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Geometrical and Physical Properties of W_2-Symmetric and -Recurrent Manifolds

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Abstract. The authors discuss mainly that the Riemannian manifold M^n admitting a unit preserving circle field ξ in the present paper. A sufficient and necessary condition is given that Riemannian manifold M^n is an Einstein manifold by imposing some conditions on W_2 curvature tensor. Further, this paper obtains the algebra representation of curvature tensors of a W_2-recurrent Riemannian manifold M^n given by $R_{\alpha\beta\gamma\delta} = \frac{1}{d^2}[d_\beta d_\gamma R_{\alpha\delta} - d_\beta d_\delta R_{\alpha\gamma} + d_\alpha d_\delta R_{\beta\gamma} - d_\alpha d_\gamma R_{\beta\delta}]$.

1. Introduction

The study of Einstein field equations, Einstein manifolds can be traced back to the 1930s. Einstein manifolds are not only interesting in themselves but are also related to many important topics of Riemannian geometry. For the study of Einstein manifold, the famous geometer Wong Yung-Chow [25–28], in the early 1940s, published earlier their research works in the top international mathematical journals such as Ann. Math., Professor Wong studied and proved that the family of totally umbilical hypersurfaces with constant mean curvatures can be contained in an Einstein space. Henceforth many scholars have devoted themselves to the study of geometric and physical characteristics of non Einstein spaces admitting the family of totally umbilical hypersurfaces. They had done a lot of researches on non Einstein space which contains all kinds of conditions being equivalent to hypersurface clusters, and had made a lot of praiseworthy achievements. For instance, Tyuzi Adati [1] introduced the idea of preserving circle vector fields via a torse-forming field η_α (not necessarily timelike vector field)

$$\nabla_\beta \eta^\alpha = h\delta^\alpha_\beta + u_\beta \eta^\alpha,\ hu_\beta - h_\beta = p\eta_\beta \tag{1.1}$$

and studied the geometry and physics of subprojective spaces via this preserving circle vector field; T. Adati and T. Miyazawa [4, 5] studied a recurrent space using the preserving circle fields, and described the flatness of such Riemannian spaces. In 1978, T. Miyazawa [19] obtained the topologies of conformal symmetric spaces and posed some relationships between this class of spaces and Einstein space.

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Later, I. Sato [23] studies the geometrical and physical of properties of manifolds with contact like structures. T. Adati and A. Handatu [2] studied the geometries of P-Sasakian manifolds by the preserving circle vector fields, and investigated the properties of recurrent P-Sasakian manifolds. Almost at the same time, T. Adati and K. Matsmoto [3] obtained the geometry of a conformally symmetric P-Sasakian manifold, and first proposed the concept of ξ-Einstein manifold(i.e. quasi-Einstein manifold). In 1983, Zhonglin Li [15] considered the conformally recurrent Riemannian space by an equivalence between a preserving circle vector field and a family of totally umbilical hypersurfaces, and arrived at M^n is conformally falt if and only if it is of ξ-Einstein. This implies that the study of Riemannian manifolds or semiriemannian manifolds with preserving circular fields is very helpful to understand the essential characteristics of ξ-Einstein spaces.

Although the study of Einstein manifolds is in full swing, the research of quasi Einstein manifolds is relatively backward. As described in [2, 15], it was not until the 1970s and 1980s that the study of quasi Einstein manifolds was published. After that, the research on the geometry and physical characteristics of quasi Einstein manifolds is more and more in-depth, and has made remarkable achievements.

Along with this and related research ideas, many experts and scholars in the field of geometry and physics have focused their attention on the problem of the properties of quasi-Einstein spaces with some geometry structure and made a series of distinctive research results in recent years. For example, Li Zhonglin [16], M. C. Chaki and R. K. Maity [6] investigated the geometry of quasi-Einstein manifolds admitting a preserving circle vector field, respectively; U. C. De et al [7–9] studied the special quasi-Einstein manifolds, and obtained some interesting results. S. Mallick et al [17] considered and arrived at some geometric properties of mixed Einstein manifolds; Zhao and Yang [30] considered and obtained the properties of quasi-Einstein manifolds by the quasi-Einstein field equations; F. Fu et al [11, 12] studied recently the quasi Einstein and mixed super-Einstein manifolds associated with W_2 curvature tensors, and got the corresponding geometric and physical characterizations, where W_2 curvature tensor plays an important role in describing the flatness of mixed super-Einstein manifolds.

