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A New Characterization of the Closure of Dirichlet Type Spaces $\mathcal{D}_s$ in Bloch Spaces and Interpolating Blaschke Product

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Abstract. In this paper, motivated by Qian, et al [20, 22], we give a new characterization for the closure of the space $\mathcal{D}_s$ in the Bloch space. Moreover, a new characterization for interpolating Blaschke product in $\mathcal{C}_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B})$ is also investigated.

1. Introduction

As usual, let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{D}_e = \mathbb{C} \setminus \overline{\mathbb{D}}$, $H(\mathbb{D})$ be the class of all functions analytic in $\mathbb{D}$ and H^∞ denote the space of all bounded analytic function. A Blaschke product B with sequence of zeros $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{D}$ is called interpolating if there exists a positive constant δ such that

$$\prod_{j\neq k} \left| \frac{a_j - a_k}{1 - \overline{a}_j a_k} \right| \geq \delta, \quad k = 1, 2, \cdots .$$

Suppose that $0 < p < \infty$, H^p denotes the Hardy space, which consists of all functions $f \in H(\mathbb{D})$ for which (see [14])

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Let $0 \leq s < \infty$. The Dirichlet type space $\mathcal{D}_s$ consists of those functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{D}_s} = |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^s dA(z) \right)^{1/2} < \infty.$$

The space $\mathcal{D}_s$ has been studied extensively. In particular, if $s = 0$, this gives the classical Dirichlet space $\mathcal{D}$. If $s = 1$, then $\mathcal{D}_s$ is the Hardy space H^2. When $s > 1$, it gives the Bergman space A^2_{s-2}. Stegenga [25] and Taylor [26] studied the multipliers of the space $\mathcal{D}_s$ respectively. Rochberg and Wu [23] studied small hankel

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operator acting on the space $\mathcal{D}_s$. Pau and Pérez [18] investigated composition operator acting on the space $\mathcal{D}_s$. For more information relate to the space $\mathcal{D}_s$, we refer to [18, 23, 25, 26] and the paper referinthere.

The Bloch space $\mathcal{B}$ ([28]) is the class of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

Recently, the problem of characterizing the closure $C_{\mathcal{B}}(H \cap \mathcal{B})$ of $H \cap \mathcal{B}$ in the Bloch norms for certain spaces H of analytic functions in $\mathbb{D}$ has attracted the interest of many scholars. In 1974, Anderson, Clunie and Pommerenke in [1] raised the problem of characterizing the closure of H^∞ in the Bloch norm? (The problem is still unsolved.) Zhao in [27] studied the closures of some Möbius invariant spaces in the Bloch space. Lou and Chen [15] generalized [27] to a more general analytic function spaces. Monreal Galán and Nicolau in [16] characterized the closure in the Bloch norm of the space $H^p \cap \mathcal{B}$, i.e., $C_{\mathcal{B}}(H^p \cap \mathcal{B})$. Galanopoulos, Monreal Galán and Pau [12] have extended this result to the whole range $0 < p < \infty$. Bao and Göğüş in [2] studied the closure of Dirichlet type spaces $\mathcal{D}_s$ in the Bloch space. Galanopoulos and Girela [13] generalized the results in [2] to a more general class of Dirichlet type spaces $\mathcal{D}^p_s$. Qian, Li and Zhu in [21, 22] studied the closure of Dirichlet type spaces $\mathcal{D}_\mu$ in the Bloch space.

Motivated by Qian and Zhu in [22], we study the closure of the Dirichlet type spaces $\mathcal{D}_s$ in the Bloch space via pseudoanalytic extension. Pseudoanalytic extension was introduced by Dyn'kin in [11]. There are many papers related to pseudoanalytic extension, we refer to [3, 6, 8, 9, 11]. Moreover, motivated by Qian and Shi in [20], a new characterization for interpolating Blaschke product in $C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B})$ is also given.

