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On Kantorovich Variant of Baskakov Type Operators Preserving Some Functions

Khursheed J. Ansari[a]

[a]*Department of Mathematics, College of Science, King Khalid University, 61413, Abha, Saudi Arabia*

Abstract. This paper deals with a generalization of Kantorovich variant of Baskakov type operators preserving constant function and e^{-2y}. We discuss uniform convergence properties and weighted approximation for this generalized Baskakov-Kantorovich type operators.

1. Introduction

In 2003, J.P. King [13] introduced a sequence of operators preserving test functions $e_j(y) = y^j,\ (j = 0, 2)$. Such type of modifications of the operators called after his name as King type modifications or King type operators in the later literatures. He claimed that the Bernstein type operators on $C[0, 1]$ modified by him has better approximation than the classical one on $[0, 1/3)$. Several such type of modifications has been done by many researchers in this direction since then. We will mention a few of them here; cf previous literature [2, 9, 15]. Not long ago, Acar et al [1] provided a modification of Śzasz-Mirakjan type operators that retain the function $e^{2\theta y}$, for some fixed positive θ. For functions $f \in C[0, \infty)$, such that the right-hand side below is absolutely convergent, authors introduced operators as

$$R^*_{m,\theta}(g; y) = e^{-m\alpha_m(y)} \sum_{i=0}^{\infty} \frac{(m\alpha_m(y))^i}{i!} g\left(\frac{i}{m}\right) \tag{1}$$

$y \geq 0,\ m \in \mathbb{N}$, such that the conditions

$$R^*_{m,\theta}(e^{2\theta t}; y) = e^{2\theta y} \tag{2}$$

are satisfied for all y and m. Using (1) and (2), authors found $\alpha_m(y)$. They proved uniform convergence, order of approximation via a certain weighted modulus of continuity, and a quantitative Voronovskaya-type theorem. A comparison with the classical Szász-Mirakyan operators was given. Some shape preservation properties of the new operators were discussed as well. Using a natural transformation, authors also presented a uniform error estimate for the operators in terms of the first and second-order moduli of smoothness.

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Email address: ansari.jkhursheed@gmail.com (Khursheed J. Ansari)

Chen [3], in 1989, presented with the help of a parameter $\theta > 0$, a different generalization of Baskakov operators as

$$H_m(g;y) = \sum_{i=0}^{\infty} \frac{m(m+\theta)(m+2\theta)\cdots(m+(i-1)\theta)}{i!} \frac{y^i}{(1+\theta y)^{\frac{m}{\theta}+i}} g\left(\frac{i}{m}\right). \tag{3}$$

Very recently, Yılmaz et al [26] introduced the following form of operators defined as:

$$H_m^{\varphi}(g;y) = \sum_{i=0}^{\infty} \frac{m(m+\theta)(m+2\theta)\cdots(m+(i-1)\theta)}{i!} \frac{\varphi_m(y)^i}{(1+\theta\varphi_m(y))^{\frac{m}{\theta}+i}} \int_{\frac{i}{m}}^{\frac{i+1}{m}} g(t)\mathrm{d}t, \tag{4}$$

where $g : [0,\infty) \to \mathbb{R}$ is such a mapping provided above operators are absolutely convergent for $y \geq 0,\ m \in \mathbb{N}$ with the condition

$$H_m^{\varphi}(e^{-t};y) = e^{-y} \tag{5}$$

keeps for all m and y. Authors discussed uniform convergence of this generalization by means of the modulus of continuity and establish quantitative asymptotic formula.

Erençin and Büyükdurakoğlu [5] introduced the following operators in 2014 as follows:

$$K_m(g;y) = \sum_{i=0}^{\infty} S_{m,a_m}(i,y) \frac{b_m}{d_m - c_m} \int_{\frac{i+c_m}{b_m}}^{\frac{i+d_m}{b_m}} g(t)\mathrm{d}t, \qquad y \geq 0,\ m \in \mathbb{N}, \tag{6}$$

where

$$S_{m,a_m}(i,y) = e^{-\frac{a_m y}{1+y}} \frac{P_i(m,a_m)}{i!} \frac{y^i}{(1+y)^{m+i}},$$

and $(a_m),\ (b_m),\ (c_m)$ and (d_m) are sequences of real numbers having the properties:

(i) $a_m \geq 0,\ b_m \geq 1,\ 0 \leq c_m \leq d_m \leq 1$;

(ii) $\lim_{m\to\infty} \frac{m}{b_m} = 1, \qquad \lim_{m\to\infty} \frac{a_m}{b_m} = 0.$

We remark that for $a_m = a$, $b_m = m$, $c_m = 0$ and $d_m = 1$, the operators $K_m(g;y)$ turn out to be the operators $H_m(g;y)$. Authors established some direct results and weighted approximation properties for a modification of generalized Baskakov-Kantorovich operators (6).
A good work can be seen in [10] where authors presented the differences of some positive linear operators, e.g., the difference between the Baskakov and the Szász-Mirakyan-Baskakov operators. Approximation of the continuous functions by a general class of Srivastava-Gupta operators can be read in [11]. Srivastava et al. constructed a Kantorovich type q-Szász-Mirakjan operators via Dunkl's generalization in [22]. For more results on the approximation by positive linear operators, we can refer to [14, 16–21, 23–25].

