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Construction of New Hadamard Matrices Using Known Hadamard Matrices

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Abstract. In this paper, by the use of Latin squares, we describe a procedure to construct Hadamard matrices using the existing Hadamard matrices of order m as input matrices. We propose constructions of Hadamard matrices of orders $m(m-1)$, $m(\frac{m}{2}-1)$ and $m(\frac{m}{k}-1)$, where k is a multiple of four that divides m into an even number. This work is a continuation of our previous one in [5].

1. Introduction

A Hadamard matrix H of order n is an $n \times n$ matrix with entries from the set $\{-1, 1\}$, which satisfies the equality $HH^\top = nI_n$. Hadamard matrices are also important combinatorial structures as they induces symmetric designs (see [3]; Part V). So, they are applied in coding, graph and cryptography theories. Moreover, they are of maximum determinant, and therefore, of analytical importance (see [4]).

The construction of Hadamard matrices has been an elusive problem, but it is well known that this matrices exist only for the orders 1, 2, and $4k$, where $k \in \mathbb{N}^*$. Many mathematicians have discussed the construction of Hadamard matrices. We mention of their works, Paley's (in[8]) who describe the construction of Hadamard matrices of order $q+1$, where $q \equiv 3\ (mod 4)$ is a power of a prime number, called *Paley type I* matrices, then the construction of Hadamard matrices of order $2(q'+1)$, where $q' \equiv 1\ (mod 4)$ is a power of a prime number, known as *Paley type II* matrices. Both of the constructions are obtained by operating the quadratic residue on the field $GF(q)$ or $GF(q')$; respectively. Paley type Hadamard matrices are used as input matrices in Scarpis' theorem [6, 7] to construct Hadamard matrices of order $q(q+1)$, where $q \equiv 3($ mod 4) a prime number. This theorem has been generalized by Dokovič in [9], to contain the cases where q is a power of prime number. Then we have given an analogue of this construction in [5] to obtain $2q'(q'+1)$ Hadamard matrices using as input matrices the Paley type II Hadamard matrices, where $q' \equiv 1($ mod 4) a power of a prime number.

The constructions of Hadamard matrices based on Latin squares is widely known and used. Among

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them we mention the work of Goethlas and Seidel (see [15]), who use 17 different classes of Latin squares of order 6 to construct Hadamard matrices. The multiplication tables of finite groups form particular Latin squares, called *Cayley tables*. While, *difference sets* are designs defined on the last multiplication tables, over a subset of the group. They induce a description of families of Hadamard matrices, which includes the Paley type matrices (see [3]; Part VI, 18).

In the early 90s a relation between Hadamard matrices and Homological groups led to the study of *cocyclic Hadamard matrices* [16]. Many families of Hadamard matrices are shown to be cocyclic, and more likely the cocyclic theory of Hadamard matrices should induce an answer to the Hadamard conjecture. The description of cocyclic Hadamard matrices is based on the description of a sets 2-cocycles operator of values in the multiplicative field $\{-1, 1\}$. In [17], the authors have shown that cocyclic Hadamard matrices are equivalent to *relative difference sets*, which are a generalization of the concept of difference sets, and based on Cayley tables.

This shows the importance of Latin squares in the construction of Hadamard matrices, and in the resolution of the Hadamard conjecture.

In this paper, by generalizing the ideas of Scarpis' theorem, we propose constructions of Hadamard matrices of orders $m(m-1)$, $m(\frac{m}{2}-1)$ and $m(\frac{m}{k}-1)$ where k is a multiple of 4 that divides m, using as input matrix a Hadamard matrix of order m. So, the rest of the paper is divided as follows: In the second section we study a set of Latin squares that shares a specific propriety, we call them *Latin Squares Eligible for a Scarpis' Construction*, or *LSESC*. The *complete set of LSESC* reveals the main idea in the constructions theorems. We define such Latin squares, and we give methods and conditions to obtain them. Then in the third section we propose the construction of Hadamard matrices in question. In the fourth section, we give the relation between Scarpis' constructions and the proposed constructions, we study the eligibility of Sylvester Hadamard matrices as input matrices, then we conclude the paper with some extracted problems.

First, let's recall some definitions and notation.

We denote the set of all $n \times n$ Hadamard matrices by $\mathcal{H}_n$. The i-th row of a matrix $A = (a_{ij})_{1\le i\le n}^{1\le j\le n}$ is denoted by $\mathbf{a}_i$. $A^\top$ denotes the transpose matrix of A. We denote by $\mathbb{A}_n$ the set $\{1, ..., n\}$. Let $\mathbf{J}_m$ be the $1 \times m$ matrix (row vector) with all entries equal to 1. $\mathrm{O}_{m,p}$ is the zero matrix of size $m \times p$, and I_n is the identity matrix of order n. Two $n \times n$ matrices X, Y are *disjoint* if there is no position where they are both nonzero.

Two vectors are orthogonal if their inner product (or dot product) is 0 (i.e., two $1 \times n$ vectors $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ are orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^n x_i y_i = \mathbf{x}\mathbf{y}^\top = 0$).

The Kronecker (or tensor) product $X \otimes Y$ of two matrices $X = (x_{ij})$ and Y is the block matrix $X \otimes Y = (x_{ij} Y)$.

By deleting the first row and column of a *normalized Hadamard matrix* H, we obtain a matrix where the inner product of every two of its rows (or columns resp) gives -1. This matrix is called the *Core* of H.

Note that the multiplicative group of permutation matrices of size n, is a group isomorphic to the symmetric group $\mathbb{S}_n$ by corresponding to each $\pi \in \mathbb{S}_n$ the permutation matrix $P_\pi = [\delta(\pi(i), j)]$, where

$$\delta(i, j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

It is generated by transposition matrices which are permutations that switch only two rows (or two columns). For all $\sigma \in \mathbb{S}_n$ on the symboles set $\{1, ..., n\}$, we write

$$\sigma = \begin{pmatrix} 1 & ... & n \\ \sigma(1) & ... & \sigma(n) \end{pmatrix}.$$

A multidimensional array is called a *tensor*. Vectors are first order tensors, matrices are second order tensors, and we call *higher order tensors*, the tensors of order greater than two. The set of all tensors of order l and of the same dimenssions $n_1 \times n_2 \times .... \times n_l$ form a vector space, and denoted by $\mathbb{T}_{n_1,n_2,...,n_l}$.

In this paper, we use the third-order tensors. A *cubic tensor* is a third-order tensor of dimensions all equal to an integer n, and we denote the set of all such tensors by $\mathbb{T}_n$. Using the terminology in [2], we denote tensors by calligraphic letters as $\mathcal{X}$, $\mathcal{Y}, ...$ *Slices* are the two-dimensional sections of a tensor. The slices of the third-order tensor $\mathcal{X}$ are: the horizontal slices $\mathbf{X}_{i::}$, the lateral slices $\mathbf{X}_{:j:}$, and the frontal slices $\mathbf{X}_{::k} = \mathbf{X}_k$.

A *Latin square* of order n is an $n \times n$ matrix with entries chosen from the n-set of *symbols*, such that each symbol appears exactly once in each row and exactly once in each column. Here, the rows and columns of a Latin square of order n are indexed by the elements of $\mathbb{A}_n$, or the elements of a group of order n. We denote the set of all Latin squares of order n over a fixed symbols set by $\mathbb{L}_n$. If we permute the rows or columns of a Latin square, then the result is a Latin square; and if we permute the symbols set of a Latin square or replace the symbols set by another symbols set, then the result is a Latin square. Any Latin square obtained from another Latin square, L, by any combination of these operations is said to be *isotopic* to L. Clearly, isotopy is an equivalence relation. Then, Latin squares are classified by *isotopy*.

A pair of Latin squares L and L', of the same order are said to be *orthogonal* if for each a in the symbols set of L and each b in the symbols set of L', there exists a unique pair i, j such that $l_{i\,j} = a$ and $l'_{i\,j} = b$.