A W_2 manifold introduced by G. P. Pokhariyal and R. S. Mishra [22] in 1970 is essentially a Weyl projective manifold [29]. G. P. Pokhariyal [21] had studied the basic geometrical characteristics of such curvature tensors. It is with W_2 curvature tensor that G. P. Pokhariyal and R. S. Mishra [22] characterized relativistic significance. C. A. Mantica and L. G. Molinari [18] derived that a Lorentzian manifold associated with W_2 curvature tensor (called briefly W_2-Lorentzian manifold) is GRW if and only if there exists a timelike torse-forming vector field being the eigenvector of Ricci tensor $R_{\alpha\beta}$. And Z. Li [15] proved that a Riemannian manifold M^n admits a family of umbilical hypersurfaces if and only if there exists a unit torse-forming field ξ with $\langle\xi,\xi\rangle = e(=\pm 1)$ on M^n.

Motivated by those celebrated works stated above, we will in this paper intend to study the geometric and physical properties of W_2-symmetric and -recurrent manifolds. With the help of the theory of circle preserving field and the theory of transformation groups, we give the fine characterizations of Einstein and ξ-Einstein properties of the W_2-recurrent and -symmetric manifolds.

The present paper is organized as follows. In Section 3 we investigate the W_2-symmetric manifolds, and discuss the Einstein properties of this W_2-symmetric manifold. Section 4 will focus on the curvature properties of a W_2-recurrent manifold. Section 5 contributes some interesting examples.

2. Preliminaries

Let M^n be a Riemannian manifold, and N^{n-1} be a hypersurface with the fundamental quadratic form $\psi = g_{ij}dx^i dx^j$ immersed in M^n with the quadratic form $\psi = a_{\alpha\beta}dy^\alpha dy^\beta$. N^{n-1} is defined by $\sigma(y^\alpha) = const$, or $y^\alpha = y^\alpha(x^1, \cdots, x^{n-1})$, $(\alpha = 1, 2, \cdots, n)$. Then we have from [10] the following

$$g_{ij} = a_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} \hat{=} a_{\alpha\beta}y^\alpha_{,i}y^\beta_{,j},\ \ g^{ij}y^\alpha_{,i}y^\beta_{,j} = a^{\alpha\beta} - e\xi^\alpha\xi^\beta, \tag{2.1}$$

and

$$a_{\alpha\beta}y^\alpha_{,i}\xi^\beta = 0,\ \ \xi^\beta_{,j} = -\Omega_{lj}g^{lm}y^\beta_{,m} - \{^{\ \beta}_{\mu\nu}\}y^\mu_{,j}\xi^\nu,\ (\alpha,\beta,\mu,\nu = 1,\cdots,n) \tag{2.2}$$

where $a_{\alpha\beta}\xi^{\alpha} = \xi_{\beta} = \frac{\sigma_{\beta}}{\sigma}$, $\sigma_{\beta} = \frac{\partial\sigma}{\partial y^{\beta}}$, and $\bar{\sigma} = \sqrt{e\sigma^{\gamma}\sigma_{\gamma}}$, $\sigma^{\gamma} = a^{\alpha\gamma}\sigma_{\gamma}$, $e = \xi^{\alpha}\xi_{\alpha} = \pm 1$.