2. Construction of operator

Inspired by the above discussed work, we have tried to construct a new generalization of Kantorovich variant of Baskakov type operators preserving constant function and e^{-2y} as follows:

$$H_m^{\theta,\varphi}(g;y) = \frac{b_m}{d_m - c_m} \sum_{i=0}^{\infty} \frac{m(m+\theta)(m+2\theta)\cdots(m+(i-1)\theta)}{i!} \frac{(\varphi_m(y))^i}{(1+\theta\varphi_m(y))^{\frac{m}{\theta}+i}} \int_{\frac{i+c_m}{b_m}}^{\frac{i+d_m}{b_m}} g(t)\mathrm{d}t, \tag{7}$$

where $g : [0, \infty) \to \mathbb{R}$ is such a mapping provided above operators are absolutely convergent for $y \geq 0,\ m \in \mathbb{N}$ with the condition

$$H_m^{\theta,\varphi}(e^{-2t}; y) = e^{-2y}, \tag{8}$$

holds for all m and y. To find (φ_m) sequence, we try the famous binomial series

$$\sum_{i=0}^{\infty} \frac{(a)_i}{i!} z^i = (1-z)^{-a}, \qquad |z| < 1, \tag{9}$$

and combining (7) and (9), we have

$$\begin{aligned}
e^{-2y} &= \frac{b_m}{d_m - c_m} \sum_{i=0}^{\infty} \frac{\left(\frac{i}{\theta}\right)_i}{i!} \frac{(\theta\varphi_m(y))^i}{(1+\theta\varphi_m(y))^{\frac{m}{\theta}+i}} \int_{\frac{i+c_m}{b_m}}^{\frac{i+d_m}{b_m}} e^{-2t} \mathrm{d}t \\
&= \frac{b_m}{d_m - c_m} \frac{e^{-\frac{2c_m}{b_m}} - e^{-\frac{2d_m}{b_m}}}{2} (1+\theta\varphi_m(y))^{-\frac{m}{\theta}} \sum_{i=0}^{\infty} \frac{\left(\frac{m}{\theta}\right)_i}{i!} \frac{\left(\theta\varphi_m(y) e^{-\frac{2}{b_m}}\right)^i}{(1+\theta\varphi_m(y))^i} \\
&= \frac{b_m}{d_m - c_m} \frac{e^{-\frac{2c_m}{b_m}} - e^{-\frac{2d_m}{b_m}}}{2} (1+\theta\varphi_m(y))^{-\frac{m}{\theta}} \left(1 - \frac{\theta\varphi_m(y) e^{-\frac{2}{b_m}}}{(1+\theta\varphi_m(y))^i}\right)^{-\frac{m}{\theta}} \\
&= \frac{b_m}{d_m - c_m} \frac{e^{-\frac{2c_m}{b_m}} - e^{-\frac{2d_m}{b_m}}}{2} \left[1 + \theta\varphi_m(y)\left(1 - e^{-\frac{2}{b_m}}\right)\right].
\end{aligned}$$

Solving above relation, $\varphi_m(y)$ becomes the function

$$\varphi_m(y) = \frac{(d_m - c_m) e^{\frac{2\theta x}{m}} \left[b_m(e^{-2c_m/b_m} - e^{-2d_m/b_m})\right]^{\theta/m} - \sqrt[m]{2^\theta}}{\sqrt[m]{2^\theta}(1 - e^{-2/b_m})\theta}. \tag{10}$$

Thus, under the consideration of (10), the sequence of operators (7) takes the form

$$\begin{aligned}
H_m^{\theta,\varphi}(g; y) &= 2b_m(1 - e^{-2/b_m})^{m/\theta} \left\{e^{\frac{2\theta y}{m}} [b_m(e^{-2c_m/b_m} - e^{-2d_m/b_m})]^{\frac{\theta}{m}} - e^{-\frac{2}{b_m}}\right\}^{-m/\theta} \\
&\quad \times \sum_{i=0}^{\infty} \frac{(\frac{m}{\theta})_i}{i!} \left(\frac{e^{\frac{2\theta y}{m}} [b_m(e^{-2c_m/b_m} - e^{-2d_m/b_m})]^{\frac{\theta}{m}} - \sqrt[m]{2^\theta}}{(d_m - c_m) e^{\frac{2\theta y}{m}} [b_m(e^{-2c_m/b_m} - e^{-2d_m/b_m})]^{\theta/m} - e^{2/b_m}}\right)^i \int_{\frac{i+c_m}{b_m}}^{\frac{i+d_m}{b_m}} g(t) \mathrm{d}t.
\end{aligned}$$

3. Auxiliary results

Lemma 3.1. 1. $H_m^{\theta,\varphi}(1; y) = 1$;

2. $H_m^{\theta,\varphi}(t; y) = \dfrac{m}{b_m}\varphi_m(y) + \dfrac{c_m + d_m}{2b_m}$;

3. $H_m^{\theta,\varphi}(t^2; y) = \dfrac{m(m+\theta)}{b_m^2}\varphi_m^2(y) + \dfrac{m(1 + c_m + d_m)}{b_m^2}\varphi_m(y) + \dfrac{c_m^2 + c_m d_m + d_m^2}{3b_m^2}$;

4. $H_m^{\theta,\varphi}(t^3; y) = \dfrac{m(m+\theta)(m+2\theta)}{b_m^3}\varphi_m^3(y) + \dfrac{3m(m+\theta)(2 + c_m + d_m)}{2b_m^3}\varphi_m^2(y)$
$+ \dfrac{m}{b_m^3}\left\{c_m^2 + c_m d_m + d_m^2 + \dfrac{3}{2}(c_m + d_m) + 2\right\}\varphi_m(y) + \dfrac{c_m^3 + c_m^2 d_m + c_m d_m^2 + d_m^3}{4b_m^3}$;

5. $H_m^{\theta,\varphi}(t^4; y) = \dfrac{m(m+\theta)(m+2\theta)(m+3\theta)}{b_m^4}\varphi_m^4(y) + \dfrac{2m(m+\theta)(m+2\theta)(3 + c_m + d_m)}{b_m^4}\varphi_m^3(y)$
$+ \dfrac{m(m+\theta)}{b_m^4}\left\{2(c_m^2 + c_m d_m + d_m^2) + 6(c_m + d_m) + 7\right\}\varphi_m^2(y)$

$$+\frac{m}{b_m^4}\left\{(c_m^3+c_m^2d_m+c_md_m^2+d_m^3)+2(c_m^2+c_md_m+d_m^2)+4(c_m+d_m)+6\right\}\varphi_m(y)$$
$$+\frac{c_m^4+c_m^3d_m+c_m^2d_m^2+c_md_m^3+d_m^4}{5b_m^4}.$$

We can write the above results of Lemma 3.1 as follows:

1. $H_m^{\theta,\varphi}(1;y)=1$;
2. $H_m^{\theta,\varphi}(t;y)=\frac{m}{b_m}\varphi_m(y)+\frac{\sigma_0(m)}{2b_m}$;
3. $H_m^{\theta,\varphi}(t^2;y)=\frac{m(m+\theta)}{b_m^2}\varphi_m^2(y)+\frac{m\sigma_1(m)}{b_m^2}\varphi_m(y)+\frac{\sigma_2(m)}{3b_m^2}$;
4. $H_m^{\theta,\varphi}(t^3;y)=\frac{m(m+\theta)(m+2\theta)}{b_m^3}\varphi_m^3(y)+\frac{3m(m+\theta)\sigma_3(m)}{2b_m^3}\varphi_m^2(y)+\frac{m\sigma_4(m)}{2b_m^3}\varphi_m(y)+\frac{\sigma_5(m)}{4b_m^3}$;
5. $H_m^{\theta,\varphi}(t^4;y)=\frac{m(m+\theta)(m+2\theta)(m+3\theta)}{b_m^4}\varphi_m^4(y)+\frac{2m(m+\theta)(m+2\theta)\sigma_6(m)}{b_m^4}\varphi_m^3(y)$
$+\frac{m(m+\theta)\sigma_7(m)}{b_m^4}\varphi_m^2(y)+\frac{m\sigma_8(m)}{b_m^4}\varphi_m(y)+\frac{\sigma_9(m)}{5b_m^4}$;

where
$\sigma_0(m)=c_m+d_m$,
$\sigma_1(m)=1+c_m+d_m$,
$\sigma_2(m)=c_m^2+c_md_m+d_m^2$,
$\sigma_3(m)=2+c_m+d_m$,
$\sigma_4(m)=2(c_m^2+c_md_m+d_m^2)+3(c_m+d_m)+2$,
$\sigma_5(m)=c_m^3+c_m^2d_m+c_md_m^2+d_m^3$,
$\sigma_6(m)=3+c_m+d_m$,
$\sigma_7(m)=2(c_m^2+c_md_m+d_m^2)+6(c_m+d_m)+7$,
$\sigma_8(m)=c_m^3+c_m^2d_m+c_md_m^2+d_m^3+2(c_m^2+c_md_m+d_m^2)+2(c_m+d_m)+1$,
$\sigma_9(m)=c_m^4+c_m^3d_m+c_m^2d_m^2+c_md_m^3+d_m^4$.

Lemma 3.2.

$$H_m^{\theta,\varphi}\left((t-y)^2;y\right)=\frac{m(m+\theta)}{b_m^2}\varphi_m^2(y)+\left\{\frac{m\sigma_1(m)}{b_m^2}-\frac{2my}{b_m}\right\}\varphi_m(y)+y^2-\frac{\sigma_0(m)}{b_m}y+\frac{\sigma_2(m)}{3b_m^2};$$

$$\begin{aligned}H_m^{\theta,\varphi}\left((t-y)^4;y\right)=&\frac{m(m+\theta)(m+2\theta)(m+3\theta)}{b_m^4}\varphi_m^4(y)+\frac{2m(m+\theta)(m+2\theta)\sigma_6(m)}{b_m^4}\varphi_m^3(y)\\&+\frac{m(m+\theta)\sigma_7(m)}{b_m^4}\varphi_m^2(y)+\frac{m\sigma_8(m)}{b_m^4}\varphi_m(y)+\frac{\sigma_9(m)}{5b_m^4}\\&-4y\left\{\frac{m(m+\theta)(m+2\theta)}{b_m^3}\varphi_m^3(y)+\frac{3m(m+\theta)\sigma_3(m)}{2b_m^3}\varphi_m^2(y)+\frac{m\sigma_4(m)}{2b_m^3}\varphi_m(y)+\frac{\sigma_5(m)}{4b_m^3}\right\}\\&+6y^2\left\{\frac{m(m+\theta)}{b_m^2}\varphi_m^2(y)+\frac{m\sigma_1(m)}{b_m^2}\varphi_m(y)+\frac{\sigma_2(m)}{3b_m^2}\right\}\\&-4y^3\left\{\frac{m}{b_m}\varphi_m(y)+\frac{\sigma_0(m)}{2b_m}\right\}+y^4.\end{aligned}$$

4. Approximation Results

We will denote all real-valued uniformly continuous and bounded functions defined on $\mathbb{R}_0^+$ by $C_B(\mathbb{R}_0^+)$, with the norm $\|g\|_{C_B}=\sup_{x\in\mathbb{R}_0^+}|g(x)|$.

Let $g \in C_B(\mathbb{R}_0^+)$ and $\delta \geq 0$, the usual and second order modulus of continuity, denoted by $\omega(g,\delta)$ and $\omega_2(g,\delta)$, respectively, are defined as

$$\omega(g,\delta) = \sup_{0<|t|\leq\delta} \sup_{y+t, y \in \mathbb{R}_0^+} |g(y+t) - g(y)| \tag{11}$$

and

$$\omega_2(g,\delta) = \sup_{0<|t|\leq\delta} \sup_{y+2t, y+t, y \in \mathbb{R}_0^+} |g(y+2t) - 2g(y+t) + g(y)|. \tag{12}$$

The Peetre's K-functional [4] of $g \in C_B(\mathbb{R}_0^+)$ is defined by

$$K_2(g,\delta) := \inf_{f \in C_B^2(\mathbb{R}_0^+)} \left\{ \|g - f\|_{C_B} + \delta \|f\|_{C_B^2} \right\}, \tag{13}$$

where

$$C_B^2(\mathbb{R}_0^+) := \left\{ f \in C_B(\mathbb{R}_0^+) :\ f',\ f'' \in C_B(\mathbb{R}_0^+) \right\},$$

and the norm $\|f\|_{C_B^2} := \|f\|_{C_B} + \|f'\|_{C_B} + \|f''\|_{C_B}$. It is clear that the following inequality:

$$K_2(g,\delta) \leq M\omega_2(g, \sqrt{\delta}), \tag{14}$$

is valid, for all $\delta \geq 0$. The constant M is independent of g and δ.

