1,p_2)}^{n}} f(\mathfrak{N}_n). \tag{4}$$

In continuation, we wish to show that if a mapping $f : V^n \longrightarrow W$ is multicubic-quartic, then f satisfies the equation

$$\sum_{s \in \{-1,1\}^k} \sum_{t \in \{-2,2\}^{n-k}} f\left(2x_1^k + sx_2^k, x_1^{n-k} + tx_2^{n-k}\right)$$
$$= \sum_{l=0}^{k} \sum_{p_2=0}^{n-k} \sum_{p_1=0}^{n-k-p_2} 2^{k-l} 12^l 4^{n-k-p_1-p_2} (-6)^{p_1} 24^{p_2} f\left(\mathbb{M}_l^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right), \tag{5}$$

for all $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1} \ldots, x_{in}) \in V^{n-k}$ where $i \in \{1, 2\}$.

It has been shown in [6, Proposition 2.2] that if a mapping $f : V^n \longrightarrow W$ is multicubic, then it satisfies the equation

$$\sum_{s\in\{-1,1\}^n} f(2x_1 + sx_2) = \sum_{l=0}^{n} 2^{n-l}12^l f\left(\mathbb{M}_l^n\right). \tag{6}$$

Furthermore, it was proved in [5, Theorem 2.2] that if a mapping $f : V^n \longrightarrow W$ is multiquartic, then it satisfies the equation

$$\sum_{t\in\{-2,2\}^n} f(x_1 + tx_2) = \sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2} 4^{n-p_1-p_2}24^{p_2}(-6)^{p_1} f\left(\mathcal{N}_{(p_1,p_2)}^n\right). \tag{7}$$

In the next proposition, we show the system of n equations defining the k-cubic and $n-k$-quartic (multicubic-quartic) mapping can be reduced to (5).

Proposition 2.1. *Let $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$. If a mapping $f : V^n \longrightarrow W$ is k-cubic and $n-k$-quartic* (*multicubic-quartic*)*, then f satisfies equation* (5).

Proof. Since for $k \in \{0, n\}$ our assertion follows from [5, Theorem 2.2] and [6, Proposition 2.2], we can assume that $k \in \{1, \ldots, n-1\}$. For any $x^{n-k} \in V^{n-k}$, define the mapping $g_{x^{n-k}} : V^k \longrightarrow W$ by $g_{x^{n-k}}(x^k) := f(x^k, x^{n-k})$ for $x^k \in V^k$. By assumption, $g_{x^{n-k}}$ is k-cubic, and hence Proposition 2.2 from [6] implies that

$$\sum_{s\in\{-1,1\}^k} g_{x^{n-k}}\left(2x_1^k + sx_2^k\right) = \sum_{l=0}^{k} 2^{k-l}12^l g_{x^{n-k}}\left(\mathbb{M}_l^k\right), \qquad (x_1^k, x_2^k \in V^k).$$

It now follows from the above equality that

$$\sum_{s\in\{-1,1\}^k} f\left(2x_1^k + sx_2^k, x^{n-k}\right) = \sum_{l=0}^{k} 2^{k-l}12^l f\left(\mathbb{M}_l^k, x^{n-k}\right), \tag{8}$$

for all $x_1^k, x_2^k \in V^k$ and $x^{n-k} \in V^{n-k}$. Similar to the above, for any $x^k \in V^k$, consider the mapping $h_{x^k} : V^{n-k} \longrightarrow W$ defined through $h_{x^k}(x^{n-k}) := f(x^k, x^{n-k})$, $x^{n-k} \in V^{n-k}$ which is $n-k$-quartic and thus we conclude from Theorem 2.2 of [5] that

$$\sum_{t\in\{-2,2\}^{n-k}} h_{x^k}\left(x_1^{n-k} + tx_2^{n-k}\right) = \sum_{p_2=0}^{n-k}\sum_{p_1=0}^{n-k-p_2} 4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2} h_{x^k}\left(\mathcal{N}_{(p_1,p_2)}^{n-k}\right), \tag{9}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$. By the definition of h_{x^k}, relation (9) is equivalent to

$$\sum_{t\in\{-2,2\}^{n-k}} f\left(x^k, x_1^{n-k} + tx_2^{n-k}\right) = \sum_{p_2=0}^{n-k}\sum_{p_1=0}^{n-k-p_2} 4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2} f\left(x^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right), \tag{10}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$ and $x^k \in V^k$. Plugging (8) into (10), we get

$$\begin{aligned}
&\sum_{s\in\{-1,1\}^k}\sum_{t\in\{-2,2\}^{n-k}} f\left(2x_1^k + sx_2^k, x_1^{n-k} + tx_2^{n-k}\right)\\
&\qquad = \sum_{s\in\{-1,1\}^k}\sum_{p_2=0}^{n-k}\sum_{p_1=0}^{n-k-p_2} 4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2} f\left(2x_1^k + sx_2^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right)\\
&\qquad = \sum_{l=0}^{k}\sum_{p_2=0}^{n-k}\sum_{p_1=0}^{n-k-p_2} 2^l 12^{k-l} 4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2} f\left(\mathbb{M}_l^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right),
\end{aligned}$$

for all $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1} \ldots, x_{in}) \in V^{n-k}$, shows that f satisfies equation (5). □

It is easily verified that the mapping $f(z_1,\dots,z_n)=\prod_{i=1}^{k}\prod_{j=k+1}^{n} z_i^3 z_j^4$ is multicubic-quartic and so by Proposition 2.1, it satisfies (5). Therefore, this equation is said to be *multicubic-quartic functional equation*.

Definition 2.2. *Given a mapping $f: V^n \longrightarrow W$. We say f*

(i) *has zero condition if $f(x)=0$ for any $x\in V^n$ with at least one component which is equal to zero;*

(ii) *satisfies (has) the r-power condition in the jth variable if*

$$f(z_1,\dots,z_{j-1},2z_j,z_{j+1},\dots,z_n)=2^r f(z_1,\dots,z_{j-1},z_j,z_{j+1},\dots,z_n),$$

for all $z_1,\dots,z_n\in V$.

