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Approximation by q-Bernstein-Stancu-Kantorovich Operators with Shifted Knots of Real Parameters

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Abstract. Our main purpose of this article is to study the convergence and other related properties of q-Bernstein-Kantorovich operators including the shifted knots of real positive numbers. We design the shifted knots of Bernstein-Kantorovich operators generated by the basic q-calculus. More precisely, we study the convergence properties of our new operators in the space of continuous functions and Lebesgue space. We obtain the degree of convergence with the help of modulus of continuity and integral modulus of continuity. Furthermore, we establish the quantitative estimates of Voronovskaja-type.

1. Introduction and Basic Definitions

In 1912, S. N. Bernstein gave a very short and easy proof of Weierstrass approximation theorem by introducing the Bernstein polynomials [3]. The q-analog of Bernstein polynomials were introduced by Lupaş [14] and Phillips [24], separately. Since then several polynomials were generalized and studied by using q-calculus. For example related to our present theme, q-Bernstein shifted operators [20], q-Bernstein-Kantorovich operators [21], Bernstein-Kantorovich operators based on (p,q)-calculus [19], and other related operators [1], [4], [13], [25], etc..

The Bernstein operators have been extended in various forms for the purpose of approximating functions of different classes by replacing the point evaluation functionals by some integrals. To deal with the approximation in $L_p[0,1]$ $(1 \leq p < \infty)$ for which Bernstein operators are unsuitable, Bernstein-Kantorovich operators [12] and Bernstein-Durrmeyer operators [6] were introduced. Jackson type estimation by Kantorovich type positive linear operators in L_p-spaces is studied in [28] while weighted L_p-approximation was studied in [7].

In 2004, the Stancu variant of Bernstein-Kantorovich operators [2] were defined as follows:

$$\mathcal{J}_r^{\mu_1,\nu_1}(g;\xi) = (r+1+\nu_1)\sum_{s=0}^{r}\binom{r}{s}\xi^s(1-\xi)^{r-s}\int_{\frac{s+\mu_1}{r+1+\nu_1}}^{\frac{s+1+\mu_1}{r+1+\nu_1}} f(t)\mathrm{d}t,\ (0 \leq \mu_1 \leq \nu_1). \tag{1}$$

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Recently, a variety of Bernstein-Kantorovich operators have been studied in [18], [23] and [17]. While the operators (1) with shifted knots were studied by Gadjiev et al. [8] for the operators $\mathcal{S}_{r,\mu_2}^{\nu_2} : C[0,1] \to C\left[\frac{\mu_2}{r+\nu_2}, \frac{r+\mu_2}{r+\nu_2}\right]$ defined by

$$\mathcal{S}_{r,\mu_2}^{\nu_2}(f;\xi) = \left(\frac{r+\nu_2}{r}\right)^r \sum_{s=0}^{r} \binom{r}{s} \left(\xi - \frac{\mu_2}{r+\nu_2}\right)_q^s \left(\frac{r+\mu_2}{r+\nu_2} - \xi\right)_q^{r-s} f\left(\frac{t+\mu_1}{r+\nu_1}\right),$$

where $\frac{\mu_2}{r+\nu_2} \leq \xi \leq \frac{r+\mu_2}{r+\nu_2}$ and $\mu_k, \nu_k (k=1,2)$ are positive real numbers satisfying $0 \leq \mu_2 \leq \mu_1 \leq \nu_1 \leq \nu_2$. Moreover, Sucu et al. [27] introduced the Bernstein-Stancu-Kantorovich operators with shifted knots by

$$\begin{aligned} \mathcal{K}_{r,\mu_1,\nu_1}^{\mu_2,\nu_2}(f;\xi) &= \left(\frac{r+\nu_2+1}{r+1}\right)^{r+1} \sum_{s=0}^{r} \left(\int_{\frac{r+\mu_1}{r+1+\nu_1}}^{\frac{r+\mu_1+1}{r+1+\nu_1}} f(\xi)\mathrm{d}\xi \right) \\ &\times \binom{r}{s} \left(\xi - \frac{\mu_2}{r+\nu_2+1}\right)^s \left(\frac{r+\mu_2+1}{r+\nu_2+1} - \xi\right)^{r-s}. \end{aligned}$$

Here we introduce the Bernstein-Stancu-Kantorovich operators with shifted knots by using q-integers and then investigate the approximation in the L_p-spaces ($p \geq 1$). Using of shifted knots of real parameters has certainly some advantage that one can do approximation on some interval as well as on its subinterval. Some basic results for approximation as well as rate of convergence of the introduced operators are established. We obtain the degree of approximation of our new operators by using the modulus of continuity and integral modulus of continuity. We give some direct theorems in L_p-spaces. Furthermore, we obtain the approximation in Lipschitz spaces and also establishe the quantitative estimates of the Voronovskaja-type.

The q-calculus emerged as a very useful tool and a very fruitful connection between Mathematics and Physics. We recall here some basic definitions and notations about the q-calculus. We take $\mathbb{C}$ as the set of complex numbers and $\mathbb{N}$ the set of positive integers.

Definition 1.1. *Let $q \in \mathbb{C} \setminus \{0,1\}$. Then the q-number is defined by*

$$[\theta]_q = \begin{cases} \dfrac{1-q^\theta}{1-q} & (\theta \in \mathbb{C} \setminus \{0\}) \\ 1 & (\theta = 0) \\ \sum\limits_{s=0}^{r-1} q^s = 1 + q + q^2 + \cdots + q^s & (\theta \in \mathbb{N}). \end{cases} \tag{2}$$

Definition 1.2. *For number $q \in \mathbb{C} \setminus \{0,1\}$, the q-factorial is defined by*

$$[\theta]_q! = \begin{cases} 1 & (\theta = 0) \\ \prod\limits_{\theta=1}^{r} [\theta]_q & (\theta \in \mathbb{N}). \end{cases} \tag{3}$$

Definition 1.3. *For $q \in \mathbb{C} \setminus \{0,1\}$, and $0 \leq s \leq r$ the q-binomial coefficient is defined by*

$$\left[{r \atop s}\right] = \frac{[r]_q!}{[s]_q!\,[r-s]_q!}.$$

The q-analogue of $(1+\xi)^r$ is the polynomial given by

$$(1+\xi)_q^{r-s} = \begin{cases} (1+\xi)(1+q\xi)\cdots(1+q^{r-s-1}\xi) & (r,s \in \mathbb{N}) \\ 1 & (r = s = 0). \end{cases} \tag{4}$$

Definition 1.4. *[9, 11] The q-Jackson integral from* 0 *to* $A \in \mathbb{R}$ *is defined by*

$$\int_0^A f(\xi)\mathrm{d}_q\xi = A(1-q)\sum_{s=0}^{\infty} f(Aq^s)q^s, \tag{5}$$

while, the more general q-Jackson integral on interval $[A,B]$ *is given by*

$$\int_A^B f(z)\mathrm{d}_q z = \int_0^B f(z)\mathrm{d}_q z - \int_0^A f(z)\mathrm{d}_q z. \tag{6}$$