Lemma 4.1. *([8]) Let $g \in C^2[0,\infty)$ and $\{J_m(g;y)\}_{m\geq1}$ be a sequence of positive linear operators with the property $J_m(1;y) = 1$. Then*

$$|J_m(g;y) - g(y)| \ \leq\ \|g'\| \sqrt{J_m\big((t-y)^2;y\big)} + \frac{1}{2}\|g''\| J_m\big((t-y)^2;y\big).$$

Lemma 4.2. *([28]) Let $g \in C[a,b]$ and $h \in \left(0, \frac{b-a}{2}\right)$. Let g_h be the second order Steklov function attached to the function g. Then the following inequalities are satisfied:*

$$\| g_h - g \| \ \leq\ \frac{3}{4}\omega_2(g;h), \tag{15}$$

$$\| g_h'' \| \ \leq\ \frac{3}{2}\frac{1}{h^2}\omega_2(g;h). \tag{16}$$

Now, we consider the following auxiliary operator

$$\widehat{H_m^{\theta,\varphi}}(g;y) = H_m^{\theta,\varphi}(g;y) - g\left(\frac{m}{b_m}\varphi_m(y) + \frac{\sigma_0(m)}{2b_m}\right) + g(y)$$

where $\sigma_0(m)$ is given as in Lemma 3.2.

Lemma 4.3. *Let $g \in C_B^2[0,\infty)$. Then, we have*

$$\left|\widehat{H_m^{\theta,\varphi}}(g;y) - g(y)\right| \ \leq\ \delta_m(x)\|g''\|,$$

where

$$\delta_m(y) = H_m^{\theta,\varphi}\left((t-y)^2;y\right) + \left(\left(\frac{m}{b_m}\varphi_m(y) - y\right) + \frac{\sigma_0(m)}{2b_m}\right)^2.$$

Proof. By the definition of the sequence of operators $\widehat{H_m^{\theta,\varphi}}(g;y)$ and Lemma 3.1, we get

$$\begin{aligned}\widehat{H_m^{\theta,\varphi}}(t-y;y) &= H_m^{\theta,\varphi}(t-y;y) - \left(\frac{m}{b_m}\varphi_m(y) + \frac{\sigma_0(m)}{2b_m} - y\right)\\ &= H_m^{\theta,\varphi}(t;y) - yH_m^{\theta,\varphi}(1;y) - \left(\frac{m}{b_m}\varphi_m(y) + \frac{\sigma_0(m)}{2b_m} - y\right)\\ &= 0.\end{aligned}$$

Let $g \in C_B^2[0,\infty)$ and $y \in C_B^2[0,\infty)$. By Taylor's formula of g,

$$g(t) - g(y) = (t-y)g'(y) + \int_y^t (t-u)g''(u)\mathrm{d}u, \quad t \in [0,\infty),$$

one may write

$$\begin{aligned}\widehat{H_m^{\theta,\varphi}}(g;y) - g(y) &= g'(y)\widehat{H_m^{\theta,\varphi}}(t-y;y) + \widehat{H_m^{\theta,\varphi}}\left(\int_y^t (t-u)g''(u)\mathrm{d}u;x\right)\\ &= \widehat{H_m^{\theta,\varphi}}\left(\int_y^t (t-u)g''(u)\mathrm{d}u;y\right)\\ &= H_m^{\theta,\varphi}\left(\int_y^t (t-u)g''(u)\mathrm{d}u;y\right) - \int_y^{\frac{m}{b_m}\varphi_m(y)+\frac{\sigma_0(m)}{2b_m}} \left(\frac{m}{b_m}\varphi_m(y) + \frac{\sigma_0(m)}{2b_m} - u\right)g''(u)\mathrm{d}u.\end{aligned}$$

Now, using the following inequalities

$$\left|\int_y^t (t-u)g''(u)\mathrm{d}u\right| \leq (t-y)^2\|g''\|$$

and

$$\begin{aligned}&\left|\int_y^{\frac{m}{b_m}\varphi_m(y)+\frac{\sigma_0(m)}{2b_m}} \left(\frac{m}{b_m}\varphi_m(y) + \frac{\sigma_0(m)}{2b_m} - u\right)g''(u)\mathrm{d}u\right|\\ &\qquad\leq \left(\frac{m}{b_m}\varphi_m(y) + \frac{\sigma_0(m)}{2b_m} - y\right)^2\|g''\| \ \leq \ \left(\left(\frac{m}{b_m}\varphi_m(y) - y\right) + \frac{\sigma_0(m)}{2b_m}\right)^2\|g''\|\end{aligned}$$

we reach to

$$\widehat{H_m^{\theta,\varphi}}(g;y) - g(y) \leq \left\{H_m^{\theta,\varphi}\left((t-y)^2;y\right) + \left(\left(\frac{m}{b_m}\varphi_m(y) - y\right) + \frac{\sigma_0(m)}{2b_m}\right)^2\right\}\|g''\| \ = \ \delta_m(y)\|g''\|.$$

We get our desired inequality. □

Theorem 4.4. *Let $g \in C_B[0,\infty)$, then for all $y \in [0,\infty)$, there exists a constant $A > 0$ such that*

$$|H_m^{\theta,\varphi}(g;y) - g(y)| \ \leq \ A\omega_2\left(g;\sqrt{\delta_m(y)}\right) + \omega\left(g;\left|\frac{m}{b_m}\varphi_m(y) - y\right| + \frac{\sigma_0(m)}{2b_m}\right),$$

where $\delta_m(y)$ is defined as in Lemma 4.3.

