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On Non-Null Relatively Normal-Slant Helices in Minkowski 3-Space

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Abstract. By using the Darboux frame $\{\xi, \zeta, \eta\}$ of a non-null curve lying on a timelike surface in Minkowski 3-space, where ξ is the unit tangent vector of the curve, η is the unit spacelike normal vector field restricted to the curve and $\zeta = \pm\eta \times \xi$, we define relatively normal-slant helices as the curves satisfying the condition that the scalar product of the fixed vector spanning their axis and the non-constant vector field ζ is constant. We give the necessary and sufficient conditions for non-null curves lying on a timelike surface to be relatively normal-slant helices. We consider the special cases when non-null relatively-normal slant helices are geodesic curves, asymptotic curves, or lines of the principal curvature. We show that an asymptotic spacelike hyperbolic helix lying on the principal normal surface over the helix and a geodesic spacelike general helix lying on the timelike cylindrical ruled surface, are some examples of non-null relatively normal-slant helices in $\mathbb{E}_1^3$.

1. Introduction

Researchers are very interested in helices since they have an important role not only in differential geometry but also in Computer Aided Geometric Design ([35]) and nature (medical sciences, engineering, biology, etc.). Namely, helices exist extensively in the structure of proteins, in particular as α-helix, and there is a lot of research on it ([5, 23, 29, 31]). Moreover, helices have been used in physics for studying different shapes of springs and helical gears as well as elastic rods ([11, 14]). The helix curve or helical structures can be found in fractal geometry ([28, 31]).

In Euclidean space $\mathbb{E}^3$, a regular curve whose tangent vector T makes a constant angle with a fixed direction, is called a *general helix* (or the curve of the constant slope) ([30]). It is well known that a regular curve α with the first curvature $\kappa \neq 0$ and second curvature τ in $\mathbb{E}^3$ is the general helix if and only if it has a constant conical curvature $\frac{\tau}{\kappa}$. The *slant helix* is a curve whose the principal normal vector N makes a constant angle with a fixed direction and has the property that the geodesic curvature

$$k_g = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)'$$

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of the spherical image of its principal normal indicatrix is a constant function ([12]). There is a nice relation between slant helices and general helices. Namely, slant helices are the successor curves of general helices ([20]). They can be found on general Hopf cylinders ([3, 18]) and helix surfaces ([18]). For the resent characterizations of the mentioned helices in different spaces, we refer to [2, 9, 12, 15, 19, 20, 24, 32]. Ali and Lopez gave some new characterizations of the slant helices in Minkowski 3-space ([1]). Also, several kinds of helices have been introduced and characterized by many researchers in [2, 8, 10, 16, 21, 26, 27, 33, 34].

General and slant helices also appear on the surfaces as special curves. For example, it is shown in [7] that geodesic isophotic curves are the slant helices. Also, asymptotic isophotic curves are the general helices. In Minkowski space $\mathbb{E}^3_1$, the spacelike, the timelike, and the null Cartan isophotic curves are characterized in [6, 22]. Macit and Düldül [19] have defined a relatively normal-slant helix lying on a surface in Euclidean space $\mathbb{E}^3$ as a curve with the Darboux frame $\{T, V, U\}$ whose vector field V makes a constant angle with a fixed vector. However, there are no references related with the relatively normal-slant helices in Minkowski space $\mathbb{E}^3_1$ yet.

In this paper, we define a non-null relatively normal-slant helix lying on a timelike surface in Minkowski 3-space $\mathbb{E}^3_1$ with the Darboux frame $\{\xi, \zeta, \eta\}$ as the spacelike, or the timelike curve with non-null principal normal having a property that the scalar product of its fixed axis and the vector field ζ is constant. First, we show that there are three relations between the Darboux frame and the Frenet frame of α. We give the necessary and sufficient conditions for the spacelike and the timelike curves lying on the timelike surface with the non-null principal normal, to be a relatively normal-slant helix in terms of their geodesic curvature, normal curvature, and geodesic torsion. We obtain parameter equations of their axes and also consider the special cases when relatively-normal slant helices are geodesic curves, asymptotic curves, or lines of the principal curvature. We obtain the next relation between relatively-normal slant helices and isophote curves - a non-null relatively-normal slant helix that is a line of principal curvature, is an isophote curve with respect to the same axis. On the other hand, every non-null geodesic relatively normal-slant helix is a silhouette curve with respect to the same axis. In particular, we prove that a non-null curve with the spacelike principal normal is a geodesic relatively normal-slant helix if and only if it is the general helix. Finally, we show that an asymptotic spacelike hyperbolic helix lying on the principal normal surface over the helix and a geodesic spacelike general helix lying on the timelike cylindrical ruled surface, are some examples of non-null relatively normal-slant helices in $\mathbb{E}^3_1$.

