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# A New Two-Step Iteration Method for Discrete Ill-Posed Problems and Image Restoration

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**Abstract.** In this study, for the augmented linear system of discrete ill-posed problems we establish a new two-step (NTS) iteration method containing a parameter and a parameter matrix, which is based on the Hermitian and skew-Hermitian splitting (HSS) and the upper and lower triangular splitting (ULT) of the coefficient matrix. Then, we theoretically study its convergence properties and determine its optimal iteration parameters. It is seen that the NTS method converges faster when the parameters are chosen properly. Experimental examples are carried out to further validate the effectiveness and accuracy of the new method compared to the newly developed methods in terms of the numerical performance and image recovering quality.

## 1. Introduction

Linear ill-posed problems arise in essentially every branch of science and engineering, including in computerized tomography [7], image restoration [10] and geoscience [31]. Due to the universal existence and significance of the linear ill-posed problems, there has been a surge of interest in the problems, and numerous solution techniques have been proposed in recent years [18, 20, 27, 29]. Discretization of these problems gives rise to linear systems of equations

$$Af = g, \quad A \in \mathbb{R}^{n^2 \times n^2}, \quad f, g \in \mathbb{R}^{n^2}, \tag{1}$$

which is commonly referred to as linear discrete ill-posed problems in the sense the singular values of $A$ gradually decay and cluster at zero. The decay rate depends on the problem, and many large-scale problems tend to have a rather slow decay, however, due to the large problem dimensions the matrix is very ill conditioned. This makes the solution $f$ of (1), if it exists, very sensitive to perturbations in the right-hand side $g$.

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As it is known, the right-hand side vector $g$ of linear discrete ill-posed problems represents available date and typically is contaminated by an error $e \in \mathbb{R}^{n^2}$ that may stem from measurement inaccuracies, discretization error, and eletronic noise in the device used (such as in computerized tomography), which we will refer to as 'noise'. Let $\hat{g} \in \mathbb{R}^{n^2}$ denote the (unknown) noise-free vector associated with $g$, i.e.,

$$g = \hat{g} + e, \tag{2}$$

and assume that the unavailable linear system of equation with the noise-free right-hand side

$$Af = \hat{g} \tag{3}$$

is consistent. Let $\hat{f}$ denote a desired least-squares solution of (3) in the sense of the minimal Euclidean norm. We would like to determine an approximation of $\hat{f}$ by computing a suitable approximate solution of the available linear system of equations (1).

Because of the ill-conditioned of $A$ and the error $e$ in $g$, the straightforward least-squares solution of minimal Euclidean norm of (1) generally does not yield a meaningful approximate solution of the system (3). Therefore, one often replaces (1) by a nearby problem that is less sensitive to the error $e$. This replacement is referred to as regularization. The possibly most popular regularization method is due to Tikhonov regularization [24, 26, 27]. This method replaces (1) by a penalized least-squares problem of the form

$$\min_{f \in \mathbb{R}^{n^2}} \|Af - g\|_2^2 + \mu^2 \|Lf\|_2^2, \tag{4}$$

where $L$ is a carefully selected regularization matrix (typically either the identity matrix or a discrete approximation of the derivative operation) and $\mu > 0$ is called the regularization parameter (generally small, i.e., $0 < \mu < 1$). In the Tikhonov method, the factor $\mu$ controls the balance between the minimization of $\|Af - g\|_2^2$ and and the regularization term $\|Lf\|_2^2$ involving a smoothing norm. The regularization parameter $\mu$ can be determined in a variety of ways [6, 12, 18, 21, 25]. Throughout this paper, $\|\cdot\|$ denotes the Euclidean vector norm or the associated induced matrix norm. The solution of this system is considered as an approximation of the solution of noise-free linear system (3). In this work, we limit our discussion to $L$ being the identity matrix. The other cases can be obtained by using the similar technique. It is easy to seen that the Tikhonov minimization problem is mathematically equivalent to solving the following equation [20]:

$$(A^T A + \mu^2 I)f = A^T g. \tag{5}$$

Several iterative methods have been proposed for investigating the solution of (5). In [22], Lv et al. presented the following equivalent $2n^2$-by-$2n^2$ augmented system

$$\underbrace{\begin{pmatrix} I & A \\ -A^T & \mu^2 I \end{pmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} e \\ f \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} g \\ 0 \end{pmatrix}}_{b}, \tag{6}$$

where the variable $e$ denotes the additive noise, i.e., $e = g - Af$. By recasting equivalently the original system (1), employing the Tikhonov regularization method, into the $2n^2$-by-$2n^2$ linear system (6), the behaviour of ill-conditioned of latter system can be greatly improved.

It is obvious that the linear system (6) is a non-Hermitian positive-definite system, which is also a saddle point problem. Many methods have been proposed in the literature to solve this linear system efficiently, we can refer to [4, 11, 28, 30]. In the existing methods, the Hermitian and skew-Hermitian splitting (HSS) method has attracted many researchers' attentions due to its promising performance and elegant mathematical properties. Bai et al. in [3] first put forward the HSS method which is a two-step iteration method. Recently, Lv et al. in [22] used the idea of the HSS method and established a special case of the HSS (SHSS) method to solve the proposed equation, and its convergence properties and the optimal value of the iteration parameter were discussed. Inspired by the idea of [22], Cheng et al. in [9] derived a new special HSS (NSHSS) iterative method and made comparisons between the proposed new method with the SHSS one. In order to improve the convergence rate of the HSS method, Benzi in [5] developed a generalization of the HSS (GHSS) iteration method for solving a class of non-Hermitian linear systems. The GHSS method split the Hermitian part of the coefficient matrix of the linear system into positive definite and positive semi-definite matrices. After that, based on the GHSS method, Aghazadeh et al. [1] proposed a restricted version of the generalized HSS (RGHSS) iteration method to solve image restoration problem, and experimental results demonstrated that the RGHSS method is more effective and accurate than the SHSS method. In [2], based on a new splitting of the Hermitian part of the coefficient matrix for the GHSS method, Aminikhah and Yousefi newly presented a new special generalized Hermitian and skew-Hermitian splitting (SGHSS) method for solving ill-posed inverse problems. Numerical experiments showed that the SGHSS iterative method can compete with direct method, Tikhonov and RGHSS methods. Lately, Fan in [13] presented a class of upper and lower triangular (ULT) splitting iteration methods, of which optimal iteration parameters and corresponding convergence factors for some special cases were derived. And numerical experiments illustrate the performance of the versions of ULT method.