In this paper, let $f \in H(\mathbb{D})$ and F be the primitive function of f with $F(0) = 0$, that is,

$$F(z) = \int_0^z f(w)dw, \quad z \in \mathbb{D}.$$

We say that $A \lesssim B$ if there exists a constant C such that $A \le CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Closure of Dirichlet type spaces in Bloch spaces

Before we go into proofs, we need some lemmas.

Lemma 1. [28] *Suppose $s > 0$ and $t > -1$. Then there exists a positive constant C such that*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\bar{z}w|^{2+t+s}} dA(w) \le \frac{C}{(1-|z|^2)^s}$$

for all $z \in \mathbb{D}$.

Lemma 2. [24] *Let $0 < s < 1$ and let n be a positive integer. Then*

$$g \in \mathcal{D}_s \Leftrightarrow \int_{\mathbb{D}} |g^{(n)}(z)|^2(1-|z|^2)^{2(n-1)}(1-|z|^2)^s dA(z) < \infty.$$

Using the same strategy as [3], we have the following result.

Lemma 3. *Let $n \ge 2$ be an integer and let f be a Bloch function. Let F be the primitive of f with $F(0) = 0$. Then the following statements are equivalent.*

(1) $f \in \mathcal{D}_s$;

(2) There exists a function $G \in C^1(\mathbb{C} \setminus \overline{\mathbb{D}})$ satisfying

$$\lim_{r\to 1^+} G(re^{i\theta}) = F(e^{i\theta}) \;\; a.e. \;\; and \; in \; L^2[0, 2\pi], \tag{a}$$

$$G(z) = O(z^n), \;\; as \;\; z \to \infty, \tag{b}$$

$$\bar{\partial} G(z) = O(z^{n-2}), \;\; as \;\; z \to \infty, \tag{c}$$

and

$$\int_{\mathbb{D}_e} \frac{|\bar{\partial} G(z)|^2}{(|z|^n-1)^2}(|z|^2-1)^s dA(z) < \infty, \tag{d}$$

where

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \ \ z = x+iy.$$

Proof. (1) ⇒ (2) Let $z \in \mathbb{D}_e$ and

$$G(z) = \sum_{i=0}^{n} \frac{(-1)^i}{i!}(z^*-z)^i F^{(i)}(z^*), \ \ z^* = \frac{1}{\bar{z}},$$

where $f \in \mathcal{D}_s$ and $F(z^*) = \int_0^{z^*} f(w)dw$. Since $f \in \mathcal{D}_s \subseteq H^2$, then F has a continuous extension to the closed unit disk. By the facts on Hardy spaces (see [4]), it follows that, for $i = 1, 2, ...,$

$$M_2(r, F^i) = o((1-r)^{1-i}), \ \ M_\infty(r, F^i) = o((1-r)^{\frac{1}{2}-i}), \ \ as \ \ r \to 1^-. \tag{e}$$

Using (e), we deduce that

$$\lim_{r\to 1^+} G(re^{i\theta}) = F(e^{i\theta}) \ \ a.e. \ and \ in \ L^2,$$

and

$$G(z) = O(z^n), \ \ as \ \ z \to \infty.$$

Note that

$$\bar{\partial} G(z) = \frac{(-1)^{n+1}}{n!}(z^*-z)^n (z^*)^2 F^{(n+1)}(z^*).$$

We have,

$$\bar{\partial} G(z) = O(z^{n-2}), \ \ as \ \ z \to \infty.$$

Making a change of variable with $z = \frac{1}{\bar{w}} = w^*$ and combining with Lemma 2, we have

$$\int_{\mathbb{D}_e} \frac{|\bar{\partial} G(z)|^2}{(|z|^n-1)^2}\left(|z|^2-1\right)^s dA(z) = \frac{1}{(n!)^2}\int_{\mathbb{D}_e} \frac{|z^*-z|^{2n}|z^*|^4|f^{(n)}(z^*)|^2}{(|z|^n-1)^2}\left(|z|^2-1\right)^s dA(z)$$
$$\approx \int_{\mathbb{D}} |f^{(n)}(w)|^2(1-|w|^2)^{2(n-1)}\left(1-|w|^2\right)^s dA(w) \lesssim \|f\|^2_{\mathcal{D}_s}.$$