Further, if N^{n-1} is a totally umbilical hypersurface, then there holds

$$\Omega_{jl} = \frac{\Omega}{n-1} g_{jl}. \tag{2.3}$$

From (2.2) and (2.3), it is not hard to show from [15] that there holds

Lemma 2.1. *Let M^n be a Riemannian manifold admitting a family of totally umbilical hypersurfaces, and $\xi^{\beta}(= a^{\alpha\beta}\xi_{\alpha})$ be an unit normal vector field to the family of hypersurfaces, then there holds*

$$\nabla_{\alpha}\xi_{\beta} = -Ha_{\alpha\beta} + \nu_{\beta}\xi_{\alpha}, \ \ (\nu_{\beta} \text{ is a vector}) \tag{2.4}$$

Proof. By a direct computation, one can achieve Lemma 2.1. □

In particular, if $H = \text{const}$, then we arrive at

$$\xi^{\alpha}R_{\alpha\beta} = T\xi_{\beta}, \ \ (\alpha, \beta, \gamma, \cdots, n). \tag{2.5}$$

$$\xi^{\alpha}\xi^{\beta}\nabla_{\lambda}R_{\alpha\beta} = eT_{\lambda}, \ \ T = \frac{1}{2}[R - \bar{R} + (n-1)(n-2)eH^2], \tag{2.6}$$

where $R, \bar{R}$ are the scalar curvatures of M^n and N^{n-1}, respectively, and T is the Ricci principal curvature corresponding to the vector ξ, $T_{\lambda} = \partial_{\lambda}T$.

Definition 2.1. *A vector field ξ^{α} is said to be a torse-forming field if it satisfies*

$$\nabla_{\beta}\xi^{\alpha} = h\delta^{\alpha}_{\beta} + u_{\beta}\xi^{\alpha}.$$

Further, a torse-forming field ξ^{β} is called a preserving circle field if satisfies $hu_{\beta} - h_{\beta} = p\xi_{\beta}$.

From Lemma 2.1, we can derive that there holds the following

Lemma 2.2. *M^n admits a unit torse-forming field ξ if and only if M^n admits a family of totally umbilical hypersurfaces, and the orthogonal trajectory is geodesic.*

In this case, ξ^{α} are exactly the normal vector fields of these hypersurfaces, and there holds

$$\nabla_{\alpha}\xi_{\beta} = -H(a_{\alpha\beta} - e\xi_{\alpha}\xi_{\beta}). \tag{2.7}$$

According to Lemma 2.1 and Lemma 2.2, it is easy to show that

Lemma 2.3. *M^n admits a unit preserving circle field ξ if and only if M^n admits a family of totally umbilical hypersurfaces with constant mean curvature $H(\neq 0)$, and the orthogonal trajectories are geodesics.*

3. W_2-Symmetric Manifolds

In this subsection, we will study the Einstein characteristics of W_2-symmetric manifolds.

As we all know W_2-curvature tensor is given by

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{n-1}[g(X, Z)QY - g(Y, Z)QX], \tag{3.1}$$

where Q is the Ricci operator, that is, $g(QX, Y) = R(X, Y)$ for all X, Y. In the local coordinate system, W_2-curvature can be written as

$$W_{2\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{n-1}\Big(a_{\alpha\gamma}R_{\beta\delta} - a_{\beta\gamma}R_{\alpha\delta}\Big). \tag{3.2}$$

The W_2 curvature tensor introduced by G. P. Pokhariyal and R. S. Mishra in [22] can describe effectively the existence of the nonnull electrovariance, and extend Pirani formulation of gravitational waves to Einstein space [21, 22]. A Riemannian manifold M^n is called W_2-flat if W_2 curvature vanishes, i.e. $W_{2\alpha\beta\gamma\delta} = 0$.

In addition, we say that M^n is a W_2-symmetric manifold if there holds

$$\nabla_{\lambda}W_{2\alpha\beta\gamma\delta} = 0. \tag{3.3}$$

Theorem 3.1. *Let M^n be a Riemannian manifold admitting a unit preserving circle field ξ, then M^n is an Einstein manifold if and only if $\xi^\alpha W_{2\alpha\beta\gamma\delta} = 0$.*

Proof. By the Ricci identity,

$$\xi^\alpha R_{\alpha\beta\gamma\delta} = \nabla_\gamma \nabla_\delta \xi_\beta - \nabla_\delta \nabla_\gamma \xi_\beta, \tag{3.4}$$

Then, by a direct computation, we have

$$\xi^\alpha I_{\alpha\beta\gamma\delta} = 0, \tag{3.5}$$

$$a^{\beta\gamma} I_{\alpha\beta\gamma\delta} = R_{\alpha\delta} - T a_{\alpha\delta}, \tag{3.6}$$

$$-H I_{\alpha\beta\gamma\delta} + \xi^\alpha \nabla_\lambda R_{\alpha\beta\gamma\delta} - \frac{T_\lambda}{n-1}(a_{\beta\gamma}\xi_\delta - a_{\beta\delta}\xi_\gamma) = 0, \tag{3.7}$$

where $I_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{T}{n-1}(a_{\beta\gamma}a_{\alpha\delta} - a_{\alpha\gamma}a_{\beta\delta})$.

