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Global Regularity for the 3D Micropolar Fluid Flows

Ahmad M. Alghamdi[a], Sadek Gala[b,c], Maria Alessandra Ragusa[c,d]

[a] *Department of Mathematical Science , Faculty of Applied Science, Umm Al-Qura University, P. O. Box 14035, Makkah 21955, Saudi Arabia*
[b] *Department of Mathematics, ENS of Mostaganem, Box 227, Mostaganem 27000, Algeria*
[c] *Dipartimento di Matematica e Informatica, Università di Catania, Viale Andrea Doria, 6 95125 Catania - Italy*
[d] *RUDN University, 6 Miklukho - Maklay St, Moscow, 117198, Russia*

Abstract. The aim of this note is to establish the global regularity of classical solutions of the 3D micropolar fluid equations for a family of large initial data with finite energy.

1. Introduction

In this paper, we consider the following Cauchy problem for the incompressible micropolar fluid equations :

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \pi - \nabla \times \omega = 0, \\ \partial_t \omega - \Delta \omega - \nabla div \omega + 2\omega + u \cdot \nabla \omega - \nabla \times u = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \ \omega(x,0) = \omega_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^3$, $\omega = \omega(x,t) \in \mathbb{R}^3$ and $p = p(x,t)$ denote the unknown velocity vector field, the micro-rotational velocity and the unknown scalar pressure of the fluid at the point $(x,t) \in \mathbb{R}^3 \times (0,T)$, respectively, while u_0, ω_0 are given initial data with $\nabla \cdot u = 0$ in the sense of distributions.

Micropolar fluid system was first proposed by Eringen [2] in 1966. Later on, Galdi and Rionero [3] considered the weak solution in the year 1977. Using linearization and an almost fixed point theorem, in 1988, Lukaszewicz [4] established the global existence of weak solutions with sufficiently regular initial data. In 1989, using the same technique, Lukaszewicz [5] proved the local and global existence and the uniqueness of the strong solutions under asymmetric condition. In 2005, Yamaguchi [8] proved the existence theorem of global in time solution for small initial data.

Inspired by the work of [6], for the 3D Navier-Stokes equations, the main purpose of this note is to study the global existence of smooth solutions to (1.1) for a family of large initial data with finite energy.

Our result is the following.

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Email addresses: amghamdi@uqu.edu.sa (Ahmad M. Alghamdi), sgala793@gmail.com (Sadek Gala), mariaalessandra.ragusa@unict.it (Maria Alessandra Ragusa)

Theorem 1.1. *Assume that $u_0, w_0 \in \mathcal{M}$ for some constant $\delta > 0$. Then there exists a positive constant δ_0 such that (1.1) with the initial data (u_0, w_0) has a unique global classical solution (u, ω) if $\delta \le \delta_0$. Here*

$$\mathcal{M} = \left\{ \begin{array}{cc} \multicolumn{2}{c}{u_0, w_0 \in H^1(\mathbb{R}^3) \text{ with } \nabla \cdot u_0 = 0,} \\ \sum_{i=1}^{2} \|u_0\|_{L^2} \left\|\partial_{x_i} u_0\right\|_{L^2} \le \delta, & \sum_{i=1}^{2} \|\omega_0\|_{L^2} \left\|\partial_{x_i} \omega_0\right\|_{L^2} \le \delta, \\ \sum_{i=1}^{2} \|u_0\|_{L^2} \left\|\partial_{x_i} \omega_0\right\|_{L^2} \le \delta, & \sum_{i=1}^{2} \|\omega_0\|_{L^2} \left\|\partial_{x_i} u_0\right\|_{L^2} \le \delta \end{array} \right\}.$$

We recall the following Serrin's type non-blow up criterion [3].

Lemma 1.2. *Assume that the initial data $u_0, \omega_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. If*

$$u \in L^q\left((0,T); L^p(\mathbb{R}^3)\right) \text{ with } \frac{2}{q} + \frac{3}{p} \le 1 \text{ and } 3 < p \le \infty,$$

then the solution (u, ω) remains smooth on $[0, T]$.

In the following calculations, we use the following interpolation inequality due to [6] :

$$\left\|f\right\|_{L^4} \le C \left\|f\right\|_{L^2}^{\frac{1}{2}} \left\|\partial_1 f\right\|_{L^2}^{\frac{1}{8}} \left\|\partial_1 \partial_3 f\right\|_{L^2}^{\frac{1}{8}} \left\|\partial_2 f\right\|_{L^2}^{\frac{1}{8}} \left\|\partial_2 \partial_3 f\right\|_{L^2}^{\frac{1}{8}}. \tag{1.2}$$

2. Proof of Theorem 1.1

Proof: Assume that u_0, ω_0 belongs to $\mathcal{M}$. The local existence theory is classical, see for instance [3, 8]. Hence there exists a unique smooth solution (u, ω) of (1.1) on some time interval $[0, T)$ with $T > 0$.