We consider a function that associate to every Latin square a cubic tensor defined as follows

$$\begin{array}{cccc} F & : & \mathbb{L}_n & \rightarrow & \mathbb{T}_n \\ & & L & \mapsto & \mathcal{X} \end{array},$$

where $\mathcal{X}$ is the cubic tensor of frontal slices

$$\mathbf{X}_k = \left[\begin{array}{c} \delta(l_{i\,k}, j) \end{array}\right].$$

We define also the functions

$$\begin{array}{cccc} G_m & : & \mathbb{T}_n & \rightarrow & \mathbb{T}_{mn,mn,n} \\ & & \mathcal{X} & \mapsto & \mathcal{Y} \end{array}$$

where, $m \in \mathbb{N}$, and $\mathcal{Y}$ is the third-order tensor of frontal slices

$$\mathbf{Y}_k = I_m \otimes \mathbf{X}_k.$$

Remark 1.1. *Let $L \in \mathbb{L}_n$, and let $F(L) = \mathcal{X}$. Then*

(1) *Every frontal slice $\mathbf{X}_t$ is a permutation matrix such that, the nonzero entry in row i is at the l_{it}-th position. So, the frontal slices form a set of disjoint permutation matrices;*

(2) *The horizontal slices $\mathbf{X}_{t::}$ also form a set of disjoint permutation matrices, where every row i contain the nonzero element in the l_{ti}-th position ;*

(3) *Furthermore, the lateral slices $\mathbf{X}_{:t:}$ form a set of disjoint permutation matrices, where*

$$L = \sum_{i=1}^{n} i\mathbf{X}_{:i:}. \tag{1}$$

Therefore, using a fixed symbols set, the notion of isotopic is equivalent in the image of F to the permutations over cubic tensors.

2. Latin squares eligible for a Scarpis' construction

In view of Remark 1.1, we see that any Latin square of size n on a symbols set $\mathbb{A}_n$ (considered as scalars from $\mathbb{Z}$) can be written as a linear combination of n disjoint permutation matrices each of which multiplied by an element of $\mathbb{A}_n$. Therefore, using this remark, we describe Latin squares by permutation matrices, then we define the set of Latin squares, which is the kernel of the construction of Hadamard matrices in the next sections. We describe them and we study their construction and proprieties.

Lemma 2.1. *An $n \times n$ matrix L of entries from the symbols set $\mathbb{A}_n$ is a Latin square if and only if there exists a set of disjoint permutation matrices of order n, $\{P_1, ..., P_n\}$, such that*

$$L = \sum_{i=1}^{n} iP_i.$$

Proof. Supposing that L is a Latin square, we conclude immediately from Remark 1.1, (3) the result, by taking the set of lateral slices of $F(L)$ and we obtain the equation (1). Conversely, as the P_is are permutation matrices, every row or column contains exactly one nonzero element, and by disjointness, every $i \in \mathbb{A}_n$ appear surely one time in every row or column of L. Then,

$$L = \sum_{i=1}^{n} iP_i$$

form a Latin square. □

Note that, if the Latin square $L = \sum_{i=1}^{n} iP_i$, an isotopic L' obtained by permuting the symbols set by a $\phi \in \mathbb{S}_n$, is equal to

$$\sum_{i=1}^{n} \phi(i)P_i = \sum_{i=1}^{n} iP_{\phi^{-1}(i)}.$$

In [5–7], the resulted matrices' construction is based on an idea of rows indexes intersections of some extracted matrix from the input Hadamard matrix. But, this can be done only if we use finite fields as we need the inverse of every element, and therefore it is used in the obtainment of Hadamard matrices larger by q times of the original order, for q a power of a prime number. Here, we will propose a generalisation of this aspect using Latin squares (or on the *Quasi groups* case, by the correspondence between the two notions).

Definition 2.2. *We say that a pair of Latin squares L and L' of the same order n are eligible for a Scarpis' construction, or LSESC, if for every i and i' there exists a unique j such that $l_{i\,j} = l'_{i'\,j}$. A complete set of LSESC is a set that contain the maximal number of pairwise LSESC Latin squares. Its cardinality is denoted by $N'(n)$.*

Remark 2.3. *Let L_1 and L_2 be two Latin squares of order n, and let L'_1, L'_2 isotopic of L_1, L_2 obtained by a rows permutations , respectively. It is clear from the definition that L_1 and L_2 are LSESC if and only if L'_1 and L'_2 are LSESC.*

As for MOLS (*Mutually orthogonal Latin squares*), the notion of complete set of MOLS can be projected on LSESC, and we define the *complete set of LSESC*. Hence, we show that $N'(n) \leq n-1$ then we propose a construction of a complete set of LSESC.

Theorem 2.4. *Let $n \geq 2$. Then, $N'(n) \leq n-1$.*

Proof. Let $L_1, L_2, ..., L_r$ a set of r LSESC. Then, by Remark 2, and with out a loss of generality we can assume that $l_{i,1\,1} = a$ for all $i \in \{1, ..., r\}$. Here, when $j \neq 1$, each j-th entry of the first row of L_i are distinct, for all $i \leq r$, none of them is equal to a. Hence, $r \leq n-1$. □

When n is a power of prime number, it is easy to find a complete set of LSESC, and next corollary give a way to construct such sets.

Corollary 2.5. *Let $GF(q) = \{x_1, x_2, ..., x_q\}$ a field of order q, and let L_b be the $q \times q$ matrix with i, j-th entry $x_i + bx_j$. Then the set $\mathcal{L} = \{L_b : b \in GF(q) - 0\}$ is a complete set of LSESC.*

Proof. Let i and i' be the row indices of two rows of different Latin squares L_b and $L_{b'}$, respectively. Considering the equation in $GF(q)$

$$x_i + br = x'_i + b'r,$$

which admits a unique solution $r = (x'_i - x_i)(b - b')^{-1}$. Assuming that $r = x_j$, the equation is equivalent to $l_{b,i\,j} = l_{b',i'\,j}$. Therefore, the set $\mathcal{L}$ is a complete set of LSESC. □

The set $\mathcal{L}$ is also a complete set of MOLS [see Theorem 1.6 in [1]]. But, for $n = 3$ the following set

$$\mathcal{S} = \left\{ L = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, L' = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \right\}$$

is a complete set of LSESC, and not of MOLS (taking $a = 1$ and $b = 2$, then for all i and j, $l_{i\,j} = 1$ implies that $l'_{i\,j} \neq 2$). So, the fact that two Latin squares are LSESC, does not imply that they are MOLS; Corollary 2.5 suggests the question whether or not a set of MOLS is a set of LSESC.

A long-standing conjecture is that a complete set of MOLS of order n and $N(n) = n - 1$ exists if and only if n is a power of prime number (see [1]). Therefore, in the search of a set of LSESC of order n and $N'(n) = n - 1$, is preferable to discuss the complete sets of LSESC that are not MOLS to avoid complexities. Next, we propose a construction of such sets, using the disjoint permutation matrices description of a Latin square.

Theorem 2.6. *Let $\{P_1, ..., P_n\}$ be a set of disjoint permutation matrices of order n, and $\sigma \in \mathbb{S}_n$ that fixes the 1, and of order $n - 1$ on each of the other symbols. Then, the set*

$$\mathcal{S}_\sigma = \{S^{(t)} = \sum_{i=1}^{n} \sigma^t(i) P_i : \ 1 \leq j \leq n-1, \ t \in \mathbb{Z}_{n-1}\}$$

is a complete set of LSESC if and only if the sets $\{P_{\sigma^t(i)} P_i^\top : 1 \leq i \leq n\}$ are sets of disjoint permutation matrices, for each $t \in \mathbb{Z}_{n-1}$.

Proof. Replacing the symbols set of $\mathcal{S}_\sigma$ by indeterminants x_i, we obtain $\mathcal{O}_\sigma = \{O^{(0)}, O^{(1)}, ..., O^{(n-2)}\}$. For instance, the Latin square $S^{(t)} = \sum_{i=1}^{n} \sigma^t(i) P_i$ is equivalently written as $O^{(t)} = \sum_{i=1}^{n} x_{\sigma^t(i)} P_i$.
Particularly, taking $\{x_1, x_2, ..., x_n\}$ as a normalized base of the vector space $GF(2)^n$ (over the field $GF(2) = \{0, 1\}$), and P_is considered as its operators. In this way, if $\mathcal{S}_\sigma$ is a set of LSESC, the inner product of any row of $O^{(t)}$ and another from $O^{(s)}$ is equal to 1, when $t \neq s$.
Therefore, checking the product of any two rows each of them from a different Latin square in $\mathcal{O}_\sigma$, defined over the normalized base, should lead to the condition on $\{P_{\sigma^t(i)} P_i^\top : 1 \leq i \leq n\}$ and yields the theorem consequently. Hence, we proceed as follows, for any $l, m \in \{1, .., n\}$ and $t \neq s \in \mathbb{Z}_{n-1}$ we have

$$\mathbf{o^{(t)}}_l \mathbf{o^{(s)}}_m{}^\top = \left(\sum_{i=1}^{n} x_{\sigma^t(i)} \mathbf{p_i}_l \right) \left(\sum_{i=1}^{n} x_{\sigma^s(i)} \mathbf{p_i}_m \right)^\top = \sum_{i,j=1}^{n} x_{\sigma^t(i)} \mathbf{p_i}_l (\mathbf{p_j}_m)^\top x_{\sigma^s(j)}{}^\top.$$

Here, $\mathbf{p_i}_l$ denotes the l-th row of the matrix P_i. We compute the product by considering two cases:

1. $l = m$:
   The disjointness propriety yields that $\mathbf{p_i}_l(\mathbf{p_j}_l)^\top = 0$, unless, $i = j$ where it is equal to 1. Then,

$$\mathbf{o^{(t)}}_l\mathbf{o^{(s)}}_l^\top \quad = \quad \sum_{i=1}^{n} x_{\sigma^t(i)}x_{\sigma^s(i)}^\top = x_{\sigma^t(1)}x_{\sigma^s(1)}^\top = 1,$$

   as σ fixes the 1, and of order $n-1$ on each of the other symbols.