2. Preliminaries

Minkowski space $\mathbb{E}^3_1$ is the real vector space $\mathbb{E}^3$ endowed with an indefinite flat metric $\langle \cdot\, , \cdot \rangle$ given by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}^3_1$. An arbitrary vector $x \in \mathbb{E}^3_1$ can be *spacelike*, *timelike*, or *null* (*lightlike*), if $\langle x, x \rangle > 0$,$\langle x, x \rangle < 0$, or $\langle x, x \rangle = 0$ and $x \neq 0$ respectively. In particular, the vector $x = 0$ is said to be spacelike. The *norm* (length) of a non-null vector $x \in \mathbb{E}^3_1$ is given by $||x|| = \sqrt{|\langle x, x \rangle|}$. If $||x|| = 1$, the vector x is called *unit*.

The *vector product* of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}^3_1$ is defined by ([36])

$$u \times v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \tag{1}$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{E}^3_1$.

Lemma 2.1. *Let u, v, and w be the vectors in $\mathbb{E}^3_1$. Then:*

(i) $\langle u \times v, w \rangle = \det(u, v, w)$,
(ii) $u \times (v \times w) = -\langle u, w \rangle v + \langle u, v \rangle w$,
(iii) $\langle u \times v, u \times v \rangle = -\langle u, u \rangle \langle v, v \rangle + \langle u, v \rangle^2$.

An arbitrary curve $\alpha : I \to \mathbb{E}_1^3$ can be the *spacelike*, the *timelike*, or the *null (lightlike)*, if all of its velocity vectors α' are spacelike, timelike, or null, respectively ([17, 25]).

The Frenet formulae of a unit speed spacelike or timelike curve α with a spacelike or a timelike principal normal N in $\mathbb{E}_1^3$ read ([17])

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_1\kappa(s) & 0 \\ -\epsilon_0\kappa(s) & 0 & -\epsilon_0\epsilon_1\tau(s) \\ 0 & -\epsilon_1\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2}$$

where $\kappa(s)$ and $\tau(s)$ are the *first curvature* and the *second curvature* of α, respectively and it holds

$$\langle T, T\rangle = \epsilon_0 = \pm 1, \quad \langle N, N\rangle = \epsilon_1 = \pm 1, \quad \langle B, B\rangle = -\epsilon_0\epsilon_1, \tag{3}$$

$$T \times N = -\epsilon_0\epsilon_1 B, \quad N \times B = \epsilon_0 T, \quad B \times T = \epsilon_1 N. \tag{4}$$

Definition 2.2. *A surface M in Minkowski space $\mathbb{E}_1^3$ is called timelike (resp. spacelike) if the induced metric on the surface M is a Lorentzian (resp. positive definite Riemannian) metric.*

A spacelike or timelike surface in Minkowski 3-space is also called a *non-degenerate surface*.

Lemma 2.3. *Let M be a timelike surface in $\mathbb{E}_1^3$ and $\alpha : I \to \mathbb{E}_1^3$ an arbitrary curve lying on M with the geodesic curvature k_g, normal curvature k_n and geodesic torsion τ_g. Then the following statements hold:*
(i) α is a geodesic curve on M if and only if $k_g = 0$;
(ii) α is an asymptotic curve on M if and only if $k_n = 0$;
(iii) α is a line of principal curvature on M if and only if $\tau_g = 0$.

Definition 2.4. *Isophote curve in Minkowski space $\mathbb{E}_1^3$ is a non-null or a null curve lying on the surface and having a property that the scalar product of the surface's normal along that curve and a constant vector spanning its axis is constant.*

The special isophote curve, along which the surface's normal is orthogonal to its axis, is called the *silhouette* curve. Throughout the next sections, let $\mathbb{R}_0$ denote $\mathbb{R}\backslash\{0\}$.

3. Darboux frame of a non-null curve lying on a timelike surface in $\mathbb{E}_1^3$

In this section, we define the Darboux frame of a non-null curve with a non-null principal normal lying on a timelike surface in Minkowski 3-space. We also show that there are three relations between the Darboux frame and the Frenet frame. In relation to that, let M be a timelike surface in Minkowski space $\mathbb{E}_1^3$ with parametrization

$$X : U \subseteq \mathbb{E}^2 \longrightarrow \mathbb{E}^3,\ X(u,t) = (x_1(u,t), x_2(u,t), x_3(u,t))\ . \tag{5}$$

