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# On Asymptotic Properties of Certain Neutral Differential Equations

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**Abstract.** The main purpose of the present paper is to investigate asymptotic properties of some neutral delay differential equations by means of Lyapunov functions. The asymptotic stability of the solutions is given in terms of delay-independent criteria. Two examples are given to illustrate the results.

## 1. Introduction

The asymptotic properties of solutions for functional differential equations have received considerable attention in recent years. Despite the permanent application of this type of equation in the natural sciences, technology and population dynamics [6, 8, 13, 14], there is an interest in obtaining sufficient new conditions allowing to know the long-term behavior of these equations. Asymptotic properties of neutral type differential equations of the form

$$\frac{d}{dt}[x(t) + px(t-\tau)] = -ax(t) + b\tanh x(t-\sigma), \quad t \geq 0, \tag{1}$$

have been investigated by several authors, see refs. [1, 2] and [3]. Delay differential equations of various types that contain (1) with $p = 0$ as a special case have been proposed by many authors for the study of characteristics of neural networks of Hopfield type. By constructing suitable Lyapunov functions, it was shown that under certain conditions, the solutions of equation (1) are stable. The asymptotic stability was proven in both the cases delay-dependent and delay-independent criteria. Here we are concerned with two more general cases. In the first stage of this paper, we will study asymptotic properties of neutral delay differential equations of the following form

$$\frac{d}{dt}[x(t) + \beta(t)x(t-\sigma)] = -a(t)x(t) - b(t)f(x(t-r)), \tag{2}$$

for all $t \geq t_1 = t_0 + \rho$, where $\rho = \max\{r, \sigma\}$ and the functions $a(t), b(t), \beta(t)$ and $f(x(t))$ are continuous in their respective arguments, with $|\beta(t)| < \beta_1 < 1$ and $f(0) = 0$. It is also supposed that the derivatives $\beta'(t)$ and $f'(x(t))$ exist and are continuous. For each solution of (2), we assume the following initial condition :

$$x(t) = \phi(t), \quad t \in [-\rho, t_0], \quad \phi \in C([-\rho, 0], \mathbb{R}).$$

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It can be easily seen that if $\beta(t) = p$, $a(t) = a$, $b(t) = -b$, and $f(x(t)) = \tanh x(t)$, then equation (2) reduces to the simpler case (1). We will in the next stage give results on boundedness of solutions for the forced equation (2) of the form

$$\frac{d}{dt}[x(t) + \beta(t)x(t-\sigma)] = -a(t)x(t) - b(t)f(x(t-r)) + e(t), \tag{3}$$

where $e(t) \in L^1[t_0, \infty)$. In the third and last stage of this paper, we give asymptotic stability results for solutions of the equation

$$\frac{d}{dt}[x(t) + \beta(t)x(t-\sigma)] = -a(t)g(x(t)) - b(t)f(x(t-r)), \tag{4}$$

where the function $g(x)$ is continuous and satisfies $g(0) = 0$. Also, it is assumed that the derivative $g'(x)$ exist and is continuous.

**Remark 1.1.** *The reader is referred to [2, 3, 15] for general references related to delay and neutral differential equations.*

## 2. Main Results

The present section is divided into three subsections. The first subsection contains assumptions and hypothesis made on the functions appearing in the treated equations. The next subsection contains the main theorems and results of this paper. In the third and last subsection, the proofs of the results are given.

### 2.1. Hypothesis

In the sequel we assume what follows hold. Suppose that there exist positive constants $a_i$, $b_i$, $f_i$, and $g_i$ for $i = 0, 1$, such that

*i)* $a_0 \leq a(t) \leq a_1$, $b_0 \leq b(t) \leq b_1$;

*ii)* $|f'(x)| \leq f_1$, and $\dfrac{f(x)}{x} \geq f_0$ $(x \neq 0)$, for all $x$;

*iii)* $|g'(x)| \leq g_1$, and $\dfrac{g(x)}{x} \geq g_0$ $(x \neq 0)$, for all $x$.

### 2.2. Results

Here, we will state our main theorems. The first theorem concerns the asymptotic properties of solutions of equation (2) and is stated as follows

**Theorem 2.1.** *Assume conditions (i) and (ii) satisfied. Suppose also that*

$$a_0 > \frac{b_1}{2}[1 + 2a_1 + \beta_1(1 + f_1^2) + f_1^2],$$

*then, the trivial solution of equation (2) is asymptoticaly stable. Moreover, each solution $x$ of equation (2) is bounded and is in $L^2[t_1, \infty)$.*

The next theorem concerns the boundedness of solutions to equation (3).