Let $W_{2\alpha\beta\gamma\delta} = I_{\alpha\beta\gamma\delta} + \frac{T}{n-1}(a_{\beta\gamma}a_{\alpha\delta} - a_{\alpha\gamma}a_{\beta\delta}) + \frac{1}{n-1}(a_{\alpha\gamma}R_{\beta\delta} - a_{\beta\gamma}R_{\alpha\delta})$. From the condition $\xi^\alpha W_{2\alpha\beta\gamma\delta} = 0$ and (3.5), we obtain

$$\xi_\gamma R_{\beta\delta} = T\xi_\gamma a_{\beta\delta}. \tag{3.8}$$

Formula (3.8) implies that there holds

$$R_{\beta\delta} = T a_{\beta\delta}. \tag{3.9}$$

In other words, Riemannian manifold M^n is an Einstein manifold.

On the other hand, if M^n is an Einstein manifold, one has

$$\begin{aligned} W_{2\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} + \frac{1}{n-1}(a_{\alpha\gamma}R_{\beta\delta} - a_{\beta\gamma}R_{\alpha\delta}) \\ &= R_{\alpha\beta\gamma\delta} - \frac{R}{n(n-1)}(a_{\beta\gamma}a_{\alpha\delta} - a_{\alpha\gamma}a_{\beta\delta}) \\ &= I_{\alpha\beta\gamma\delta}. \end{aligned} \tag{3.10}$$

Formula (3.10) shows that Theorem 3.1 is tenable. □

Theorem 3.2. *Let M^n be a Riemannian manifold admitting a unit preserving circle vector field ξ, then M^n is a W_2-flat manifold if and only if $\xi^\alpha W_{2\alpha\beta\gamma\delta} = 0$.*

Proof. According to $\xi^\alpha W_{2\alpha\beta\gamma\delta} = 0$, and Theorem 3.1, we get $R_{\alpha\beta} = T a_{\alpha\beta} = \frac{R}{n} a_{\alpha\beta}$. This implies that there hold the following

$$T_\lambda = 0, \ \xi^\alpha \nabla_\lambda R_{\alpha\beta\gamma\delta} = 0.$$

By Ricci identity (3.4), i.e., $\xi^\alpha R_{\alpha\beta\gamma\delta} = \nabla_\gamma \nabla_\delta \xi_\beta - \nabla_\delta \nabla_\gamma \xi_\beta$, Formula (3.7), and notice that $H \neq 0$, it is not hard to see that there holds

$$W_{2\alpha\beta\gamma\delta} = 0. \tag{3.11}$$

This shows that M^n is a W_2-flat manifold.

On the other hand, if M^n is W_2-flat, then one has

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{n-1}(a_{\alpha\gamma}R_{\beta\delta} - a_{\beta\gamma}R_{\alpha\delta}).$$

which means that M^n is an Einstein manifold. Further, it's not hard for us to verify that Formula $\xi^\alpha W_{2\alpha\beta\gamma\delta} = 0$ is tenable. □

By Formula (3.3), it is easy to see that there holds

Theorem 3.3. *If M^n is a symmetric Riemannian manifold, then it is a W_2-symmetric manifold.*

A Riemannian manifold M^n is said to be a quasi-Einstein manifold[6, 16, 20] if its Ricci tensor $R_{\alpha\beta}$ satisfies

$$R_{\alpha\beta} = Aa_{\alpha\beta} + B\xi_\alpha\xi_\beta, \ \ (1 \le \alpha, \beta, \cdots, \lambda, \mu, \nu, \cdots, \le n), \tag{3.12}$$

where ξ is a unit vector field (also called a fundamental element), and A, B are two scalar functions. A quasi-Einstein manifold is also called a ξ-Einstein manifold, or a Robertson-Walker (RW) spacetime.