(2) ⇒ (1). Using the Cauchy-Green's formula and (a), we obtain

$$F(z) = \frac{1}{2\pi i}\int_{|\xi|=1} \frac{F(\xi)}{\xi - z}d\xi = \frac{1}{2\pi i}\int_{|\xi|=R} \frac{G(\xi)}{\xi - z}d\xi - \frac{1}{\pi}\int_{1<|\xi|<R} \frac{\bar{\partial} G(\xi)}{\xi - z}dA(\xi), \ \ z \in \mathbb{D}. \tag{f}$$

Combine with (b) and (c), we have that

$$\int_{|\xi|=R} \frac{G(\xi)}{(\xi - z)^{n+2}}d\xi \to 0, \ \ as \ \ R \to \infty,$$

and

$$\int_{\mathbb{D}_e} \left|\frac{\bar{\partial} G(\xi)}{(\xi - z)^{n+2}}\right| dA(\xi) < \infty.$$

Using these facts and differentiating $n+1$ times in (f), we get

$$F^{(n+1)}(z) = -\frac{(n+1)!}{\pi}\int_{\mathbb{D}_e} \frac{\bar{\partial} G(\xi)}{(\xi - z)^{n+2}}dA(\xi).$$

Using Hölder's inequality, we deduce that

$$|F^{(n+1)}(z)|^2 \lesssim \int_{\mathbb{D}_e} \frac{1}{|\xi - z|^4} dA(\xi) \int_{\mathbb{D}_e} \frac{|\overline{\partial} G(\xi)|^2}{|\xi - z|^{2n}} dA(\xi).$$

Making the change of variables $\xi = \frac{1}{\overline{w}} = w^*$ ($w \in \mathbb{D}$) and combining with Lemma 1, we have

$$\int_{\mathbb{D}_e} \frac{1}{|\xi - z|^4} dA(\xi) \lesssim \int_{\mathbb{D}} \frac{1}{|1 - \overline{w}z|^4} dA(w) \lesssim \frac{1}{(1 - |z|^2)^2}.$$

Hence,

$$|F^{(n+1)}(z)|^2 \lesssim \frac{1}{(1 - |z|^2)^2} \int_{\mathbb{D}_e} \frac{|\overline{\partial} G(w^*)|^2}{|w^* - z|^{2n}} dA(w^*).$$

Using Lemma 1 and (d), we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |f^{(n)}(z)|^2 (1 - |z|^2)^{2(n-1)} (1 - |z|^2)^s dA(z) \\
&= \int_{\mathbb{D}} |F^{(n+1)}(z)|^2 (1 - |z|^2)^{2(n-1)} (|z|^2 - 1)^s dA(z) \\
&\lesssim \int_{\mathbb{D}} \int_{\mathbb{D}_e} \frac{|\overline{\partial} G(w^*)|^2}{|w^* - z|^{2n}} dA(w^*) (1 - |z|^2)^{2(n-2)} (1 - |z|^2)^s dA(z) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|\overline{\partial} G(w^*)|^2}{|1 - \overline{w}z|^{2n}} |w|^{2n-4} dA(w) (1 - |z|^2)^{2(n-2)} (1 - |z|^2)^s dA(z) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |z|^2)^{2(n-2)} (1 - |z|^2)^s}{|1 - \overline{w}z|^{2n}} dA(z) |\overline{\partial} G(w^*)|^2 |w|^{2n-4} dA(w) \\
&\lesssim \int_{\mathbb{D}} \frac{1}{(1 - |w|^2)^{2-s}} |\overline{\partial} G(w^*)|^2 |w|^{2n-4} dA(w) \\
&\lesssim \int_{\mathbb{D}_e} \frac{|\overline{\partial} G(\xi)|^2}{(|\xi|^n - 1)^2} \left(|z|^2 - 1\right)^s dA(\xi) < \infty,
\end{aligned}$$

which implies that $f \in \mathcal{D}_s$ by Lemma 2. The proof is complete. □

We also need the following lemma.