Proof. By means of the definitions of the operators $\widehat{H_m^{\theta,\varphi}}$ and $H_m^{\theta,\varphi}$, we have

$$\left|H_m^{\theta,\varphi}(g;y)-g(y)\right| \le \left|\widehat{H_m^{\theta,\varphi}}(g-f;y)\right| + \left|(g-f)(y)\right| + \left|\widehat{H_m^{\theta,\varphi}}(f;y)-f(y)\right| + \left|g\left(\frac{m}{b_m}\varphi_m(y)+\frac{\sigma_0(m)}{2b_m}\right)-g(y)\right|$$

and

$$\begin{aligned}\left|\widehat{H_m^{\theta,\varphi}}(g;y)\right| &\le \left|H_m^{\theta,\varphi}(g;y)\right| + 2\|g\| \\ &\le \|g\|H_m^{\theta,\varphi}(1;y) + 2\|g\| = 3\|g\|.\end{aligned}$$

Thus, we may conclude

$$\left|H_m^{\theta,\varphi}(g;y)-g(y)\right| \le 4\|g-f\| + \left|\widehat{H_m^{\theta,\varphi}}(f;y)-f(y)\right| + \left|g\left(\frac{m}{b_m}\varphi_m(y)+\frac{\sigma_0(m)}{2b_m}\right)-g(y)\right|.$$

In the light of Lemma 4.3, one gets

$$\left|H_m^{\theta,\varphi}(g;y)-g(y)\right| \le 4\|g-f\| + \delta_m(y)\|f''\| + \omega\left(g;\left|\frac{m}{b_m}\varphi_m(y)-y\right|+\frac{\sigma_0(m)}{2b_m}\right).$$

Therefore, taking the infimum overall $f \in C_B^2[0,\infty)$ on the right hand side of the inequality and considering (14), we find that

$$\begin{aligned}\left|H_m^{\theta,\varphi}(g;y)-g(y)\right| &\le 4K_2\left(g;\sqrt{\delta_m(y)}\right) + \omega\left(g;\left|\frac{m}{b_m}\varphi_m(y)-y\right|+\frac{\sigma_0(m)}{2b_m}\right) \\ &\le 4C\omega_2\left(g;\sqrt{\delta_m(y)}\right) + \omega\left(g;\left|\frac{m}{b_m}\varphi_m(y)-y\right|+\frac{\sigma_0(m)}{2b_m}\right) \\ &= A\omega_2\left(g;\sqrt{\delta_m(y)}\right) + \omega\left(g;\left|\frac{m}{b_m}\varphi_m(y)-y\right|+\frac{\sigma_0(m)}{2b_m}\right)\end{aligned}$$

which completes the proof of the theorem. □

Theorem 4.5. *Let* $0<\gamma<1$ *and* $g \in C_B[0,\infty)$. *Then, if* $g \in Lip_M(\gamma)$, *i.e. the inequality*

$$|g(t)-g(y)| \;\le\; M|t-y|^{\gamma}, \quad y,t\in[0,\infty)$$

holds, then for each $y \in [0,\infty)$, *we have*

$$\left|H_m^{\theta,\varphi}(g;y)-g(y)\right| \;\le\; \delta_m^{\gamma/2}(y),$$

where $\delta_m(y) = H_m^{\theta,\varphi}\left((t-y)^2;y\right)$, *and* $M>0$ *is a constant.*

Proof. Let $g \in C_B[0,\infty)\cap Lip_M(\gamma)$. By the linearity and monotonicity of the operators $H_m^{\theta,\varphi}$, we get

$$\begin{aligned}\left|H_m^{\theta,\varphi}(g;y)-g(y)\right| &\le H_m^{\theta,\varphi}\left(|g(t)-g(y)|;y\right) \\ &\le MH_m^{\theta,\varphi}\left(|t-y|^{\gamma};y\right) \\ &\le MH_m^{\theta,\varphi}\left((t-y)^2;y\right)^{\gamma/2} H_m^{\theta,\varphi}\left(1^2;y\right)^{(2-\gamma)/2} \\ &= MH_m^{\theta,\varphi}\left((t-y)^2;y\right)^{\gamma/2} \le M\delta_n^{\gamma/2}(y)\end{aligned}$$

which completes the proof of the theorem. □

Now, we compute the rate of convergence of the operators $H_m^{\theta,\varphi}$ with the help of the second order modulus of smoothness.

Theorem 4.6. *Let g be defined on $[0,\infty)$ and $g \in C[0,a]$, then the rate of convergence of the sequence of operators $H_m^{\theta,\varphi}$ is governed by*

$$|H_m^{\theta,\varphi}(g;y) - g(y)| \leq \frac{2}{a}h^2 \parallel g \parallel + \frac{3}{4}(2 + a + h^2)\omega_2(g;h)$$

where $h = \sqrt[4]{H_m^{\theta,\varphi}\left((t-y)^2;y\right)}$.

Proof. Let us denote the second order Steklov function of g as g_h. Because of $H_m^{\theta,\varphi}(1;y) = 1$, one can write

$$|H_m^{\theta,\varphi}(g;y) - g(y)| \leq 2 \parallel g_h - g \parallel + |H_m^{\theta,\varphi}(g_h;y) - g_h(y)|. \tag{17}$$

From the Landau inequality and applying (16), we may derive the following inequality

$$\begin{aligned} \parallel g_h' \parallel &\leq \frac{2}{a} \parallel g_h \parallel + \frac{a}{2} \parallel g_h'' \parallel \\ &\leq \frac{2}{a} \parallel g \parallel + \frac{3a}{4}\frac{1}{h^2}\omega_2(g;h). \end{aligned} \tag{18}$$

By virtue of $g_h \in C^2[0,a]$, if we use the Lemma 4.1, equations (16) and (18) we obtain the estimate

$$\begin{aligned} &|H_m^{\theta,\varphi}(g_h;y) - g_h(y)| \\ &\leq \left(\frac{2}{a} \parallel g_h \parallel + \frac{3a}{4}\frac{1}{h^2}\omega_2(g;h)\right)\sqrt{H_m^{\theta,\varphi}\left((t-y)^2;y\right)} + \frac{3}{4}\frac{1}{h^2}H_m^{\theta,\varphi}\left((t-y)^2;y\right)\omega_2(g;h). \end{aligned} \tag{19}$$

Choosing $h = \sqrt[4]{H_m^{\theta,\varphi}\left((t-y)^2;y\right)}$ in inequality (19) and then considering the last term in (17), so the proof is completed. □

Furthermore, in the case, g is a smooth function, the following theorem gives the estimation of an approximation to the function g.