2. $l \neq m$:
   When $i = j$, the sum $\sum_{i=1}^n x_{\sigma^t(i)}\mathbf{p_i}_l(\mathbf{p_i}_m)^\top x_{\sigma^s(i)}^\top = 0$ as, $\mathbf{p_i}_l(\mathbf{p_i}_m)^\top = 0$ by the definition of permutation matrices. We also should omit the factors when's $\sigma^t(i) \neq \sigma^s(j)$, because $x_{\sigma^t(i)}x_{\sigma^s(j)}^\top = 0$, and then, $x_{\sigma^t(i)}\mathbf{p_i}_l(\mathbf{p_j}_m)^\top x_{\sigma^s(j)}^\top = 0$, whatever the value of $\mathbf{p_i}_l(\mathbf{p_j}_m)^\top$.
   It remains just to consider the case when $\sigma^t(i) = \sigma^s(j)$. Thus, $i = \sigma^{s-t}(j)$. The product in result is equal to

$$\sum_{j=1, i=\sigma^{s-t}(j)}^{n} x_{\sigma^t(i)}\mathbf{p_i}_l(\mathbf{p_j}_m)^\top x_{\sigma^s(j)}^\top = \sum_{j=1}^{n} x_{\sigma^s(j)}\mathbf{p}_{\sigma^{\mathbf{s-t}}(\mathbf{j})_l}(\mathbf{p_j}_m)^\top x_{\sigma^s(j)}^\top.$$

   To compute it's result we need to know the values of $\mathbf{p}_{\sigma^{\mathbf{s-t}}(\mathbf{j})_l}(\mathbf{p_j}_m)^\top$.
   If $P_{\sigma^{s-t}(j)}P_j^\top$s are disjoint permutation matrices, then there exist a unique symbol j' such that $\mathbf{p}_{\sigma^{\mathbf{s-t}}(\mathbf{j'})_l}\mathbf{p_{j'}}_m^\top = 1$. So,

$$\mathbf{o^{(t)}}_l\mathbf{o^{(s)}}_m^\top = x_{\sigma^t(j')}\mathbf{p}_{\sigma^{\mathbf{s-t}}(\mathbf{j'})_l}(\mathbf{p_{j'}}_m)^T x_{\sigma^t(j')}^\top = 1.$$

   Otherwise, it must exist m and l such that, either there is no symbol j' where $\mathbf{p}_{\sigma^{\mathbf{s-t}}(\mathbf{j'})_l}\mathbf{p_{j'}}_m^\top = 1$ or a multiple symbols that lead to the result, and consequently the inner product leads to 0 or an integer different than 1, respectively.

This completes the proof. □

From the theorem we notice that the set $\mathcal{S}_\sigma$ is a set of isotopes of $S^{(0)} = \sum_{i=1}^n iP_i$ obtained by permuting the symbols set following the powers of σ. So, the construction of the set aims in the construction of $S^{(0)}$. More precisely, the obtainment of the particular set of disjoint permutation matrices. Moreover, the set revealed by Theorem 2.6 is not a set of MOLS (taking 1 and $b \neq 1$, there is no indices i and j such that $l_{ij} = 1$ and $l'_{ij} = b$, for any $L, L' \in \mathcal{S}_\sigma$).

Additionally to the example given for the case $n = 3$, we conclude this section with the following two examples

Example 2.

(1) For $n = 4$, let's consider the following disjoint permutation matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ P_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

It is easy to check that the set satisfies the conditions in Theorem 2.6, taking $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$. The disjoint permutations matrices induces the Latin square

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$

together with σ form a set of isotopic Latin squares

$$\mathcal{S}_\sigma = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 4 & 2 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{pmatrix} \right\},$$

which is a complete set of LSESC and not of MOLS.

(2) For $n = 5$, we consider the following set:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, P_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$P_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix};$$

the set satisfies the conditions in Theorem 2.6, taking $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$. Therefore, the Latin square

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 2 & 5 & 3 \\ 5 & 4 & 1 & 3 & 2 \end{pmatrix},$$

together with σ form a set of isotopic Latin squares

$$\mathcal{S}_\sigma = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 2 & 5 & 3 \\ 5 & 4 & 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 4 & 5 & 2 \\ 3 & 4 & 2 & 1 & 5 \\ 4 & 2 & 5 & 3 & 1 \\ 5 & 1 & 3 & 2 & 4 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 5 & 2 & 3 \\ 4 & 5 & 3 & 1 & 2 \\ 5 & 3 & 2 & 4 & 1 \\ 2 & 1 & 4 & 3 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 5 & 2 & 3 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 2 & 4 & 3 & 5 & 1 \\ 3 & 1 & 5 & 4 & 2 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix} \right\},$$

which is a complete set of LSESC and not of MOLS.

3. Construction of the Hadamard matrices

We describe three procedure whose input is a Hadamard matrix H of order m and output is a Hadamard matrix of orders $m(m-1)$, or $m(\frac{m}{2}-1)$, or $m(\frac{m}{k}-1)$, where k is a multiple of four and $\frac{m}{k}$ is even. Except the first matrix, the others can be constructed only if the input matrix H satisfies some conditions. These procedures use the complete sets of LSESC. The constructions are based on permutations of rows of matrices and insertions.
This section is divided into three subsections, each containing the conditions on the input matrix, and the construction Hadamard matrix of appropriate order.
Before starting the description of the main results, let's for every complete set of LSESC associate by F a tensors set, $\mathcal{X}$. We call it a *complete tensors set of LSESC*. By each function G_k applied on $\mathcal{X}$ we associate a set of tensors $\mathcal{X}^{(k)}$. For the Latin squares sets obtained by Theorem 2.6, it is sufficient to apply F only on $S^{(0)}$, and then permute the lateral slices following the powers of σ to obtain the set $\mathcal{X}$.

3.1. Construction of the Hadamard matrices of order $m(m-1)$

Here, we describe a procedures whose input is a Hadamard matrix H of order m, and output is a Hadamard matrix B of order $m(m-1)$. So, we obtain

$$\Phi_{\mathcal{X}} : \mathcal{H}_m \to \mathcal{H}_{m(m-1)}.$$

$\Phi_{\mathcal{X}}(H)$ generates a family of Hadamard matrices which depends on the choice of the complete set of LSESC, and whose sizes depend on variation of m. Thus, we have

Theorem 3.1 (Construction theorem). *Let H be a normalized Hadamard matrix of order m and a Core C. $\mathcal{X} = \{\chi^{(1)}, ..., \chi^{(m-2)}\}$ is a complete tensors set of LSESC of size $m-1$, such that $N'(m-1) = m-2$. Let $B_0 = H' \otimes \mathbf{J}_{m-1}$, with, H' is obtained from H by deleting the first row, and*

$$B = \left[\begin{array}{c|cccc} {\mathbf{J}_{m-1}}^\top \otimes \mathbf{c}_1 & C & C & ... & C \\ {\mathbf{J}_{m-1}}^\top \otimes \mathbf{c}_2 & \mathbf{X}_1^{(1)}C & \mathbf{X}_2^{(1)}C & ... & \mathbf{X}_{m-1}^{(1)}C \\ {\mathbf{J}_{m-1}}^\top \otimes \mathbf{c}_3 & \mathbf{X}_1^{(2)}C & \mathbf{X}_2^{(2)}C & ... & \mathbf{X}_{m-1}^{(2)}C \\ \vdots & \vdots & \vdots & ... & \vdots \\ {\mathbf{J}_{m-1}}^\top \otimes \mathbf{c}_{m-1} & \mathbf{X}_1^{(m-2)}C & \mathbf{X}_2^{(m-2)}C & ... & \mathbf{X}_{m-1}^{(m-2)}C \end{array}\right].$$

Then, the matrix

$$\Phi_{\mathcal{X}}(H) = \begin{bmatrix} B_0 \\ B \end{bmatrix},$$

is a Hadamard matrix of order $m(m-1)$.