Lemma 4. [3] *Let $n \geq 2$ be an integer and let $f \in H(\mathbb{D})$. Let $F \in H^2$ be the primitive of f with $F(0) = 0$. Then the following statements are equivalent.*

(1) $f \in \mathcal{B}$;

(2) There exists a function $G \in C^1(\mathbb{C} \setminus \overline{\mathbb{D}})$ satisfying

$$\lim_{r \to 1^+} G(re^{i\theta}) = F(e^{i\theta}) \ \text{a.e. and in } L^2[0, 2\pi], \tag{g}$$

$$G(z) = O(z^n), \ \text{as } z \to \infty, \tag{h}$$

$$\overline{\partial} G(z) = O(z^{n-2}), \ \text{as } z \to \infty, \tag{i}$$

and

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}_e} \frac{|\overline{\partial} G(z)|^2}{(|z|^n - 1)^2} \left(1 - \frac{1}{|\varphi_a(z)|^2}\right)^p s dA(z) < \infty, \tag{j}$$

where $1 < p < 2$ and $\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}$.

Theorem 1. *Let $f \in \mathcal{B}$, $n \geq 2$, $0 < s < 1$ and $1 < p < 2$. For any $\epsilon > 0$, the following are equivalent.*

(1)$f \in C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B})$.

(2)

$$\int_{\Omega_\epsilon(f)} \frac{1}{(1-|w|^2)^{2-p}} dA(w) < \infty,$$

where

$$\Omega_\epsilon(f) = \left\{ w \in \mathbb{D} : (1-|w|^2)^n |f^{(n)}(w)| \geq \epsilon \right\}.$$

(3)

$$\int_{\Delta_\epsilon(G)} \frac{1}{(1-|w|^2)^{2-p}} dA(w) < \infty,$$

where

$$\Delta_\epsilon(G) = \left\{ w \in \mathbb{D} : \int_{\mathbb{D}_e} \frac{|\bar\partial G(z)|^2}{(|z|^n-1)^2} \left(1 - \frac{1}{|\varphi_w(z)|^2}\right)^p dA(z) \geq \epsilon^2 \right\},$$

where G is the function in Lemma 3.

Proof. (1) $\Leftrightarrow$ (2). See [2].

(1) $\Rightarrow$ (3). Suppose that $f \in C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B}) \subseteq \mathcal{B}$. For any $h \in \mathcal{B}$, from the proof of Lemma 4 (Theorem 2.1 in [3]), there exists a constant $C > 0$, such that

$$\left(\int_{\mathbb{D}_e} \frac{|\bar\partial G(z) - \bar\partial G_1(z)|^2}{(|z|^n-1)^2} \left(1 - \frac{1}{|\varphi_w(z)|^2}\right)^p dA(z) \right)^{1/2} \leq C\|f-h\|_{\mathcal{B}},$$

where G, G_1 are its pseudoanalytic extension of f and h, respectively. Since $f \in C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B}) \subseteq \mathcal{B}$, for any $\epsilon > 0$, there exists a function $g \in \mathcal{D}_s \cap \mathcal{B}$, such that

$$\|f-g\|_{\mathcal{B}} \leq \frac{\epsilon}{2C},$$

where C is the constant stated as above. Thus,

$$\left(\int_{\mathbb{D}_e} \frac{|\bar\partial G(z) - \bar\partial G_2(z)|^2}{(|z|^2-1)^2} \left(1 - \frac{1}{|\varphi_w(z)|^2}\right)^p dA(z) \right)^{1/2} \leq \frac{\epsilon}{2}.$$