Theorem 4.7. *For $g \in C_B^2(\mathbb{R}_0^+)$, we have*

$$|H_m^{\theta,\varphi}(g;y) - g(y)| \leq \frac{1}{b_m}\mu(y) \parallel g \parallel_{C_B^2} \tag{20}$$

where

$$\mu(y) = \left|\frac{m(m+\theta)}{b_m}\varphi_m^2(y) - 2my\varphi_m(y) + \frac{m}{\varphi_m}\left(\sigma_1(m) + b_m\right)\varphi_m(y) + \left(b_m(y-1) - \sigma_0(m)\right)y + \frac{3b_m\sigma_0(m) + 2\sigma_2(m)}{6b_m}\right|.$$

Proof. From the Taylor's formula

$$g(t) = g(y) + g'(y)(t-y) + \frac{g''(\eta)}{2}(t-y)^2$$

where $\eta \in (y,t)$. Due to linearity property of the operators $H_m^{\theta,\varphi}$, one can write

$$H_m^{\theta,\varphi}(g;y) - g(y) = g'(y)H_m^{\theta,\varphi}\left((t-y);y\right) + \frac{g''(\eta)}{2}H_m^{\theta,\varphi}\left((t-y)^2;y\right).$$

From this fact and using Lemma 3.1, we obtain

$$\begin{aligned} |H_m^{\theta,\varphi}(g;y) - g(y)| \leq &\left|\left(\frac{m}{b_m}\varphi_m(y) - y\right) + \frac{\sigma_0(m)}{2b_m}\right| \parallel g' \parallel_{C_B} \\ &+ \left|\frac{m(m+\theta)}{b_m^2}\varphi_m^2(y) + \left(\frac{m\sigma_1(m)}{b_m^2} - \frac{2my}{b_m}\right)\varphi_m(y) + y^2 - \frac{\sigma_0(m)}{b_m}y + \frac{\sigma_2(m)}{3b_m^2}\right| \parallel g'' \parallel_{C_B}. \end{aligned}$$

We immediately get (20) by just a simple calculation of the above inequality. □

5. Weighted Approximation

Now with the mean of weighted Korovkin type theorem [6, 7], we will develop convergence properties of the operators $H_m^{\theta,\varphi}$. Aiming this, we recall some notations and definitions.

Let $\rho(y) = 1 + y^2$ and $B_\rho[0,\infty)$ be the space of all functions with the property

$$|g(y)| \leq M_g \rho(y),$$

where $y \in [0,\infty)$ and M_g, depending only on g, is a positive constant. $B_\rho[0,\infty)$ is provided with the norm

$$\|g\|_\rho = \sup_{y\in[0,\infty)} \frac{|g(y)|}{1+y^2}.$$

The space of all continuous functions belonging to $B_\rho[0,\infty)$ is denoted by $C_\rho[0,\infty)$. By $C_\rho^0[0,\infty)$, the subspace of all functions $g \in C_\rho[0,\infty)$ is denoted by for which

$$\lim_{y\to\infty} \frac{|g(y)|}{\rho(y)} < \infty.$$

Theorem 5.1. *([6, 7]) Let $\{A_m\}$ be linear positive operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ and satisfying the conditions*

$$\lim_{m\to\infty} \left\|A_m\left(t^\nu; y\right) - y^\nu\right\|_\rho = 0, \qquad \nu = 0, 1, 2.$$

Then for any function $g \in C_\rho^0[0,\infty)$,

$$\lim_{m\to\infty} \left\|A_m\left(g; y\right) - g(y)\right\|_\rho = 0.$$

Note that positive linear operators A_m acts from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ if and only if

$$\left\|A_m\left(\rho; y\right)\right\|_\rho \leq M_\rho,$$

where M_ρ is a positive constant. This fact is a simple result of the necessary and sufficient condition that $A_m(\rho; y) \leq M_\rho(y)$ given in [6, 7].

Theorem 5.2. *Let $\{H_m^{\theta,\varphi}\}$ be the sequence of positive linear operators defined by* (7)*. Then, for each $g \in C_\rho^0[0,\infty)$, we have*

$$\lim_{m\to\infty} \left\|H_m^{\theta,\varphi}\left(g; y\right) - g(y)\right\|_\rho = 0.$$

Proof. Using Lemma 3.1, we may write

$$\begin{aligned}
\frac{H_m^{\theta,\varphi}(\rho; y)}{1+y^2} &= \frac{H_m^{\theta,\varphi}(1; y) + H_m^{\theta,\varphi}(t^2; y)}{1+y^2} \\
&= \frac{1}{1+y^2}\left\{1 + \frac{m(m+\theta)}{b_m^2}\varphi_m^2(y) + \frac{m\sigma_1(m)}{b_m^2}\varphi_m(y) + \frac{\sigma_2(m)}{3b_m^2}\right\} \\
&\leq 1 + \frac{m(m+\theta)}{b_m^2} + \frac{m\sigma_1(m)}{b_m^2} + \frac{\sigma_2(m)}{3b_m^2}.
\end{aligned}$$

Thus

$$\sup_{0\leq<\infty} \frac{|H_m^{\theta,\varphi}(\rho; y)|}{1+y^2} \leq 1 + \frac{m(m+\theta)}{b_m^2} + \frac{m\sigma_1(m)}{b_m^2} + \frac{\sigma_2(m)}{3b_m^2}.$$