Taking the inner products of $(1.1)_1$ with u and $(1.1)_2$ with ω, adding the results and integrating by parts, we obtain

$$\begin{aligned} &\|u(\cdot,t)\|_{L^2}^2 + \|\omega(\cdot,t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\cdot,s)\|_{L^2}^2 \, ds + 2\int_0^t \|\nabla \omega(\cdot,s)\|_{L^2}^2 \, ds \\ &+2\int_0^t \|\nabla \cdot \omega(\cdot,s)\|_{L^2}^2 \, ds + 2\int_0^t \|\omega(\cdot,s)\|_{L^2}^2 \, ds \le \|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2, \end{aligned}$$

for all $t \ge 0$.

Applying the derivatives $\partial_i = \frac{\partial}{x_i}$ $(i = 1, 2)$ on either sides of the equations (1.1) yields to

$$\left\{ \begin{array}{l} \partial_t \partial_i u + (u \cdot \nabla) \partial_i u - \Delta \partial_i u + \nabla \partial_i \pi - \nabla \times \partial_i \omega = 0, \\ \partial_t \partial_i \omega - \Delta \partial_i \omega - \nabla \mathrm{div} \partial_i \omega + 2\partial_i \omega + (u \cdot \nabla) \partial_i \omega - \nabla \times \partial_i u = 0. \end{array} \right. \tag{2.1}$$

Considering the scalar products with $\partial_i u, \partial_i \omega$, respectively, and adding them, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\partial_i u\|_{L^2}^2 + \|\partial_i \omega\|_{L^2}^2\right) + \|\nabla \partial_i u\|_{L^2}^2 + \|\nabla \partial_i \omega\|_{L^2}^2 + \|\nabla \cdot \partial_i \omega\|_{L^2}^2 + 2\|\partial_i \omega\|_{L^2}^2 \\ = \; & -\int_{\mathbb{R}^3} \partial_i u \cdot \nabla u \cdot \partial_i u dx + \int_{\mathbb{R}^3} (\nabla \times \partial_i \omega) \cdot \partial_i u dx + \int_{\mathbb{R}^3} (\nabla \times \partial_i u) \cdot \partial_i \omega dx - \int_{\mathbb{R}^3} \partial_i u \cdot \nabla \omega \cdot \partial_i \omega dx \\ = \; & A_1 + A_2 + A_3 + A_4. \end{aligned}$$

To bound A_1, we integrate by parts and apply Hölder's inequality to obtain by (1.2)

$$\begin{aligned} A_1 &= -\int_{\mathbb{R}^3} \partial_i u \cdot \nabla u \cdot \partial_i u dx = \int_{\mathbb{R}^3} \partial_i u \cdot u \cdot \nabla \partial_i u dx \\ &\le \|u\|_{L^4} \|\partial_i u\|_{L^4} \|\nabla \partial_i u\|_{L^2} \\ &\le C \|u\|_{L^2}^{\frac{1}{2}} \|\partial_1 u\|_{L^2}^{\frac{1}{8}} \|\partial_1 \partial_3 u\|_{L^2}^{\frac{1}{8}} \|\partial_2 u\|_{L^2}^{\frac{1}{8}} \|\partial_2 \partial_3 u\|_{L^2}^{\frac{1}{8}} \|\partial_i u\|_{L^2}^{\frac{1}{4}} \|\nabla \partial_i u\|_{L^2}^{\frac{7}{4}} \\ &\le C \|u\|_{L^2}^{\frac{1}{2}} \|\partial_i u\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_i u\|_{L^2}^2. \end{aligned}$$

Using the integration by parts and the Cauchy–Schwarz inequality, we estimate

$$A_2 + A_3 \le 2\,\|\partial_i\omega\|_{L^2}\,\|\nabla\partial_i u\|_{L^2} \le \|\partial_i\omega\|_{L^2}^2 + \|\nabla\partial_i u\|_{L^2}^2 .$$

To bound A_4, we integrate by parts and apply Hölder's inequality to get by (1.2)

$$\begin{aligned}
A_4 &= -\int_{\mathbb{R}^3} \partial_i u\cdot\nabla\omega\cdot\partial_i\omega dx = -\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_j\omega_k\partial_j\partial_i\omega_k dx\\
&\le \|\omega\|_{L^4}\,\|\partial_i u\|_{L^4}\,\|\nabla\partial_i\omega\|_{L^2}\\
&\le C\,\|\omega\|_{L^2}^{\frac12}\,\|\partial_1\omega\|_{L^2}^{\frac18}\,\|\partial_1\partial_3\omega\|_{L^2}^{\frac18}\,\|\partial_2\omega\|_{L^2}^{\frac18}\,\|\partial_2\partial_3\omega\|_{L^2}^{\frac18}\,\|\partial_i u\|_{L^2}^{\frac14}\,\|\nabla\partial_i u\|_{L^2}^{\frac34}\,\|\nabla\partial_i\omega\|_{L^2}\\
&\le C\,\|\omega\|_{L^2}^{\frac12}\,\|\partial_i\omega\|_{L^2}^{\frac14}\,\|\nabla\partial_i\omega\|_{L^2}^{\frac54}\,\|\partial_i u\|_{L^2}^{\frac14}\,\|\nabla\partial_i u\|_{L^2}^{\frac34}\\
&\le C\,\|\omega\|_{L^2}^{\frac12}\,\|\partial_i\omega\|_{L^2}^{\frac14}\,\|\partial_i u\|_{L^2}^{\frac14}\left(\|\nabla\partial_i\omega\|_{L^2}^2 + \|\nabla\partial_i u\|_{L^2}^2\right)\\
&= C\left(\|\omega\|_{L^2}^{\frac12}\,\|\partial_i\omega\|_{L^2}^{\frac12} + \|\omega\|_{L^2}^{\frac12}\,\|\partial_i u\|_{L^2}^{\frac12}\right)\left(\|\nabla\partial_i\omega\|_{L^2}^2 + \|\nabla\partial_i u\|_{L^2}^2\right).
\end{aligned}$$