It is obvious that ξ is the isotropic Ricci principal direction with Ricci principal curvature $\frac{R}{n}$.

Furthermore, if M^n is a quasi-Einstein manifold, one has

Theorem 3.4. *Let M^n be a Riemannian manifold, then M^n is W_2- symmetric quasi-Einstein manifold if and only if M^n is an Einstein manifold or ξ is a parallel vector field.*

From [16], we know that Theorem 3.4 is tenable if the following Proposition 3.5 is tenable.

Proposition 3.5. *Let M^n be a ξ-Einstein manifold, then a vector η is the Ricci principal direction vector if and only if $\eta \perp \xi$ or $\eta \parallel \xi$.*

Proof. In fact, if η is a Ricci principal direction, then we get

$$\eta^\alpha R_{\alpha\beta} = T\eta_\beta, \tag{3.13}$$

Making a contraction with η to ξ-Einstein equation $R_{\alpha\beta} = Aa_{\alpha\beta} + B\xi_\alpha\xi_\beta$, we have

$$T\eta_\beta = A\eta_\beta + B\eta^\alpha\xi_\alpha\xi_\beta, \tag{3.14}$$

Considering the contraction with ξ^β to (3.14), we get

$$(T - A - B)\xi^\beta\eta_\beta = 0.$$

This implies that $\eta \perp \xi$ $(T \neq A + B)$, where A, B are defined as (3.12).

Similarly, making a contraction with ξ to ξ-Einstein equation $R_{\alpha\beta} = Aa_{\alpha\beta} + B\xi_\alpha\xi_\beta$, then we get

$$\xi^\alpha R_{\alpha\beta} = A\xi_\beta + B\xi_\beta = (A + B)\xi_\beta. \tag{3.15}$$

From (3.13), (3.15) means that ξ is also a Ricci principal direction, i.e., $\eta \parallel \xi$.

On the other hand, if $\eta \perp \xi$, then we know

$$0 = a(\eta, \xi) = \eta^\alpha\xi^\beta a_{\alpha\beta} = \eta^\alpha\xi_\alpha.$$

Making a contraction with η to ξ-Einstein equation, we obtain

$$\eta^\alpha R_{\alpha\beta} = Aa_{\alpha\beta}\eta^\alpha + B\xi_\alpha\xi^\beta\eta^\alpha = A\eta_\beta. \tag{3.16}$$

Formula (3.16) shows that η is a Ricci principal direction.

Further, if $\eta \parallel \xi$, one can assume that $\eta = \lambda\xi$ without loss of generality, then we have

$$\begin{aligned} \eta^\alpha R_{\alpha\beta} &= Aa_{\alpha\beta}\eta^\alpha + B\xi_\alpha\xi_\beta\eta^\alpha \\ &= A\eta_\beta + B\lambda\xi_\alpha\xi^\alpha\xi_\beta \\ &= A\eta_\beta + B\lambda\xi_\beta = (A + B)\eta_\beta. \end{aligned} \tag{3.17}$$

In other words, η is a Ricci principal direction. □

Next, the present paper refers to Wong's idea in [25], and considers the general curvature tensor defined below, then we can make the following

Theorem 3.6. *A semi-Riemannian manifold* (M^n, a) *associated with a curvature tensor* $\mathcal{W}$ *by*

$$\begin{aligned} \mathcal{W}_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} + d\Big(a_{\alpha\gamma}R_{\beta\delta} - a_{\beta\gamma}R_{\alpha\delta} + a_{\beta\delta}R_{\alpha\gamma} - a_{\alpha\delta}R_{\beta\gamma}\Big) \\ &+ pR\Big(a_{\alpha\delta}a_{\beta\gamma} - a_{\beta\delta}a_{\alpha\gamma}\Big) \end{aligned} \tag{3.18}$$

admits a family of totally umbilical hypersurfaces, where d, p *are two constants,* R *is the scalar curvature, then* (M^n, a) *is a quasi-Einstein manifold if and only if* $\xi^\alpha \mathcal{W}_{\alpha\beta\gamma\delta} = 0$.