Since $\lim_{m\to\infty}\frac{m}{b_m}=1$, we have $\lim_{m\to\infty}\frac{1}{b_m}=0$. Thus under the conditions (*i*) and (*ii*), there exists a positive constant M^* such that

$$\frac{m(m+\theta)}{b_m^2}+\frac{m\sigma_1(m)}{b_m^2}+\frac{\sigma_2(m)}{3b_m^2}<M^*$$

for each m. Hence, we get

$$\|H_m^{\theta,\varphi}(\rho;y)\|\leq 1+M^*$$

which shows that $H_m^{\theta,\varphi}$ are positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$. In order to complete the proof, it is enough to prove that the conditions of theorem A

$$\lim_{m\to\infty}\|H_m^{\theta,\varphi}(t^\nu;y)-y^\nu\|_\rho=0,\quad \nu=0,1,2$$

are satisfied. It is clear that

$$\lim_{m\to\infty}\|H_m^{\theta,\varphi}(1;y)-1\|_\rho=0.$$

By Lemma 2.2, we have

$$\begin{aligned}\|H_m^{\theta,\varphi}(t;y)-y\|_\rho &= \sup_{0\leq y<\infty}\left|\frac{m}{b_m}\frac{\varphi_m(y)}{1+y^2}-\frac{y}{1+y^2}+\frac{\sigma_0(m)}{2b_m}\frac{1}{1+y^2}\right|\\ &\leq \sup_{0\leq y<\infty}\left|\left(\frac{m}{b_m}-1\right)\frac{y}{1+y^2}+\frac{\sigma_0(n)}{2b_m}\frac{1}{1+y^2}\right|\\ &\leq \left|\frac{m}{b_m}-1\right|+\frac{\sigma_0(m)}{2b_m}.\end{aligned}$$

Thus, taking into consideration the conditions (*i*) and (*ii*), we can conclude that

$$\lim_{m\to\infty}\|H_m^{\theta,\varphi}(t;y)-y\|_\rho=0.$$

Similarly, one gets

$$\begin{aligned}\|H_m^{\theta,\varphi}(t^2;y)-y^2\|_\rho &= \sup_{0\leq y<\infty}\left|\frac{m(m+\theta)}{b_m^2}\frac{\varphi_m^2(y)}{1+y^2}-\frac{y^2}{1+y^2}+\frac{m\sigma_1(m)}{b_m^2}\frac{\varphi_m(y)}{1+y^2}+\frac{\sigma_2(m)}{3b_m^2}\frac{1}{1+y^2}\right|\\ &\leq \sup_{0\leq y<\infty}\left|\left(\frac{m(m+\theta)}{b_m^2}-1\right)\frac{y^2}{1+y^2}+\frac{m\sigma_1(m)}{b_m^2}\frac{y}{1+y^2}+\frac{\sigma_2(m)}{3b_m^2}\frac{1}{1+y^2}\right|\\ &\leq \left|\frac{m(m+\theta)}{b_m^2}-1\right|+\frac{m\sigma_1(m)}{b_m^2}+\frac{\sigma_2(m)}{3b_m^2}\end{aligned}$$

which leads to

$$\lim_{n\to\infty}\|H_m^{\theta,\varphi}(t^2;y)-y^2\|_\rho=0.$$

Thus the proof is completed. □

It is well-known that the first and second order modulus of continuity in general do not tend to zero with $\delta\to 0$ on $\mathbb{R}_0^+$, so we use the following weighted modulus of continuity [27]:

$$\Omega(g,\delta)=\sup_{y\geq 0}\sup_{0<|t|\leq\delta}\frac{|g(y+t)-g(y)|}{1+(y+t)^2}. \tag{21}$$

We have the following lemma:

Lemma 5.3. *([27]) If $g \in C_2^*(\mathbb{R}_0^+)$, then*

1. *$\Omega(g,\delta)$ is a monotonic increasing function of δ,*
2. *$\lim_{\delta\to 0}\Omega(g,\delta) = 0$,*
3. *For any $\lambda \in (0,\infty)$, $\Omega(g,\lambda\delta) \le (1+\lambda)\Omega(g,\delta)$.*

For $t, y \in \mathbb{R}_0^+$ and $\delta \ge 0$, using (21) and Lemma 5.3, we get

$$|g(t)-g(y)| \le 2(1+y^2)(1+(t-y)^2)\left(1+\frac{|t-y|}{\delta}\right)\Omega(g,\delta). \tag{22}$$

Theorem 5.4. *Let $\{H_m^{\theta,\varphi}\}$ be a sequence of positive linear operators. Then, for each $g \in C_\rho^0[0,\infty)$, we have*

$$\sup_{0\le y<\infty}\frac{|H_m^{\theta,\varphi}(g;y)-g(y)|}{1+y^2} \le C^*\Omega_2(g;\delta_m)$$

where C^* is a positive constant and $\delta_m = \sqrt[4]{H_m^{\theta,\varphi}((t-y)^4;y)}$.

Proof.

$$\begin{aligned}|g(t)-g(y)| &\le \left(1+(y+|t-y|)^2\right)\left(1+\frac{|t-y|}{\delta_m}\right)\Omega_2(g;\delta_m)\\ &\le 2(1+y^2)\left(1+(t-y)^2\right)\left(1+\frac{|t-y|}{\delta_m}\right)\Omega_2(g;\delta_m).\end{aligned}$$

By using the monotonicity of $H_m^{\theta,\varphi}$ and the following inequality (see [12])

$$\left(1+(t-y)^2\right)\left(1+\frac{|t-y|}{\delta_m}\right) \le 2(1+\delta_m^2)\left(1+\frac{(t-y)^4}{\delta_m^4}\right)$$

one gets

$$\begin{aligned}|H_m^{\theta,\varphi}(g;y)-g(y)| &\le 2(1+y^2)H_m^{\theta,\varphi}\left(\left(1+(t-y)^2\right)\left(1+\frac{|t-y|}{\delta_m}\right)\right)\Omega_2(g;\delta_m)\\ &\le 4(1+\delta_m^2)(1+y^2)H_m^{\theta,\varphi}\left(1+\frac{(t-y)^4}{\delta_m^4}\right)\Omega_2(g;\delta_m)\\ &= 4(1+\delta_m^2)(1+y^2)\left\{1+\frac{1}{\delta_m^4}H_m^{\theta,\varphi}\left((t-y)^4;y\right)\right\}\Omega_2(g;\delta_m)\\ &\le C_1(1+y^2)\left\{1+\frac{1}{\delta_m^4}H_m^{\theta,\varphi}\left((t-y)^4;y\right)\right\}\Omega_2(g;\delta_m)\end{aligned}$$

if $\delta_n^4 = H_m^{\theta,\varphi}\left((t-y)^4;y\right)$ implies $\delta_m = \sqrt[4]{H_m^{\theta,\varphi}((t-y)^4;y)}$ which implies that

$$|H_m^{\theta,\varphi}(g;y)-g(y)| \le 2C_1(1+y^2)\Omega_2(g;\delta_m)$$

$$\frac{|H_m^{\theta,\varphi}(g;y)-g(y)|}{1+y^2} \le C^*(1+y^2)\Omega_2(g;\delta_m),$$

where $C^* = 2C_1$. □

In Theorem 5.2, we have shown that $H_m^{\theta,\varphi}$ converges to g in the weighted space $C_\rho[0,\infty)$. But in Theorem 5.4, we have computed the rate of convergence for these operators in the weighted space $C_\rho[0,\infty)$.