Note that 3*-power and* 4*-power conditions are also called the cubic and quartic conditions, respectively.*

We remember that the binomial coefficient for all $n,r\in\mathbb{N}_0$ with $n\geq r$ is defined and denoted by $\binom{n}{r}:=\frac{n!}{r!(n-r)!}$. We shall show that if a mapping $f:V^n\longrightarrow W$ satisfies equation (5), then it is multicubic-quartic. In order to do this, we need the next result. Since the proof is similar to the proof of [5, Lemma 2.1], is omitted.

Lemma 2.3. *Let a mapping $f: V^n\longrightarrow W$ satisfies equation* (5)*. Then, f satisfying zero condition.*

Theorem 2.4. *Suppose that a mapping $f: V^n\longrightarrow W$ satisfies equation* (5)*. If f satisfies the cubic condition in the first k variables, then it is multicubic-quartic.*

Proof. It follows from Lemma 2.3 that f satisfies zero condition. Putting $x_2^{n-k}=(0,\dots,0)$ in (5), we have

$$\begin{aligned}
&2^{n-k}\sum_{s\in\{-1,1\}^k} f\left(2x_1^k+sx_2^k,x_1^{n-k}\right)\\
&=\sum_{l=0}^{k}\sum_{p_1=0}^{n-k}2^{k-l}12^l\binom{n-k}{p_1}4^{n-k-p_1}(-6)^{p_1}2^{n-k-p_1}f\left(\mathbb{M}_l^k,x_1^{n-k}\right)\\
&=\sum_{l=0}^{k}2^{k-l}12^l\sum_{p_1=0}^{n-k}\binom{n-k}{p_1}8^{n-k-p_1}(-6)^{p_1}f\left(\mathbb{M}_l^k,x_1^{n-k}\right)\\
&=2^{n-k}\sum_{l=0}^{k}2^{k-l}12^l f\left(\mathbb{M}_l^k,x_1^{n-k}\right),
\end{aligned}\tag{11}$$

for all $x_1^k,x_2^k\in V^n$ and $x_1^{n-k}\in V^{n-k}$. It now follows from (11) that

$$\sum_{s\in\{-1,1\}^k} f\left(2x_1^k+sx_2^k,x_1^{n-k}\right)=\sum_{l=0}^{k}2^{k-l}12^l f\left(\mathbb{M}_l^k,x_1^{n-k}\right),$$

for all $x_1^k,x_2^k\in V^n$ and $x_1^{n-k}\in V^{n-k}$. In light of [6, Proposition 2.3], we see that f is cubic in each of the k first variables. Furthermore, by putting $x_2^k=(0,\cdots,0)$ in (5) and applying the hypotheses, the left side of (5) will be

$$2^k\times 2^{3k}\sum_{t\in\{-2,2\}^{n-k}} f\left(x_1^k,x_1^{n-k}+tx_2^{n-k}\right)=2^k\sum_{t\in\{-2,2\}^{n-k}} f\left(2x_1^k,x_1^{n-k}+tx_2^{n-k}\right),\tag{12}$$

for all $x_1^k\in V^k$ and $x_1^{n-k},x_2^{n-k}\in V^{n-k}$. On the other hand, the right side of (5) converts to

$$\begin{aligned}
&\sum_{l=0}^{k}\binom{k}{l}2^{k-l}12^l2^{k-l}\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2}f\left(x_1^k,\mathcal{N}_{(p_1,p_2)}^{n-k}\right)\\
&=(4+12)^k\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2}f\left(x_1^k,\mathcal{N}_{(p_1,p_2)}^{n-k}\right),
\end{aligned}\tag{13}$$

for all $x_1^k \in V^k$ and $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$. Comparing (12) and (13), we find

$$\sum_{t\in\{-2,2\}^{n-k}} f\left(x_1^k, x_1^{n-k} + tx_2^{n-k}\right) = \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}(-6)^{p_1} 24^{p_2} f\left(x_1^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right),$$

for all $x_1^k \in V^k$ and $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$ and thus [5, Theorem 2.2] now completes the proof. □

3. Characterization of multimixed cubic-quartic mappings

In this section, we introduce the multimixed cubic-quartic mappings and then characterize them as an equation. We start this section with a definition as follows.

Definition 3.1. *Let V and W be vector spaces over $\mathbb{Q}$, $n \in \mathbb{N}$. A mapping $f : V^n \longrightarrow W$ is called n-mixed cubic-quartic or briefly multimixed cubic-quartic if f fulfills mixed cubic-quartic functional equation* (3) *in each variable.*

Let $n \in \mathbb{N}$ with $n \geq 2$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. For $x_1, x_2 \in V^n$ and $q \in \mathbb{N}_0$ with $0 \leq q \leq n$, put

$$\mathcal{M} = \left\{\mathfrak{M}_n = (M_1, \ldots, M_n) |\ M_j \in \{x_{1j} \pm 2x_{2j}, 2x_{2j}\}\right\},$$

where $j \in \{1, \ldots, n\}$. Consider the subset $\mathcal{M}_q^n$ of $\mathcal{M}$ as follows:

$$\mathcal{M}_q^n := \left\{\mathfrak{M}_n \in \mathcal{M} |\ \mathrm{Card}\{M_j : M_j = x_{2j}\} = q\right\}.$$

Hereafter, for the multimixed cubic-quartic mappings, we use the following notations:

$$f\left(\mathcal{M}_q^n\right) := \sum_{\mathfrak{M}_n \in \mathcal{M}_q^n} f(\mathfrak{M}_n), \tag{14}$$

and

$$f\left(\mathcal{M}_q^n, z\right) := \sum_{\mathfrak{M}_n \in \mathcal{M}_q^n} f(\mathfrak{M}_n, z) \qquad (z \in V).$$

For each $x_1, x_2 \in V^n$, we consider the equation

$$\sum_{q=0}^{n} (-3)^q f\left(\mathcal{M}_q^n\right) = \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} (-24)^{p_2} f\left(\mathcal{N}_{(p_1,p_2)}^n\right), \tag{15}$$

where $f\left(\mathcal{M}_q^n\right)$ and $f\left(\mathcal{N}_{(p_1,p_2)}^n\right)$ are defined in (14) and (4), respectively. Next, we reduce the system of n equations defining the multimixed cubic-quartic mapping in obtaining the single functional equation (15).