Here G_2 is its pseudoanalytic extension of g. Note that

$$\begin{aligned}
&\int_{\mathbb{D}_e} \frac{|\bar\partial G(z)|^2}{(|z|^n-1)^2} \left(1 - \frac{1}{|\varphi_w(z)|^2}\right)^p dA(z) \\
\leq& 2\int_{\mathbb{D}_e} \frac{|\bar\partial G(z) - \bar\partial G_2(z)|^2}{(|z|^n-1)^2} \left(1 - \frac{1}{|\varphi_w(z)|^2}\right)^p dA(z) + 2\int_{\mathbb{D}_e} \frac{|\bar\partial G_2(z)|^2}{(|z|^n-1)^2} \left(1 - \frac{1}{|\varphi_w(z)|^2}\right)^p dA(z).
\end{aligned}$$

Hence $\Delta_\epsilon(G) \subseteq \Delta_{\frac{\epsilon}{2}}(G_2)$. Then

$$\begin{aligned}
&\int_{\Delta_\epsilon(G)} \frac{1}{(1-|w|^2)^{2-s}} dA(w) \\
\leq& \int_{\Delta_{\frac{\epsilon}{2}}(G_2)} \frac{1}{(1-|w|^2)^{2-s}} dA(w) \\
\leq& \frac{4}{\epsilon^4} \int_{\Delta_{\frac{\epsilon}{2}}(G_2)} \int_{\mathbb{D}_e} \frac{|\bar\partial G_2(z)|^2}{(|z|^n-1)^2} \left(1 - \frac{1}{|\varphi_w(z)|^2}\right)^p dA(z) \frac{1}{(1-|w|^2)^{2-s}} dA(w) \\
=& \frac{4}{\epsilon^4} \int_{\Delta_{\frac{\epsilon}{2}}(G_2)} \left(1 - \frac{1}{|\varphi_w(z)|^2}\right)^p \frac{1}{(1-|w|^2)^{2-s}} dA(w) \int_{\mathbb{D}_e} \frac{|\bar\partial G_2(z)|^2}{(|z|^n-1)^2} dA(z) \\
\lesssim& \int_{\mathbb{D}_e} \int_{\mathbb{D}} \left(1 - \frac{1}{|\varphi_w(z)|^2}\right)^p \frac{1}{(1-|w|^2)^{2-s}} dA(w) \frac{|\bar\partial G_2(z)|^2}{(|z|^n-1)^2} dA(z).
\end{aligned}$$

Making a change of variable with $z = \frac{1}{\nu}$ and using Lemma 1, we obtain

$$\begin{aligned}&\int_{\mathbb{D}}\left(1-\frac{1}{|\varphi_w(z)|^2}\right)^p\frac{1}{(1-|w|^2)^{2-s}}dA(w)\\=&\int_{\mathbb{D}}\frac{(1-|w|)^p(|z|^2-1)^p}{|z-w|^{2p}}\frac{1}{(1-|w|^2)^{2-s}}dA(w)\\=&\int_{\mathbb{D}}\frac{(1-|w|)^p(1-|\nu|^2)^p}{|1-\overline{\nu}w|^{2p}}\frac{1}{(1-|w|^2)^{2-s}}dA(w)\\\lesssim&(1-|\nu|^2)^s\lesssim(|z|^2-1)^s.\end{aligned}$$

By Lemma 3, we have

$$\int_{\Delta_\epsilon(G)}\frac{1}{(1-|w|^2)^{2-s}}dA(w)\lesssim\int_{\mathbb{D}_e}\frac{|\overline{\partial}G_2(z)|^2}{(|z|^n-1)^2}\left(|z|^2-1\right)^s dA(z)\lesssim\|g\|^2_{\mathcal{D}_s}.$$