Theorem 1.5. *[11]* ***(Fundamental theorem of*** q***-calculus)*** *Let* φ *be the anti q-derivative of the function* f*, namely* $D_q\varphi = f$*, be continuous at* $z = A$*. Then*

$$\int_A^B f(\xi)\mathrm{d}_q\xi = \varphi(B) - \varphi(A), \tag{7}$$

$$D_q\left(\int_A^{\xi} f(t)\mathrm{d}_q t\right) = f(\xi). \tag{8}$$

2. Construction of operators and basic properties

In this section, taking into account the operators introduced by [2, 12], we focus on these operators and then construct the new generalized operators in L_p-spaces by q-analogue and investigate the convergence results. Thus, with the basic definitions of q-integers for $0 < q < 1$, we define new construction of Bernstein-Kantorovich polynomials with shifted knots of positive real numbers μ_s, ν_s for $(s = 1,2)$ with $\frac{\mu_2}{[r+1]_q+\nu_2} \leq \xi \leq \frac{[r+1]_q+\mu_2}{[r+1]_q+\nu_2}$, provided $0 \leq \mu_2 \leq \mu_1 \leq \nu_1 \leq \nu_2$. For this purpose we let $1 \leq p < \infty$, $\mathcal{J}_r = \left[\frac{\mu_2}{[r+1]_q+\nu_2}, \frac{[r+1]_q+\mu_2}{[r+1]_q+\nu_2}\right]$ and define the operators $\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1} : L_p[0,1] \to L_p(\mathcal{J}_r)$ by

$$\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) = ([r+1]_q+\nu_1)\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q \sum_{s=0}^{r} P^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\int_{\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}}^{\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}} f(t)\mathrm{d}_q t, \tag{9}$$

for $f \in L_p[0,1]$ and $r = 1,2,3,\cdots$, where

$$P^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi) = \begin{bmatrix} r \\ s \end{bmatrix}_q \left(\xi - \frac{\mu_2}{[r+1]_q+\nu_2}\right)^s_q \prod_{i=0}^{r-s-1}\left(\frac{[r+1]_q+\mu_2}{[r+1]_q+\nu_2} - q^i\xi\right)$$

and

$$\prod_{i=0}^{r-s-1}\left(\frac{[r+1]_q+\mu_2}{[r+1]_q+\nu_2} - q^i\xi\right) = \left(\frac{[r+1]_q+\mu_2}{[r+1]_q+\nu_2} - \xi\right)^{r-s}_q.$$

In addition, it is very clear that for $q = 1$, these operators reduce to [27] and for $\mu_2 = \nu_2 = 0$ with $q = 1$ we get operators (1) by [2] . If we take $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$ with $q = 1$, then the operators (1) reduce to the classic Bernstein-Kantorovich operators [12].

Lemma 2.1. *Take the test functions $\eta_j(t) = t^j$ for $j = 0, 1, 2$. Then for all $\xi \in \mathcal{J}_r = \left[\frac{\mu_2}{[r+1]_q+\nu_2}, \frac{[r+1]_q+\mu_2}{[r+1]_q+\nu_2}\right]$, we have:*

$$
\begin{aligned}
(1)\quad \mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_0(t);\xi) &= \left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q;\\
(2)\quad \mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_1(t);\xi) &= \frac{2q}{[2]_q}\frac{[r]_q}{[r+1]_q+\nu_1}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)_q\\
&+ \frac{(1+2\mu_1)}{[2]_q([r+1]_q+\nu_1)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q;\\
(3)\quad \mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_2(t);\xi) &= \frac{3q^3}{[3]_q}\frac{[r]_q[r-1]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^3_q\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)^2_q\\
&+ \frac{3q(q+1+2\mu_1)}{[3]_q}\frac{[r]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)_q\\
&+ \frac{1}{[3]_q}\frac{(1+3\mu_1+3\mu_1^2)}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q.
\end{aligned}
$$

Proof. From q-Jackson integral, clearly we have

$$
\begin{aligned}
\int_{\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}}^{\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}} t^\alpha d_q t &= \int_0^{\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}} t^\alpha d_q t - \int_0^{\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}} t^\alpha d_q t\\
&= (1-q)\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}\sum_{m=0}^{\infty}\left(\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}q^m\right)^\alpha q^m\\
&- (1-q)\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}\sum_{m=0}^{\infty}\left(\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}q^m\right)^\alpha q^m\\
&= \frac{(1-q)}{([r+1]_q+\nu_1)^{\alpha+1}}\Big(([s+1]_q+\mu_1)^{\alpha+1}-(q[s]_q+\mu_1)^{\alpha+1}\Big)\sum_{m=0}^{\infty}q^{m(1+\alpha)}.
\end{aligned}
$$

Thus we easily get

$$
\int_{\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}}^{\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}} t^\alpha d_q t = \begin{cases} \frac{1}{[r+1]_q+\nu_1} & \text{for } \alpha=0;\\ \frac{1}{[2]_q\left([r+1]_q+\nu_1\right)^2}\left(1+2\mu_1+2q[s]_q\right) & \text{for } \alpha=1;\\ \frac{1}{[3]_q\left([r+1]_q+\nu_1\right)^3}\Big(1+3\mu_1+3\mu_1^2+3q(1+2\mu_1)[s]_q+3q^2[s]_q^2\Big) & \text{for } \alpha=2, \end{cases} \tag{10}
$$

where we used $[s+1]_q = 1 + q[s]_q$.

Thus in the view of (10) for $\alpha = 0, 1, 2$, $\eta_\alpha(t) = t^\alpha$, we get

$$
\begin{aligned}
\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_0(t);\xi) &= ([r+1]_q+\nu_1)\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q\frac{1}{([r+1]_q+\nu_1)}\sum_{s=0}^{r}P^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\\
&= \left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q\left(\frac{[r+1]_q}{[r+1]_q+\nu_2}\right)^{r}_q.
\end{aligned}
$$