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References

[1] T. Acar, A. Aral, H. Gonska, On Szász-Mirakyan operators preserving e^{2ax}, $a > 0$. Mediterr J. Math. 14(1) (2017) 1-14.
[2] M. Birou, A note about some general King-type operators. Ann Tiberiu Popoviciu Semin Funct Equ Approx Convexity. 12 (2014) 3-16.
[3] W.Z. Chen, Approximation Theory of Operators, Xiamen University Publishing House. China: Xia-men, 1989.
[4] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer-Verlag, New York, 1987.
[5] A. Erençin, S. Büyükdurakoğlu, A modification of generalized Baskakov-Kantorovich operators, Stud. Univ. Babeş-Bolyai Math. 59(3) (2014) 351-364.
[6] A.D. Gadjiev, The convergence problem for a sequence of positive linear operators on bounded sets and theorems analogous to that of P.P. Korovkin, Dokl. Akad. Nauk SSSR 218 (5) (1974); Transl. in Soviet Math. Dokl. 15(5) (1974) 1433-1436.
[7] A.D. Gadjiev, On P.P. Korovkin type theorems, Mat. Zametki 20 (1976) 781-786; Transl. in Math. Notes (5-6) (1978) 995-998.
[8] I. Gavrea, I. Rasa, Remarks on some quantitative Korovkin-type results, Rev. Anal. Numer. Theor. Approx. 22(2) (1993) 173-176.
[9] H. Gonska, P. Pitul, I. Rąsa, General King-type operators. Results Math. 53 (2009) 279-286.
[10] V. Gupta, A.M. Acu, H.M. Srivastava, Difference of Some Positive Linear Approximation Operators for Higher-Order Derivatives, Symmetry, 12(6) (2020) 915.
[11] V. Gupta, H.M. Srivastava, A General Family of the Srivastava-Gupta Operators Preserving Linear Functions. European Journal of Pure and Applied Mathematics, 11(3) (2018) 575-579.
[12] N. İspir, On modified Baskakov operators on weighted spaces, Turkish J. Math. 25(3) (2001) 355-365.
[13] J.P. King, Positive linear operators which preserve x^2 . Acta Math Hungar. 99(3) (2003) 203-208.
[14] V.N. Mishra, R.B. Gandhi, Study of sensitivity of parameters of Bernstein–Stancu operators, Iranian Journal of Science and Technology, Transactions A: Science, 43 (2019) 2891-2897.
[15] M. Mursaleen, K.J. Ansari, Approximation of q-Stancu-Beta operators which preserve x^2, Bull. Malays. Math. Sci. Soc. 40 (2017) 1479-1491.
[16] M. Mursaleen, K.J. Ansari, A. Khan, Approximation properties and error estimation of q-Bernstein shifted operators, Numerical Algorithms, 84 (2020) 207–227.
[17] M. Mursaleen, Md. Nasiruzzaman, H.M. Srivastava, Approximation by bicomplex beta operators in compact $\mathbb{BC}$-disks, Math. Methods Appl. Sci. 39 (2016) 2916-2929.
[18] F. Özger, Weighted statistical approximation properties of univariate and bivariate λ-Kantorovich operators, Filomat, 33(11) (2019) 473-3486.
[19] F. Özger, On new Bézier bases with Schurer polynomials and corresponding results in approximation theory, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat, 69(1) (2020) 376-393.
[20] H.M. Srivastava, V. Gupta, Rate of convergence for the Bézier variant of the Bleimann–Butzer–Hahn operators, Applied Mathematics Letters, 18 (2005) 849–857.
[21] H.M. Srivastava, G. İçöz, B. Çekim, Approximation Properties of an Extended Family of the Szász–Mirakjan Beta-Type Operators, Axioms, 8(4) (2019) 111.
[22] H.M. Srivastava, M. Mursaleen, Abdullah M. Alotaibi, Md. Nasiruzzaman, A.A.H. Al-Abied, Some approximation results involving the q-Szász–Mirakjan–Kantorovich type operators via Dunkl's generalization, Math. Methods Appl. Sci. 40 (2017) 5437-5452.
[23] H.M. Srivastava, M. Mursaleen, M. Nasiruzzaman, Approximation by a Class of q-Beta Operators of the Second Kind Via the Dunkl-Type Generalization on Weighted Spaces, Complex Anal. Oper. Theory, 13 (2019) 1537–1556.
[24] H.M. Srivastava, F. Özger, S.A. Mohiuddine, Construction of Stancu-Type Bernstein Operators Based on Bézier Bases with Shape Parameter λ, Symmetry, 11(3) (2019) 316.
[25] H.M. Srivastava, X-M Zeng, Approximation by means of the Szász-Bézier integral operators, Internat. J. Pure Appl. Math. 14 (2004), 283-294.
[26] Ö.G. Yılmaz, V. Gupta, A. Aral, A note on Baskakov-Kantorovich type operators preserving e^{-x}, Math Meth Appl Sci. (2018) 1-7.
[27] I. Ÿuksel, N. Ispir, Weighted approximation by a certain family of summation integral-type operators, Comput. Math. Appl., 52(10-11) (2006) 1463-1470.
[28] V.V. Zhuk, Functions of the Lip1 class and S. N. Bernstein's polynomials, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. 1 (1989) 25-30.