Proof. We prove the theorem by showing that $\Phi_{\mathcal{X}}(H)$ satisfies row orthogonality requirement.
First, let's divide B into $m-1$ block matrices, each of size $(m-1) \times m(m-1)$, by a row division as follows

$$\Gamma_0 = \left[\begin{array}{c|cccc} {\mathbf{J}_{m-1}}^\top \otimes \mathbf{c}_1 & C & C & ... & C \end{array}\right]$$

of rows

$$\left[\mathbf{c}_1 |\ \mathbf{c}_j\ \mathbf{c}_j\ ...\ \mathbf{c}_j\right], \tag{2}$$

and

$$\Gamma_k = \left[\begin{array}{c|cccc} {\mathbf{J}_{m-1}}^\top \otimes \mathbf{c}_{k+1} & \mathbf{X}_1^{(k)}C & \mathbf{X}_2^{(k)}C & ... & \mathbf{X}_{m-1}^{(k)}C \end{array}\right],$$

of rows

$$\left[\mathbf{c}(k+1) |\ \mathbf{c}({s^{(k)}}_{j\,1})\ \ \mathbf{c}({s^{(k)}}_{j\,2})\ ...\ \mathbf{c}({s^{(k)}}_{j\,m-1})\right], \tag{3}$$

where $1 \le k \le m-2$. Then, we have four cases to consider.

1. Two distinct rows of B_0 are evidently orthogonal as Kronecker product preserves orthogonality.

2. A row l of B_0 can be written on the form $\mathbf{h'}_l \otimes \mathbf{J}_{m-1}$, and a row s of B should be from one of the Γ_ks. When $k \neq 0$, considering (3) the two rows are orthogonal. Because,

$$\begin{aligned}\mathbf{b_0}_l\mathbf{b}_s{}^\top &= h'_{1\,l}\sum_{i=1}^{m-1}[\mathbf{c}(k+1)^\top]_i + \sum_{i=1}^{m-1}h'_{(i+1)\,l}\sum_{r=1}^{m-1}[\mathbf{c}(s^{(k)}{}_{ji})^\top]_r \\ &= -\sum_{i=1}^{m}h'_{i\,l} = 0.\end{aligned}$$

Here $[\mathbf{c}(s^{(k)}{}_{ji})]_r$ are components of the row vector $\mathbf{c}(s^{(k)}{}_{ji})$. If the second row is from Γ_0, using (2), we can in the same way conclude that the expression leads to 0.

3. Two distinct rows of Γ_k are orthogonal.
For instance, two distinct rows l and r of Γ_k when $k \neq 0$ inner product is equal to

$$\mathbf{c}(k+1)\mathbf{c}(k+1)^\top + \sum_{i=1}^{m-1}\mathbf{c}(s^{(k)}{}_{li})\mathbf{c}(s^{(k)}{}_{ri})^\top = m-1-(m-1) = 0. \tag{4}$$

Using (2), taking two distinct rows of Γ_0 leads to similar equations, and then similar results.

4. It remains to verify that when $0 \leq s \neq k \leq m-1$, a row from Γ_k and another from Γ_s are orthogonal. We have two subcases to consider:

s and k are different from 0. So, taking a row l of Γ_k and another r of Γ_s, their inner product is equal to

$$\mathbf{c}(k+1)\mathbf{c}(s+1)^\top + \sum_{i=1}^{m-1}\mathbf{c}(s^{(k)}{}_{li})\mathbf{c}(s^{(s)}{}_{ri})^\top \tag{5}$$

By the definition 2.1, two rows from different LSESC shares exactly one position, namely i'. Then, $\mathbf{c}(s^{(k)}{}_{li'}) = \mathbf{c}(s^{(l)}{}_{ri'})$, and the inner product becomes $-1 + (m-1-(m-2)) = 0$. Hence, the two rows are orthogonal.

Taking $s = 0$, the inner product of the two rows is equal to

$$\mathbf{c}(k+1)\mathbf{c}_1{}^\top + \sum_{i=1}^{m-1}\mathbf{c}(s^{(k)}{}_{li})\mathbf{c}_r{}^\top \tag{6}$$

As every row of a Latin square contain each element of $\mathbb{A}_n$ exactly once, then, (6) becomes $-1 + (m-1-(m-2)) = 0$.

This finishes showing that $\Phi_{\mathcal{X}}(H)$ is a Hadamard matrix. □

Note that, in this construction any known Hadamard matrix can be used as input matrix. Therefore, we obtain the following.

Corollary 3.2. *Let H be a normalized Hadamard matrix of order m, and let's define the sequence*

$$\begin{cases} U_0 &= m \\ U_{t+1} &= U_t^2 - U_t; \quad t \in \mathbb{N}^* \end{cases}.$$

If for some integer n there exists a complete set of LSESC of order $U_r - 1$ and $N'(U_r - 1) = U_r - 2$, for each $0 \leq r \leq n$, then there exists a Hadamard matrix of orders U_{r+1} of the form $\Phi^{r+1}(H)$, respectively to each r.

3.2. Construction of the Hadamard matrices of order $m(\frac{m}{2}-1)$

Now we present the construction of the Hadamard matrix of order $m(\frac{m}{2}-1)$ from a Hadamard matrix H of order m. H is normalized, and of second column $(1,-1,1,...,1,-1,...,-1)$ (see Lemma 2.1 in [5]). Furthermore, this construction is done only if H satisfies the following conditions:

C.1 By deleting the first two rows and columns of H, then using a column permutation N on the resulted matrix, we get a matrix $T = \begin{bmatrix} C \\ D \end{bmatrix}$ such that C and D are of sizes $(\frac{m}{2}-1)\times(m-2)$ that satisfies the following: For each row of C, the sum of the first $\frac{m}{2}-1$ entries is -1, and the sum of the remaning $\frac{m}{2}-1$ entries is also -1. While, for each row of D, the sum of the first $\frac{m}{2}-1$ entries is -1, and the sum of the remaning $\frac{m}{2}-1$ entries is 1.

C.2 We denote by H'' the matrix obtained from H by deleting the first two rows. There exists a column permutation matrix M that rearranges the rows of H'' in such away: starting from the first entry, any two consecutive entries appear as the elements of the set $\mathcal{A} = \{\pm(1,1), \pm(1,-1)\}$, where each row of H'' contains as many $(1,1)$s as $(-1,-1)$s and as many $(-1,1)$s as $(1,-1)$s.

We describe then a procedure whose input is H that satisfies **C.1** and **C.2**, and output is a Hadamard matrix of order $m(\frac{m}{2}-1)$. So, we define

$$\Phi'_{\mathcal{X}} : \mathcal{H}'_m \to \mathcal{H}_{m(\frac{m}{2}-1)},$$

where $\mathcal{H}'_m$ is the set of all Hadamard matrices of order m that satisfies the conditions **C.1** and **C.2**. $\Phi'_{\mathcal{X}}(H)$ generates a family of Hadamard matrices which depends on the choice of the complete set of LSESC, and whose orders depend on variation of m. Then, we obtain the following

Theorem 3.3 (Construction theorem). *Let H be a normalized Hadamard matrix of order m that has a second column on the form $(1,-1,1,...,1,-1,...,-1)$, and satisfies the conditions* **C.1** *and* **C.2**. *Let $\mathcal{X}$ be a complete tensors set of LSESC of order $\frac{m}{2}-1$ such that $N'(\frac{m}{2}-1) = \frac{m}{2}-2$. By G_2, we associate to $\mathcal{X}$ the tensors set $\mathcal{X}^{(2)} = \{\chi'^{(1)}, ..., \chi'^{(\frac{m}{2}-2)}\}$. Let $B'_0 = H' \otimes \mathbf{J}_{\frac{m}{2}-1}$, $H' = H''M$, H'' and M as mentioned in* **C.2**, *and*

$$B' = \left[\begin{array}{c|cccc} \widetilde{T}_1 & T & T & \dots & T \\ \widetilde{T}_2 & \mathbf{X}_1'^{(1)}T & \mathbf{X}_2'^{(1)}T & \dots & \mathbf{X}_{\frac{m}{2}-1}'^{(1)}T \\ \widetilde{T}_3 & \mathbf{X}_1'^{(2)}T & \mathbf{X}_2'^{(2)}T & \dots & \mathbf{X}_{\frac{m}{2}-1}'^{(2)}T \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \widetilde{T}_{\frac{m}{2}-1} & \mathbf{X}_1'^{(\frac{m}{2}-2)}T & \mathbf{X}_2'^{(\frac{m}{2}-2)}T & \dots & \mathbf{X}_{\frac{m}{2}-1}'^{(\frac{m}{2}-2)}T \end{array}\right]$$

where $\widetilde{T}_k = \begin{bmatrix} \mathbf{J}^{\top}_{\frac{m}{2}-1} \otimes \mathbf{c}_k \\ \mathbf{J}^{\top}_{\frac{m}{2}-1} \otimes \mathbf{d}_k \end{bmatrix}$ and T as in **C.1**. *Then, the matrix*

$$\Phi'_{\mathcal{X}}(H) = \begin{bmatrix} B'_0 \\ B' \end{bmatrix},$$

is a Hadamard matrix of order $m(\frac{m}{2}-1)$.