$$\begin{aligned}
\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_1(t);\xi) &= \frac{(1+2\mu_1)([r+1]_q+\nu_1)}{[2]_q}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q \frac{1}{([r+1]_q+\nu_1)^2}\sum_{s=0}^{r} P^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\\
&+ \frac{2q([r+1]_q+\nu_1)}{[2]_q}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q \frac{1}{([r+1]_q+\nu_1)^2}\sum_{s=0}^{r}[s]_q P^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\\
&= \frac{(1+2\mu_1)}{[2]_q([r+1]_q+\nu_1)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q\left(\frac{[r+1]_q}{[r+1]_q+\nu_2}\right)^{r}_q\\
&+ \frac{2q}{[2]_q}\frac{[r]_q}{([r+1]_q+\nu_1)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q \sum_{s=0}^{r} P^{\mu_2,\nu_2}_{r-1,q,\mu_1,\nu_1}(\xi)\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)_q\\
&= \frac{(1+2\mu_1)}{[2]_q([r+1]_q+\nu_1)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q\\
&+ \frac{2q}{[2]_q}\frac{[r]_q}{([r+1]_q+\nu_1)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)_q\left(\frac{[r+1]_q}{[r+1]_q+\nu_2}\right)^{r-1}_q;
\end{aligned}$$

$$\begin{aligned}
&\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_2(t);\xi)\\
&= \frac{1}{[3]_q}\frac{(1+3\mu_1+3\mu_1^2)}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q\sum_{s=0}^{r} P^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\\
&+ \frac{3q(1+2\mu_1)}{[3]_q}\frac{[r]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q\sum_{s=0}^{r} P^{\mu_2,\nu_2}_{r-1,q,\mu_1,\nu_1}(\xi)\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)_q\\
&+ \frac{3q^2}{[3]_q}\frac{[r]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q\sum_{s=0}^{r}[1+s]_q P^{\mu_2,\nu_2}_{r-1,q,\mu_1,\nu_1}(\xi)\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)_q\\
&= \frac{1}{[3]_q}\frac{(1+3\mu_1+3\mu_1^2)}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q\\
&+ \frac{3q(1+2\mu_1)}{[3]_q}\frac{[r]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{2}_q\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)_q\\
&+ \frac{3q^2}{[3]_q}\frac{[r]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q\sum_{s=0}^{r} P^{\mu_2,\nu_2}_{r-1,q,\mu_1,\nu_1}(\xi)\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)_q\\
&+ \frac{3q^3}{[3]_q}\frac{[r]_q[r-1]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q\sum_{s=0}^{r} P^{\mu_2,\nu_2}_{r-2,q,\mu_1,\nu_1}(\xi)\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)^{2}_q\\
&= \frac{1}{[3]_q}\frac{(1+3\mu_1+3\mu_1^2)}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q\\
&+ \frac{3q(1+2\mu_1)}{[3]_q}\frac{[r]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{2}_q\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)_q\\
&+ \frac{3q^2}{[3]_q}\frac{[r]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{2}_q\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)_q\\
&+ \frac{3q^3}{[3]_q}\frac{[r]_q[r-1]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{3}_q\left(\xi-\frac{\mu_2}{[r+1]_q+\nu_2}\right)^{2}_q.
\end{aligned}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *For the operators $\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(.\,;\,.)$ defined by (9), we have the following central moments:*

(1) $\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_1(t)-\xi;\xi)$

$$= \Big[\frac{2q}{[2]_q}\frac{[r]_q}{[r+1]_q+\nu_1}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q-\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q\Big]\xi$$

$$+ \Big[\frac{1+2\mu_1}{[2]_q([r+1]_q+\nu_1)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q-\frac{2q\mu_2}{[2]_q}\frac{[r]_q}{([r+1]_q+\nu_1)([r+1]_q+\nu_2)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q\Big];$$

(2) $\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)$

$$\begin{aligned}
= &\Big[\frac{3q^4}{[3]_q}\frac{[r]_q[r-1]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^3_q\\
-&\frac{4q}{[2]_q}\frac{[r]_q}{([r+1]_q+\nu_1)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q+\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q\Big]\xi^2\\
+&\Big[\frac{3q(q+1+2\mu_1)}{[3]_q}\frac{[r]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q\\
+&\frac{4q\mu_2}{[2]_q}\frac{[r]_q}{([r+1]_q+\nu_1)([r+1]_q+\nu_2)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q\\
-&\frac{3q^3(1+q)\mu_2}{[3]_q([r+1]_q+\nu_2)}\frac{[r]_q[r-1]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^3_q\\
-&\frac{2(1+2\mu_1)}{[2]_q([r+1]_q+\nu_1)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q\Big]\xi\\
+&\frac{3q^3}{[3]_q}\frac{[r]_q[r-1]_q}{([r+1]_q+\nu_1)^2}\frac{\mu_2}{([r+1]_q+\nu_2)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q\\
-&\frac{3q\mu_2(1+q+2\mu_1)}{[3]_q([r+1]_q+\nu_2)}\frac{[r]_q}{([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q\\
+&\frac{1+3\mu_1+3\mu_1^2}{[3]_q([r+1]_q+\nu_1)^2}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q.
\end{aligned}$$

Remark 2.3. *It was mentioned in [5] that the integral operators defined by q-Jackson integral are linear but may not remain positive. So to overcome this problem, the function f must be strictly monotonic increasing.*

Theorem 2.4. *Let $\xi\in\mathcal{J}_r$. Then, for strictly monotonic increasing function $f\in C(\mathcal{J}_r)$, we get*

$$\lim_{r\to\infty}\|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)-f(\xi)\|_{C(\mathcal{J}_r)}=0,$$

where $\mathcal{J}_r=\left[\frac{\mu_2}{[r+1]_q+\nu_2},\frac{[r+1]_q+\mu_2}{[r+1]_q+\nu_2}\right]$, and $C(\mathcal{J}_r)$ is the set of all continuous functions f on $\mathcal{J}_r$.

Proof. In the view of Lemma 2.1 for $\kappa=0,1,2$, we easily get

$$\lim_{r\to\infty}\max_{\xi\in\mathcal{J}_r}|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_\kappa(t);\xi)-\xi^\kappa|=0. \tag{11}$$

We construct operators $C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu}$ as follows

$$C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) = \begin{cases} \mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) & \text{if } \xi \in \mathcal{J}_r, \\ f(\xi) & \text{if } \xi \in [0,1] \setminus \mathcal{J}_r. \end{cases} \tag{12}$$

Then, it is easy to see that

$$\|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)\|_{C[0,1]} = \max_{\xi\in\mathcal{J}_r} |\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)|. \tag{13}$$

From (11) and (13), we get

$$\lim_{r\to\infty} \|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_\kappa(t);\xi) - \xi^\kappa\|_{C[0,1]} = 0, \ \kappa = 0,1,2.$$

On applying the well-known Korovkin's theorem, we get

$$\lim_{r\to\infty} \|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)\|_{C[0,1]} = 0.$$

Therefore, from (13) we get

$$\lim_{r\to\infty} \max_{\xi\in\mathcal{J}_r} |C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)| = 0.$$

This completes proof of Theorem 2.4. □

Theorem 2.5. *For every strictly monotonic increasing function $f \in L_p[0,1]$, $p \geq 1$, operators* (9) *satisfy*

$$\lim_{r\to\infty} \|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)\|_{L_P[0,1]} = 0.$$