Proposition 3.2. *If a mapping $f : V^n \longrightarrow W$ is multimixed cubic-quartic, it satisfies functional equation* (15).

Proof. The proof is by induction on n. For $n = 1$, it is clear that f satisfies equation (3). Assume that (15) is true for some positive integer $n > 1$. Then

$$
\begin{aligned}
&\sum_{q=0}^{n+1} (-3)^{q+1} f\left(\mathcal{M}_q^{n+1}\right) = \sum_{q=0}^{n} (-3)^q f\left(\mathcal{M}_q^n, x_{1,n+1} + 2x_{2,n+1}\right) \\
&+ \sum_{q=0}^{n} (-3)^q f\left(\mathcal{M}_q^n, x_{1,n+1} - 2x_{2,n+1}\right) + \sum_{q=0}^{n} (-3)^{q+1} f\left(\mathcal{M}_q^n, 2x_{2,n+1}\right) \\
&= \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} (-24)^{p_2} f\left(\mathcal{N}_{(p_1,p_2)}^n, x_{1,n+1} + 2x_{2,n+1}\right) \\
&+ \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} (-24)^{p_2} f\left(\mathcal{N}_{(p_1,p_2)}^n, x_{1,n+1} - 2x_{2,n+1}\right) \\
&+ \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} (-24)^{p_2} (-3) f\left(\mathcal{N}_{(p_1,p_2)}^n, 2x_{2,n+1}\right) \\
&= 4 \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \sum_{t\in\{-1,1\}} 4^{n-p_1-p_2}(-6)^{p_1} (-24)^{p_2} f\left(\mathcal{N}_{(p_1,p_2)}^n, x_{1,n+1} + tx_{1,n+1}\right) \\
&- 6 \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} (-24)^{p_2} f\left(\mathcal{N}_{(p_1,p_2)}^n, x_{1,n+1}\right) \\
&- 24 \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} (-24)^{p_2} f\left(\mathcal{N}_{(p_1,p_2)}^n, x_{2,n+1}\right) \\
&= \sum_{p_1=0}^{n+1} \sum_{p_2=0}^{n+1-p_1} 4^{n+1-p_1-p_2}(-6)^{p_1} (-24)^{p_2} f\left(\mathcal{N}_{(p_1,p_2)}^{n+1}\right).
\end{aligned}
$$

This means that (15) holds for $n + 1$. $\square$

By a mathematical computation, one can check that the mapping $f(z_1, \ldots, z_n) = \prod_{j=1}^{n}(a_j z_j^4 + b_j z_j^3)$ is multimixed cubic-quartic and thus (15) is valid for it by Proposition 3.2. Therefore, equation (15) is said to be *multimixed cubic-quartic functional equation*.

Similar to Lemma 2.1 from [5], we need the upcoming lemma in obtaining our goal in this section. The proof is similar but we include some parts for the sake of completeness.

Lemma 3.3. *If a mapping $f : V^n \longrightarrow W$ satisfies the equation* (15)*, then f has zero condition.*

Proof. Putting $x_1 = x_2 = (0, \ldots, 0)$ in (15), we have

$$
\begin{aligned}
&\sum_{q=0}^{n}(-3)^q 2^{n-q} \binom{n}{n-q} f(0, \ldots, 0) \\
&= \sum_{p_2=0}^{n} \sum_{p_1=0}^{n-p_2} 4^{n-p_1-p_2}(-6)^{p_1}(-24)^{p_2} \binom{n}{n-p_1-p_2} \binom{p_1+p_2}{p_1} 2^{n-p_1-p_2} f(0, \ldots, 0).
\end{aligned}
\tag{16}
$$

It is easy to check that

$$
\binom{n-k}{n-k-p_1-p_2} \binom{p_1+p_2}{p_1} = \binom{n-k}{p_2} \binom{n-k-p_2}{p_1} \tag{17}
$$

for $0 \leq k \leq n-1$. Using (17) for $k=0$, the right side of (16) will be as follows:

$$\begin{aligned}
&\sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2} 4^{n-p_1-p_2}(-6)^{p_1}(-24)^{p_2}\binom{n}{n-p_1-p_2}\binom{p_1+p_2}{p_1}2^{n-p_1-p_2} f(0,\dots,0)\\
&= 2^n\left[\sum_{p_2=0}^{n}\binom{n}{p_2}(-12)^{p_2}\sum_{p_1=0}^{n-p_2}\binom{n-p_2}{p_1}4^{n-p_1-p_2}(-3)^{p_1}\right] f(0,\dots,0)\\
&= 2^n\left[\sum_{p_2=0}^{n}\binom{n}{p_2}(-12)^{p_2}(4-3)^{n-p_2}\right] f(0,\dots,0)\\
&= 2^n(-12+1)^n f(0,\dots,0) = (-22)^n f(0,\dots,0). \qquad (18)
\end{aligned}$$

On the other hand, by a simple computation, the left side of (16) is

$$(-1)^n f(0,\dots,0). \qquad (19)$$

It follows from the relations (16), (18) and (19) that $f(0,\cdots,0) = 0$. We continue in this fashion obtaining $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero. □

Definition 3.4. *Recall from* [28] *that a mapping $f : V^n \longrightarrow W$ is*

(i) *is even in the jth variable if*

$$f(z_1,\dots,z_{j-1},-z_j,z_{j+1},\dots,z_n) = f(z_1,\dots,z_{j-1},z_j,z_{j+1},\dots,z_n);$$

(ii) *is odd in the jth variable if*

$$f(z_1,\dots,z_{j-1},-z_j,z_{j+1},\dots,z_n) = -f(z_1,\dots,z_{j-1},z_j,z_{j+1},\dots,z_n).$$

Proposition 3.5. *Let a mapping $f : V^n \longrightarrow W$ satisfying equation* (15)*. Then, it is multimixed cubic-quartic. In particular,*