Proof. As in Theorem 3.1 we divide B' into $\frac{m}{2}-1$ block matrices, each of size $(m-2)\times m(\frac{m}{2}-1)$, by a row division as follows

$$\Gamma'_0 = \left[\begin{array}{c|cccc} \widetilde{T}_1 & T & T & \dots & T \end{array}\right]$$

of rows of the form

$$\left[\mathbf{c}_1 | \ \mathbf{c}_j \ \mathbf{c}_j \ \dots \ \mathbf{c}_j\right], \tag{7}$$

or

$$\left[\mathbf{d}_1 |\ \mathbf{d}_j\ \mathbf{d}_j\ \dots\ \mathbf{d}_j\right], \tag{8}$$

where $1 \le j \le \frac{m}{2} - 1$, and when $1 \le k \le m-2$,

$$\Gamma'_k = \left[\begin{array}{c|cccc} \widetilde{T}_{k+1} & \mathbf{X}'^{(k)}_1 T & \mathbf{X}'^{(k)}_2 T & \dots & \mathbf{X}'^{(k)}_{m-1} T \end{array}\right],$$

of rows of the form

$$\left[\mathbf{c}(k+1) |\ \mathbf{c}(s^{(k)}{}_{j\,1})\ \mathbf{c}(s^{(k)}{}_{j\,2})\ \dots\ \mathbf{c}(s^{(k)}{}_{j\,m-1})\right], \tag{9}$$

or

$$\left[\mathbf{d}(k+1) |\ \mathbf{d}(s^{(k)}{}_{j\,1})\ \mathbf{d}(s^{(k)}{}_{j\,2})\ \dots\ \mathbf{d}(s^{(k)}{}_{j\,m-1})\right], \tag{10}$$

then we discuss the row orthogonality of the constructed matrix by considering the four cases as claimed in Theorem 3.1.
From **C.1**, the inner product of a row of C and another from D is 0. Therefore, the inner product of a row of $\Phi'_{\mathcal{X}}(H)$ obtained by C rows ((7) or (9)) and another obtained from D rows ((8) or (10)) is 0. So, it stays just to consider the cases where both rows in test are constructed from either C or D, respectively. Thus, we have:

1. Two rows of B'_0 are evidently orthogonal by Kronecker product proprieties.

2. The inner product of two rows of Γ'_k of the same form is equal to the expression (4) substituting the entries by rows of C or D; respectively. It gives zero consequently.

3. Taking a row of Γ'_k and another from Γ'_s, when $k \neq s$. The inner product gives a substitute by the rows of C or D of the expression (5) (or (6) if k or s is equal to 0), and leads to zero by Definition 1 (or by the construction of Latin squares, resp).

4. It remains to show the orthogonality of a row of B'_0 and another from B'. A row k of B'_0 has the form $[\mathbf{h}'_k \otimes \mathbf{J}_{\frac{m}{2}-1}]$, considering a row l from Γ'_n, $l \equiv j\ (mod(\frac{m}{2}-1))$. Then, we discuss two cases:

   If the row of B' has the form (9), the inner product of the two rows is the result of summing the terms of the following:

$$h'_{k\,1} \sum_{u=1}^{\frac{m}{2}-1} c_{n+1\ u} + h'_{k\,2} \sum_{u=\frac{m}{2}}^{m-2} c_{n+1\ u} = -(h'_{k\,1} + h'_{k\,2}).$$

   For $0 < v < \frac{m}{2} - 1$:

$$h'_{k\,2v+1} \sum_{u=1}^{\frac{m}{2}-1} [\mathbf{c}(s^{(n)}{}_{j\,2v-1})]_u + h'_{k\,2v+2} \sum_{u=\frac{m}{2}}^{m-2} [\mathbf{c}(s^{(n)}{}_{j\,2v})]_u = -(h'_{k\,2v+1} + h'_{k\,2v+2}).$$

   We obtain the terms $-(h'_{k\,1} + h'_{k\,2}), -(h'_{k\,3} + h'_{k\,4}), \dots, -(h'_{k\,2q+1} + h'_{k\,2q+2})$. Summing all of them we get $-\sum_{v=1}^{n} h'_{k\,v} = 0$. Hence, the orthogonality is shown. In the same way we conclude the orthogonality if the second row is from Γ'_0 and has the form (7).

   Similarly, considering (10) we have

$$h'_{k\,1} \sum_{u=1}^{\frac{m}{2}-1} d_{n+1\ u} + h'_{k\,2} \sum_{u=\frac{m}{2}}^{m-2} d_{n+1\ u} = h'_{k\,1} - h'_{k\,2}$$

and for $0 < v < \frac{m}{2} - 1$:

$$h'_{k\,2v+1}\sum_{u=1}^{\frac{m}{2}-1}[\mathbf{d}(s^{(n)}{}_{j1})]_u + h'_{k\,2v+2}\sum_{u=\frac{m}{2}}^{m-2}[\mathbf{d}(s^{(n)}{}_{j1})]_u = h'_{k\,2v+1} - h'_{k\,2v+2}$$

yielding the terms $(h'_{k\,1} - h'_{k\,2}), (h'_{k\,3} - h'_{k\,4}), ..., (h'_{k\,2q+1} - h'_{k\,2q+2})$. If $h'_{k\,2v+1} = -h'_{k\,2v+2} = 1$, the v-th term has the value 2, if $h'_{k\,2v+1} = -h'_{k\,2v+2} = -1$, then it yields -2, and zero otherwise. By **C.2**, we have as many $(1,-1)$s as $(-1,1)$s. Therefore,

$$\sum_{v=0}^{\frac{m}{2}-1} h'_{k\,2v+1} - \sum_{v=1}^{\frac{m}{2}} h'_{k\,2v} = 0$$

implying the orthogonality. The result must be concluded in the same way if the second row is of Γ'_0 of the form (8).

The proof is now complete. □

From Lemma 2.2 in [5], Paley type II Hadamard matrices of size $2(q+1)$, where $q \equiv 1\ (mod 4)$ a power of prime number, satisfies **C.1** and **C.2**. Then, they can be used as input matrices to construct a family of $2q(q+1)$ Hadamard matrices using this construction. Furthermore, the construction of Theorem 3.3 induces the following result:

Corollary 3.4. *Let H be a normalized Hadamard matrix of size m and satisfies **C.**1 and **C.**2, and let's define the sequence*

$$\begin{cases} V_0 &= m \\ V_{t+1} &= \dfrac{V_t^2}{2} - V_t; \quad t \in \mathbb{N}^* \end{cases}.$$

If for some integer n there exists a complete set of LSESC of order $V_r - 1$ and $N'(V_r - 1) = V_r - 2$, for each $0 \le r \le n$, then there exists a Hadamard matrix of orders V_{r+1} of the form $\Phi'^{r+1}(H)$, whenever $\Phi'^{s}(H)$ satisfies also the two conditions for each $s \le r$; respectively to each r.

3.3. Construction of the Hadamard matrices of order $m(\frac{m}{k} - 1)$

By extending the conditions **C.1** and **C.2** on the input matrix, we pass to the construction of Hadamard matrices of other orders. Mainly, for a normalized Hadamard matrix H of size m, taking a positive integer k that divides m, we construct a Hadamard matrix of order $m(\frac{m}{k} - 1)$.

Next, we discuss the conditions and we conclude the constraints on k.
Assume that H has $m - k$ rows that form T' and $m - k$ rows that construct H'' such that the following two conditions are satisfied:

$\mathbf{C_k.1}$

$$T' = \left[\begin{array}{c|c} D^{(k)} & T'^{(k)} \end{array}\right]$$

where, $D^{(k)}$ is a matrix of size $(m-k) \times k$ and $T'^{(k)}$ is a square matrix of order $m - k$. The last two matrices are as follows:

i.

$$D^{(k)} = \begin{bmatrix} \mathbf{J}^T_{\frac{m}{k}-1} \otimes \mathbf{w}_1 \\ \vdots \\ \mathbf{J}^T_{\frac{m}{k}-1} \otimes \mathbf{w}_k \end{bmatrix},$$

with $\{\mathbf{w}_1, ..., \mathbf{w}_k\}$ mutually orthogonal rows vectors of size k. So, k is a multiple of 4; and

ii. For a permutation matrix N, $T'^{(k)}N = T^{(k)}$, with $T^{(k)}$ is equal to

$$\begin{bmatrix} C_1 \\ \vdots \\ C_k \end{bmatrix},$$

where each matrix C_u is of size $(\frac{m}{k} - 1) \times (m - k)$ such that, in each row of C_u, computing from the first entry, and summing every $\frac{m}{k} - 1$ entries, then putting the sum results in a row vector according to their order. We obtain the same $\{1, -1\}$ row vector $\mathbf{r}_u$ of size k.