Proof. In order to prove the result, we consider Theorem 2.4 and operators $C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}$ by (12). By Luzin theorem [15], for a given $\epsilon > 0$, a continuous function ψ on $[0,1]$ exists and satisfies $\|f - \psi\|_{L_p[0,1]} < \epsilon$. Then for all $r \in \mathbb{N}$, it is enough to prove that there exists a number $R > 0$ such that $\|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\|_{L_P[0,1]} \leq R$. For this purpose, taking into account Theorem 2.4, for given $\epsilon > 0$, there exists a positive number $r_0 \in \mathbb{N}$ with $r \geq r_0$ such that $\|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\psi;\xi) - \psi(\xi)\|_{L_P[0,1]} < \epsilon$. Consider the inequality

$$\begin{aligned} \|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)\|_{L_p[0,1]} &\leq \|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\psi;\xi) - \psi(\xi)\|_{C[0,1]} + \|f-\psi\|_{L_p[0,1]} \\ &+ \|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\psi;\xi)\|_{L_p[0,1]}. \end{aligned} \tag{14}$$

Apply the Jensen's inequality to operators (9), we immediately get
$|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)|^p$

$$\begin{aligned} &\leq \left\{([r+1]_q+\nu_1)\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r+1}_q \sum_{s=0}^{r} P^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi) \int_{\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}}^{\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}} |f(t)|\mathrm{d}t\right\}^p \\ &\leq \sum_{s=0}^{r}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r}_q P^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\left\{([r+1]_q+\nu_1)\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q \int_{\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}}^{\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}} |f(t)|\mathrm{d}t\right\}^p \\ &\leq \sum_{s=0}^{r}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{r}_q P^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)([r+1]_q+\nu_1)\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^p_q \int_{\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}}^{\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}} |f(t)|^p\mathrm{d}t. \end{aligned}$$

Taking integral over $\mathcal{J}_r = \left[\frac{\mu_2}{[r+1]_q+\nu_2}, \frac{[r+1]_q+\mu_2}{[r+1]_q+\nu_2}\right]$, we obtain

$$\begin{aligned}
\int_{\mathcal{J}_r} |\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)|^p &\leq \sum_{s=0}^{r} \left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^r_q \left(\frac{[r+1]_q}{[r+1]_q+\nu_2}\right)^{r+1}_q \frac{1}{[r+1]_q} \\
&\times ([r+1]_q+\nu_1)\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^p_q \int_{\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}}^{\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}} |f(t)|^p \mathrm{d}t \\
&= \frac{[r+1]_q+\nu_1}{[r+1]_q}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^{p-1}_q \sum_{s=0}^{r} \int_{\frac{q[s]_q+\mu_1}{[r+1]_q+\nu_1}}^{\frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}} |f(t)|^p \mathrm{d}t \\
&\leq \left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^p_q \|f\|^p_{L_p[0,1]}.
\end{aligned}$$

From (12) and using the inequality $\int_{[0,1]\setminus\mathcal{J}_r} |f(\xi)|^p \mathrm{d}\xi \leq \|f\|^p_{L_p[0,1]}$, we get that

$$\int_0^1 |C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)|^p \mathrm{d}\xi \leq \left[1+\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^p_q\right]\|f\|^p_{L_p[0,1]}. \tag{15}$$

Thus

$$\|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f)\|_{L_p[0,1]} \leq (2+\nu_2)\|g\|_{L_p[0,1]} \leq R\|f\|_{L_p[0,1]}.$$

Therefore, for all $r \in \mathbb{N}$, if $\|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\|_{L_P[0,1]} \leq R$, then (14) gives us

$$\begin{aligned}
\|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f)-f\|_{L_p[0,1]} &\leq \|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\psi)-\psi\|_{L_p[0,1]} \\
&+ \|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\| \, \|f-\psi\|_{L_p[0,1]} + \|f-\psi\|_{L_p[0,1]} \\
&\leq \epsilon R + 2\epsilon.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
\|C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f)-f\|_{L_p[0,1]} &= \left(\int_0^1 |C^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)|^p \mathrm{d}\xi\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathcal{J}_r} |\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)-f(\xi)|^p \mathrm{d}\xi\right)^{\frac{1}{p}} \\
&= \|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)-f(\xi)\|_{L_p(\mathcal{J}_r)} \\
&\leq \epsilon R + 2\epsilon.
\end{aligned}$$

Thus we get $\lim_{r\to\infty} \|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)-f(\xi)\|_{L_p(\mathcal{J}_r)} = 0$. This completes the proof of Theorem 2.5. □

3. Convergence properties of operators $\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}$

We write $\tilde{C}[0,1]$ for the space of uniformly continuous functions on $[0,1]$ and $E_f = \{f \mid f \in \tilde{C}[0,1]\}$. For $\tilde{\delta} > 0$, let $\tilde{\omega}(f;\tilde{\delta})$ be the modulus of continuity of the function f of order one and $\lim_{\tilde{\delta}\to 0+} \tilde{\omega}(f;\tilde{\delta}) = 0$. Thus we immediately see

$$\tilde{\omega}(f;\tilde{\delta}) = \sup_{|t_1-t_2|\leq\tilde{\delta}} |\, f(t_1)-f(t_2)\,|; \quad t_1,t_2 \in [0,1], \tag{16}$$

$$| f(t_1) - f(t_2) |\leq \left(1+\frac{| t_1 - t_2 |}{\tilde{\delta}}\right)\tilde{\omega}(f;\tilde{\delta}). \tag{17}$$

Theorem 3.1. *[26] Let $[u, v] \subseteq [x, y]$ and $\{K_r\}_{r\geq 1}$ be the sequence of positive linear operators acting from $C[x, y]$ to $C[u, v]$. Then*

1. *if $f \in C[x, y]$ and $\xi \in [u, v]$, then we have*

$$\begin{aligned} |K_r(f;\xi) - f(\xi)| &\leq |f(\xi)||K_r(\eta_0(t);\xi) - 1| \\ &+ \left\{K_r(\eta_0(t);\xi) + \frac{1}{\tilde{\delta}}\sqrt{K_r((\eta_1(t)-\xi)^2;\xi)}\sqrt{K_r(\eta_0(t);\xi)}\right\}\tilde{\omega}(f;\tilde{\delta}),\end{aligned}$$

2. *for any $f' \in C[x, y]$ and $\xi \in [u, v]$, we have*

$$\begin{aligned} |K_r(f;\xi) - f(\xi)| &\leq |f(\xi)||K_r(\eta_0(t);\xi) - 1| + |f'(\xi)||K_r(\eta_1(t)-\xi;\xi)| \\ &+ K_r((\eta_1(t)-\xi)^2;\xi)\left\{\sqrt{K_r(\eta_0(t);\xi)} + \frac{1}{\tilde{\delta}}\sqrt{K_r((\eta_1(t)-\xi)^2;\xi)}\right\}\tilde{\omega}(f';\tilde{\delta}).\end{aligned}$$

Theorem 3.2. *For all strictly monotonic increasing functions $f \in E_f$ and $\xi \in \mathcal{J}_r$ operators (9) satisfy the inequality*

$$|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)| \leq \frac{\nu_2}{[r+1]_q}|f(\xi)| + 2\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q \tilde{\omega}\left(f;\sqrt{\tilde{\delta}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)}\right),$$

where $\tilde{\delta}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi) = \left(\frac{[r+1]_q}{[r+1]_q+\nu_2}\right)_q \mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)$.