**Theorem 2.2.** *Suppose conditions of Theorem 2.1 beeing satisfied. Assume that there exists a positive constant $e_1$ such that*

*iv)* $\displaystyle\int_{t_1}^{t} |e(s)|ds < e_1$, *for all* $t \geq t_1$,

*then, there exists a positive constant $N$, such that any solution of (3) satisfies*

$$|x(t)| \leq N, \quad \textit{for all } t \geq t_1.$$

The last result of this paper is :

**Theorem 2.3.** *Assume* $(i)-(iii)$ *satisfied. Suppose also that*

$$2a_0g_0 > b_1 + \beta_1(a_1 + b_1 + a_1g_1^2) + b_1f_1^2(1+\beta_1),$$

*then, the trivial solution of equation (4) is asymptoticaly stable.*

*2.3. Proofs*

Before procceding to the proofs of the previous theorems, we rewrite equations (2), (3) and (4) in the following equivalent descriptor system

$$\begin{cases} 0 = -z(t) + x(t) + \beta(t)x(t-\sigma), \\ z'(t) = -a(t)g(x(t)) - b(t)f(x(t-r)) + e(t), \end{cases} \tag{5}$$

where $g(x) = x$ and $e(t) = 0$ for equation (2), $g(x) = x$ and $e(t) \neq 0$ for equation (3). In all of our proofs, we make use of the Lyapunov function

$$V = V(t,x) = \frac{1}{2}z^2 + \lambda\int_{t-\sigma}^{t} x^2(s)\,ds + \mu\int_{t-r}^{t} x^2(s)\,ds, \tag{6}$$

where $\lambda$ and $\mu$ are positive constants to be determined later with respect to the appropriate case. It can be easily seen that

$$V(t,x) \geq \frac{1}{2}z^2, \tag{7}$$

since $\lambda\int_{t-\sigma}^{t} x^2(s)\,ds + \mu\int_{t-r}^{t} x^2(s)\,ds \geq 0$. In addition, we have $V(t,0) = 0$, for all $t \geq t_1$.

*Proof.* [Proof of Theorem 2.1] The time derivative of the Lyapunov function (6) along trajectories of equation (2) is

$$\begin{aligned} V'_{(2)} =& \left(-a(t) + \lambda + \mu\right)x^2(t) - \lambda x^2(t-\sigma) - \mu x^2(t-r) \\ & - b(t)f(x(t-r))x(t) - \beta(t)a(t)x(t)x(t-\sigma) \\ & - \beta(t)b(t)f(x(t-r))x(t-\sigma). \end{aligned}$$

With the use of the conditions $(i)$ and $(ii)$ toghether with inequality $2|uv| \leq u^2 + v^2$, one obtain

$$\begin{aligned} V'_{(2)} \leq & \left(-a_0 + \lambda + \mu + \frac{b_1}{2} + \frac{b_1a_1}{2}\right)x^2(t) \\ & + \left(-\lambda + \frac{b_1}{2}[a_1 + \beta_1]\right)x^2(t-\sigma) \\ & + \left(-\mu + \frac{b_1f_1^2}{2}[1+\beta_1]\right)x^2(t-r). \end{aligned}$$

Choose

$$\lambda = \frac{b_1}{2}[a_1 + \beta_1],$$

and

$$\mu = \frac{b_1f_1^2}{2}[1+\beta_1].$$

With the previous choice of constants $\lambda$ and $\mu$, we get

$$V'_{(2)} \leq \left(-a_0 + \frac{b_1}{2}\left[1 + 2a_1 + \beta_1\left(1 + f_1^2\right) + f_1^2\right]\right) x^2(t).$$

Using the fact

$$a_0 > \frac{b_1}{2}\left[1 + 2a_1 + \beta_1\left(1 + f_1^2\right) + f_1^2\right],$$

we get

$$V'_{(2)} \leq -kx^2, \tag{8}$$

where $k = a_0 - \frac{b_1}{2}\left[1 + 2a_1 + \beta_1\left(1 + f_1^2\right) + f_1^2\right] > 0$. From properties of the Lyapunov function (6), namely (7) and (8), we conclude that the trivial solution of equation (2) is asymptoticaly stable.
It follows from (8) that

$$V(t) + k\int_{t_1}^{t} x^2(s)\,ds \leq V(t_1), \text{ for } t > t_1.$$

Since $V(t) \geq 0$, and $k > 0$, the above inequality lead to

$$\int_{t_1}^{\infty} x^2(s)\,ds \leq \infty. \tag{9}$$

Hence $x \in L^2[t_1, \infty)$. From (8), there exists a positive constant $K$ such that

$$[x(t) + \beta(t)x(t-\sigma)]^2 < K^2 \text{ for all } t \geq 0,$$

thus

$$|x(t)| \leq \beta_1 |x(t-\sigma)| + K, t \geq 0. \tag{10}$$

We claim that $|x(t)|$ is bounded. Suppose not. Then there exists a subsequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$|x(t_n)| = \sup\{|x(t)|, t \leq t_n\}, n = 1, 2, ...$$