(3) ⇒ (2). From the proof of Lemma 4 (see [3]), for any $z \in \mathbb{D}$, we have

$$\int_{\mathbb{D}}|f^{(n)}(w)|^2(1-|w|^2)^{2n-2}\left(1-|\varphi_z(w)|^2\right)^p dA(w)\lesssim\int_{\mathbb{D}_e}\frac{|\overline{\partial}G(w)|^2}{(|w|^n-1)^2}\left(1-\frac{1}{|\varphi_z(w)|^2}\right)^p dA(w).$$

Using sub-mean inequality of $|f^{(n)}|^2$, we have

$$|f^{(n)}(z)|^2\lesssim(1-|z|^2)^{-2}\int_{D(z,r)}|f^{(n)}(w)|^2dA(w),$$

where $D(z, r) = \{w \in \mathbb{D} : |\varphi_w(z)| < r\}$. Hence,

$$\begin{aligned}(1-|z|^2)^{2n}|f^{(n)}(z)|^2\lesssim&(1-|z|^2)^{2n-2}\int_{D(z,r)}|f^{(n)}(w)|^2dA(w)\\\lesssim&\int_{\mathbb{D}}|f^{(n)}(w)|^2(1-|w|^2)^{2n-2}\left(1-|\varphi_z(w)|^2\right)^p dA(w)\\\lesssim&\int_{\mathbb{D}_e}\frac{|\overline{\partial}G(w)|^2}{(|w|^n-1)^2}\left(1-\frac{1}{|\varphi_z(w)|^2}\right)^p dA(w).\end{aligned}$$

Thus there exists a constant $C > 1$ such that

$$(1-|z|^2)^{2n}|f^{(n)}(z)|^2\le C\int_{\mathbb{D}_e}\frac{|\overline{\partial}G(w)|^2}{(|w|^n-1)^2}\left(1-\frac{1}{|\varphi_z(w)|^2}\right)^p dA(w).$$

Thus,

$$\Omega_\epsilon(f)\subseteq\Delta_{\frac{\epsilon}{\sqrt{C}}}(G)$$

and

$$\int_{\Omega_\epsilon(f)}\frac{1}{(1-|w|^2)^{2-s}}dA(w)\le\int_{\Delta_{\frac{\epsilon}{\sqrt{C}}}(G)}\frac{1}{(1-|w|^2)^{2-s}}dA(w).$$

The proof is complete. □

3. Inner function in $C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B})$

In this section, we will give some equivalent characterizations of inner function in $C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B})$. An analytic function in the unit disc $\mathbb{D}$ is called an inner function if it is bounded and modulus equals 1 almost everywhere on the boundary $\partial\mathbb{D}$. Let us recall the following notion [10].

Let X and Y be two classes of analytic functions on $\mathbb{D}$, and $X \subseteq Y$. Suppose that θ is an inner function, θ is said to be (X, Y)-improving, if every function $f \in X$ satisfying $f\theta \in Y$ must actually satisfy $f\theta \in X$.

Theorem 2. *Let $0 < s < 1$ and θ be an interpolating Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$. Then*

(1) $\theta \in C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B})$.
(2) $\sum_{k=1}^{\infty}(1 - |a_k|^2)^s < \infty$.
(3) $\theta \in \mathcal{D}_s$.
(4) θ is $(C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B}) \cap BMOA, BMOA)$-improving.

Proof. (1) ⇔ (2). See [2].

(2) ⇔ (3). See [19].