Combining the estimates for A_1, A_2, A_3 and A_4, we find

$$\begin{aligned}
&\frac12\frac{d}{dt}\sum_{i=1}^{2}\left(\|\partial_i u\|_{L^2}^2 + \|\partial_i\omega\|_{L^2}^2\right) + \sum_{i=1}^{2}\left(\|\nabla\partial_i u\|_{L^2}^2 + \|\nabla\partial_i\omega\|_{L^2}^2 + \|\nabla\cdot\partial_i\omega\|_{L^2}^2 + 2\,\|\partial_i\omega\|_{L^2}^2\right)\\
\le\; & C\sum_{i=1}^{2}\left(\|\nabla\partial_i\omega\|_{L^2}^2 + \|\nabla\partial_i u\|_{L^2}^2\right)\left(\|u\|_{L^2}^{\frac12}\,\|\partial_i u\|_{L^2}^{\frac12} + \|\omega\|_{L^2}^{\frac12}\,\|\partial_i\omega\|_{L^2}^{\frac12} + \|\omega\|_{L^2}^{\frac12}\,\|\partial_i u\|_{L^2}^{\frac12}\right).
\end{aligned}$$

Hence, if the initial data belongs to $\mathcal{M}$ and taking $\delta_0 = \frac{1}{2C}$, we have

$$\frac12\frac{d}{dt}\sum_{i=1}^{2}\left(\|\partial_i u\|_{L^2}^2 + \|\partial_i\omega\|_{L^2}^2\right) + \sum_{i=1}^{2}\left(\|\nabla\partial_i u\|_{L^2}^2 + \|\nabla\partial_i\omega\|_{L^2}^2 + \|\nabla\cdot\partial_i\omega\|_{L^2}^2 + 2\,\|\partial_i\omega\|_{L^2}^2\right) \le 0,$$

for all $t \ge 0$. In particular, there holds

$$\sum_{i=1}^{2}\|\partial_i u\|_{L^2}^2 \le \sum_{i=1}^{2}\left(\|\partial_i u_0\|_{L^2}^2 + \|\partial_i\omega_0\|_{L^2}^2\right). \tag{2.2}$$

By using (2.2), it yields

$$\begin{aligned}
\int_0^t \|u(\cdot,s)\|_{L^4}^6\,ds &\le C\int_0^t \|\partial_1 u(\cdot,s)\|_{L^2}^{\frac43}\,\|\partial_2 u(\cdot,s)\|_{L^2}^{\frac43}\,\|\partial_3 u(\cdot,s)\|_{L^2}^{\frac43}\,ds\\
&\le C\left(\sup_{0\le s\le t}\|\partial_1 u(\cdot,s)\|_{L^2}^{\frac43}\,\|\partial_2 u(\cdot,s)\|_{L^2}^{\frac23}\right)\int_0^t \|\partial_2 u(\cdot,s)\|_{L^2}^{\frac23}\,\|\partial_3 u(\cdot,s)\|_{L^2}^{\frac43}\,ds\\
&\le C\left(\sum_{i=1}^{2}\|\partial_i u\|_{L^2}^2\right)\int_0^t \|\nabla u(\cdot,s)\|_{L^2}^2\,ds\\
&\le C\sum_{i=1}^{2}\left(\|\partial_i u_0\|_{L^2}^2 + \|\partial_i\omega_0\|_{L^2}^2\right)\left(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2\right) < \infty,
\end{aligned}$$

where we have used the following interpolation inequality [1] :

$$\left\|f\right\|_{L^4} \le C\left\|\partial_1 f\right\|_{L^2}^{\frac13}\left\|\partial_2 f\right\|_{L^2}^{\frac13}\left\|\partial_3 f\right\|_{L^2}^{\frac13}.$$

Hence, by Lemma 1.2, we have proved that u, ω is a smooth solution. This completes the proof of Theorem 1.1. □

Remark 2.1. *It should be added that at the time the paper was accepted, the authors learnt that Y. Wang and L. Gu [7] have also obtained a similar result for the three dimensional magneto-micropolar fluid equations for a family of large initial data with finite energy.*

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