Proof. In fact, by Lemma 2.1, Lemma 2.3 and Formula (3.5), we derive that there holds

$$\xi^\alpha \mathcal{W}_{\alpha\beta\gamma\delta} = d(R_{\beta\delta}\xi_\gamma - R_{\beta\gamma}\xi_\delta) - (pR + \frac{T}{n-1} - dT)(a_{\beta\delta}\xi_\gamma - a_{\beta\gamma}\xi_\delta). \tag{3.19}$$

Formula (3.19) implies that (3.12) is equivalent to the condition $\xi^\alpha \mathcal{W}_{\alpha\beta\gamma\delta} = 0$. This ends the proof of Theorem 3.6. □

Remark 3.1. *It is obvious that Theorem 3.6 is also tenable for a* W_2 *Lorentzian manifold. For the general curvature tensor (3.18), if* $d = p = 0$*, then* ${\mathcal{W}_{\alpha\beta\gamma}}^\mu = {R_{\alpha\beta\gamma}}^\mu$*; if* $d = 0, p = -\frac{1}{n(n-1)}$*, then* ${\mathcal{W}_{\alpha\beta\gamma}}^\mu$ *is a concircle curvature tensor; if* $d = \frac{1}{n-2}, p = \frac{1}{(n-1)(n-2)}$*,* ${\mathcal{W}_{\alpha\beta\gamma}}^\mu$ *is a conformal curvature tensor.*

4. W_2-Recurrent manifolds

In this subsection, we will investigate the Einstein properties of W_2-recurrent manifolds.

Definition 4.1. *If the* W_2 *curvature of Riemannian manifold* M^n *satisfies the following*

$$\nabla_\lambda W_{2\alpha\beta\gamma\delta} = d_\lambda W_{2\alpha\beta\gamma\delta} \ \ (d_\lambda \neq 0), \tag{4.1}$$

then we call the Riemannian manifold M^n *a* W_2*-recurrent manifold, and* d_λ *the* W_2*-recurrent vector, and denote this manifold by RW shortly.*

Theorem 4.1. *Assume that* M^n *is a RW Riemannian manifold, then its curvature tensor can be written as*

$$R_{\alpha\beta\gamma\delta} = \frac{1}{d^2}[d_\beta d_\gamma R_{\alpha\delta} - d_\beta d_\delta R_{\alpha\gamma} + d_\alpha d_\delta R_{\beta\gamma} - d_\alpha d_\gamma R_{\beta\delta}], \tag{4.2}$$

where $d^\gamma d_\gamma \hat{=} d^2$.

Proof. From $\nabla_\lambda W_{2\alpha\beta\gamma\delta} = d_\lambda W_{2\alpha\beta\gamma\delta}$, we have

$$\begin{aligned} \nabla_\lambda R_{\alpha\beta\gamma\delta} &+ \frac{1}{n-1}(a_{\alpha\gamma}\nabla_\lambda R_{\beta\delta} - a_{\beta\gamma}\nabla_\lambda R_{\alpha\delta}) \\ &= d_\lambda R_{\alpha\beta\gamma\delta} + \frac{1}{n-1}d_\lambda(a_{\alpha\gamma}R_{\beta\delta} - a_{\beta\gamma}R_{\alpha\delta}). \end{aligned} \tag{4.3}$$

Making a contraction operation to $a^{\beta\gamma}$ for Equation (4.3), one gets

$$\nabla_\lambda R_{\alpha\delta} = d_\lambda R_{\alpha\delta}, \ \ \nabla_\lambda R = d_\lambda R. \tag{4.4}$$

Formula (4.4) confirms the following facts

$$\nabla_\lambda R_{\alpha\beta\gamma\delta} = d_\lambda R_{\alpha\beta\gamma\delta}. \tag{4.5}$$

Using Bianchi identity,

$$d_\lambda R_{\alpha\beta\gamma\delta} + d_\gamma R_{\alpha\beta\delta\lambda} + d_\delta R_{\alpha\beta\lambda\gamma} = 0. \tag{4.6}$$