Proof. From (1) of Theorem 3.1 and Lemma 2.2, we can write

$$\begin{aligned} |\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)| &\leq |f(\xi)||\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_0(t);\xi) - 1| + \left\{\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_0(t);\xi)\right. \\ &+ \left.\frac{1}{\tilde{\delta}}\sqrt{\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)}\sqrt{\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_0(t);\xi)}\right\}\tilde{\omega}(f;\tilde{\delta}).\end{aligned}$$

If we choose $\tilde{\delta} = \sqrt{\tilde{\delta}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)} = \sqrt{\left(\frac{[r+1]_q}{[r+1]_q+\nu_2}\right)_q}\sqrt{\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)}$, then we get

$$|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)| \leq \frac{\nu_2}{[r+1]_q}|f(\xi)| + 2\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q \tilde{\omega}\left(f;\sqrt{\tilde{\delta}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)}\right).$$

□

Theorem 3.3. *For all strictly monotonic increasing functions $f \in E_f$ and $\xi \in \mathcal{J}_r$ operators (9) satisfy the inequality*

$$|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)| \leq \frac{\nu_2}{[r+1]_q}R_r + 2\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q \tilde{\omega}\left(f;\sqrt{\tilde{\delta}}\right),$$

where $R_r = \max_{\xi\in\mathcal{J}_r}|f(\xi)|$ and $\tilde{\delta} = \max_{\xi\in\mathcal{J}_r}\tilde{\delta}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)$.

Proof. In the view of monotonicity of the modulus of continuity, we easily get the desired result. □

Remark 3.4. *For all $\xi \in \mathcal{J}_r$ Theorem 3.2 estimates the local order approximation, while Theorem 3.3 allows to estimate the global order approximation.*

Theorem 3.5. *If $\varphi \in C'[0,1]$ which is also strictly monotonic increasing, then for every $\xi \in \mathcal{J}_r$ we have*

$$\begin{aligned}
&|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\varphi;\xi)-\varphi(\xi)| \\
&\leq \frac{\nu_2}{[r+1]_q}|\varphi(\xi)| + \Bigg\|\Bigg[\frac{2q}{[2]_q}\frac{[r]_q}{[r+1]_q+\nu_1}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q - \left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q\Bigg]\xi \\
&+ \Bigg[\frac{1+2\mu_1}{[2]_q([r+1]_q+\nu_1)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q \\
&- \frac{2q\mu_2}{[2]_q}\frac{[r]_q}{([r+1]_q+\nu_1)([r+1]_q+\nu_2)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q\Bigg]\Bigg\||\varphi'(\xi)| \\
&+ 2\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q\sqrt{\tilde{\delta}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}}\,\tilde{\omega}\left(\varphi';\sqrt{\tilde{\delta}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)}\right),
\end{aligned}$$

where $\tilde{\delta}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}$ is defined in Theorem 3.2.

Proof. Taking into consideration (2) of Theorem 3.1 and Lemma 2.2, we get

$$\begin{aligned}
&|\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\varphi;\xi)-\varphi(\xi)| \\
&\leq \frac{\nu_2}{[r+1]_q}|\varphi(\xi)| + \Bigg\|\Bigg[\frac{2q}{[2]_q}\frac{[r]_q}{[r+1]_q+\nu_1}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q - \left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q\Bigg]\xi \\
&+ \Bigg[\frac{1+2\mu_1}{[2]_q([r+1]_q+\nu_1)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)_q \\
&- \frac{2q\mu_2}{[2]_q}\frac{[r]_q}{([r+1]_q+\nu_1)([r+1]_q+\nu_2)}\left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^2_q\Bigg]\Bigg\||\varphi'(\xi)| \\
&+ \sqrt{\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)}\sqrt{\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_0(t);\xi)}\Bigg\{1+\frac{1}{\tilde{\delta}}\frac{\sqrt{\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)}}{\sqrt{\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_0(t);\xi)}}\Bigg\}\tilde{\omega}\left(\varphi';\tilde{\delta}\right).
\end{aligned}$$

In the view of theorem 3.2 if we choose $\tilde{\delta} = \sqrt{\tilde{\delta}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)} = \sqrt{\left(\frac{[r+1]_q}{[r+1]_q+\nu_2}\right)_q}\sqrt{\mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)}$, then we get the result easily. □

By virtue of some earlier information we want to study approximation by positive linear operators (9) in L_p spaces.

In our main instruments we use space $L_p(\mathcal{J}_r)$ of integral modification in terms of modulus of continuity for all $\psi \in L_p(\Phi_\lambda)$ by

$$\tilde{\omega}_{1,p}(\psi,t) = \sup_{z\in[0,1]}\sup_{0<\lambda\leq t} \| \psi(z+\lambda)-\psi(z) \|_{L_p(\Phi_\lambda)}, \quad (1\leq p<\infty), \tag{18}$$

where $\| \,.\, \|_{L_p(\Phi_\lambda)}$ is the L_p-norm defined over $\Phi_\lambda = [0,1-\lambda]$. In addition to measure the quantitative estimates by the Peetre's K-functional we let ψ be absolutely continuous function and $\mathcal{F}_{1,p}(\Phi_\lambda) = \{\psi,\psi' \in L_p(\Phi_\lambda)\}$. For any $\varphi \in L_p(\Phi_\lambda)$ and $1 \leq p < \infty$, the Peetre's K-functional is given by

$$K_{1,p}(\psi;t) = \inf_{\varphi\in\mathcal{F}_{1,p}(\Phi_\lambda)}\Big(\| \psi-\varphi \|_{L_p(\Phi_\lambda)} + t \| \varphi' \|_{L_p(\Phi_\lambda)} \Big). \tag{19}$$

Next, the connection between the Peetre's K-functional and integral modulus of continuity is given by the inequality [10] as follows

$$M_1\tilde{\omega}_{1,p}(\psi;t) \leq K_{1,p}(\psi;t) \leq M_2\tilde{\omega}_{1,p}(\psi;t). \tag{20}$$

We denote $\mathcal{J}_s = \left[\frac{[s]_q+\mu_1}{[r+1]_q+\nu_1}, \frac{[s+1]_q+\mu_1}{[r+1]_q+\nu_1}\right]$ and taking into account the operators (9), let us consider the following auxiliary operators

$$\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) = \left(\frac{[r+1]_q}{[r+1]_q+\nu_2}\right)_q \mathcal{B}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi). \tag{21}$$

Theorem 3.6. *For all strictly monotonic increasing functions $\varphi \in \mathcal{F}_{1,p}[0,1]$ and $p > 1$, the operators $\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}$ (21) satisfy the inequality*

$$\left\|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\varphi) - \varphi\right\|_{L_p(\mathcal{J}_s)} \leq 2^{\frac{1}{p}}\left(1+\frac{1}{p-1}\right)\max_{\xi\in\mathcal{J}_s}\left(\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\left((\eta_1(t)-\xi)^2;\xi\right)\right)^{\frac{1}{2}}\left\|\varphi'\right\|_{L_p[0,1]}$$

where $\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\left(\eta_1(t)-\xi)^2;\xi\right)$ is defined by (21).