(i) *if f is odd in a variable, then it is cubic in the same variable;*

(ii) *if f is even in a variable, then it is quartic in the same variable.*

Proof. Let $j \in \{1,\dots,n\}$ be arbitrary and fixed. Set

$$f_j^*(z) := f\left(z_1,\dots,z_{j-1},z,z_{j+1},\dots,z_n\right).$$

Putting $x_{2k} = 0$ for all $k \in \{1, \cdots, n\}\backslash\{j\}$ in (15) and using Lemma 3.3, we get

$$\begin{aligned}
&2^{n-1}[f_j^*(z+2w) + f_j^*(z-2w) - 3f_j^*(2w)] \\
&= \left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-p_1}(-6)^{p_1} 2^{n-p_1-1}\right](f_j^*(z+w) + f_j^*(z-w)) \\
&+ \left[\sum_{p_1=1}^{n} \binom{n-1}{p_1-1} 4^{n-p_1}(-6)^{p_1} 2^{n-p_1}\right] f_j^*(z) \\
&+ \left[\sum_{p_1=0}^{n} \binom{n-1}{p_1} 4^{n-p_1-1}(-6)^{p_1}(-24) 2^{n-p_1-1}\right] f_j^*(w) \\
&= 4\left[2^{n-1}\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-1-p_1}(-3)^{p_1}\right](f_j^*(z+w) + f_j^*(z-w)) \\
&- 6\left[2^{n-1}\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-1-p_1}(-3)^{p_1}\right] f_j^*(z) \\
&- 24\left[2^{n-1}\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-1-p_1}(-3)^{p_1}\right] f_j^*(w) \\
&= 4 \times 2^{n-1}(f_j^*(z+w) + f_j^*(z-w)) - 6 \times 2^{n-1} f_j^*(z) - 24 \times 2^{n-1} f_j^*(w).
\end{aligned}$$

The above equalities show that

$$f_j^*(z+2w) + f_j^*(z-2w) - 3f_j^*(2w) = 4[f_j^*(z+w) + f_j^*(z-w)] - 6f_j^*(z) - 24f_j^*(w).$$

In other words, (3) is true for f_j^*. Since j is arbitrary, f is a multimixed cubic-quartic mapping.

(i) Repeating the proof of Lemma 2.2 from [13] for f_j^*, we see that

$$f_j^*(2z+w) + f_j^*(2z-w) = 2f_j^*(z+w) + 2f_j^*(z-w) + 12f_j^*(z).$$

This means that f is cubic in the jth variable (see equation (1)).

(ii) Similar to the previous part, it follows from the proof of [13, Lemma 2.1] that

$$f_j^*(z+2w) + f_j^*(z-2w) = 4[f_j^*(z+w) + f_j^*(z-w)] - 6f_j^*(z) + 24f_j^*(w).$$

Therefore, f is quartic in the jth variable. □

Corollary 3.6. *Suppose that a mapping $f : V^n \longrightarrow W$ satisfies equation* (15).

(i) *If f is odd in each variable, then it is multicubic;*
(ii) *If f is even in each variable, then it is multiquartic;*
(iii) *If f is odd in each of some k variables and is even in each of the other variables, then it is multicubic-quartic. In particular, f fulfilling equation* (5).

4. Various stability results

Throughout this section, for two sets E and F, the set of all mappings from E to F is denoted by F^E. Here, we express some basic facts concerning non-Archimedean spaces and some preliminary results. Let us recall that a metric d on a nonempty set X is said to be non-Archimedean (or an ultrametric) provided

$$d(x,z) \le \max\{d(x,y), d(y,z)\}$$

for $x, y, z \in X$. By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|a| = 0$ if and only if $a = 0$, $|ab| = |a||b|$, and $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let $\mathcal{X}$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : \mathcal{X} \longrightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|ax\| = |a|\|x\|, \quad (x \in \mathcal{X}, a \in \mathbb{K})$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad (x, y \in \mathcal{X}).$$

Then, $(\mathcal{X}, \|\cdot\|)$ is called a non-Archimedean normed space. Due to the fact that

$$\|x_n - x_m\| \leq \max\left\{\|x_{j+1} - x_j\|; m \leq j \leq n-1\right\}, \quad (n \geq m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $\mathcal{X}$. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent. If $(\mathcal{X}, \|\cdot\|)$ is a non-Archimedean normed space, then it is easily verified that the function $d_{\mathcal{X}} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}_+$, given by $d_{\mathcal{X}}(x, y) := \|x - y\|$, is a non-Archimedean metric on $\mathcal{X}$ that is invariant (i.e., $d_{\mathcal{X}}(x + z, y + z) = d_{\mathcal{X}}(x, y)$ for $x, y, z \in X$). Hence, non-Archimedean normed spaces are also special cases of metric spaces with invariant metrics.

The most important examples of non-Archimedean normed spaces are the p-adic numbers, which have gained the interest of physicists because of their connections with some problems coming from quantum physics, p-adic strings and superstrings [21]. Indeed, Hensel [16] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most interesting example of non-Archimedean normed spaces is p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$.

We recall that for a field $\mathbb{K}$ with multiplicative identity 1, the characteristic of $\mathbb{K}$ is the smallest positive number n such that $\overbrace{1 + \ldots + 1}^{n-\text{times}} = 0$. In this section, we prove the generalized Hyers-Ulam stability of equation (15) in non-Archimedean normed spaces. The proof is based on the upcoming fixed point theorem that can be derived from [8, Theorem 1]. This result plays a key tool in obtaining our aim in this section.