Considering a contraction to d^λ for Equation (4.6), one has

$$d^2 R_{\alpha\beta\gamma\delta} + d_\gamma d^\lambda R_{\alpha\beta\delta\lambda} + d_\delta d^\lambda R_{\alpha\beta\lambda\gamma} = 0. \tag{4.7}$$

By the following identity

$$\nabla_\lambda R_{\alpha\delta} - \nabla_\delta R_{\alpha\lambda} = a^{\beta\gamma}\nabla_\gamma R_{\beta\alpha\delta\lambda}. \tag{4.8}$$

we get

$$d^\beta R_{\beta\alpha\delta\lambda} = d_\lambda R_{\alpha\delta} - d_\delta R_{\alpha\lambda}. \tag{4.9}$$

Substituting (4.9) into (4.7), we see that Equation (4.2) is tenable. □

Corollary 4.2. *By Theorem 4.2, we know that if a RW Riemannian manifold is also an Einstein manifold, i.e., if $R_{\beta\gamma} = \frac{R}{n}a_{\beta\gamma}$, by a direct computation then we know $R_{kjih} = 0$, that is, a RW-Einstein manifold is flat. But if M^n is RW-quasi-Einstein manifold, we can't derive that it is flat! According to Lemma 2.2, if M^n admits a torse-forming field ξ, then by a direct computation we know that M^n is also flat.*

Corollary 4.2 implies that there holds the following

Theorem 4.3. *A ξ-Einstein manifold can't be a RW manifold.*

Theorem 4.4. *Let M^n be a RW Riemannian manifold admitting a preserving circle vector field ξ, then M^n is of subprojective and the family of corresponding hypersurfaces is of constant curvature.*

Proof. By Definition 4.1, (3.7), (3.8) and notice that $\nabla_\lambda R^\lambda_\alpha = \frac{1}{2}\nabla_\alpha R$, we can derive that

$$d_\lambda R_{\alpha\beta\gamma}{}^\lambda + \frac{d_\lambda}{n-1}(a_{\alpha\gamma}R^\lambda_\beta - a_{\beta\gamma}R^\lambda_\alpha) = d_\lambda R_{\alpha\beta\gamma}{}^\lambda + \frac{1}{2(n-1)}(a_{\alpha\gamma}d_\beta R - a_{\beta\gamma}d_\alpha R), \tag{4.10}$$

From Equation (4.10), and by a direct computation, one can obtain that $R = 0, T = 0$. Then it is not hard to show that there holds

$$W_{2\alpha\beta\gamma\delta} = I_{\alpha\beta\gamma\delta}.$$

By a similar argument to [15], we know that Theorem 4.4 is tenable. □

5. Examples

Example 5.1. *Let $\bar{a}_{\alpha\beta} = \sigma^{-2}a_{\alpha\beta}$ be a conformal transformation, if there exists a function ρ such that $\nabla_\alpha\nabla_\beta\sigma \hat{=} \sigma_{\alpha\beta} = \rho a_{\alpha\beta}$, then from [14] it is exactly a concircle transformation. If there exists a non-trivial concircle transformation mapping a RW-manifold to a Riemannian space(where $\rho \neq 0$), then we know by Theorem 3 in [14] that this W_2 recurrent manifold is an Einstein manifold.*

Example 5.2. *Consider a ξ-quasi-concircle map as*

$$h_{\alpha\beta} = U a_{\alpha\beta} + V\xi_\alpha\xi_\beta,\ h = -\frac{1}{\sigma}, \tag{5.1}$$

where $h_{\alpha\beta} = \nabla_\beta h_\alpha - h_\alpha h_\beta + \frac{1}{2}a^{\mu\nu}h_\mu h_\nu a_{\alpha\beta}$, U, V are two scalar functions. From [15] we know that if a manifold (M^n, a) associated with $\mathcal{W}$ curvature tensor is recurrent and flat, then it is a ξ-Einstein manifold.

Example 5.3. *A semi-Riemannian manifold (M, a) with Ricci curvature $R_{\alpha\beta}$ and the energy momentum tensor $T_{\alpha\beta}$ satisfy a quasi-Einstein field equation [30], it is obvious that (M, a) is, of course, a quasi-Einstein manifold.*

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