Denote by

$$n(u,t) = \frac{X_u \times X_t}{\|X_u \times X_t\|} \tag{6}$$

a unit spacelike normal vector field of M and by $\alpha : I \subset \mathbb{R} \to M$ a non-null curve with a non-null principal normal N lying on M. The *Darboux frame* of α is positively oriented an orthonormal frame $\{\xi, \zeta, \eta\}$, consisting of the tangential vector field $T = \xi$, the unit spacelike normal vector field $\eta = n|_\alpha$ and a unit vector field $\zeta = \pm\eta \times \xi$, where the sign + or − is chosen in such way that $det(\xi, \zeta, \eta) = [\xi, \zeta, \eta] = 1$. The Darboux's frame equations of α read

$$\begin{bmatrix} \xi'(s) \\ \zeta'(s) \\ \eta'(s) \end{bmatrix} = \begin{bmatrix} 0 & -\epsilon_0 k_g(s) & k_n(s) \\ -\epsilon_0 k_g(s) & 0 & \tau_g(s) \\ -\epsilon_0 k_n(s) & \epsilon_0\tau_g(s) & 0 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \zeta(s) \\ \eta(s) \end{bmatrix}, \tag{7}$$

where $k_n(s)$, $k_g(s)$ and $\tau_g(s)$ are the *normal curvature*, the *geodesic curvature*, and the *geodesic torsion* of α respectively, defined by

$$k_n(s) = \langle \eta(s), \xi'(s) \rangle, \quad k_g(s) = \langle \zeta(s), \xi'(s) \rangle, \quad \tau_g(s) = \langle \eta(s), \zeta'(s) \rangle. \tag{8}$$

The Darboux frame of α satisfies the relations

$$\langle \xi, \xi \rangle = \epsilon_0 = \pm 1, \quad \langle \zeta, \zeta \rangle = -\epsilon_0, \quad \langle \eta, \eta \rangle = 1, \tag{9}$$

$$\xi \times \zeta = \eta, \quad \zeta \times \eta = \epsilon_0 \xi, \quad \eta \times \xi = -\epsilon_0 \zeta. \tag{10}$$

We show that there are three relations between the Darboux frame and the Frenet frame of α:

(i) If α is a timelike curve, the Frenet frame and Darboux frame are related by the Euclidean rotation in the spacelike normal plane $T^{\perp}$ of α, given by

$$\begin{bmatrix} \xi \\ \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{11}$$

where $\theta(s) = \angle(\zeta, N)$ is an angle between two spacelike vectors. By using relations (2), (7), and (11), we get

$$k_g = \kappa\cos\theta, \quad k_n = \kappa\sin\theta, \quad \tau_g = \tau - \theta'. \tag{12}$$

(ii) If α is a spacelike curve with the timelike principal normal N, the Frenet frame and Darboux frame are related by the hyperbolic rotation in the timelike normal plane $T^{\perp}$ of α, given by

$$\begin{bmatrix} \xi \\ \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{13}$$

where $\theta(s) = \angle(\zeta, N)$ is an angle between two timelike vectors. Now relations (2), (7), and (13) give

$$k_g = \kappa\cosh\theta, \quad k_n = \kappa\sinh\theta, \quad \tau_g = \tau + \theta'. \tag{14}$$

(iii) If α is a spacelike curve with the spacelike principal normal N, the Frenet frame, and Darboux frame are related by the composition of the hyperbolic rotation for an angle $-\theta$ and symmetry with respect to the null straight line $x_1 = -x_2$, given by

$$\begin{bmatrix} \xi \\ \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh(-\theta) & \sinh(-\theta) \\ 0 & \sinh(-\theta) & \cosh(-\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{15}$$

where $\theta(s) = \angle(\eta, N)$ is an angle between a timelike and a spacelike vector. In this case, relations (2), (7), and (15) imply

$$k_g = \kappa\sinh\theta, \quad k_n = -\kappa\cosh\theta, \quad \tau_g = \tau - \theta'. \tag{16}$$

Theorem 3.1. *Let α be a spacelike or a timelike curve with a spacelike or a timelike principal normal and the curvatures k_g, k_n and τ_g lying on the timelike surface M in $\mathbb{E}^3_1$. Then the following statements hold:*
(i) If $k_n = k_g = 0$, then α is the straight line;
(ii) If $k_n = \tau_g = 0$, then α is the straight line if B is timelike, or the straight line or a plane curve if B is spacelike;
(iii) If $k_g = \tau_g = 0$, then α is the straight line if N is timelike, or the straight line or a plane curve if N is spacelike.

Proof. **(i)** If $k_n = k_g = 0$, substituting this in (5) we get $T' = 0$. Thus α is the straight line.

(ii) If $k_n = \tau_g = 0$, substituting this in (14), we obtain $\kappa = 0$. Hence α is the straight line if B is timelike. Substituting $k_n = \tau_g = 0$ in (5) and using (10) and (12), α is the straight line or plane curve if B is spacelike.