Theorem 4.1. *Let the following hypotheses hold.*

(H1) *E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j \in \mathbb{N}$, $g_1, \ldots, g_j : E \longrightarrow E$ and $L_1, \ldots, L_j : E \longrightarrow \mathbb{R}_+$,*
(H2) *$\mathcal{T} : Y^E \longrightarrow Y^E$ is an operator satisfying the inequality*

$$\left\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\right\| \leq max_{i\in\{1,\ldots,j\}} L_i(x) \left\|\lambda(g_i(x)) - \mu(g_i(x))\right\|,$$

for all $\lambda, \mu \in Y^E, x \in E$,
(H3) *$\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ is an operator defined through*

$$\Lambda\delta(x) := max_{i\in\{1,\ldots,j\}} L_i(x)\delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^E, x \in E.$$

Moreover, the function $\theta : E \longrightarrow \mathbb{R}_+$ and the mapping $\varphi : E \longrightarrow Y$ fulfill the following two conditions:

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \theta(x), \qquad \lim_{l\to\infty} \Lambda^l \theta(x) = 0, \quad (x \in E).$$

Then, for every $x \in E$, the limit $\lim_{l\to\infty} \mathcal{T}^l\varphi(x) =: \psi(x)$ exists and the mapping $\psi \in Y^E$, defined in this way, is a fixed point of $\mathcal{T}$ with

$$\left\|\varphi(x) - \psi(x)\right\| \leq \sup_{l\in\mathbb{N}_0} \Lambda^l \theta(x) \qquad (x \in E).$$

For the simplicity of computations, from now on, for a mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathbf{D}_{CQ}f : V^n \times V^n \longrightarrow W$ by

$$\mathbf{D}_{CQ}f(x_1, x_2) = \sum_{q=0}^{n} (-3)^q f\left(\mathcal{M}_q^n\right) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1}(-24)^{p_2} f\left(\mathcal{N}^n_{(p_1,p_2)}\right),$$

where $f\left(\mathcal{M}_q^n\right)$ and $f\left(\mathcal{N}^n_{(p_1,p_2)}\right)$ are defined in (14) and (4), respectively. In the sequel, all mappings $f : V^n \longrightarrow W$ are assumed that satisfy zero condition. With this assumption, we have the next stability result for functional equation (15) in the odd case.

Theorem 4.2. *Let $\beta \in \{-1, 1\}$ be fixed, V be a linear space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. Suppose that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying the equality*

$$\lim_{l\to\infty}\left(\frac{1}{|2|^{3n\beta}}x\right)^l \varphi(2^{l\beta}x_1, 2^{l\beta}x_2) = 0, \tag{20}$$

for all $x_1, x_2 \in V^n$. Assume also $f : V^n \longrightarrow W$ is an odd mapping in each variable fulfilling the inequality

$$\|\mathbf{D}_{CQ}f(x_1, x_2)\| \leq \varphi(x_1, x_2), \tag{21}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathcal{C} : V^n \longrightarrow W$ of (15) such that

$$\left\|f(x) - \mathcal{C}(x)\right\| \leq \sup_{l\in\mathbb{N}_0} \frac{1}{|3|^n \times |2|^{3n\frac{\beta+1}{2}}}\left(\frac{1}{|2|^{3n\beta}}\right)^l \varphi\left(0, 2^{l\beta+\frac{\beta-1}{2}}x\right), \tag{22}$$

for all $x \in V^n$. Moreover, if $\mathcal{C}$ is odd in each variable, then it is a multicubic mapping.

Proof. Replacing (x_1, x_2) by $(0, x_1)$ in (21) and using the assumptions, we have

$$\left\|(-3)^n f(2x) - (-24)^n f(x)\right\| \leq \varphi(0, x),$$

for all $x = x_1 \in V^n$ and so

$$\left\|f(2x) - 2^{3n}f(x)\right\| \leq \frac{1}{|3|^n}\varphi(0, x), \tag{23}$$

for all $x \in V^n$. Inequality (23) implies that

$$\left\|f(x) - \mathcal{T}f(x)\right\| \leq \theta(x), \tag{24}$$

for all $x \in V^n$, where

$$\theta(x) := \frac{1}{|3|^n \times |2|^{3n\frac{\beta+1}{2}}}\varphi\left(0, 2^{\frac{\beta-1}{2}}x\right), \qquad \mathcal{T}\xi(x) := \frac{1}{2^{3n\beta}}\xi\left(2^\beta x\right),$$

for all $\xi \in W^{V^n}$ and $x \in V^n$. Define $\Lambda\eta(x) := \frac{1}{|2|^{3n\beta}}\eta\left(2^\beta x\right)$ for all $\eta \in \mathbb{R}_+^{V^n}, x \in V^n$. It is easy to see that Λ has the form described in (H3) with $E = V^n$, $g_1(x) := 2^\beta x$ for all $x \in V^n$ and $L_1(x) = \frac{1}{|2|^{3n\beta}}$. Moreover, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we get

$$\begin{aligned}\left\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\right\| &= \left\|\frac{1}{2^{3n\beta}}\lambda(2^\beta x) - \frac{1}{2^{3n\beta}}\mu(2^\beta x)\right\| \\ &\leq L_1(x)\left\|\lambda(g_1(x)) - \mu(g_1(x))\right\|.\end{aligned}$$

The above relation shows that the hypothesis (H2) is valid. By induction on l, one can check that for any $l \in \mathbb{N}$ and $x \in V^n$ that

$$\Lambda^l \theta(x) := \left(\frac{1}{|2|^{3n\beta}}\right)^l \theta\left(2^{l\beta}x\right)$$
$$= \frac{1}{|3|^n \times |2|^{3n\frac{\beta+1}{2}}} \left(\frac{1}{|2|^{3n\beta}}\right)^l \varphi\left(0, 2^{l\beta+\frac{\beta-1}{2}}x\right), \tag{25}$$

for all $x \in V^n$. Now, relations (24) and (25) imply that all assumptions of Theorem 4.1 are satisfied. Hence, there exists a unique mapping $C : V^n \longrightarrow W$ such that $C(x) = \lim_{l\to\infty}\left(\mathcal{T}^l f\right)(x)$ for all $x \in V^n$, and also (22) holds. We also can verify by induction on l that

$$\left\|\mathbf{D}_{CQ}\left(\mathcal{T}^l f\right)(x_1, x_2)\right\| \leq \left(\frac{1}{|2|^{3n\beta}}\right)^l \varphi\left(2^{l\beta}x_1, 2^{l\beta}x_2\right), \tag{26}$$

for all $x_1, x_2 \in V^n$. Letting $l \to \infty$ in (26) and applying (20), we arrive at $\mathbf{D}_{CQ}C(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping satisfies equation (15) and the proof is now completed by part (i) of Corollary 3.6. □

Next, in analogy with Theorem 4.2, we bring the following stability result for functional equation (3.2) in the even case.