Proof. We consider

$$Q^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi) = \left(\frac{[r+1]_q+\nu_2}{[r+1]_q}\right)^r_q \left[\begin{array}{c} r \\ s \end{array}\right]_q \left(\xi - \frac{\mu_2}{[r+1]_q+\nu_2}\right)^s_q \left(\frac{[r+1]_q+\mu_2}{[r+1]_q+\nu_2} - \xi\right)^{r-s}_q. \tag{22}$$

For any $\xi \in \mathcal{J}_s$, we can write

$$\begin{aligned}
\left|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\varphi;\xi) - \varphi(\xi)\right| &= ([r+1]_q+\nu_1)\left|\sum_{s=0}^{r} Q^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\int_{\mathcal{J}_s}(\varphi(t)-\varphi(\xi))\,\mathrm{d}t\right| \\
&\leq ([r+1]_q+\nu_1)\sum_{s=0}^{r} Q^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\int_{\mathcal{J}_s}\int_{\xi}^{t}\left|\varphi'(\zeta)\right|\mathrm{d}\zeta\mathrm{d}t \\
&\leq \Theta_{\varphi'}(\xi)([r+1]_q+\nu_1)\sum_{s=0}^{r} Q^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\int_{\mathcal{J}_s}|t-\xi|\,\mathrm{d}t,
\end{aligned}$$

where $\Theta_{\varphi'}(\xi) = \sup_{t\in[0,1]}\frac{1}{t-\xi}\int_{\xi}^{t}\left|\varphi'(\lambda)\right|\mathrm{d}\lambda$ $(t \neq \xi)$ denotes the Hardy-Littlewood majorant of φ'. By virtue of the well-known Cauchy-Schwarz's inequality, we immediately get

$$\begin{aligned}
\left|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\varphi;\xi) - \varphi(\xi)\right| &\leq \Theta_{\varphi'}(\xi)([r+1]_q+\nu_1)^{\frac{1}{2}}\left(\sum_{s=0}^{r} Q^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\right)^{\frac{1}{2}} \\
&\times \left(\sum_{s=0}^{r} Q^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\int_{\mathcal{J}_s}(\eta_1(t)-\xi)^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq \Theta_{\varphi'}(\xi)\max_{\xi\in\mathcal{J}_s}\left(\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\left((\eta_1(t)-\xi)^2;\xi\right)\right)^{\frac{1}{2}}.
\end{aligned}$$

For $1 < p < \infty$, the Hardy-Littlewood theorem [29] gives the inequality

$$\int_0^1 \Theta_{\varphi'}(\xi)\mathrm{d}\xi \leq 2\left(\frac{p}{p-1}\right)^p\int_0^1\left|\varphi'(\xi)\right|^p\mathrm{d}\xi$$

Thus we get

$$\left\|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\varphi)-\varphi\right\|_{L_p(\mathcal{J}_s)}\leq 2^{\frac{1}{p}}\left(1+\frac{1}{p-1}\right)\max_{z\in\mathcal{J}_s}\left(\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\left((\eta_1(t)-\xi)^2;\xi\right)\right)^{\frac{1}{2}}\left\|\varphi'\right\|_{L_p[0,1]}.$$

□

Theorem 3.7. *For all strictly monotonic increasing functions $\varphi\in L_p[0,1]$, the operators (21) satisfy*

$$\left\|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\varphi)-\varphi\right\|_{L_p(\mathcal{J}_s)}\leq 2M_2\Big(1+2^{\frac{1-p}{p}}\left(1+\frac{1}{p-1}\right)\tilde{\omega}_{1,p}\left(\varphi;\rho^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\right)\Big),$$

where M_2 is positive constant and $\rho^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)=\max_{\xi\in\mathcal{J}_s}\left(\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\left((\eta_1(t)-\xi)^2;\xi\right)\right)^{\frac{1}{2}}$ by Theorem 3.6.

Proof. We consider

$$\left\|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\varphi)-\varphi\right\|_{L_p[0,1]}\leq\begin{cases}2\left\|\varphi\right\|_{L_p[0,1]} & \text{if } \varphi\in L_p[0,1],\\ 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\rho^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\left\|\varphi\right\|_{L_p[0,1]} & \text{if } \varphi\in\mathcal{F}_{1,p}[0,1],\end{cases}\tag{23}$$

where as in Theorem 3.6 we suppose $\rho^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)=\max_{\xi\in\mathcal{J}_s}\left(\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\left((\eta_1(t)-\xi)^2;\xi\right)\right)^{\frac{1}{2}}$.

Thus for an arbitrary function $\psi\in\mathcal{F}_{1,p}[0,1]$, for operators we have

$$\begin{aligned}&\left\|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\varphi)-\varphi\right\|_{L_p(\mathcal{J}_s)}\\ \leq\;& 2\Big\{\left\|\varphi-\psi\right\|_{L_p[0,1]}+2^{\frac{1-p}{p}}\left(1+\frac{1}{p-1}\right)\rho^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\left\|\psi'\right\|_{L_p[0,1]}\Big\}\\ \leq\;& 2K_{1,p}\Big\{\varphi;2^{\frac{1-p}{p}}\left(1+\frac{1}{p-1}\right)\rho^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\Big\}\\ \leq\;& 2M_2\tilde{\omega}_{1,p}\Big\{\varphi;2^{\frac{1-p}{p}}\left(1+\frac{1}{p-1}\right)\rho^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\Big\}\\ \leq\;& 2M_2\Big\{1+2^{\frac{1-p}{p}}\left(1+\frac{1}{p-1}\right)\Big\}\tilde{\omega}_{1,p}\left(\varphi;\rho^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\right).\end{aligned}$$

□

Now we give the local direct estimate for the operators $\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}$ defined by (21) via the well-known Lipschitz-type maximal function involving the parameters $\alpha,\ \beta>0$ and number $\sigma\in(0,1]$. Thus from [22] we recall that

$$Lip_K^{(\alpha,\beta)}(\sigma):=\Big\{f\in C[0,1]:|f(t)-f(\xi)|\leq K\frac{|t-\xi|^{\sigma}}{(\alpha\xi^2+\beta\xi+t)^{\frac{\sigma}{2}}};\ \xi,\ t\in[0,1]\Big\},$$

where K is a positive constant.