(iii) If $k_g = \tau_g = 0$, substituting this in (12), we find $\kappa = 0$. Therefore, α is the straight line if N is timelike. On the other hand, substituting $k_g = \tau_g = 0$ in (5) and using (10) and (14), we get that α is the straight line or plane curve if N is spacelike. □

4. Non-null relatively normal-slant helices in Minkowski space $\mathbb{E}^3_1$

In this section, we introduce a non-null relatively-normal slant helix lying on a timelike surface in Minkowski 3-space. We give the necessary and sufficient conditions in terms of the geodesic curvature, normal curvature, and geodesic torsion of non-null curves with a non-null principal normal lying on the timelike surface, to be a relatively normal-slant helix and determine their axes. We also consider the special cases when relatively-normal slant helices are geodesic curves, asymptotic curves, or lines of the principal curvature. We obtain the next relation between relatively-normal slant helices and isophote curves - a non-null relatively-normal slant helix that is a line of principal curvature, is an isophote curve with respect to the same axis. On the other hand, every non-null geodesic relatively normal-slant helix is a silhouette curve with respect to the same axis. Throughout this section, let M denote a timelike surface.

Definition 4.1. *A non-null curve α with a non-null principal normal N and the Darboux frame $\{\xi, \zeta, \eta\}$ lying on M in $\mathbb{E}^3_1$ is called a relatively normal-slant helix, if there exists a non-zero fixed vector $U \in \mathbb{E}^3_1$ such that holds*

$$\langle \zeta \,, U \rangle = c,$$

where $c \in \mathbb{R}$.

The fixed direction U spans an *axis* of the helix and can be spacelike, timelike, or null (lightlike). We exclude the case when the vector field ζ is constant. According to Theorem 3.1, we also exclude the cases when two of the curvatures k_g, k_n and τ_g, are equal to zero, since then α is the straight line.

Assume that α is a non-null relatively normal-slant helix. According to the Definition 4.1, there exists a non-zero fixed vector U such that holds

$$\langle \zeta \,, U \rangle = c, \quad c \in \mathbb{R}. \tag{17}$$

Assume that $c \neq 0$. By using the relation (17), the fixed direction U can be written as

$$U = u_1 \xi - \epsilon_0 c \zeta + u_3 \eta \,, \tag{18}$$

where $u_1(s)$ and $u_3(s)$ are some differentiable functions in the arc-length parameter s of α. Differentiating the equation (18) with respect to s and using the equations (7), we obtain the following system of differential equations

$$\begin{cases} u_1' - \epsilon_0 k_n u_3 + c k_g = 0, \\ \quad -k_g u_1 + \tau_g u_3 = 0, \\ u_3' + k_n u_1 - \epsilon_0 c \tau_g = 0. \end{cases} \tag{19}$$

If $k_g \neq 0$ and $\tau_g \neq 0$, from the second and the third equation of (19) we get

$$\begin{cases} u_1 = \epsilon_0 c \frac{\tau_g}{k_g} e^{-\int \frac{\tau_g k_n}{k_g} ds} \left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds \right), \\ u_3 = \epsilon_0 c e^{-\int \frac{\tau_g k_n}{k_g} ds} \left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds \right). \end{cases} \tag{20}$$

Substituting (20) in the first equation of (19), we obtain that the curvature functions of α satisfy the relation

$$u_0\left[\left(\frac{\tau_g}{k_g}\right)' - k_n\left(\frac{\tau_g}{k_g}\right)^2 - \epsilon_0 k_n\right] + \frac{\tau_g^2}{k_g} + \epsilon_0 k_g = 0, \tag{21}$$

where

$$u_0 = e^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right). \tag{22}$$

Conversely, assume that the relation (21) holds. Consider the vector U given by

$$U = \epsilon_0 c \frac{\tau_g}{k_g} u_0 \xi - \epsilon_0 c \zeta + \epsilon_0 c u_0 \eta\,, \tag{23}$$

where $\tau_g \neq 0$ and $k_g \neq 0$, $c \in \mathbb{R}_0$ and $u_0(s)$ is given by (22). Differentiating the equation (23) with respect to s and using (7), we find $U' = 0$. Hence U is a fixed direction. It can be easily checked that $\langle \zeta\,, U\rangle = c$, $c \in \mathbb{R}_0$. According to the Definition 4.1, α is a non-null relatively normal-slant helix whose axis is spanned by U. This proves the next theorem.

Theorem 4.2. *Let α be a non-null curve with a spacelike or a timelike principal normal lying on M in $\mathbb{E}^3_1$ with the curvatures $k_g \neq 0$, k_n and $\tau_g \neq 0$. Then α is the relatively normal-slant helix if and only if*

$$u_0\left[\left(\frac{\tau_g}{k_g}\right)' - k_n\left(\frac{\tau_g}{k_g}\right)^2 - \epsilon_0 k_n\right] + \frac{\tau_g^2}{k_g} + \epsilon_0 k_g = 0, \tag{24}$$

where $u_0 = e^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right)$.