Theorem 4.3. *Given $\beta \in \{-1, 1\}$ be fixed. Let V be a linear space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. Suppose that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying the equality*

$$\lim_{l\to\infty}\left(\frac{1}{|2|^{4n\beta}}\right)^l \varphi(2^{l\beta}x_1, 2^{l\beta}x_2) = 0, \tag{27}$$

for all $x_1, x_2 \in V^n$. Assume also $f : V^n \longrightarrow W$ is an even mapping in each variable fulfilling the inequality

$$\|\mathbf{D}_{CQ}f(x_1, x_2)\| \leq \varphi(x_1, x_2), \tag{28}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (15) such that

$$\left\|f(x) - \mathcal{Q}(x)\right\| \leq \sup_{l\in\mathbb{N}_0} \frac{1}{|2|^{4n\frac{\beta+1}{2}}}\left(\frac{1}{|2|^{4n\beta}}\right)^l \varphi\left(0, 2^{l\beta+\frac{\beta-1}{2}}x\right), \tag{29}$$

for all $x \in V^n$. In particular, if $\mathcal{Q}$ is even in each variable, then it is a multiquartic mapping.

Proof. Replacing (x_1, x_2) by $(0, x_1)$ in (28) and applying the hypotheses, we get

$$\left\|\sum_{q=0}^{n}\binom{n}{q}(-3)^q\, 2^{n-q} f(2x) - \sum_{p_2=0}^{n}\binom{n}{p_2}(-24)^{p_2}\, 4^{n-p_2} \times 2^{n-p_2} f(x)\right\| \leq \varphi(0, x), \tag{30}$$

where $x = x_1 \in V^n$. On the other hand

$$\sum_{q=0}^{n}\binom{n}{q}(-3)^q\, 2^{n-q} = (-3+2)^n = (-1)^n$$

and

$$\sum_{p_2=0}^{n}\binom{n}{p_2}(-24)^{p_2}\, 4^{n-p_2} \times 2^{n-p_2} = (-24+8)^n = (-16)^n\,.$$

It follows the above relations that (30) is equivalent to

$$\left\|f(2x)-2^{4n}f(x)\right\| \leq \varphi(0,x), \tag{31}$$

for all $x \in V^n$. Set

$$\theta(x) := \frac{1}{|2|^{4n\frac{\beta+1}{2}}}\varphi\left(0, 2^{\frac{\beta-1}{2}}x\right), \qquad \mathcal{T}\xi(x) := \frac{1}{2^{4n\beta}}\xi\left(2^{\beta}x\right),$$

for all $\xi \in W^{V^n}$ and $x \in V^n$. Now, inequality (31) implies that

$$\left\|f(x)-\mathcal{T}f(x)\right\| \leq \theta(x), \tag{32}$$

for all $x \in V^n$, Define $\Lambda\eta(x) := \frac{1}{|2|^{4n\beta}}\eta\left(2^{\beta}x\right)$ for all $\eta \in \mathbb{R}_+^{V^n}, x \in V^n$. It is easily checked that Λ has the form described in (H3) with $E = V^n$, $g_1(x) := 2^{\beta}x$ for all $x \in V^n$ and $L_1(x) = \frac{1}{|2|^{4n\beta}}$. In addition, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we obtain

$$\begin{aligned}\left\|\mathcal{T}\lambda(x)-\mathcal{T}\mu(x)\right\| &= \left\|\frac{1}{2^{4n\beta}}\lambda(2^{\beta}x)-\frac{1}{2^{4n\beta}}\mu(2^{\beta}x)\right\| \\ &\leq L_1(x)\left\|\lambda(g_1(x))-\mu(g_1(x))\right\|.\end{aligned}$$

The above relation implies that the hypothesis (H2) is true. By induction on l, one can check for any $l \in \mathbb{N}$ and $x \in V^n$ that

$$\begin{aligned}\Lambda^l\theta(x) &:= \left(\frac{1}{|2|^{4n\beta}}\right)^l \theta\left(2^{l\beta}x\right) \\ &= \frac{1}{|2|^{4n\frac{\beta+1}{2}}}\left(\frac{1}{|2|^{4n\beta}}\right)^l \varphi\left(0, 2^{l\beta+\frac{\beta-1}{2}}x\right),\end{aligned} \tag{33}$$

for all $x \in V^n$. It follows from relations (32) and (33) that all assumptions of Theorem 4.1 are satisfied. Thus, there exists a unique mapping $Q : V^n \longrightarrow W$ such that $Q(x) = \lim_{l\to\infty}\left(\mathcal{T}^l f\right)(x)$ for all $x \in V^n$, and (29) holds as well. On the other hand, by induction on l we reach

$$\left\|\mathbf{D}_{CQ}\left(\mathcal{T}^l f\right)(x_1, x_2)\right\| \leq \left(\frac{1}{|2|^{4n\beta}}\right)^l \varphi\left(2^{l\beta}x_1, 2^{l\beta}x_2\right), \tag{34}$$

for all $x_1, x_2 \in V^n$. Taking $l \to \infty$ in (34) and using (27), we find $\mathbf{D}_{CQ}Q(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$, and therefore the mapping satisfies equation (15). In addition, if Q is even in each variable, then it is multiquartic by part (ii) of Corollary 3.6. □

For the rest of paper, we assume that $|2| < 1$. The following corollaries are the direct consequence of Theorems 4.2 and 4.3 concerning the stability of (15).

Corollary 4.4. *Let $\delta > 0$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from* 2. *Suppose that $f : V^n \longrightarrow W$ satisfies the inequality*

$$\left\|\mathbf{D}_{CQ}f(x_1, x_2)\right\| \leq \delta.$$

for all $x_1, x_2 \in V^n$.