In this paper, we continue research on solving the $2n^2$-by-$2n^2$ linear system (6) and propose a new two-splitting (NTS) iteration method in which the parameter $\alpha$ and the parameter matrix $Q$ are incorporated. The NTS iteration method is based on the Hermitian and skew-Hermitian splitting (HSS) and the upper and lower triangular splitting (ULT) of the coefficient matrix $\mathcal{A}$ in (6). We analyze the convergence of the NTS method and obtain the optimal parameters that minimize the spectral radius of the iteration matrix of the NTS method. A few numerical examples are given to illustrate the behavior of the method of the present paper and compare it with the SHSS method in [22] and the ULT method in [13] to solve ill-posed problems and image restoration. And the NTS iteration method can result in more rapid convergence rate with suitable choices of the parameters and parameter matrix.

The arrangement of this paper is organized as follows. In Section 2, we construct a new two-splitting (NTS) iteration method in which the parameter $\alpha$ and the parameter matrix $Q$ are incorporated. The convergence properties of the NTS method for solving (6) are analyzed and the optimal parameters are also given here in Section 3. Section 4 is devoted to presenting numerical examples to examine the feasibility and effectiveness of the versions of NTS method.

## 2. The NTS iteration method

Motivated by the ideas of [13, 22], we construct a new two-splitting (NTS) iteration method for the augmented system (6). For a given symmetric positive definite matrix $Q \in \mathbb{R}^{n^2\times n^2}$, we propose the following two new splittings for the coefficient matrix $\mathcal{A}$:

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} I & 0 \\ 0 & \mu^2 I \end{pmatrix} + \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} = H + S, \\ \mathcal{A} &= \begin{pmatrix} I & A \\ 0 & \mu^2 I + Q \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ A^T & Q \end{pmatrix} = K_1 - K_2, \end{aligned}$$

where $I$ is the unit matrix with appropriate dimension. The first splitting is the Hermitian and skew-Hermitian splitting (HSS) of the coefficient matrix $\mathcal{A}$ in (6), and the second one is on the basis of the upper and lower triangular splitting of the coefficient matrix $\mathcal{A}$. So based on the above two splittings, we design a new iteration method as follows

$$\begin{cases} (\alpha I + H) z^{(k+\frac{1}{2})} = (\alpha I - S) z^{(k)} + b \\ K_1 z^{(k+1)} = K_2 z^{(k+\frac{1}{2})} + b \end{cases}, \tag{7}$$

where $\alpha$ is a positive scalar. The iteration method (7) is referred to as the new two-splitting (NTS) iteration method, which is essentially obtained by combining the first step of the SHSS method in [22] and the second step of the ULT-I method in [13]. By eliminating the intermediate vector $z^{(k+\frac{1}{2})}$, the NTS iteration method (7) can be written in a fixed-point form

$$z^{(k+1)} = L(\alpha) z^{(k)} + c, \; k = 0, 1, 2, \cdots,$$

where

$$\begin{cases} L(\alpha) = K_1^{-1}K_2(\alpha I + H)^{-1}(\alpha I - S) \\ c = K_1^{-1}(K_2 + \alpha I + H)(\alpha I + H)^{-1}b \end{cases}, \tag{8}$$

and $L(\alpha)$ is the iteration matrix of the NTS method. It is well known that the preconditioner $M(\alpha)$ can be chosen by the splitting $\mathcal{A}$=$M(\alpha) - N(\alpha)$ with a reversible matrix $M(\alpha)$. From (8), by simple computations, we have

$$M(\alpha) = (\alpha I + H)(K_2 + \alpha I + H)^{-1}K_1, \; N(\alpha) = M(\alpha) - \mathcal{A}. \tag{9}$$

Then $M(\alpha)$ can be used as a preconditioner for the matrix $\mathcal{A}$, which is referred to as the NTS preconditioner.

## 3. Convergence of the NTS method

In this section, we study the convergence of the NTS method for solving (6). To this end, we first investigate the eigenvalues of the iteration matrix $L(\alpha)$ of the NTS iteration method.

**Theorem 3.1.** *For the system* (6), *let $\alpha$ be a positive scalar and the parameter matrix $Q \in \mathbb{R}^{n^2\times n^2}$ be a symmetric positive definite matrix. If $\lambda$ is an eigenvalue of the iteration matrix $L(\alpha)$ of the NTS method in* (7), *then $\lambda = 0$ with algebraic multiplicity at least $n^2$, and other $n^2$ eigenvalues of the matrix $L(\alpha)$ satisfy $\lambda_i = 1-\xi_i$. Here $\xi_i$ $(i = 1, 2, \cdots n^2)$ is the eigenvalue of the matrix $(I + \frac{1}{\alpha+\mu^2}Q)(\mu^2 I + A^TA)(\mu^2 I + Q)^{-1}$.*

*Proof.* Since the iteration matrix $L(\alpha)$ defined as in (8) is similar to

$$\tilde{L}(\alpha) = K_2(\alpha I + H)^{-1}(\alpha I - S)K_1^{-1}.$$

In the following, we only need to study the spectral properties of the matrix $\tilde{L}(\alpha)$ instead of $L(\alpha)$. By some simple operations, we obtain

$$\begin{aligned}
\tilde{L}(\alpha) &= \begin{pmatrix} 0 & 0 \\ A^T & Q \end{pmatrix}\begin{pmatrix} \frac{1}{\alpha+1}I & 0 \\ 0 & \frac{1}{\alpha+\mu^2}I \end{pmatrix}\begin{pmatrix} \alpha I & -A \\ A^T & \alpha I \end{pmatrix}\begin{pmatrix} I & A \\ 0 & \mu^2 I + Q \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0 & 0 \\ A^T & Q \end{pmatrix}\begin{pmatrix} \frac{1}{\alpha+1}I & 0 \\ 0 & \frac{1}{\alpha+\mu^2}I \end{pmatrix}\begin{pmatrix} \alpha I & -A \\ A^T & \alpha I \end{pmatrix}\begin{pmatrix} I & -A(\mu^2 I + Q)^{-1} \\ 0 & (\mu^2 I + Q)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ \frac{1}{\alpha+1}A^T & \frac{1}{\alpha+\mu^2}Q \end{pmatrix}\begin{pmatrix} \alpha I & -(\alpha+1)A(\mu^2 I + Q)^{-1} \\ A^T & (\alpha I - A^TA)(\mu^2 I + Q)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ (\frac{\alpha}{\alpha+1}I + \frac{1}{\alpha+\mu^2}Q)A^T & I - (I + \frac{1}{\alpha+\mu^2}Q)(\mu^2 I + A^TA)(\mu^2 I + Q)^{-1} \end{pmatrix}.
\end{aligned}$$

From the structure of the matrix $\tilde{L}(\alpha)$ and the fact that $\tilde{L}(\alpha)$ and $L(\alpha)$ have same spectrum, $\lambda = 0$ is the eigenvalue of the iteration matrix $L(\alpha)$ with algebraic multiplicity at least $n^2$, and other $n^2$ eigenvalues of the matrix $L(\alpha)$ satisfy $\lambda_i = 1 - \xi_i$. Here $\xi_i$ $(i = 1, 2, \cdots n^2)$ is the eigenvalue of the matrix $(I + \frac{1}{\alpha+\mu^2}Q)(\mu^2 I + A^TA)(\mu^2 I + Q)^{-1}$. The completes the proof. □

It is well known that the NTS iteration method is convergent if and only if the spectral radius of its iteration matrix $L(\alpha)$ is less than one (i.e., $\rho(L(\alpha)) < 1$). The following theorem describes the necessary and sufficient conditions for guaranteeing the convergence of the NTS method.