Theorem 3.8. *For any strictly monotonic increasing function $f\in Lip_K^{(\alpha,\beta)}(\sigma)$ and $\sigma\in(0,1]$, there exists a positive constant K such that*

$$|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)-f(\xi)|\leq K.(\alpha\xi^2+\beta\xi)^{-\sigma/2}\left[\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)\right]^{\frac{\sigma}{2}},$$

where $\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\left(\eta_1(t)-\xi)^2;\xi\right)$ defined by (21).

Proof. For any strictly monotonic increasing function $f \in Lip_K^{(\alpha,\beta)}(\sigma)$ and $\sigma \in (0,1]$, first we check the statement holds for $\sigma = 1$. Then, in conclusion (22) we can see that

$$\begin{aligned}|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)| &\leq |\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(|f(t) - f(\xi)|;\xi)| + f(\xi)\,|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_0(t);\xi) - 1| \\ &\leq ([r+1]_q + \nu_1)\sum_{s=0}^{r}\left|f(t) - f(\xi)\right| \mathcal{Q}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi) \\ &\leq K([r+1]_q + \nu_1)\sum_{s=0}^{r}\frac{|t-\xi|}{(\alpha\xi^2 + \beta\xi + t)^{\frac{1}{2}}}\,\mathcal{Q}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi).\end{aligned}$$

For any $\alpha,\ \beta \geq 0$, we use the inequality $(\alpha\xi^2 + \beta\xi + t)^{-1/2} \leq (\alpha\xi^2 + \beta\xi)^{-1/2}$ and apply the well-known Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)| &\leq K([r+1]_q + \nu_1)(\alpha\xi^2 + \beta\xi)^{-1/2}\sum_{s=0}^{r}|t-\xi|\,\mathcal{Q}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi) \\ &= K.(\alpha\xi^2 + \beta\xi)^{-1/2}|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_1(t) - \xi;\xi)| \\ &\leq K\left|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t) - \xi)^2;\xi)\right|^{1/2}(\alpha\xi^2 + \beta\xi)^{-1/2}.\end{aligned}$$

These conclusions imply that it is true for $\sigma = 1$. Now we want to show the statement is valid for $\eta \in (0,1)$. We apply the monotonicity property to operators $\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}$ and use the Hölder's inequality two times with $c = 2/\sigma$ and $d = 2/(2-\sigma)$. Thus, we get

$$\begin{aligned}\left|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi)\right| &\leq ([r+1]_q + \nu_1)\sum_{s=0}^{r}\left|f(t) - f(\xi)\right|\mathcal{Q}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi) \\ &\leq \left(\sum_{s=0}^{r}\left|f(t) - f(\xi)\right|^{\frac{2}{\sigma}}([r+1]_q + \nu_1)\mathcal{Q}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\right)^{\frac{\sigma}{2}} \\ &\times \left(\sum_{s=0}^{r}([r+1]_q + \nu_1)\mathcal{Q}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\right)^{\frac{2-\sigma}{2}} \\ &\leq K\left(\sum_{s=0}^{r}\frac{\left(t-\xi\right)^2([r+1]_q + \nu_1)\mathcal{Q}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)}{t + \alpha\xi^2 + \beta\xi}\right)^{\frac{\sigma}{2}} \\ &\leq K(\alpha\xi^2 + \beta\xi)^{-\sigma/2}\left\{\sum_{s=0}^{r}\left(t-\xi\right)^2([r+1]_q + \nu_1)\mathcal{Q}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\xi)\right\}^{\frac{\sigma}{2}} \\ &\leq K(\alpha\xi^2 + \beta\xi)^{-\sigma/2}\left[\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t) - \xi)^2;\xi)\right]^{\frac{\sigma}{2}}.\end{aligned}$$

This completes the proof. □

Here, we establish a quantitative Voronovskaja-type theorem for the operators $\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)$. For any $f \in C[0,1]$, the modulus of smoothness is defined by

$$\omega_\beta(f,\tilde{\delta}) = \sup_{0<|\rho|\leq\tilde{\delta}}\left\{\left|f\left(\xi + \frac{\rho\beta(\xi)}{2}\right) - f\left(\xi - \frac{\rho\beta(\xi)}{2}\right)\right|, \xi \pm \frac{\rho\beta(\xi)}{2} \in [0,1]\right\}, \tag{24}$$

where $\beta(\xi) = (\xi - \xi^2)^{\frac{1}{2}}$ and related Peetre's K-functional is given as

$$\mathcal{K}_\beta(f,\tilde{\delta}) = \inf_{g\in W_\beta[0,1]}\left\{\|f-g\| + \tilde{\delta}\|\beta g'\| : g \in C^1[0,1], \tilde{\delta} > 0\right\},$$

and $\mathcal{W}_\beta[0,1] = \{g : g \in C_A[0,1], \|\beta g'\| < \infty\}$. Let $C_A[0,1]$ be the class of absolutely continuous functions defined on $[0,1]$. There is a positive constant C such that

$$\mathcal{K}_\beta(f,\tilde{\delta}) \leq C\,\omega_\beta(f,\tilde{\delta}).$$

Theorem 3.9. *Let f be strictly monotonic increasing function and $f, f', f'' \in C[0,1]$, then for every $\xi \in [0,1]$ we have*

$$\begin{aligned}&\left|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi) - \mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_1(t)-\xi;\xi)f'(\xi) - \frac{f''(\xi)}{2}\Big(\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)+1\right|\\ \leq\ & C.O\left(\frac{1}{[r]_q}\right)\lambda^2(\xi)\omega_\lambda\Big(f'',[r]_q^{-\frac{1}{2}}\Big),\end{aligned}$$

where $\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}$ is defined by (21) *and $\lambda(\xi) = \xi + 1$.*

Proof. For any strictly monotonic increasing function $f \in C[0,1]$, we consider

$$f(t) - f(\xi) - (t-\xi)f'(\xi) = \int_\xi^t (t-\gamma)f''(\gamma)\mathrm{d}\gamma.$$

Therefore, we can write

$$f(t) - f(\xi) - (t-\xi)f'(\xi) - \frac{f''(\xi)}{2}\Big((t-\xi)^2+1\Big) \leq \int_\xi^t (t-\gamma)[f''(\gamma)-f''(\xi)]\mathrm{d}\gamma.$$