$\mathbf{C_k.2}$ There exists a column permutation matrix $M_{H''}$ that rearranges the rows of H'' to obtain H' in such a way that, starting from the first entry, every k consecutive entries appear as the elements of the set $\mathcal{A}^{(k)} = \{\pm\mathbf{v} : \mathbf{v} \text{ is a } \{1, -1\}\text{-row vector of size } k\}$, where each row of H' contains as many $\mathbf{v}$s as $-\mathbf{v}$s. Consequently, $\frac{m}{k}$ must be an even number.

Remark 3.5. *From $\mathbf{C_k.1}$ i., and $\mathbf{C_k.2}$, the integer k must be a multiple of 4 that divides m into an even number.*

We describe a procedure whose input is a Hadamard matrix H of order m that satisfies $\mathbf{C_k.1}$ and $\mathbf{C_k.2}$, and output is a Hadamard matrix of order $m(\frac{m}{k} - 1)$. So, we define

$$\Phi^{(k)}_{\mathcal{X}} : \mathcal{H}^{(k)}_m \to \mathcal{H}_{m(\frac{m}{k}-1)},$$

where $\mathcal{H}^{(k)}_m$ is the set of all Hadamard matrices of order m that satisfies the conditions $\mathbf{C_k.1}$ and $\mathbf{C_k.2}$. $\Phi^{(k)}_{\mathcal{X}}(H)$ generates a family of Hadamard matrices which depends on the choice of the complete set of LSESC, and whose orders depend on variation of m.

Theorem 3.6 (Construction theorem). *Let H be a normalized Hadamard matrix of a order m divisible by 8, and let $k \in \mathbb{N}^*$ such that $k \equiv 0 \ (mod 4)$ and $\frac{m}{k}$ is an even number. Moreover, suppose that H satisfies the conditions $\mathbf{C_k.1}$ and $\mathbf{C_k.2}$. Let $\mathcal{X}$ be a complete tensors set of LSESC of order $\frac{m}{k}-1$ such that, $N'(\frac{m}{k}-1) = \frac{m}{k}-2$. $\mathcal{X}^{(k)} = \{\chi^{(k)^{(1)}}, ..., \chi^{(k)^{(\frac{m}{k}-2)}}\}$ is the set of tensors obtained by G_k applied on $\mathcal{X}$. Let $B^{(k)}_0 = H' \otimes \mathbf{J}_{\frac{m}{k}-1}$, where H' is defined as in $\mathbf{C_k.2}$, and*

$$B' = \left[\begin{array}{c|cccc} \widetilde{T^{(k)}}_1 & T^{(k)} & T^{(k)} & \dots & T^{(k)} \\ \widetilde{T^{(k)}}_2 & \mathbf{X}^{(k)^{(1)}}_1 T^{(k)} & \mathbf{X}^{(k)^{(1)}}_2 T^{(k)} & \dots & \mathbf{X}^{(k)^{(1)}}_{\frac{m}{k}-1} T^{(k)} \\ \widetilde{T^{(k)}}_3 & \mathbf{X}^{(k)^{(2)}}_1 T^{(k)} & \mathbf{X}^{(k)^{(2)}}_2 T^{(k)} & \dots & \mathbf{X}^{(k)^{(2)}}_{\frac{m}{k}-1} T^{(k)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \widetilde{T^{(k)}}_{\frac{m}{k}-1} & \mathbf{X}^{(k)^{(\frac{m}{k}-2)}}_1 T^{(k)} & \mathbf{X}^{(k)^{(\frac{m}{k}-2)}}_2 T^{(k)} & \dots & \mathbf{X}^{(k)^{(\frac{m}{k}-2)}}_{\frac{m}{k}-1} T^{(k)} \end{array}\right]$$

where $\widetilde{T^{(k)}}_r = \begin{bmatrix} \mathbf{J}^T_{\frac{m}{k}-1} \otimes \mathbf{c_1}(r) \\ \vdots \\ \mathbf{J}^T_{\frac{m}{k}-1} \otimes \mathbf{c_k}(r) \end{bmatrix}$, $\mathbf{c_u}(r)$ are rows of C_u in $\mathbf{C_k.1}$, and $T^{(k)}$ is the matrix defined in $\mathbf{C_k.1}$. Then, the matrix

$$\Phi^{(k)}_{\mathcal{X}}(H) = \begin{bmatrix} B^{(k)}_0 \\ B^{(k)} \end{bmatrix},$$

is a Hadamard matrix of order $m(\frac{m}{k} - 1)$.

Proof. In the same way as in Theorems 3.1 and 3.3, we construct the proof by checking row orthogonality. Firstly, we divide $B^{(k)}$ into $\frac{m}{k} - 1$ block matrices, each of size $(m - k) \times m(\frac{m}{k} - 1)$, by a row division and we denote them by $\Gamma_t^{(k)}$, where

$$\Gamma_0^{(k)} = \left[\begin{array}{c|cccc} \widetilde{T^{(k)}}_1 & T^{(k)} & T^{(k)} & \dots & T^{(k)} \end{array}\right]$$

and when $t \neq 0$

$$\Gamma_t^{(k)} = \left[\begin{array}{c|cccc} \widetilde{T^{(t+1)}} & \mathbf{X}_1^{(k)^{(t)}} T^{(k)} & \mathbf{X}_2^{(k)^{(t)}} T^{(k)} & \dots & \mathbf{X}_{\frac{m}{k}-1}^{(k)^{(t)}} T^{(k)} \end{array}\right],$$

then we consider the four cases and we discuss the orthogonality.
Rows of $\Gamma_t^{(k)}$ (where, $0 \leq t \leq \frac{m}{k} - 2$) are the following:
Every row l of $\Gamma_0^{(k)}$ has one of the forms:

$$\left[\mathbf{c_r}1|\ \mathbf{c_r}_j\ \mathbf{c_r}_j\ \dots\ \mathbf{c_r}_j\right]; \tag{11}$$

where $1 \leq r \leq k$, and $l \equiv j\ (mod(m-k))$.
When $t \neq 0$, every l row of $\Gamma_t^{(k)}$ is one of the following rows

$$\left[\mathbf{c_r}(t+1)|\ \mathbf{c_r}(s^{(t)}{}_{j\,1})\ \mathbf{c_r}(s^{(t)}{}_{j\,2})\ \dots\ \mathbf{c_r}(s^{(t)}{}_{j\,\frac{m}{k}-1})\right], \tag{12}$$

where, $1 \leq r \leq k$, and $l \equiv j\ (mod(m-k))$.
From the construction of T' in $\mathbf{C_k.1}$, the inner product of every row of C_i and another from C_j, when $i \neq j$ is 0. Thus, every two rows of $\Phi_{\mathcal{X}}^{(k)}(H)$ constructed from different C_rs are mutually orthogonal.

Since the rows of $\Phi_{\mathcal{X}}^{(k)}(H)$ have the same forms as the ones obtained in Theorems 3.1 and 3.3, we can conclude easily that the proof of this theorem is similar to the proof of Theorem 3.3. So, following the same steps we find that: every two rows of $B_0^{(k)}$ are mutually orthogonal, every two rows of $\Gamma_t^{(k)}$ are mutually orthogonal, and for all $0 \leq t \leq \frac{m}{k} - 2$, and taking any row of $\Gamma_t^{(k)}$ and another from $\Gamma_s^{(k)}$, when $s \neq t$ the two rows are mutually orthogonal.
Here, we show only the case of a row from $B_0^{(k)}$ and another from $B^{(k)}$, as its demonstration is obtained differently. Take a row r of $B_0^{(k)}$ and another l of $\Gamma_n^{(k)}$, such that $l \equiv j\ (mod(m-k))$. Then, the inner product is the sum of the following expressions

$$\begin{aligned} h'_{r\,1}\sum_{i=1}^{\frac{m}{k}-1}[C_u]_{n+1\,i} \quad + \quad h'_{r\,2}\sum_{i=\frac{m}{k}}^{2(\frac{m}{k}-1)}[C_u]_{n+1\,i} + \dots + h'_{r\,k}\sum_{i=(k-1)(\frac{m}{k}-1)}^{(m-k)}[C_u]_{n+1\,i} \\ = \quad \sum_{i=1}^{k} h'_{r\,i}[\mathbf{r}_u]_i, \end{aligned} \tag{13}$$

for $1 \leq v \leq \frac{m}{k} - 1$,

$$\begin{aligned} h'_{r\,vk+1}\sum_{i=1}^{\frac{m}{k}-1}[\mathbf{c_u}(s^{(n)}{}_{j\,(v-1)k+1})]_i \quad + \quad h'_{r\,vk+2}\sum_{i=\frac{m}{k}}^{2(\frac{m}{k}-1)}[\mathbf{c_u}(s^{(n)}{}_{j\,(v-1)k+2})]_i + \\ \dots + h'_{r\,(v+1)k}\sum_{i=(k-1)(\frac{m}{k}-1)}^{(m-k)}[\mathbf{c_u}(s^{(n)}{}_{j\,vk})]_i \quad = \quad \sum_{i=1}^{k} h'_{r\,vk+i}[\mathbf{r}_u]_i, \end{aligned} \tag{14}$$

for some $u \in \{1, \dots, k\}$. It yields the terms

$$\sum_{i=1}^{k} h'_{r\,i}[\mathbf{r}_u]_i,\ \sum_{i=1}^{k} h'_{r\,k+i}[\mathbf{r}_u]_i, \dots, \sum_{i=1}^{k} h'_{r\,(\frac{m}{k}-1)k+i}[\mathbf{r}_u]_i.$$