Corollary 4.3. *An axis of the relatively normal-slant helix α with the curvatures $k_g \neq 0$, k_n and $\tau_g \neq 0$ is spanned by*

$$U = \epsilon_0 c \frac{\tau_g}{k_g} u_0 \xi - \epsilon_0 c \zeta + \epsilon_0 c u_0 \eta\,, \tag{25}$$

where $u_0 = e^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right)$ and $c \in \mathbb{R}_0$.

In particular, if the axis U is orthogonal to ζ, substituting $c = 0$ in the relation (19), we get

$$\begin{cases} u_1' - \epsilon_0 k_n u_3 = 0, \\ -k_g u_1 + \tau_g u_3 = 0, \\ u_3' + k_n u_1 = 0. \end{cases} \tag{26}$$

From the second equation and third equation of (26), we have

$$\begin{aligned} u_1 &= \tfrac{\tau_g}{k_g} e^{-\int \frac{\tau_g k_n}{k_g} ds}, \\ u_3 &= e^{-\int \frac{\tau_g k_n}{k_g} ds}. \end{aligned} \tag{27}$$

Substituting (27) in the first equation of (26), we obtain that the curvature functions of α satisfy the relation

$$\left(\frac{\tau_g}{k_g}\right)' - k_n\left(\frac{\tau_g}{k_g}\right)^2 - \epsilon_0 k_n = 0. \tag{28}$$

Thus, we can give the next corollary.

Corollary 4.4. *If an axis of the non-null relatively normal-slant helix with the curvatures $k_g \neq 0$, k_n and $\tau_g \neq 0$ is orthogonal to ζ, then*

$$\left(\frac{\tau_g}{k_g}\right)' - k_n\left(\frac{\tau_g}{k_g}\right)^2 - \epsilon_0 k_n = 0, \tag{29}$$

and its axis is spanned by the vector U given by

$$U = \frac{\tau_g}{k_g} e^{-\int \frac{\tau_g k_n}{k_g} ds}\xi + e^{-\int \frac{\tau_g k_n}{k_g} ds}\eta\,. \tag{30}$$

In what follows, we consider the next three subcases: (A.1) $k_n = 0, k_g \neq 0, \tau_g \neq 0$; (A.2) $\tau_g = 0, k_g \neq 0, k_n \neq 0$; (A.3) $k_g = 0, k_n \neq 0, \tau_g \neq 0$.

(A.1) If $k_n = 0$, $k_g \neq 0$, $\tau_g \neq 0$, then the relations (12) and (14) gives $k_g = \kappa$, $\tau_g = \tau$ and $\theta = 0$. Also, the relation (19) gives

$$\epsilon_0\left(\int \tau_g ds\right)^2 - \left(\int k_g ds\right)^2 = constant. \tag{31}$$

Conversely, assume that the relation (31) holds. Consider the vector $U = -c\left(\int k_g ds\right)\xi - \epsilon_0 c\zeta + \epsilon_0 c\left(\int \tau_g ds\right)\eta$, where $c \in \mathbb{R}_0$. Differentiating the last equation with respect to s and using the relation (7), we find $U' = 0$. Since $\langle \zeta\,, U\rangle = c$, the Definition 4.1 implies that α is an asymptotic relatively normal-slant helix. This proves the following theorem.

Theorem 4.5. *Let α be a non-null curve with the spacelike binormal B lying on M in $\mathbb{E}^3_1$ with the curvatures $k_g \neq 0$, $k_n = 0$ and $\tau_g \neq 0$. Then α is an asymptotic relatively normal-slant helix if and only if*

$$\epsilon_0\left(\int \tau_g ds\right)^2 - \left(\int k_g ds\right)^2 = constant.$$

Corollary 4.6. *An axis of the asymptotic relatively normal-slant helix α lying on M in $\mathbb{E}^3_1$ is spanned by the vector U given by*

$$U = -c\left(\int k_g ds\right)\xi - \epsilon_0 c\zeta + \epsilon_0 c\left(\int \tau_g ds\right)\eta.$$

where $c \in \mathbb{R}_0$.

(A.2) If $\tau_g = 0$, $k_g \neq 0$, $k_n \neq 0$, then the relation (19) gives $\frac{k_g}{k_n} = constant \neq 0$. Conversely, if $\frac{k_g}{k_n} = constant \neq 0$, consider the vector $U = -\epsilon_0 c\zeta + \epsilon_0 c\frac{k_g}{k_n}\eta$, where $c \in \mathbb{R}_0$. Differentiating the last equation with respect to s and using the relation (7), we find $U' = 0$. Consequently, U is a fixed direction satisfying $\langle \zeta, U\rangle = c$. In this way, the next theorem is proved.