**Theorem 3.2.** *For the system* (6), *suppose that $\alpha$ is a positive scalar and the parameter matrix $Q \in \mathbb{R}^{n^2\times n^2}$ is a symmetric positive definite matrix. Let $\mathcal{J} = (I + \frac{1}{\alpha+\mu^2}Q)(\mu^2 I + A^TA)(\mu^2 I + Q)^{-1}$, then the NTS iteration method converges to the exact solution of the system* (6) *if and only if*

$$\xi_{\min}(\mathcal{J}) > 0 \text{ and } \xi_{\max}(\mathcal{J}) < 2,$$

*where $\xi_{\max}$ and $\xi_{\min}$ are the largest and smallest eigenvalues of the matrix $\mathcal{J}$, respectively.*

*Proof.* It follows from Theorem 3.1 that $\rho(L(\alpha)) < 1$ if and only if $|\lambda_i| = |1-\xi_i| < 1$ $(i = 1, 2, \cdots, n^2)$, where $\xi_i$ is the eigenvalue of the matrix $\mathcal{J}$. Thus, we have $0 < \xi_i < 2$, which results in $\xi_{\min}(\mathcal{J}) > 0$ and $\xi_{\max}(\mathcal{J}) < 2$. □

In particular, we choose the parameter matrix as $Q = sI$, the above convergence result can be simplified and the optimal parameters minimizing the spectral radius of $L(\alpha, s)$ of the NTS method with $Q = sI$ are also derived in the following theorem.

**Theorem 3.3.** *Let $\alpha, s$ be positive constants and the parameter matrix be defined as $Q = sI$, let $\sigma_i$ $(i = 1, 2, \cdots, n^2)$ satisfying $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n^2}$ be the singular value of the matrix $A$. Then the NTS iteration method with the parameter matrix $Q = sI$ is convergent for any initial vector if and only if*

$$\sigma_1^2 < \frac{(\mu^2+\alpha)(\mu^2+s)+\alpha s}{\alpha+\mu^2+s}. \tag{10}$$

*The the optimal parameters $\alpha^*$ and $s^*$ minimizing the spectral radius $\rho(L(\alpha, s))$ satisfy the following equation*

$$(\alpha+\mu^2+s)(\sigma_1^2+\sigma_{n^2}^2) - 2\alpha s = 0.$$

*As a consequence, the optimal convergence factor can be obtained as follows*

$$\rho(L(\alpha^*, s^*)) = \frac{\sigma_1^2 - \sigma_{n^2}^2}{\sigma_1^2+\sigma_{n^2}^2+2\mu^2}.$$

*Proof.* From Theorem 3.2, it can be seen that the NTS iteration method with $Q = sI$ is convergent if and only if

$$\xi_{\min}\left(\frac{\alpha+\mu^2+s}{(\alpha+\mu^2)(\mu^2+s)}(\mu^2 I + A^T A)\right) > 0 \ \text{ and } \ \xi_{\max}\left(\frac{\alpha+\mu^2+s}{(\alpha+\mu^2)(\mu^2+s)}(\mu^2 I + A^T A)\right) < 2.$$

It follows immediately that

$$\frac{(\alpha+\mu^2+s)(\mu^2+\sigma_{n^2}^2)}{(\alpha+\mu^2)(\mu^2+s)} > 0 \ \text{ and } \ \frac{(\alpha+\mu^2+s)(\mu^2+\sigma_1^2)}{(\alpha+\mu^2)(\mu^2+s)} < 2. \tag{11}$$

The first inequality of (11) holds true for all $\alpha, s > 0$, and the second inequality of (11) holds if and only if

$$\sigma_1^2 < \frac{(\mu^2+\alpha)(\mu^2+s)+\alpha s}{\alpha+\mu^2+s}.$$

Moreover,

$$\begin{aligned}\rho(L(\alpha, s)) &= \max_{\sigma_i \in \sigma(A)} \left\{\left|1 - \frac{(\alpha+\mu^2+s)(\mu^2+\sigma_i^2)}{(\alpha+\mu^2)(\mu^2+s)}\right|\right\} \\ &= \max\left\{\left|1 - \frac{(\alpha+\mu^2+s)(\mu^2+\sigma_1^2)}{(\alpha+\mu^2)(\mu^2+s)}\right|, \left|1 - \frac{(\alpha+\mu^2+s)(\mu^2+\sigma_{n^2}^2)}{(\alpha+\mu^2)(\mu^2+s)}\right|\right\},\end{aligned}$$

where $\sigma(A)$ denotes the set of the singular values of the matrix $A$. Then the optimal parameters $\alpha^*$ and $s^*$ minimizing $\rho(L(\alpha, s))$ must satisfy the following equation

$$1 - \frac{(\alpha+\mu^2+s)(\mu^2+\sigma_{n^2}^2)}{(\alpha+\mu^2)(\mu^2+s)} = \frac{(\alpha+\mu^2+s)(\mu^2+\sigma_1^2)}{(\alpha+\mu^2)(\mu^2+s)} - 1.$$

After some calculations, the above equation is equivalent to

$$(\alpha+\mu^2+s)(\sigma_1^2+\sigma_{n^2}^2) - 2\alpha s = 0. \tag{12}$$

In the sequel, we examine the fact that the optimal parameters $\alpha^*$ and $s^*$ determined by Equation (12) satisfy the convergent condition (10) in details. It follows from Equation (12) that

$$\alpha^* + \mu^2 + s^* = \frac{2\alpha^* s^*}{\sigma_1^2 + \sigma_{n^2}^2},$$

and the convergent condition (10) with the optimal parameters $\alpha^*, s^*$ can be validated by

$$\begin{aligned} \frac{(\mu^2 + \alpha^*)(\mu^2 + s^*) + \alpha^* s^*}{\alpha^* + \mu^2 + s^*} - \sigma_1^2 &= \frac{\left[(\mu^2 + \alpha^*)(\mu^2 + s^*) + \alpha^* s^*\right](\sigma_1^2 + \sigma_{n^2}^2)}{2\alpha^* s^*} - \sigma_1^2 \\ &= \frac{\mu^2(\mu^2 + s^* + \alpha^*)(\sigma_1^2 + \sigma_{n^2}^2) + 2\alpha^* s^* \sigma_{n^2}^2}{2\alpha^* s^*} > 0. \end{aligned}$$