On applying the operators $\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi)$, we obtain

$$\begin{aligned}&\Big|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi) - \mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_1(t)-\xi;\xi)f'(\xi)\\ &-\frac{f''(\xi)}{2}\Big(\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)+\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_0(t);\xi)\Big)\Big| \qquad (25)\\ &\leq \mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}\Big(\Big|\int_\xi^t |t-\gamma|\,|f''(\gamma)-f''(\xi)|\,\mathrm{d}\gamma\Big|;\xi\Big). \qquad (26)\end{aligned}$$

We can estimate the right hand side expression such as

$$\Big|\int_\xi^t |t-\gamma|\,|f''(\gamma)-f''(\xi)|\,\mathrm{d}\gamma\Big| \leq 2\|f''-g\|(t-\xi)^2 + 2\|\lambda g'\|\lambda^{-1}(\xi)|t-\xi|^3.$$

We easily conclude that

$$\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi) \leq O\left(\frac{1}{[r]_q}\right)\lambda^2(\xi) \text{ and } \mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^4;\xi) \leq O\left(\frac{1}{[r]_q^2}\right)\lambda^4(\xi),$$

where $\lambda(\xi) = \xi + 1$. From Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\Big|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi) - \mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_1(t)-\xi;\xi)f'(\xi)\\ &-\frac{f''(\xi)}{2}\Big(\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)+\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_0(t);\xi)\Big)\Big|\\ &\leq 2\|f''-g\|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi) + 2\|\lambda g'\|\lambda^{-1}(\xi)\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(|\eta_1(t)-\xi|^3;\xi)\\ &\leq 2.O\left(\frac{1}{[r]_q}\right)\gamma^2(\xi)\|f''-g\|\\ &+2\|\lambda g'\|\lambda^{-1}(\xi)\{\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi)\}^{1/2}\{\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^4;\xi)\}^{1/2}\\ &\leq 2.O\left(\frac{1}{[r]_q}\right)\lambda^2(\xi)\Big\{\|f''-g\| + [r]_q^{-\frac{1}{2}}\|\lambda g'\|\Big\}.\end{aligned}$$

By taking infimum over all $g \in W_\lambda[0,1]$, we easily deduce that

$$\left|\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(f;\xi) - f(\xi) - \mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}(\eta_1(t)-\xi;\xi)f'(\xi) - \frac{f''(\xi)}{2}\Big(\mathcal{D}^{\mu_2,\nu_2}_{r,q,\mu_1,\nu_1}((\eta_1(t)-\xi)^2;\xi+1\right|$$
$$\leq \quad C.O\left(\frac{1}{[r]_q}\right)\lambda^2(\xi)\omega_\gamma\Big(f'', [r]_q^{-\frac{1}{2}}\Big),$$

where ω_λ is the modulus of smoothness defined by (24). Thus we complete the proof of Theorem 3.9. □

4. Graphical analysis

In this section, we will give a numerical example with illustrative graphics with the help of MATLAB.

Example 4.1. *Let $f(\xi) = \xi^2 + 1$, $\mu_1 = 4, \mu_2 = 3, \nu_1 = 1.5, \nu_2 = 4, q = 0.8$ and $r \in \{10, 40, 80\}$. The convergence of the operators towards the function $f(\xi)$ is shown in Figure 1. We observe that our operator comes closer and closer to the function as r increases.*

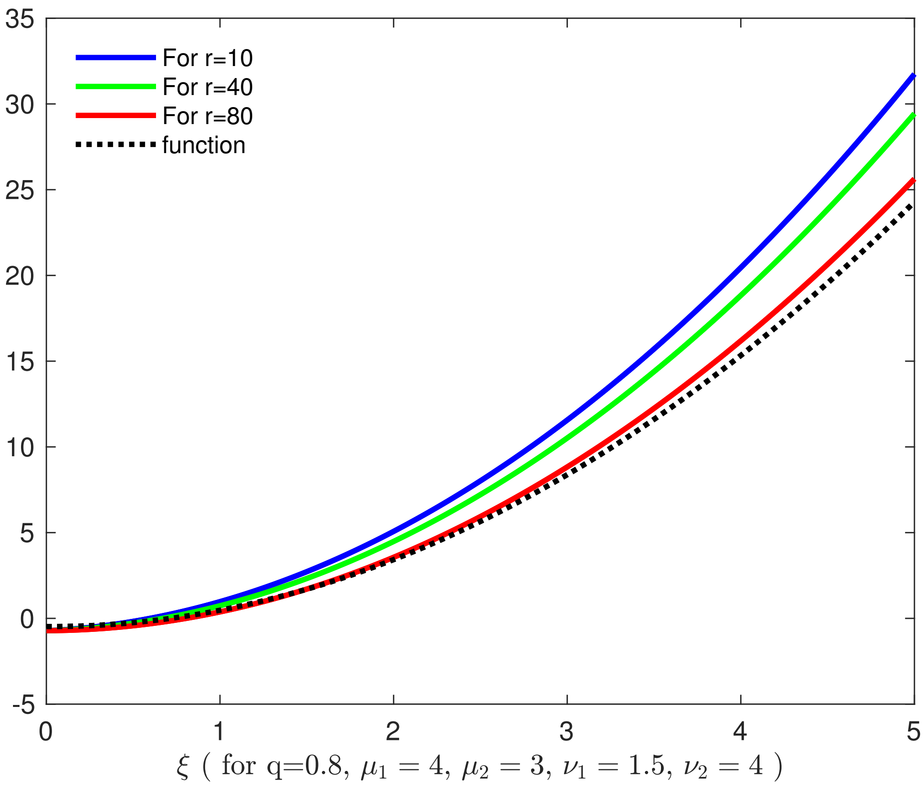

Figure 1: Convergence of the operators towards the function $f(\xi) = \xi^2 + 1$

5. Concluding remarks

It is very clear that, the choice for $q = 1$ in the operators (9), coincide with the recent operators [27] and the polynomials of our new q-Bernstein-Stancu-Kantorovich variant of shifted knots operators become the polynomials of Bernstein-Stancu-Kantorovich polynomials by [27]. Further, in case of $\mu_2 = \nu_2 = 0$ with $q = 1$ then, operators (9) reduce to (1) by [2]. Moreover, for $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$ with $q = 1$ our new operators (9) coincide with the classic Bernstein-Kantorovich operators by [12]. Thus in our investigation, we can say that the operators defined by [2, 12, 27] are special cases of our q-Bernstein-Stancu-Kantorovich variant of shifted knots operators (9). To overcome the drawback of q-Jakson integral, Marinković et al. [16] defined the Riemann type q-integral which contains only points within the interval of integral. It was shown in [5] that Riemann type q-integral is a linear and positive operator. One can construct another form of operators (9) by using Riemann type q-integrals and compare them with the existing one because operators constructed via Riemann type q-integrals do not require function to be 'strictly monotonic increasing function.

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