Theorem 4.7. *Let α be a spacelike or a timelike curve with the non-null principal normal N lying on M in $\mathbb{E}^3_1$ with the curvatures $k_g \neq 0$, $k_n \neq 0$ and $\tau_g = 0$. Then α is a relatively normal-slant helix that is a line of principal curvature if and only if $\frac{k_g}{k_n} = constant \neq 0$.*

Corollary 4.8. *An axis of the relatively normal-slant helix α that is a principal curvature line on M in $\mathbb{E}^3_1$ is spanned by $U = -\epsilon_0 c\zeta + \epsilon_0 c\frac{k_g}{k_n}\eta$, $c \in \mathbb{R}_0$.*

From the Definition 2.4, we can give the following corollary.

Corollary 4.9. *Every non-null relatively normal-slant helix, which is a line of principal curvature, is also an isophote curve with respect to the same axis.*

(A.3) If $k_g = 0$, $k_n \neq 0$, $\tau_g \neq 0$, then the relation (19) gives $\frac{\tau_g}{k_n} = constant \neq 0$. Conversely, if $\frac{\tau_g}{k_n} = constant \neq 0$, the vector $U = \epsilon_0 c \frac{\tau_g}{k_n}\xi - \epsilon_0 c\zeta$ is constant. This proves the following statement.

Theorem 4.10. *Let α be a non-null curve with the spacelike principal normal N lying on M in $\mathbb{E}^3_1$ with the curvatures $k_g = 0$, $k_n \neq 0$ and $\tau_g \neq 0$. Then α is a geodesic relatively normal-slant helix on M if and only if $\frac{\tau_g}{k_n} = constant \neq 0$.*

Corollary 4.11. *An axis of the geodesic relatively normal-slant helix α with the spacelike principal normal N on M in $\mathbb{E}^3_1$ is spanned by $U = \epsilon_0 c \frac{\tau_g}{k_n}\xi - \epsilon_0 c\zeta$, $c \in \mathbb{R}_0$.*

From the Definition 2.4, we can give the following corollary.

Corollary 4.12. *Every non-null geodesic relatively normal-slant helix with spacelike principal normal is silhouette curve with respect to the same axis.*

If a non-null curve α with the spacelike principal normal N lying on M is a geodesic curve, i.e. $k_g = 0$ for all s, we have $k_n = \kappa$ and $\tau_g = \tau$. Then we can give the following corollary.

Corollary 4.13. *A non-null curve α with the spacelike principal normal lying on M is a geodesic relatively normal-slant helix if and only if α is a general helix.*

5. Some examples of non-null relatively normal-slant helices in $\mathbb{E}^3_1$

Example 5.1. *Let us consider the principal normal surface M in $\mathbb{E}^3_1$ parameterized by*

$$x(s,t) = \alpha(s) + tN(s), \tag{32}$$

where the base curve α has parameter equation $\alpha(s) = \frac{\sqrt{2}}{2}(\cosh s, \sinh s, s)$ and $N(s)$ is the principal normal vector of α (see Figure 1).

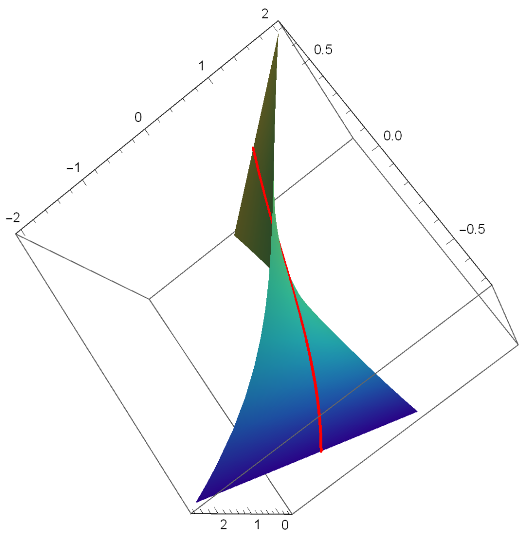

Figure 1: The timelike principal normal surface and the spacelike relatively normal-slant helix α