(3) ⇒ (4). Supposed that $\theta \in \mathcal{D}_s$, $f \in C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B}) \cap BMOA$, $f\theta \in BMOA$, we only need to prove $f\theta \in C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B})$. That is, for any $\epsilon > 0$,

$$\int_{\Lambda_\epsilon(f\theta)} \frac{1}{(1 - |z|^2)^{2-s}} dA(z) < \infty,$$

where

$$\Lambda_\epsilon(f\theta) = \{z \in \mathbb{D} : (1 - |z|^2)|(f\theta)'(z)| \geq \epsilon\}.$$

Since $f \in C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B}) \cap BMOA \subseteq C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B})$, for any $\epsilon > 0$, there exists $g \in \mathcal{D}_s \cap \mathcal{B}$, such that

$$\|f - g\|_{\mathcal{B}} \leq \frac{\epsilon}{2}.$$

Since

$$\begin{aligned}
(1 - |z|^2)|(f\theta)'(z)| =&(1 - |z|^2)|f'(z)\theta(z) + f(z)\theta'(z)| \\
\leq&(1 - |z|^2)|f'(z)\theta(z)| + (1 - |z|^2)|f(z)\theta'(z)| \\
\leq&(1 - |z|^2)|f'(z)| + (1 - |z|^2)|f(z)\theta'(z)| \\
\leq&(1 - |z|^2)|f'(z) - g'(z)| + (1 - |z|^2)|g'(z)| + (1 - |z|^2)|f(z)\theta'(z)|,
\end{aligned}$$

we see that

$$\Lambda_\epsilon(f\theta) \subseteq \Gamma_{f,g,\theta} = \{z \in \mathbb{D} : (1 - |z|^2)|g'(z)| + (1 - |z|^2)|f(z)\theta'(z)| \geq \frac{\epsilon}{2}\}.$$

Then

$$\begin{aligned}
\int_{\Lambda_\epsilon(f\theta)} \frac{1}{(1 - |z|^2)^{2-s}} dA(z) \;&\lesssim\; \int_{\Gamma_{f,g,\theta}} \frac{1}{(1 - |z|^2)^{2-s}} dA(z) \\
&\lesssim\; \frac{4}{\epsilon^2} \int_{\Gamma_{f,g,\theta}} \Big((1 - |z|^2)|g'(z)| + (1 - |z|^2)|f(z)\theta'(z)|\Big)^2 \frac{1}{(1 - |z|^2)^{2-s}} dA(z) \\
&\lesssim\; A_1 + A_2,
\end{aligned}$$

where

$$A_1 := \int_{\Gamma_{f,g,\theta}} (1 - |z|^2)^2 |g'(z)|^2 \frac{1}{(1 - |z|^2)^{2-s}} dA(z)$$

and

$$A_2 := \int_{\Gamma_{f,g,\theta}} (1 - |z|^2)^2 |f(z)|^2 |\theta'(z)|^2 \frac{1}{(1 - |z|^2)^{2-s}} dA(z).$$

It is obvious that $A_1 \lesssim \|g\|^2_{\mathcal{D}_s}$. We only need to prove that $A_2 < \infty$. Since $f\theta \in BMOA$, by [5, Theorem 1], we have

$$\sup_{z\in\mathbb{D}}(1-|\theta(z)|^2)|f(z)|^2 < \infty,$$

and hence

$$\begin{aligned} A_2 &\lesssim \int_{\Gamma_{f,g,\theta}} |f(z)|^2|\theta'(z)|^2\left(1-|z|^2\right)^s dA(z) \\ &\lesssim \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f(z)|^2\frac{(1-|\theta(z)|^2)^2}{(1-|z|^2)^2}\left(1-|z|^2\right)^s dA(z) \\ &\lesssim \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|\theta(z)|^2)}{(1-|z|^2)^2}\left(1-|z|^2\right)^s dA(z) \lesssim \|\theta\|^2_{\mathcal{D}_s}, \end{aligned}$$

where the last inequality due to [7].

(4) $\Rightarrow$ (1). Since $1 \in C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B}) \cap BMOA$ and $1\cdot\theta \in H^\infty \subseteq BMOA$. Then $\theta \in C_{\mathcal{B}}(\mathcal{D}_s \cap \mathcal{B})$. The proof is complete. □

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