(i) *If $f : V^n \longrightarrow W$ is an odd mapping in each variable, then there exists a unique solution mapping $C : V^n \longrightarrow W$ of* (15) *such that*

$$\left\|f(x)-C(x)\right\| \leq \frac{1}{|3|^n}\delta.$$

(ii) *If $f : V^n \longrightarrow W$ is an even mapping in each variable, then there exists a unique solution mapping $\mathcal{Q} : V^n \longrightarrow W$ of* (15) *such that*

$$\left\| f(x) - \mathcal{Q}(x) \right\| \leq \delta.$$

Proof. (i) We firstly note that $|2| < 1$. Letting $\varphi(x_1, x_2) = \delta$ in the case $\beta = -1$ of Theorem 4.2, we have $\lim_{l\to\infty} |2|^{3nl}\delta = 0$. Therefore, one can obtain the desired result.

(ii) The proof is similar part (i) by applying Theorem 4.3. □

Corollary 4.5. *Let $p \in \mathbb{R}$ fulfills $p \neq 3n$. Let V be a non-Archimedean normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from* 2. *If $f : V^n \longrightarrow W$ is an odd mapping in each variable fulfilling the inequality*

$$\left\| \mathbf{D}_{CQ} f(x_1, x_2) \right\| \leq \sum_{k=1}^{2} \sum_{j=1}^{n} \|x_{kj}\|^p,$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution mapping $\mathcal{C} : V^n \longrightarrow W$ of (15) *such that*

$$\left\| f(x) - \mathcal{C}(x) \right\| \leq \begin{cases} \frac{1}{|3|^n |2|^{3n}} \sum_{j=1}^n \|x_{1j}\|^p & p > 3n, \\ \\ \frac{1}{|3|^n |2|^p} \sum_{j=1}^n \|x_{1j}\|^p & p < 3n, \end{cases}$$

for all $x = x_1 \in V^n$.

Proof. Putting $\varphi(x_1, x_2) = \sum_{k=1}^2 \sum_{j=1}^n \|x_{kj}\|^p$, we have $\varphi\left(2^l x_1, 2^l x_2\right) = |2|^{lp}\varphi(x_1, x_2)$. It now follows from Theorem 4.2 the first and second inequalities in the cases $\beta = 1$ and $\beta = -1$, respectively. □

Theorem 4.3 has a consequence as follows.

Corollary 4.6. *Let $p \in \mathbb{R}$ fulfills $p \neq 4n$. Let V be a non-Archimedean normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from* 2. *If $f : V^n \longrightarrow W$ is an even mapping in each variable satisfying the inequality*

$$\left\| \mathbf{D}_{CQ} f(x_1, x_2) \right\| \leq \sum_{k=1}^{2} \sum_{j=1}^{n} \|x_{kj}\|^p$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution mapping $\mathcal{Q} : V^n \longrightarrow W$ of (15) *such that*

$$\left\| f(x) - \mathcal{Q}(x) \right\| \leq \begin{cases} \frac{1}{|2|^{4n}} \sum_{j=1}^n \|x_{1j}\|^p & p > 4n, \\ \\ \frac{1}{|2|^p} \sum_{j=1}^n \|x_{1j}\|^p & p < 4n, \end{cases}$$

for all $x = x_1 \in V^n$.

Let A be a nonempty set, (X, d) a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathcal{F}_1, \mathcal{F}_2$ operators mapping a nonempty set $D \subset X^A$ into X^{A^n}. We say that operator equation

$$\mathcal{F}_1\varphi(a_1, \ldots, a_n) = \mathcal{F}_2\varphi(a_1, \ldots, a_n) \tag{35}$$

is ψ-hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d\left(\mathcal{F}_1\varphi_0(a_1, \ldots, a_n), \mathcal{F}_2\varphi_0(a_1, \ldots, a_n)\right) \leq \psi(a_1, \ldots, a_n),$$

for all $a_1, \ldots, a_n \in A$, fulfils (35); this definition is introduced in [7]. In other words, a functional equation $\mathcal{F}$ is hyperstable if any mapping f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$. Under some conditions the functional equation (15) is hyperstable as follows.

Corollary 4.7. *Suppose that $p_{kj} > 0$ for $k \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$ fulfill $\sum_{k=1}^{2} \sum_{j=1}^{n} p_{kj} \neq 3n, 4n$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\left\| \mathbf{D}_{CQ} f(x_1, x_2) \right\| \leq \prod_{k=1}^{2} \prod_{j=1}^{n} \left\| x_{kj} \right\|^{p_{kj}}$$

for all $x_1, x_2 \in V^n$, then f satisfies (15).

It is easily verified that if a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies (2), then it has the form $g(x) = cx^4$, for all $x \in \mathbb{R}$, where $c = f(1)$. In the next proposition, we extend this result for several variables functions.

Proposition 4.8. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous n-quartic function. Then, there exists a constant $c \in \mathbb{R}$ such that*

$$f(x_1, \ldots, x_n) = c \prod_{j=1}^{n} x_j^4 \tag{36}$$

for all $x_1, \ldots, x_n \in \mathbb{R}$.

Proof. We will prove (36) by induction on n. For $n = 1$, (36) holds in virtue of the explanations above. Suppose (36) is true for a $n \in \mathbb{N}$, and let $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be a continuous $(n + 1)$-quartic function. Fix the n variables $x_1, \ldots, x_n$. Then, the function $f(x_1, \ldots, x_n, z)$ as a function of z is quartic and continuous, and so there exists a constant $c \in \mathbb{R}$ such that

$$f(x_1, \ldots, x_n, z) = cz^4, \qquad (z \in \mathbb{R}). \tag{37}$$

Note that c depends on $x_1, \ldots, x_n$, and indeed

$$c = c(x_1, \ldots, x_n). \tag{38}$$

Taking $z = 1$ in (37) and using (38), we obtain

$$c = c(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, 1).$$

Since f is $(n + 1)$-quartic, c is an n-quartic function and hence by the induction hypothesis there exists a real number c_0 such that

$$c = c(x_1, \ldots, x_n) = c_0 \prod_{j=1}^{n} x_j^4. \tag{39}$$

Now, the result follows from (37) and (39). □

We close the paper by the following counterexample for multiquartic mappings on $\mathbb{R}^n$ that its idea is taken from [15]. In fact, we show the hypothesis $p \neq 4n$ can not be removed in Corollary 4.6 when $V = W = \mathbb{R}$.