The Frenet frame of α has the form

$$\begin{aligned} T(s) &= \tfrac{\sqrt{2}}{2}(\sinh s, \cosh s, 1), \\ N(s) &= (\cosh s, \sinh s, 0), \\ B(s) &= \tfrac{\sqrt{2}}{2}(\sinh s, \cosh s, -1), \end{aligned}$$

and the Frenet curvatures of α read

$$\kappa(s) = \frac{\sqrt{2}}{2}, \quad \tau(s) = \frac{\sqrt{2}}{2}.$$

Therefore, α is a spacelike hyperbolic helix. By taking the partial derivatives of (32) with respect to s and t and using (4), we get $x_s \times x_t = -t\tau T + (1 - t\kappa)B$. The last equation and the relation (3) imply $\langle x_s \times x_t, x_s \times x_t \rangle > 0$ for all s and t. Consequently, M is a timelike surface. The Darboux frame of α is given by

$$\begin{aligned} \xi(s) &= \tfrac{\sqrt{2}}{2}(\sinh s, \cosh s, 1), \\ \zeta(s) &= -\eta(s) \times \xi(s) = (\cosh s, \sinh s, 0), \\ \eta(s) &= \tfrac{\sqrt{2}}{2}(\sinh s, \cosh s, -1). \end{aligned} \tag{33}$$

By using the relations (8) and (33), we obtain

$$k_g(s) = -\kappa(s) = -\frac{\sqrt{2}}{2}, \quad k_n(s) = 0, \quad \tau_g(s) = \tau(s) = \frac{\sqrt{2}}{2}.$$

By the Corollary 4.4, α is an asymptotic spacelike relatively normal-slant helix lying on M whose axis is spanned by $U = c\left(0, 0, -\sqrt{2}\right)$, $c \in \mathbb{R}_0$. Note that it is also an isophotic curve with respect to the same axis.

Example 5.2. *Let us consider the cylindrical ruled surface in $\mathbb{E}^3_1$ parametrized by (see Figure 2)*

$$x(s,t) = \alpha(s) + t(1, 1, 0), \tag{34}$$

with the base curve α is given by

$$\alpha(s) = \left(-\frac{s^5}{40}, -\frac{s^5}{40} + s, \frac{s^3}{6}\right).$$

The Frenet frame of α reads

$$\begin{aligned} T(s) &= \left(-\frac{s^4}{8}, -\frac{s^4}{8} + 1, \frac{s^2}{2}\right), \\ N(s) &= \left(-\frac{s^2}{2}, -\frac{s^2}{2}, 1\right), \\ B(s) &= \left(\frac{s^4}{8} + 1, \frac{s^4}{8}, -\frac{s^2}{2}\right), \end{aligned}$$

and the Frenet curvatures of α are given by

$$\kappa(s) = s, \quad \tau(s) = s. \tag{35}$$

Since $\tau(s)/\kappa(s) = constant \neq 0$, α is a spacelike general helix. By taking the partial derivatives of (34) with respect to s and t and using (4) and (35), we get $\langle x_s \times x_t, x_s \times x_t \rangle > 0$ for all s and t. Consequently, M is a timelike cylindrical

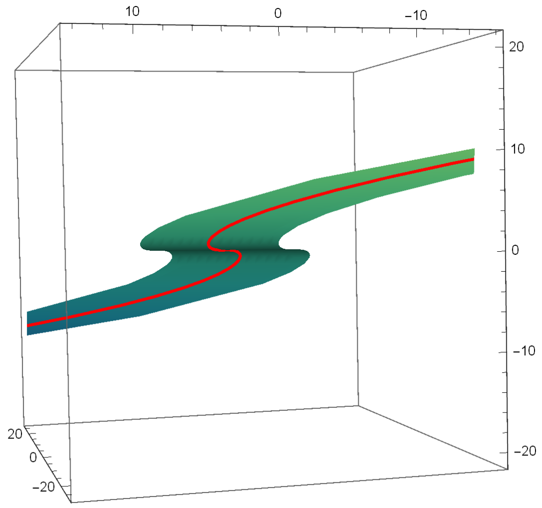

Figure 2: The timelike cylindrical ruled surface and the silhouette curve α

ruled surface. The Darboux frame of α has the form

$$\xi(s) = \left(-\frac{s^4}{8}, -\frac{s^4}{8}+1, \frac{s^2}{2}\right),$$
$$\zeta(s) = -\eta(s)\times\xi(s) = \left(\frac{s^4}{8}+1, \frac{s^4}{8}, -\frac{s^2}{2}\right),$$
$$\eta(s) = \left(\frac{s^2}{2}, \frac{s^2}{2}, -1\right).$$

Since $\eta = -N$ and $\theta = \angle(\eta, N) = 0$, the relation (16) implies that the curvatures k_g, k_n and τ_g of α read

$$k_g(s) = 0, \quad k_n(s) = -s, \quad \tau_g(s) = s.$$

According to the Theorem 4.10, the curve α is a geodesic relatively normal-slant helix. By the Corollary 4.11, an axis of α is spanned by the null vector $U = c(-1,-1,0)$, $c \in \mathbb{R}_0$. Note that according to Corollary 4.12 it is also a silhouette curve with respect to the same axis.