Furthermore, it is easy to see that $s^* \neq \frac{\sigma_1^2+\sigma_{n^2}^2}{2}$. If $s^* = \frac{\sigma_1^2+\sigma_{n^2}^2}{2}$, then from (12) we obtain $\mu^2 + \frac{\sigma_1^2+\sigma_{n^2}^2}{2} = 0$, which is a contradiction. Thus, it follows from Equation (12) that $\alpha^* = \frac{(\mu^2+s^*)(\sigma_1^2+\sigma_{n^2}^2)}{2s^*-(\sigma_1^2+\sigma_{n^2}^2)}$, then after a few computations, we have

$$\alpha^* + \mu^2 = \frac{s^*(\sigma_1 + \sigma_{n^2}^2 + 2\mu^2)}{2s^* - (\sigma_1^2 + \sigma_{n^2}^2)} \text{ and } \alpha^* + \mu^2 + s^* = \frac{2s^*(\mu^2 + s^*)}{2s^* - (\sigma_1^2 + \sigma_{n^2}^2)}.$$

By substituting the above equations into $1 - \frac{(\alpha+\mu^2+s)(\mu^2+\sigma_{n^2}^2)}{(\alpha+\mu^2)(\mu^2+s)}$, the optimal convergence factor can be obtained as follows:

$$\rho(L(\alpha^*, s^*)) = \frac{\sigma_1^2 - \sigma_{n^2}^2}{\sigma_1^2 + \sigma_{n^2}^2 + 2\mu^2},$$

which completes the proof. □

As in Theorem 3.3, the convergence result and the optimal parameters of the NTS iteration method are given by the following theorem when the parameter matrix is defined as $Q = sI + A^T A$.

**Theorem 3.4.** *Let $\alpha, s$ be positive constants and parameter matrix be defined as $Q = sI + A^T A$, let $\sigma_i$ $(i = 1, 2, \cdots, n^2)$ satisfying $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n^2}$ be the singular value of the matrix $A$. Then the NTS iteration method is convergent for any initial vector if and only if*

$$\alpha > \max\left\{\frac{(\mu^2 + s + \sigma_1^2)(\sigma_1^2 - \mu^2)}{\mu^2 + \sigma_1^2 + 2s}, 0\right\}. \tag{13}$$

*Furthermore, the optimal convergence factor $\rho(L(\alpha^*, s^*))$ is*

$$\rho(L(\alpha^*, s^*)) = 1 - \frac{(\alpha^* + s^* + b)b}{(\alpha^* + \mu^2)(s^* + b)}, \tag{14}$$

*where the optimal parameters $\alpha^*$ and $s^*$ satisfy the relation*

$$(a + s)(b + s)(\sigma_1^2 + \sigma_{n^2}^2) - \alpha s(a + b + 2s) = 0, \tag{15}$$

*with $a = \mu^2 + \sigma_1^2, b = \mu^2 + \sigma_{n^2}^2$.*

*Proof.* It follows from Theorem 3.2 that $\rho(L(\alpha, s)) < 1$ if and only if

$$\xi_{\min}\left(\frac{1}{\alpha + \mu^2}\left[(\alpha + \mu^2 + s)I + A^T A\right](\mu^2 I + A^T A)\left[(\mu^2 + s)I + A^T A\right]^{-1}\right) > 0$$

and

$$\xi_{\max}\left(\frac{1}{\alpha+\mu^2}\left[(\alpha+\mu^2+s)I+A^TA\right](\mu^2 I+A^TA)\left[(\mu^2+s)I+A^TA\right]^{-1}\right)<2.$$

The above inequalities result in

$$\xi_{\min}\left(\frac{(\alpha+\mu^2+s+\sigma_i^2)(\mu^2+\sigma_i^2)}{(\alpha+\mu^2)(\mu^2+s+\sigma_i^2)}\right)>0 \text{ and } \xi_{\max}\left(\frac{(\alpha+\mu^2+s+\sigma_i^2)(\mu^2+\sigma_i^2)}{(\alpha+\mu^2)(\mu^2+s+\sigma_i^2)}\right)<2. \tag{16}$$

It is not difficult to check the function $\psi(x)=\frac{(\alpha+\mu^2+s+x)(\mu^2+x)}{(\alpha+\mu^2)(\mu^2+s+x)}$ is monotonic increasing about $x>0$. Then Inequality (16) is equivalent to

$$\frac{(\alpha+\mu^2+s+\sigma_{n^2}^2)(\mu^2+\sigma_{n^2}^2)}{(\alpha+\mu^2)(\mu^2+s+\sigma_{n^2}^2)}>0 \text{ and } \frac{(\alpha+\mu^2+s+\sigma_1^2)(\mu^2+\sigma_1^2)}{(\alpha+\mu^2)(\mu^2+s+\sigma_1^2)}<2. \tag{17}$$

The first inequality of (17) is obviously true for all $\alpha, s>0$, and the second inequality of (17) holds if and only if

$$\alpha>\frac{(\mu^2+s+\sigma_1^2)(\sigma_1^2-\mu^2)}{\mu^2+\sigma_1^2+2s},$$

which together with $\alpha>0$ can immediately deduce the convergent condition (13).

Notice that

$$\begin{aligned}\rho(L(\alpha,s)) &= \max_{\sigma_i\in\sigma(A)}\left\{\left|1-\frac{(\alpha+\mu^2+s+\sigma_i^2)(\mu^2+\sigma_i^2)}{(\alpha+\mu^2)(\mu^2+s+\sigma_i^2)}\right|\right\}\\ &= \max\left\{\left|1-\frac{(\alpha+\mu^2+s+\sigma_1^2)(\mu^2+\sigma_1^2)}{(\alpha+\mu^2)(\mu^2+s+\sigma_1^2)}\right|,\left|1-\frac{(\alpha+\mu^2+s+\sigma_{n^2}^2)(\mu^2+\sigma_{n^2}^2)}{(\alpha+\mu^2)(\mu^2+s+\sigma_{n^2}^2)}\right|\right\}.\end{aligned}$$

To find the optimal parameters $\alpha^*$ and $s^*$ of the NTS iteration method, we minimize the spectral radius of the iteration matrix $\rho(L(\alpha,s))$. Then the optimal points $\alpha^*$ and $s^*$ must satisfy the following equation

$$\frac{(\alpha+\mu^2+s+\sigma_1^2)(\mu^2+\sigma_1^2)}{(\alpha+\mu^2)(\mu^2+s+\sigma_1^2)}-1=1-\frac{(\alpha+\mu^2+s+\sigma_{n^2}^2)(\mu^2+\sigma_{n^2}^2)}{(\alpha+\mu^2)(\mu^2+s+\sigma_{n^2}^2)}.$$