$\mathbf{C_k.2}$ states that, computing from the first entry, any k consecutive entries appears as $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_s} \in \mathcal{A}^{(k)}$, where each row of H' contains as many $\mathbf{v_i}$s as $-\mathbf{v_i}$s. Therefore, for every $0 \leq v \leq \frac{m}{k} - 1$ there exists necessarily a $0 \leq v' \leq \frac{m}{k} - 1$ such that

$$\sum_{i=1}^{k} h'_{r\ v'k+i}[\mathbf{r}_u]_i = -\sum_{i=1}^{k} h'_{r\ vk+i}[\mathbf{r}_u]_i.$$

Consequently, the sum of all the terms is zero, and the orthogonality follows.

We have shown the row orthogonality, and then the matrix in question is a Hadamard matrix. □

We now give an example of input matrices. The next proposition ensures the existence of matrices obtained by this construction.

Proposition 3.7. *Let A and B be two normalized Hadamard matrices of orders m and k greater than or equal to* 2*, respectively. Then, there exist a Hadamard matrix $\Phi_{\mathcal{X}}^{(m)}(A \otimes B)$, or $\Phi_{\mathcal{X}'}^{(k)}(B \otimes A)$, whenever, $N'(m-1) = m-2$, or $N'(k-1) = k-2$, for some LSESCs $\mathcal{X}$, or $\mathcal{X}'$; respectively.*

Proof. We show the existence and construction of the Hadamard matrix of order $mk(m-1)$, and the one of order $mk(k-1)$ can be shown in the same way. Let

$$H = A \otimes B.$$

Clearly, H is a Hadamard matrix of size mk as the tensor product preserve orthogonality. By deleting the k first rows of H we obtain the following matrix

$$\overline{T}^{(k)} = A' \otimes B,$$

where A' is the matrix obtained by deleting the first row of A. We can immediately obtain $\mathbf{C_k.2}$ as A' consists of $\frac{m}{2}$ 1s and $\frac{m}{2}$ -1s. So, every row of $\overline{T}^{(k)}$ is composed of a row of B, say $\mathbf{b}$, where we have as many $\mathbf{b}$s as $-\mathbf{b}$s.

On the other hand, we have

$$\overline{T}^{(k)} = \left[\begin{array}{c|c} \mathbf{J}_{m-1}{}^T \otimes B & C_A \otimes B \end{array}\right],$$

where C_A is the core of A. Using rows permutations on $\overline{T}^{(k)}$ we obtain the matrix

$$\begin{aligned} \widehat{T}^{(k)} &= \left[\begin{array}{c|c} \mathbf{J}_{m-1}{}^T \otimes \mathbf{b}_1 & C_A \otimes \mathbf{b}_1 \\ \mathbf{J}_{m-1}{}^T \otimes \mathbf{b}_2 & C_A \otimes \mathbf{b}_2 \\ \vdots & \vdots \\ \mathbf{J}_{m-1}{}^T \otimes \mathbf{b}_k & C_A \otimes \mathbf{b}_k \end{array}\right] \\ &= \left[\begin{array}{c|c} D^{(k)} & T'^{(k)} \end{array}\right], \end{aligned}$$

The first part of $\mathbf{C_k.1}$ follows from the rows of $D^{(k)}$. Using columns permutations on $T'^{(k)}$ we obtain

$$T^{(k)} = B \otimes C_A.$$

Then, for $i \in \{1, ..., k\}$, we divide it to block matrices C_i of size $(m-1) \times (mk - k)$ in the following way

$$T^{(k)} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_k \end{bmatrix}.$$

The sum of every $m-1$ entries staring from the first entry of each row of C_i gives either -1 if $b_{i\,j}=1$, or 1 if $b_{i\,j}=-1$, by the core proprieties. Then puting the obtained results in a row vector, according to their order, we obtain for every row of C_i, the row vector $-\mathbf{b}_i$. Therefore, the second part of $\mathbf{C_k.1}$ follows.
Finally, the matrix H satisfies $\mathbf{C_k.1}$ and $\mathbf{C_k.2}$. Then, H is eligible as input matrix in the construction of Theorem 3.3, and then we obtain a Hadamard matrix of order $mk(m-1)$.
If we suppose A as a Hadamard matrix of a random order m, and B the Hadamard matrix of order 2. In the same way we find that H satisfies $\mathbf{C.1}$ and $\mathbf{C.2}$. Then, its an input matrix of Φ', and we obtain a Hadamard matrix of order $2m(m-1)$. $\square$

Remark 3.8. *The constructions given in the last proposition are Hadamard equivalent to the matrices $\Phi_{\mathcal{X}}(A)\otimes B$ and $A\otimes\Phi_{\mathcal{X}'}(B)$, respectively. Therefore, it is more important to search about eligible input Hadamard matrices out of the ones obtained by Kronecker products. The Proposition in [7] gives an important result that can be useful in this investigation. It states that any Hadamard matrix of order m is Hadamard equivalent to the form*

$$\begin{bmatrix} J_{a,b} & X \\ Y & Z \end{bmatrix},$$

where $J_{a,b}$ is the $a\times b$ all ones block. It follows that $ab\leq m$, and that either every column sum of X is zero, or every row sum of Y is zero. Thus, it is very likely that $X^\top$ or Y induces a discription of $D^{(k)}$ in $\mathbf{C_k.1}$, when $a=b=k$.

Similarly to the case Φ', we conclude this part with the following result.

Corollary 3.9. *Let H be a normalized Hadamard matrix of order m divisible by 8, let k be a multiple of four that divides m into an even number, and H satisfies $\mathbf{C_k.1}$ and $\mathbf{C_k.2}$. Let's define the sequence*

$$\begin{cases} W_0 & = & m \\ W_{t+1} & = & \dfrac{W_t^2}{k}-W_t; \quad t\in\mathbb{N}^* \end{cases}.$$

If for some integer n there exists a complete set of LSESC of order W_r-1 and $N'(W_r-1)=W_r-2$, for each $0\leq r\leq n$, then there exists a Hadamard matrix of orders W_{r+1} of the form ${\Phi^{(k)}}^{r+1}(H)$, whenever ${\Phi^{(k)}}^{s}(H)$ satisfies also the two conditions for each $s\leq r$; respectively to each r.

4. Results using Paley and Sylvester Hadamard matrices

In this section, we discuss the relation between the constructions described in [5, 6, 9] and the proposed constructions, then we discuss the eligibility of Sylvester Hadamard matrices as input matrices in the forms.