We have $\lim\limits_{n\to\infty} |x(t_n)| = \infty$ and $|x(t_n)| \geq |x(t_n - \sigma)|$ for $n = 1, 2, ....$ Inequality (10) yields

$$\begin{aligned} |x(t_n)| &< \beta_1 |x(t_n - \sigma)| + K \\ &< \beta_1 |x(t_n)| + K, n = 1, 2, ... \end{aligned}$$

Then

$$|x(t_n)| < \frac{K}{1 - \beta_1},$$

thus, as $n \to \infty$, we get $\infty < \frac{K}{1-\beta_1}$. This contradiction implies that $|x(t)|$ is bounded. The proof of Theorem (2.1) is over. □

*Proof.* [Proof of Theorem 2.2] Consider the Lyapunov function (6). In this case the time derivative along trajectories of equation (3) is given by

$$V'_{(3)} = V'_{(2)} + [x(t) + \beta(t) x(t-\sigma)] e(t).$$

From (8), we get

$$V'_{(3)} \leq \left| x(t) + \beta(t) x(t-\sigma) \right| |e(t)|.$$

Use inequality $|u| \leq 1 + u^2$, to obtain

$$V'_{(3)} \leq \left( [x(t) + \beta(t) x(t-\sigma)]^2 + 1 \right) |e(t)|.$$

Using (7) and integrating from $t_1$ to $t$, lead to

$$V(t) \leq V(t_1) + \int_{t_1}^{t} |e(s)| ds + \int_{t_1}^{t} 2V(s) |e(s)| ds.$$

By Gronwall's inequality, it follows that

$$V(t) \leq K_1 \exp\left( 2 \int_{t_1}^{t} |e(s)| ds \right) \leq K_2,$$

where $K_1 = V(t_1) + e_1$, and $K_2 = K_1 \exp(2e_1)$ .This result implies the existance of a positive constant $N$, such that

$$\left| x(t) + \beta(t) x(t-\sigma) \right| \leq N,$$

and as our previous proof, we conclude that $|x(t)|$ is bounded. This fact completes the proof of Theorem (2.2). □

*Proof.* [Proof of Theorem 2.3] The time derivative of (6) along trajectories of equation (4) is given by

$$\begin{aligned} V'_{(4)} &= (\lambda + \mu) x^2(t) - \lambda x^2(t-\sigma) - \mu x^2(t-r) \\ &\quad -a(t) g(x(t)) x(t) - b(t) f(x(t-r)) x(t) \\ &\quad -\beta(t) a(t) g(x(t)) x(t-\sigma) - \beta(t) b(t) f(x(t-r)) x(t-\sigma) \end{aligned}$$

With the use of the conditions $(i) - (iii)$ toghether with inequality $2|uv| \leq u^2 + v^2$, one obtain

$$\begin{aligned} V'_{(4)} \leq & \frac{1}{2} \left( -2a_0 g_0 + b_1 + \beta_1 a_1 g_1^2 + 2\lambda + 2\mu \right) x^2(t) \\ & + \frac{1}{2} \left( -2\lambda + \beta_1 (a_1 + b_1) \right) x^2(t-\sigma) \\ & + \frac{1}{2} \left( -2\mu + b_1 f_1^2 (1 + \beta_1) \right) x^2(t-r). \end{aligned}$$

Put

$$2\lambda = \beta_1 (a_1 + b_1),$$

and

$$2\mu = b_1 f_1^2 (1 + \beta_1),$$

to get

$$V'_{(4)} \leq \frac{1}{2} \left( -2a_0 g_0 + b_1 + \beta_1 \left( a_1 + b_1 + a_1 g_1^2 \right) + b_1 f_1^2 (1 + \beta_1) \right) x^2(t).$$

Hence

$$V'_{(4)} \leq -k_0 x^2,$$

where $k_0 = 2a_0 g_0 - b_1 - \beta_1 \left(a_1 + b_1 + a_1 g_1^2\right) - b_1 f_1^2 (1 + \beta_1) > 0$. We conclude that the trivial solution of equation (4) is asymptoticaly stable. This fact completes the proof. □

**Remark 2.4.** *It is also possible to show that for equation (4), the solutions are bounded and square integrable. The proofs are the same in this case and hence being ommited.*

## 3. Examples

In this section, we give two examples illustrating the obtained results.