After some manipulations, we deduce that the optimal parameters $\alpha^*$ and $s^*$ satisfy the relation

$$(a+s)(b+s)(\sigma_1^2+\sigma_{n^2}^2)-\alpha s(a+b+2s)=0, \tag{18}$$

where $a=\mu^2+\sigma_1^2, b=\mu^2+\sigma_{n^2}^2$. In the sequel, we show that the optimal parameters $\alpha^*$ and $s^*$ determined by (18) satisfy the convergent condition (13). From (18) we can easily get

$$\alpha^*=\frac{(a+s^*)(b+s^*)(\sigma_1^2+\sigma_{n^2}^2)}{s^*(a+b+2s^*)}. \tag{19}$$

It can be seen that $\alpha^*>0$, and we only prove that

$$\alpha^*>\frac{(\mu^2+s^*+\sigma_1^2)(\sigma_1^2-\mu^2)}{\mu^2+\sigma_1^2+2s^*}=\frac{(a+s^*)(\sigma_1^2-\mu^2)}{a+2s^*}. \tag{20}$$

If $\sigma_1^2 \leq \mu^2$, it is obvious that (20) always holds due to $\alpha^* > 0$. Next, we prove the Inequality (20) also holds for the case of $\sigma_1^2 > \mu^2$. Checking Inequality (20) can be translated to validate $\frac{\alpha^*(a+2s^*)}{(a+s^*)(\sigma_1^2-\mu^2)} > 1$. It follows from (19) that

$$\frac{\alpha^*(a+2s^*)}{(a+s^*)(\sigma_1^2-\mu^2)} = \frac{(b+s^*)(\sigma_1^2+\sigma_{n^2}^2)(a+2s^*)}{s^*(a+b+2s^*)(\sigma_1^2-\mu^2)},$$

then $\frac{\alpha^*(a+2s^*)}{(a+s^*)(\sigma_1^2-\mu^2)} > 1$ is true inasmuch as

$$(b+s^*)(\sigma_1^2+\sigma_{n^2}^2)(a+2s^*) - s^*(a+b+2s^*)(\sigma_1^2-\mu^2) = s^*(a+b+2s^*)(\sigma_{n^2}^2+\mu^2) + b(a+s^*)(\sigma_1^2+\sigma_{n^2}^2) > 0$$

holds for all $s^* > 0$. Substituting $\alpha^*$ and $s^*$ directly into $1 - \frac{(\alpha+\mu^2+s+\sigma_{n^2}^2)(\mu^2+\sigma_{n^2}^2)}{(\alpha+\mu^2)(\mu^2+s+\sigma_{n^2}^2)}$ leads to (14). This completes our proof of Theorem 3.4. □

**Remark 3.5.** *It follows from Theorem 3.4 that*

$$\rho(L(\alpha^*, s^*)) = 1 - \frac{(\alpha^*+s^*+b)b}{(\alpha^*+\mu^2)(s^*+b)} = \frac{(\mu^2-s^*-b)b+(\alpha^*+u^2)s}{(\alpha^*+\mu^2)(s^*+b)}. \tag{21}$$

*From* (19)*, we can deduce that*

$$\alpha^* + \mu^2 = \frac{(a+s^*)(b+s^*)(\sigma_1^2+\sigma_{n^2}^2) + \mu^2 s^*(a+b+2s^*)}{s^*(a+b+2s^*)}.$$

*Substituting the above equation into* (21) *yields*

$$\rho(L(\alpha^*, s^*)) = \psi(s^*) = \frac{s^*\left[b(a+b+2s^*)(\mu^2-s^*-b) + (a+s^*)(b+s^*)(\sigma_1^2+\sigma_{n^2}^2) + \mu^2 s^*(a+b+2s^*)\right]}{\left[(a+s^*)(b+s^*)(\sigma_1^2+\sigma_{n^2}^2) + \mu^2 s^*(a+b+2s^*)\right](s^*+b)},$$

*which implies that the optimal convergence factor $\psi(s^*)$ will be sufficiently close to zero as $s^* \to 0^+$. However, $s^*$ can't be equal to zero. Since $s^* = 0$ is the contradiction with Equation* (15)*. It makes sense to choose small $s^*$ to obtain rapid convergence rate. Therefore, for the NTS method with $Q = sI + A^TA$ the optimal parameter $s^*$ is adopted as a small positive constant and then the other parameter $\alpha^*$ is computed by* (19)*. Besides, after a few computations and by making use of the properties of limit, one may deduce the following result*

$$\lim_{s^*\to\infty} \rho(L(\alpha^*, s^*)) = \lim_{s^*\to\infty} \psi(s^*) = \frac{\sigma_1^2+\sigma_{n^2}^2+2(\mu^2-b)}{\sigma_1^2+\sigma_{n^2}^2+2\mu^2} = \frac{\sigma_1^2-\sigma_{n^2}^2}{\sigma_1^2+\sigma_{n^2}^2+2\mu^2}.$$

*Here, the second equation is due to the symbol $b = \mu^2 + \sigma_{n^2}^2$ defined as in Theorem 3.4. It can be seen that the optimal convergence factor of the NTS method with the parameter matrix $Q = sI + A^TA$ is the same with ones of the NTS method with $Q = sI$ and $ULT-II_{Q_1}$ in [13] as $s^*$ is chosen sufficiently large. Then the NTS method with $Q = sI + A^TA$ can have a fast convergence rate than NTS method with $Q = sI$ and $ULT-II_{Q_1}$ by choosing small $s^*$. The fact is further verified by the experimental examples in Section 4. However, NTS method with $Q = sI$ brings smaller computation costs than NTS method with $Q = sI + A^TA$, which can be observed from the following Algorithms 3.1 and 3.2.*

The eigenvalue distributions of the preconditioned matrix relate closely to the convergence rates of the Krylov subspace methods. A tightly clustered spectrum or positive real spectrum of the preconditioned matrix may result in fast convergence of Krylov subspace acceleration. In light of this, we will discuss the spectral properties of the preconditioned matrix. In the sequel, $sp(\bullet)$ represents the spectrum of the one matrix.

Since

$$M(\alpha)^{-1}\mathcal{A} = M(\alpha)^{-1}(M(\alpha) - N(\alpha)) = I - L(\alpha),$$

then, it has

$$\lambda(M(\alpha)^{-1}\mathcal{A}) = 1 - \lambda(L(\alpha)). \tag{22}$$

From Theorem 3.1, the bounds for the eigenvalues of the NTS preconditioned matrix $M(\alpha)^{-1}\mathcal{A}$ are given in the following theorem indicating that the eigenvalues of $M(\alpha)^{-1}\mathcal{A}$ lie in a positive box, which may result in fast convergence of Krylov subspace acceleration.