4.1. Relation between Scarpis' constructions and the new constructions

In [5, 6, 9], the authors have used finite fields to construct Hadamard matrices from the known Hadamard matrices, such constructions are known as Scarpis' constructions. From the construction theorem in [6], and respectively to the notation used in it, the matrix B can also be written as

$$\left[\begin{array}{c|cccc} \mathbf{J}_{m-1}{}^\top\otimes\mathbf{c}_1 & C & C & \dots & C \\ \mathbf{J}_{m-1}{}^\top\otimes\mathbf{c}_2 & \mathbf{X}_1^{(1)}C & \mathbf{X}_2^{(1)}C & \dots & \mathbf{X}_{m-1}^{(1)}C \\ \mathbf{J}_{m-1}{}^\top\otimes\mathbf{c}_3 & \mathbf{X}_1^{(2)}C & \mathbf{X}_2^{(2)}C & \dots & \mathbf{X}_{m-1}^{(2)}C \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \mathbf{J}_{m-1}{}^\top\otimes\mathbf{c}_{m-1} & \mathbf{X}_1^{(m-2)}C & \mathbf{X}_2^{(m-2)}C & \dots & \mathbf{X}_{m-1}^{(m-2)}C \end{array}\right],$$

where $\mathbf{X}_k^{(r)}; k\in\{1,\dots,q\}$ are the k frontal slices of the image by F of a Latin square $L=(a_rx+y)_{x\in GF(q)}^{y\in GF(q)}$ for $a_r\in GF(q)-0$, and $GF(q)$ follows a fixed order. Therefore, by Corollary 2.5, we obtain the following

Corollary 4.1. *The Hadamard matrix H obtained by Scarpis' theorem of order $q(q+1)$, where $q \equiv 3\ (mod4)$, is a Hadamard matrix obtained by the complete set of LSESC $\mathcal{L} = \{L_b : b \in GF(q) - 0\}$ following the construction $\Phi_{\mathcal{L}}(H)$.*

Similarly, for the construction in [5], and respectively to the notations and conditions on the input matrix, mentioned in the last paper, the matrix B' is equivalent to a matrix

$$\left[\begin{array}{c|cccc} \widetilde{T}_1 & T & T & \dots & T \\ \widetilde{T}_2 & \mathbf{X}'^{(1)}_1 T & \mathbf{X}'^{(1)}_2 T & \dots & \mathbf{X}'^{(1)}_{\frac{m}{2}-1} T \\ \widetilde{T}_3 & \mathbf{X}'^{(2)}_1 T & \mathbf{X}'^{(2)}_2 T & \dots & \mathbf{X}'^{(2)}_{\frac{m}{2}-1} T \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \widetilde{T}_{\frac{m}{2}-1} & \mathbf{X}'^{(\frac{m}{2}-2)}_1 T & \mathbf{X}'^{(\frac{m}{2}-2)}_2 T & \dots & \mathbf{X}'^{(\frac{m}{2}-2)}_{\frac{m}{2}-1} T \end{array}\right]$$

where $\mathbf{X}'^{(r)}_k, k \in \{1, ..., q\}$ are the k frontal slices of the image by $G_2 \circ F$ of a Latin square $L = (ax+y)^{y \in GF(q)}_{x \in GF(q)}$ for $a \in GF(q) - 0$, and $GF(q)$ follows a fixed order. Therefore, by Corollary 2.1, we obtain the following

Corollary 4.2. *The Hadamard matrix H obtained from Paley type II Hadamard matrix as in [5], of order $2q(q+1)$, where $q \equiv 1\ (mod4)$, is a Hadamard matrix obtained by the complete set of LSESC $\mathcal{L} = \{L_b : b \in GF(q) - 0\}$ following the construction $\Phi'_{\mathcal{L}}(H)$.*

4.2. Case of Sylvester type Hadamard matrices

Sylvester Hadamard matrices [14] are the first known Hadamard matrices, since 1867. They are recursively defined by Kronecker products that gives Hadamard matrices of orders powers of 2. For example

$$S_0 = 1,\ S_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ S_n = \begin{bmatrix} S_{n-1} & S_{n-1} \\ S_{n-1} & -S_{n-1} \end{bmatrix}$$

are Sylvester Hadamard matrices of orders $1, 2, 2^n$, respectively. We define the set of all Sylvester Hadamard matrices by

$$\mathcal{S} = \{S_n = \otimes^n S_1 : n \in \mathbb{N}\}. \tag{15}$$

Therefore, the following Corollary holds.

Corollary 4.3. *Let $n \in \mathbb{N}^*$. Then, there exist a Hadamard matrix $\Phi^{(2^{n-t})}_{\mathcal{X}}(S_n)$ for every complete set of LSESC, $\mathcal{X}$ of order $2^t - 1$ and $N'(2^t - 1) = 2^t - 2$, when $1 \le t \le n$. (Here, $\Phi^{(2)}_{\mathcal{X}} = \Phi'_{\mathcal{X}}$ and $\Phi^{(1)}_{\mathcal{X}} = \Phi_{\mathcal{X}}$).*

Proof. This result is immediate, using Proposition 3.1, as every Sylvestr Hadamard matrix S_n is equal to

$$S_{n-k} \otimes S_k.$$

for each $k \in \{0, ..., \lfloor \frac{n}{2} \rfloor\}$, and then we obtain the stated corollary by applying the constructions in Theorems 3.1, 3.3 and 3.6. □

Finally, we conclude with remarks and open problems:

1. As the Sylvester Hadamard matrices are defined by Kronecker products, it is very likely that the obtained matrices by Corollary 4.3 satisfies the conditions **C.1** , **C.2** or $\mathbf{C_k}$**.1** , $\mathbf{C_k}$**.2**, for some ks. Hence, we can extract a recursive construction of Hadamard matrices of orders $2^s a_1 \cdots a_l$, when a_is are powers of Mersenne prime numbers less than 2^s.

2. An open problem is to find a way to construct a complete set of LSESC of order n not a power of prime number. Or equivalently, a method to find a set of disjoint permutation matrices and a permutation of order $n-1$ that satisfies the conditions in Theorem 2.6.

3. Finally, the construction given in this work yield different Hadamard matrices according to different choice of the complete set of LSESC. Then, they can be classified more properly under Hadamard equivalence (for Hadamard equivalence methods see [11–13]).

5. Conclusion

We have given a description of the constructions of Hadamard matrices of orders $m(m-1)$, $m(\frac{m}{2}-1)$, and $m(\frac{m}{k}-1)$, obtained using known Hadamard matrices as input matrices, and we illustrated the conditions on the input matrices. These constructions are based on Latin squares, we described the nature of the needed Latin squares, and we have given ways to construct them for certain orders, we studied also their relation to MOLS. Finally, we studied the relation between these constructions and Scarpis' theorem on the construction of Hadamard matrices, and then we focussed on Sylvester Hadamard matrices case, and given related open problems.

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References

[1] Anthony B. Evans, Orthogonal Latin Squares Based on Groups, Springer International Publishing AG, part of Springer Nature, 2018.
[2] Tamara G. Kolda, Brett W. Bader, Tensor Decompositions and applications, SIAM REVIEW, 51(3) (2009) 455–500.
[3] Charles. J. Colbourn, Jefferey. H. Dinitz, Handbook of combinatorial Designs, Discret mathematics and applications, 2007.
[4] J. Hadamard, Résolution d'une question relative aux déterminants [Resolution of a question about determinants], Bull. Sciences Math, 17(2) (1893) 240–246.
[5] A. Farouk, Q.W. Wang, An infinite family of Hadamard matrices constructed from Paley type matrice, To appear in FILOMAT.
[6] U. Scarpis, Sui determinanti di valore massimo [On determinants of maximal value], Rendic R Istit Lombardo Sci Lett, 31 (1898) 1441–1446.
[7] W. Orrick, Maximal determinant blog, Hadamard matrices: the construction of Scarpis, 2012 Nov 17, Available from: http://willorrick.wordpress.com/2012/11/17/.
[8] Paley Reac, On orthogonal matrices, J Math Phys, 12 (1933) 311–320.
[9] Dragomir-Ž. Dokoviċ, Generalization of Scarpis' theorem on Hadamard matrices, Linear and Multilinear Algebra, 65(10) (2017) 1985–1987.
[10] Jennifer Seberry, Beata J.W.ysocki, Tadeusz A.W.ysocki, On some applications of Hadamard matrices, Metrika, 62 (2005) 221–239.
[11] J. Cooper, J. Milas, W. D. Wallis, Hadamard equivalence, In: Combinatorial Mathematics, Lecture notes in mathematics, Springer-Verlag, Berlin, Heidelberg, New york, 686 (1978) 126–135.
[12] C. Lin, W. Wallis, L. Zhu, Generalized 4−profiles of Hadamard matrices, J. Comb. Inf. Syst. Sci, 18 (1993) 397–400.
[13] J. Cooper, J. Milas, W. D. Wallis, Hadamard equivalence, Combinatorial Mathematics, Proceeding the international confirence on combinatorial theory, Camberra. August 16-27, (1977) 126–135.
[14] J. J. Sylvester, Thoughts on inverse orthogonal matrices, simultaneous sign successions and tesselated pavements in two or more colours, with applications to Newton's rule, Ornamental tile work and the theory of numbers. Phil. Mag, 34 (1867) 461–475.
[15] J. M. Goethlas, J. J. Seidel, Strongly regular derived from combinatorial designs, Cannadian J, (1970) 597–614.
[16] K. J. Horadam, W. de Launey, Cocyclic development of a design, Jornal of Algebric Combinatorics, 2 (1993) 267–290.
[17] W. de Launey, D. L. Flanney, K. J. Horadam, Cocyclic Hadamard matrices and difference sets, Discret Applied Mathematics, 102 (2000) 47–61.