### *3.1. Example 1*

As a special case of equation (3) (respectively eq. (2) for $e(t) = 0$), consider the following equation

$$\begin{aligned} \frac{d}{dt}\left[x(t) + \frac{1}{10+t^2}x(t-\sigma)\right] = -\left(4.5 + \frac{1}{4+t^2}\right)x(t) - \left(0.4 + \frac{4}{10+t^2}\right) \\ \times\left(0.1x(t-r) + \frac{x(t-r)}{10+\mid x(t-r)\mid}\right) + \frac{1}{1+t^2}, \end{aligned} \tag{11}$$

Observing the functions over the equation (11), one can deduce the following

$$a_0 = 4.5 \leq a(t) = 4.5 + \frac{1}{4+t^2} \leq 4.75 = a_1,$$

$$b_0 = 0.4 \leq b(t) = 0.4 + \frac{4}{10+t^2} \leq 0.8 = b_1,$$

$$|\beta(t)| = \left|\frac{1}{10+t^2}\right| \leq \frac{1}{10} = \beta_1,$$

and

$$f(x) = 0.1x + \frac{x}{10+\mid x\mid}.$$

It is clear, from the relation of $f(x)$, that $f(0) = 0$, besides, since $0 \leq \dfrac{1}{10+\mid x\mid} \leq 1$, for all $x$, we have that

$$\frac{f(x)}{x} \geq 0.1 = f_0, \quad \text{for all } x \neq 0.$$

Moreover

$$|f'(x)| = \left|0.1 + \frac{1}{(10+\mid x\mid)^2}\right| \leq 0.2 = f_1.$$

A simple calculation give

$$-a_0 + \frac{b_1}{2}[1 + 2a_1 + \beta_1(1 + f_1^2) + f_1^2] = -0.24 = -k < 0.$$

Hence the trivial solution of (11) is asymptotically stable by assumptions of Theorem (2.1) with respect to $e(t) = 0$. On the other side, observe also that

$$e(t) = \frac{1}{1+t^2},$$

then

$$\int_{t_1}^{t} |e(s)|ds < \infty, \quad \text{for all} \quad t \geq t_1.$$

All assumptions of Theorem (2.2) are satisfied, so every solution of (11) is bounded and belong to $L^2[t_1, +\infty)$.

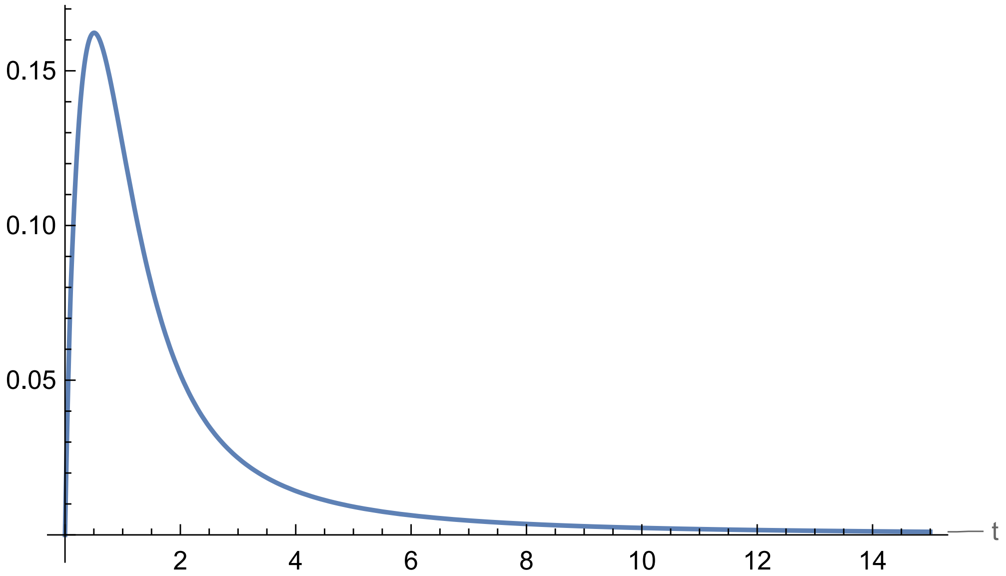


Figure 1: Numerical simulation for example (11). $x(t) = t^2$ for $t \leq 0$.