**Theorem 3.6.** *Let $\mathcal{A} \in \mathbb{R}^{2n^2\times 2n^2}$ be defined as in* (6)*, $Q \in \mathbb{R}^{n^2\times n^2}$ be a symmetric positive definite matrix, and $\alpha > 0$ be a given constant. Suppose that $sp(Q) \subseteq [\tau_{n^2}, \tau_1]$ and $\sigma_i$ $(i = 1, 2, \cdots, n^2)$ satisfying $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n^2}$ are the singular value of the matrix $A$. Then for any $\alpha > 0$, $\lambda(M(\alpha)^{-1}\mathcal{A})$ has eigenvalue 1 with multiplicity at least $n^2$, and the remaining eigenvalues are located in the positive interval*

$$\left[\frac{(\alpha+\mu^2+\tau_{n^2})(\mu^2+\sigma_{n^2}^2)}{(\alpha+\mu^2)(\mu^2+\tau_1)}, \frac{(\alpha+\mu^2+\tau_1)(\mu^2+\sigma_1^2)}{(\alpha+\mu^2)(\mu^2+\tau_{n^2})}\right].$$

*Proof.* It follows from the relation (22) and Theorem 3.1 that the eigenvalues of the preconditioned matrix $M(\alpha)^{-1}\mathcal{A}$ are given by 1 with multiplicity at least $n^2$ and that the remaining $n^2$ nonunit eigenvalues are the same as the ones of the matrix $\mathcal{J} = (I + \frac{1}{\alpha+\mu^2}Q)(\mu^2 I + A^TA)(\mu^2 I + Q)^{-1}$, which is similar to the matrix $\hat{\mathcal{J}} = (\mu^2 I + Q)^{-1}(I + \frac{1}{\alpha+\mu^2}Q)(\mu^2 I + A^TA)$. Inasmuch as $(\mu^2 I + Q)^{-1}(I + \frac{1}{\alpha+\mu^2}Q)$ and $\mu^2 I + A^TA$ are both symmetric positive definite matrices, the eigenvalues of the matrix $\hat{\mathcal{J}}$ are real, then the eigenvalues of the matrix $(I + \frac{1}{\alpha+\mu^2}Q)(\mu^2 I + A^TA)(\mu^2 I + Q)^{-1}$ are also real.

Since the matrices $Q$ and $A^TA$ are symmetric positive definite and symmetric positive semi-definite, respectively, we know that $\tau_1 \geq \cdots \geq \tau_{n^2} > 0$ and $sp(A^TA) \subseteq [\sigma_{n^2}^2, \sigma_1^2]$. It follows immediately that

$$sp(\mu^2 I + Q) \subseteq \left[\mu^2 + \tau_{n^2}, \mu^2 + \tau_1\right], sp(\mu^2 I + A^TA) \subseteq \left[\mu^2 + \sigma_{n^2}^2, \mu^2 + \sigma_1^2\right] \tag{23}$$

and

$$sp(I + \frac{1}{\alpha+\mu^2}Q) \subseteq \left[\frac{\alpha+\mu^2+\tau_{n^2}}{\alpha+\mu^2}, \frac{\alpha+\mu^2+\tau_1}{\alpha+\mu^2}\right]. \tag{24}$$

The first property of (23) implies that

$$sp((\mu^2 I + Q)^{-1}) \subseteq \left[\frac{1}{\mu^2+\tau_1}, \frac{1}{\mu^2+\tau_{n^2}}\right]. \tag{25}$$

Thus, it follows from (23)-(25) that the remaining eigenvalues of the preconditioned matrix $M(\alpha)^{-1}\mathcal{A}$ are real and located in the positive interval

$$\left[\frac{(\alpha+\mu^2+\tau_{n^2})(\mu^2+\sigma_{n^2}^2)}{(\alpha+\mu^2)(\mu^2+\tau_1)}, \frac{(\alpha+\mu^2+\tau_1)(\mu^2+\sigma_1^2)}{(\alpha+\mu^2)(\mu^2+\tau_{n^2})}\right],$$

and the proof is completed. □

In the sequel, two cases are given for the parameter matrix $Q$ and the implementing processes for image restoration problem are summarized in the following two concrete algorithms. The versions of NTS iteration method with parameter matrices $Q = sI$ and $Q = sI + A^TA$ are denoted by $\mathrm{NTS}-\mathrm{Q}_1$ and $\mathrm{NTS}-\mathrm{Q}_2$ methods, respectively. The $\mathrm{NTS}-\mathrm{Q}_1$ and $\mathrm{NTS}-\mathrm{Q}_2$ iterative methods can be written as the following Algorithms 3.1 and 3.2 respectively. Algorithm 3.1 only requires the computing of the matrix-vector multiplications $Af^{(k)}$, $Af^{(k+1)}$, $A^Te^{(k)}$ and $A^Te^{(k+\frac{1}{2})}$, as well as scalar-vector multiplications. Note that the matrix $A \in \mathbb{R}^{n^2\times n^2}$ that arises in image restoration is highly structured, such as block circulant, block Toeplitz, block Toeplitz-plus-Hankel matrices and so forth. Hence, arithmetic operations of matrix-vector multiplications with the blurring matrix $A \in \mathbb{R}^{n^2\times n^2}$ is $O(n^2 \log n)$ by fast Fourier transforms (FFTs). In Algorithm 3.2, except for computing of the matrix-vector multiplications, the linear sub-system with the coefficient matrix $(u^2 + s)I + A^TA$ is solved. Due to the fact that $(u^2+s)I + A^T A$ is symmetric positive definite, the sparse Cholesky factorization can be efficiently applied to solve the linear sub-systems. Moreover, for image restoration $A \in \mathbb{R}^{n^2\times n^2}$ is highly structured, we can use the FFTs to solve $\left[(u^2+s)I + A^T A\right] f^{(k+1)} = A^T e^{(k+\frac{1}{2})} + (sI + A^T A) f^{(k+\frac{1}{2})}$ in step 7 of Algorithm 3.2.

(1) $\mathrm{NTS}-\mathrm{Q}_1$ method

**Algorithm 3.1.**

1. *Given an initial value* $f^{(0)} = g$ *and* $e^{(0)} = g - Af^{(0)}$.
2. *Given a very small positive value* $\tau$, *and* $N$ *is the maximum prescribed number of outer iterations.*
3. $r^{(0)} = b - \mathcal{A}x^{(0)}$.
4. *For* $k = 0, 1, 2, \cdots$, *until* $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} > \tau$ *or* $k < N$,
5. $e^{(k+\frac{1}{2})} = \frac{\alpha e^{(k)} - Af^{(k)} + g}{\alpha+1}$.
6. $f^{(k+\frac{1}{2})} = \frac{A^T e^{(k)} + \alpha f^{(k)}}{\alpha+\mu^2}$.
7. $f^{(k+1)} = \frac{A^T e^{(k+\frac{1}{2})} + s f^{(k+\frac{1}{2})}}{s+\mu^2}$.
8. $e^{(k+1)} = g - Af^{(k+1)}$.
9. $r^{(k+1)} = b - \mathcal{A}x^{(k+1)}$.
10. *end for*