Example 4.9. *Let $\delta > 0$ and $n \in \mathbb{N}$. Put $\mu = \frac{2^{4n}-1}{2^{8n}S}\delta$, where*

$$S = \sum_{q=0}^{n} 3^q + \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2} \times 6^{p_1} \times 24^{p_2}.$$

Consider the function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ *defined by*

$$\psi(r_1, \ldots, r_n) = \begin{cases} \mu \prod_{j=1}^{n} r_j^4 & \text{for all } r_j \text{ with } |r_j| < 1, \\ \mu & \text{otherwise.} \end{cases}$$

Define the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *through*

$$f(r_1, \ldots, r_n) = \sum_{l=0}^{\infty} \frac{\psi(2^l r_1, \ldots, 2^l r_n)}{2^{4nl}}, \qquad (r_j \in \mathbb{R}).$$

Clearly, f is an even function in each variable and non-negative. Moreover, ψ is continuous and bounded by μ. Since f is a uniformly convergent series of continuous functions, it is continuous and bounded. Indeed, for each $(r_1, \ldots, r_n) \in \mathbb{R}^n$, we have $f(r_1, \ldots, r_n) \le \frac{2^{4n}}{2^{4n}-1}\mu$. Put $x_i = (x_{i1}, \ldots, x_{in})$, where $i \in \{1, 2\}$. We wish to show that

$$\left|\mathbf{D}_{CQ} f(x_1, x_2)\right| \le \delta \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{4n}, \tag{40}$$

for all $x_1, x_2 \in \mathbb{R}^n$. Obviously, (40) *holds for $x_1 = x_2 = 0$. Assume that $x_1, x_2 \in \mathbb{R}^n$ with $\sum_{i=1}^{2}\sum_{j=1}^{n} x_{ij}^{4n} < \frac{1}{2^{4n}}$. Thus, there exists a positive integer N such that*

$$\frac{1}{2^{4n(N+1)}} < \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{4n} < \frac{1}{2^{4nN}}. \tag{41}$$

Hence, $x_{ij}^{4n} < \sum_{i=1}^{2}\sum_{j=1}^{n} x_{ij}^{4n} < \frac{1}{2^{4nN}}$ and thus $2^N|x_{ij}| < 1$ for all $i \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$. Therefore, $2^{N-1}|x_{ij}| < 1$. If $y_1, y_2 \in \{x_{ij} | i \in \{1, 2\},\ j \in \{1, \ldots, n\}\}$, then

$$2^{N-1}|y_1 \pm y_2| < 1, \ 2^{N-1}|y_1 \pm 2y_2| < 1.$$

Since, ψ is multiquartic function on $(-1, 1)^n$, $\mathbf{D}_{CQ}\psi\left(2^l x_1, 2^l x_2\right) = 0$ for all $l \in \{0, 1, 2, \ldots, N-1\}$. The last equality and relation (41) *necessitate that*

$$\begin{aligned}
\frac{\left|\mathbf{D}_{CQ} f\left(2^l x_1, 2^l x_2\right)\right|}{\sum_{i=1}^{2}\sum_{j=1}^{n} x_{ij}^{4n}} &\le \sum_{l=N}^{\infty} \frac{\left|\mathbf{D}_{CQ}\psi\left(2^l x_1, 2^l x_2\right)\right|}{2^{4nl}\sum_{i=1}^{2}\sum_{j=1}^{n} x_{ij}^{4n}} \\
&\le \sum_{l=0}^{\infty} \frac{\mu S}{2^{4n(l+N)}\sum_{i=1}^{2}\sum_{j=1}^{n} x_{ij}^{4n}} \\
&\le \mu S 2^{4n} \sum_{l=0}^{\infty} \frac{1}{2^{4nl}} \\
&= \mu S \frac{2^{8n}}{2^{4n}-1} = \delta,
\end{aligned}$$

for all $x_1, x_2 \in \mathbb{R}^n$. If $\sum_{i=1}^{2}\sum_{j=1}^{n} x_{ij}^{4n} \ge \frac{1}{2^{4n}}$, then

$$\frac{\left|\mathbf{D}_{CQ} f\left(2^l x_1, 2^l x_2\right)\right|}{\sum_{i=1}^{2}\sum_{j=1}^{n} x_{ij}^{4n}} \le 2^{4n} \frac{2^{4n}}{2^{4n}-1} \mu S = \delta.$$

Therefore, f satisfies in (40) *for all $x_1, x_2 \in \mathbb{R}^n$. Now, suppose contrary to our claim, that there exists a number $b \in [0, \infty)$ and a multiquartic function $\mathcal{Q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $|f(r_1, \ldots, r_n) - \mathcal{Q}(r_1, \ldots, r_n)| < b \prod_{j=1}^{n} r_j^4$ for all $(r_1, \ldots, r_n) \in \mathbb{R}^n$. By Proposition 4.8, there is a constant $c \in \mathbb{R}$ such that $\mathcal{Q}(r_1, \ldots, r_n) = c \prod_{j=1}^{n} r_j^4$, and thus*

$$f(r_1, \ldots, r_n) \leq (|c| + b) \prod_{j=1}^{n} r_j^4, \tag{42}$$

for all $(r_1, \ldots, r_n) \in \mathbb{R}^n$. On the other hand, consider $p \in \mathbb{N}$ such that $(p + 1)\mu > |c| + b$. If $r = (r_1, \ldots, r_n)$ belongs to $\mathbb{R}^n$ such that $r_j \in \left(0, \frac{1}{2^p}\right)$ for all $j \in \{1, \ldots, n\}$, then $2^l r_j \in (0, 1)$ for all $l = 0, 1, \ldots, p$. Thus, we get

$$f(r_1, \ldots, r_n) = \sum_{l=0}^{\infty} \frac{\psi\left(2^l r_1, \ldots, 2^l r_2\right)}{2^{4nl}} = \sum_{l=0}^{p} \frac{\mu 2^{4nl} \prod_{j=1}^{n} r_j^4}{2^{4nl}} = (p + 1)\mu \prod_{j=1}^{n} r_j^4 > (|c| + b) \prod_{j=1}^{n} r_j^4.$$

The relation above contradicts (42).

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