Example 5.3. *Let us consider a timelike ruled surface M in $\mathbb{E}^3_1$ parameterized by (see Figure 3)*

$$x(s,t) = \left(\frac{\sqrt{5}}{2}s + \frac{\sqrt{2}}{4}t, \cos\frac{s}{2} - \frac{\sqrt{2}}{2}t\left(\cos\frac{s}{2} + \frac{\sqrt{5}}{2}\sin\frac{s}{2}\right), \sin\frac{s}{2} + \frac{\sqrt{2}}{2}t\left(\frac{\sqrt{5}}{2}\cos\frac{s}{2} - \sin\frac{s}{2}\right)\right), \tag{36}$$

with the timelike base curve α is given by

$$\alpha(s) = \left(\frac{\sqrt{5}}{2}s, \cos\frac{s}{2}, \sin\frac{s}{2}\right).$$

The Frenet frame of α reads

$$T(s) = \left(\tfrac{\sqrt{5}}{2}, -\tfrac{1}{2}\sin\tfrac{s}{2}, \tfrac{1}{2}\cos\tfrac{s}{2}\right),$$
$$N(s) = \left(0, -\cos\tfrac{s}{2}, -\sin\tfrac{s}{2}\right),$$
$$B(s) = \left(-\tfrac{1}{2}, \tfrac{\sqrt{5}}{2}\sin\tfrac{s}{2}, -\tfrac{\sqrt{5}}{2}\cos\tfrac{s}{2}\right),$$

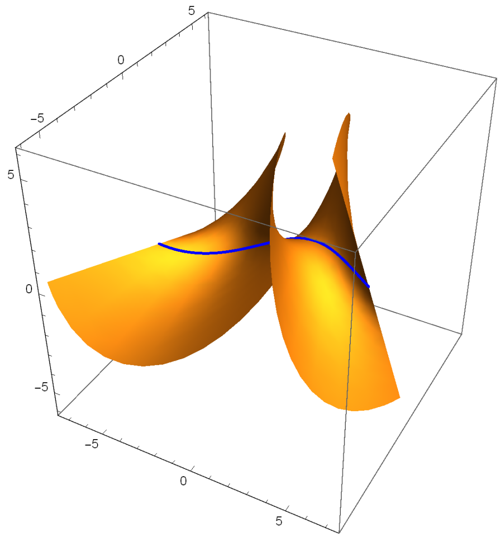

Figure 3: The timelike ruled surface and the timelike relatively normal-slant helix α

and the Frenet curvatures of α are given by

$$\kappa(s) = \frac{1}{4}, \quad \tau(s) = \frac{\sqrt{5}}{4}. \tag{37}$$

Since $\tau(s) = constant$ and $\kappa(s) = constant$, α is a timelike helix. By taking the partial derivatives of (36) with respect to s and t and using (4) and (37), we get $\langle x_s \times x_t, x_s \times x_t \rangle > 0$ for all s and t. The Darboux frame of α has the form

$$\begin{aligned}
\xi(s) &= \left(\tfrac{\sqrt{5}}{2}, -\tfrac{1}{2}\sin\tfrac{s}{2}, \tfrac{1}{2}\cos\tfrac{s}{2}\right),\\
\zeta(s) &= \eta(s) \times \xi(s) = \left(\tfrac{\sqrt{2}}{4}, -\tfrac{\sqrt{10}}{4}\sin\tfrac{s}{2} - \tfrac{\sqrt{2}}{2}\cos\tfrac{s}{2}, -\tfrac{\sqrt{2}}{2}\sin\tfrac{s}{2} + \tfrac{\sqrt{10}}{4}\cos\tfrac{s}{2}\right),\\
\eta(s) &= \left(-\tfrac{\sqrt{2}}{4}, \tfrac{\sqrt{10}}{4}\sin\tfrac{s}{2} - \tfrac{\sqrt{2}}{2}\cos\tfrac{s}{2}, -\tfrac{\sqrt{2}}{2}\sin\tfrac{s}{2} - \tfrac{\sqrt{10}}{4}\cos\tfrac{s}{2}\right).
\end{aligned}$$

The relation (12) implies that the curvatures k_g, k_n and τ_g of α read

$$k_g(s) = \frac{1}{4\sqrt{2}}, \quad k_n(s) = \frac{1}{4\sqrt{2}}, \quad \tau_g(s) = \frac{\sqrt{5}}{4}.$$

According to the Theorem 4.2, the curve α is a timelike relatively normal-slant helix. By the Corollary 4.3, an axis of α is spanned by timelike vector $U = c\left(-2\sqrt{2}, 0, 0\right)$, $c \in \mathbb{R}_0$. Note that it is also a general helix and isophotic curve with respect to the same axis.

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