(2) $\mathrm{NTS}-\mathrm{Q}_2$ method

**Algorithm 3.2.**

1. *Given an initial value* $f^{(0)} = g$ *and* $e^{(0)} = g - Af^{(0)}$.
2. *Given a very small positive value* $\tau$, *and* $N$ *is the maximum prescribed number of outer iterations.*
3. $r^{(0)} = b - \mathcal{A}x^{(0)}$.
4. *For* $k = 0, 1, 2, \cdots$, *until* $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} > \tau$ *or* $k < N$,
5. $e^{(k+\frac{1}{2})} = \frac{\alpha e^{(k)} - Af^{(k)} + g}{\alpha+1}$.
6. $f^{(k+\frac{1}{2})} = \frac{A^T e^{(k)} + \alpha f^{(k)}}{\alpha+\mu^2}$.
7. $\left[(u^2+s)I + A^T A\right] f^{(k+1)} = A^T e^{(k+\frac{1}{2})} + (sI + A^T A) f^{(k+\frac{1}{2})}$.
8. $e^{(k+1)} = g - Af^{(k+1)}$.
9. $r^{(k+1)} = b - \mathcal{A}x^{(k+1)}$.
10. *end for*

## 4. Numerical example

In this section, we shall test the newly proposed Algorithms 3.1 and 3.2 with examples from ill-posed inverse problems and image restoration to illustrate the performance of the algorithms. We compare the $\mathrm{NTS}-\mathrm{Q}_1$ and $\mathrm{NTS}-\mathrm{Q}_2$ methods with the classical Tikhonov, SHSS, ULT-I and ULT-II ones in terms of the number of iterations (denoted by 'IT') and the total computing times in seconds (denoted by 'CPU'). The following tests are carried out in MATLAB R2016b on a personal computer with 2.40-GHz central processing unit (Intel(R) Core(TM) Q6600), 4.00 GB of memory, and Windows 7 operating system.

The error vector $e$ in $g$ has normally distributed entries with zero mean and is scaled so that the contaminated $g$, defined by (2), has a specified noise level relative error

$$\epsilon = \|e\|/\|\hat{g}\|,$$

where the noise-free right-hand side $\hat{g}$ is defined as in (3). The initial approximate solution $f^{(0)} = 0$ is used for all the iterative methods in Examples 4.1 and 4.2, while for image restoration problems in Examples 4.3 and 4.4 the initial approximate solution $f^{(0)} = g$ is adopted. The parameter $\epsilon$ is set to 0.001 in all examples.

In actual computations, the parameters of the test iteration methods are always chosen to be optimal. The optimal values of unknown parameters for the SHSS, ULT-I and ULT-II methods have been presented in

[13, 22], and the optimal parameters of $\mathrm{NTS}-\mathrm{Q}_1$ and $\mathrm{NTS}-\mathrm{Q}_2$ methods are computed by Theorem 3.3 and Theorem 3.4, respectively. The parameters that are not involved in the iteration methods is denoted by '-' in the tables of numerical examples. In our implementations, the linear sub-systems are solved by the sparse Cholesky factorization when the coefficient matrix is symmetric positive definite for Examples 4.1-4.3, and for image restoration problem in Examples 4.4 we can use the FFTs to solve them. The regularization parameter $\mu$ is determined by generalized cross validation (GCV) method [14]. Because compared with the L-curve criterion method [16] and the discrepancy principle method [23], the parameter choice in GCV method does not depend on priori knowledge about the noise variance and GCV method is very practical to approximate the regularization parameter.

There are two quantities, the relative error (RES) and peek signal-to-noise ratio (PSNR), are commonly applied to measure the accuracy of these methods for ill-posed problems and image restoration problems. The proposed quantities are defined as follows:

$$\mathrm{RES} = \frac{\|f_{numerical} - f_{exact}\|_2}{\|f_{exact}\|_2}, \quad \mathrm{PSNR} = 10\log_{10}\frac{255^2 \times n^2}{\|f_{numerical} - f_{exact}\|_2^2},$$

where the size of the image is $n \times n$ and $f_{numerical}, f_{exact}$ are the numerical solutions (or restored images) and exact solutions (or the original images), respectively. In image restoration problems, a larger PSNR-value usually implies that the restoration is of higher quality.

**Example 4.1.** *This is a one-dimensional model of an image reconstruction problem. It arises from discretization of the Fredholm integral equation of the first kind [15]*

$$\int_a^b K(s,t)f(t)dt = g(s), \quad c \le s \le d \tag{26}$$

*with the kernel $K(s,t)$ being the point spread function, right-hand side $g(s)$ and the exact solution $f(t)$*

$$K(s,t) = \begin{cases} s(t-1), \ s < t \\ t(s-1), \ s \ge t \end{cases}, \quad g(s) = \begin{cases} (4s^3 - 3s)/24, \ s < \frac{1}{2} \\ (-4s^3 + 12s^2 - 9s + 1)/24, \ s \ge \frac{1}{2} \end{cases} \text{ and } f(t) = \begin{cases} t, \ t < \frac{1}{2} \\ 1 - t, \ t \ge \frac{1}{2} \end{cases}.$$

*This integral equation is discussed by deriv2 in Hansen's Regularization Tools package [17].*

We discretize the integral equation by the Galerkin method with orthonormal box functions as test and trial functions by the MATLAB program *deriv*2(500, 3) from Regularization Tools [19] and obtain the matrix $A \in \mathbb{R}^{500\times 500}$ and the discretized solution $\hat{f}$ of the error-free linear system. The associated contaminated vector $g$ in (3) is obtained by adding Gaussian white noise $\epsilon = 0.001$ to $\hat{g}$ in (2). We use Tikhonov regularization method to compute a stable solution which is less sensitive to errors. The tested iteration methods are terminated once the current residual satisfies $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-6}$ or if the iteration step exceeds the largest prescribed iteration step $N = 100$, where $r^{(k)} = b - \mathcal{A}z^{(k)}$ is the residual at the $k$th iteration. We adop GCV to select the regularization parameter $\mu$, and for Examples 4.1 $\mu = 0.0148$.

The exact and numerical solutions are shown in Figure 1 for $N = 100$. The relative error of Tikhonov method is 0.0864, and IT, CPU times and relative errors of the SHSS, ULT-I, ULT-II and the versions of NTS methods are reported in Table 1. From Table 1, it can be seen that $\mathrm{ULT}-\mathrm{I}_{\mathrm{Q}_1}$ can achieve the smallest relative error, while it requires more IT and CPU times. Moreover, the relative errors of $\mathrm{ULT}-\mathrm{II}_{\mathrm{Q}_1}$, $\mathrm{ULT}-\mathrm{II}_{\mathrm{Q}_2}$ and $\mathrm{NTS}-\mathrm{Q}_2$ are the same, however $\mathrm{ULT}-\mathrm{II}_{\mathrm{Q}_2}$ and $\mathrm{NTS}-\mathrm{Q}_2$ methods require less iteration steps. Due to the fact that no linear system needs to be solved in the NTS method, CPU times of the new method is substantially lower than other ones. The plots of relative error with respect to iterations $k$ are drawn in Figure 2 to illustrate the convergence behaviour of our methods. From Figure 2, the convergence rates of $\mathrm{ULT}-\mathrm{II}_{\mathrm{Q}_2}$ and $\mathrm{NTS}-\mathrm{Q}_2$ methods are faster than other ones. Besides, the two methods have a little semi-convergence. If we adopt smaller iteration step $k$, $\mathrm{ULT}-\mathrm{II}_{\mathrm{Q}_2}$ and $\mathrm{NTS}-\mathrm{Q}_2$ methods may can achieve a smaller relative error.