### 3.2. *Example 2*

As a special case of equation (4), consider the following equation

$$\begin{aligned}\frac{d}{dt}\left[x(t) + \frac{1}{10+t^2}x(t-\sigma)\right] = &-\left(4 + \frac{1}{2+t^2}\right)\times\left(x(t) + \frac{2x(t)}{10+\mid x(t)\mid}\right)\\ &-\left(0.2 + \frac{2}{10+t^2}\right)\times\left(0.5x(t-r) + \frac{x(t-r)}{10+\mid x(t-r)\mid}\right),\end{aligned} \tag{12}$$

Observing the functions over the equation (11), one can deduce the following

$$a_0 = 4 \leq a(t) = 4 + \frac{1}{2+t^2} \leq 4.5 = a_1,$$

$$b_0 = 0.2 \leq b(t) = 0.2 + \frac{2}{10+t^2} \leq 0.4 = b_1,$$

$$|\beta(t)| = \left|\frac{1}{10+t^2}\right| \leq \frac{1}{10} = \beta_1,$$

$$g(x) = x + \frac{2x}{10+\mid x\mid},$$

and

$$f(x) = 0.5x + \frac{x}{10+\mid x\mid}.$$

It is clear, from the relation of $f(x)$ and $g(x)$, that $f(0) = g(0) = 0$, besides, since $0 \leq \dfrac{1}{10+ \mid x \mid} \leq 1$, for all $x$, we have that

$$\frac{g(x)}{x} \geq 1 = g_0, \quad \text{for all } x \neq 0,$$

and

$$\frac{f(x)}{x} \geq 0.5 = f_0, \quad \text{for all } x \neq 0.$$

Moreover

$$|g'(x)| = \left|1 + \frac{2}{(10+ \mid x \mid)^2}\right| \leq 1.2 = g_1.$$

$$|f'(x)| = \left|0.5 + \frac{1}{(10+ \mid x \mid)^2}\right| \leq 0.6 = f_1.$$

A simple calculation give

$$-2a_0g_0 + b_1 + \beta_1\left(a_1 + b_1 + a_1g_1^2\right) + b_1f_1^2\left(1 + \beta_1\right) = -6.1 = -k_0 < 0.$$

Hence the trivial solution of (12) is asymptotically stable by assumptions of Theorem (2.3).

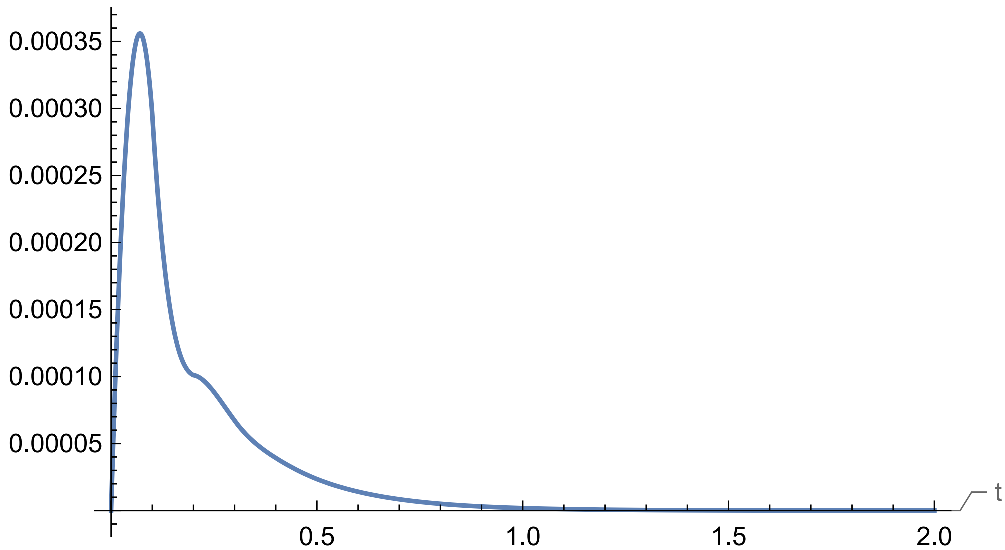


Figure 2: Numerical simulation for example (12). $x(t) = t^2$ for $t \leq 0$.

## 4. Conclusion

The asymptotic properties of certain classes of neutral delay differential equations have been investigated by constructing a suitable Lyapunov function. Delay-independent criteria have been obtained to guarantee the asymptotic stability, boundedness and square integrability of the considered classes of equations. Joint numerical simulations confirm the validity of the results.

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