As this results, the proposed NTS method are more effective for solving the ill-posed problems.

Table 1: Numerical results of iteration methods for *deriv*2(500, 3) of Example 4.1.

| Method | $s$ | $\alpha$ | IT | CPU | RES |
|---|---|---|---|---|---|
| SHSS | - | 0.0051 | 100 | 3.0413 | 0.0880 |
| $\mathrm{ULT}-\mathrm{I}_{Q_1}$ | 0.0152 | - | 100 | 1.2224 | 0.0827 |
| $\mathrm{ULT}-\mathrm{I}_{Q_2}$ | 0.8 | - | 100 | 1.4373 | 0.3319 |
| $\mathrm{ULT}-\mathrm{II}_{Q_1}$ | 0.0105 | - | 100 | 0.5169 | 0.0861 |
| $\mathrm{ULT}-\mathrm{II}_{Q_2}$ | 0.0015 | - | 34 | 0.4834 | 0.0861 |
| $\mathrm{NTS}-\mathrm{Q}_1$ | 10 | 0.0051 | 100 | 0.0611 | 0.0885 |
| $\mathrm{NTS}-\mathrm{Q}_2$ | 0.0015 | 1.0018 | 40 | 0.2575 | 0.0861 |

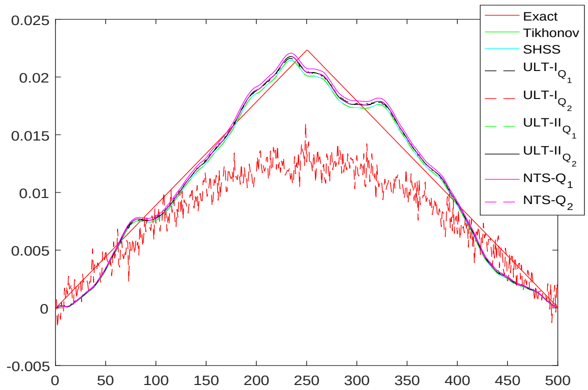


Figure 1: Comparison of the exact solution of *deriv*2(500, 3) problem and its numerical solutions.

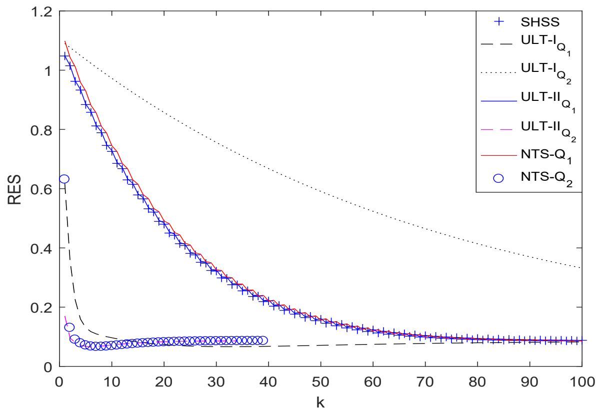


Figure 2: Residual errors versus iteration $k$ for *deriv*2(500, 3).

Table 2: Numerical results of iteration methods for $foxgood(500)$ of Example 4.2.

| Method | $s$ | $\alpha$ | IT | CPU | RES |
|---|---|---|---|---|---|
| SHSS | - | 0.2474 | 100 | 3.0280 | 0.1830 |
| $ULT-I_{Q_1}$ | 0.6584 | - | 100 | 1.1680 | 0.1424 |
| $ULT-I_{Q_2}$ | 0.8 | - | 100 | 1.4989 | 0.1083 |
| $ULT-II_{Q_1}$ | 0.6575 | - | 100 | 0.4134 | 0.1842 |
| $ULT-II_{Q_2}$ | 0.0015 | - | 73 | 1.0767 | 0.0082 |
| $NTS-Q_1$ | 10 | 0.3399 | 100 | 0.0739 | 0.1756 |
| $NTS-Q_2$ | 0.0001 | 1.0017 | 53 | 0.3185 | 0.0081 |

**Example 4.2.** *Discretization of a Fredholm integral equation of the first kind*

$$\int_a^b K(s,t)f(t)dt = g(s), \quad c \le s \le d$$

*with both integration intervals* $[0,1]$, *with kernel* $K$ *and right-hand side* $g$ *given by*

$$K(s,t) = (s^2+t^2)^{\frac{1}{2}}, \quad g(s) = \frac{1}{3}\left((1+s^2)^{\frac{3}{2}} - s^3\right),$$

*and with the solution* $f = t$. *This integral equation is discussed by* $foxgood$.

Regularization Tools [19] provides the severely ill-posed test problem $foxgood(500)$ with 500 being the matrix size. The disturbed right-hand side vector $g$ is constructed in the same way as in Example 4.1. We use Tikhonov regularization method to compute a stable solution which is less sensitive to errors. $\tau$ in Algorithms 3.1 and 3.2 is set to $10^{-6}$ and $N = 100$. We also adop GCV to select the regularization parameter $\mu$, and for Examples 4.2 $\mu = 0.0018$.

In Table 2, we disclose IT, CPU times and relative errors of the SHSS, the ULT-I and ULT-II and versions of the NTS methods. The relative error of Tikhonov method is 0.0206. The exact and numerical solutions are shown in Figure 3 for $N = 100$ and the relative errors with respect to iterations $k$ are drawn in Figure 4. From Table 2, it is seen that our proposed algorithms surpass other ones in terms of both the required number of iterations and CPU times for the convergence. The NTS iteration method needs the least CPU times and the $NTS-Q_2$ method achieve smallest relation error with least iteration steps. It converges after 53 iteration for $foxgood(500)$ while other ones are not terminated for $N = 100$ expect for $ULT-II_{Q_2}$ method needing 73 iteration steps. As Figure 4 shows that the $NTS-Q_2$ method is the most effective one as its residual reduces the fastest and it costs the less CPU times.

We would like to comment here that from the reported results from Table 2 and Figure 4, one may come to this conclusion that our presented methods outperform the SHSS, the ULT-I and ULT-II ones.

**Example 4.3.** *(Image restoration) The following examples are concerned with the restoration of images that have been contaminated by blur and noise. Let the entries of the vector* $\hat{f}$ *be pixel values for a desired, but unknown, image. The matrix* $A$ *is a discretization of a blurring operator and equation* (3) *shows that* $\hat{g}$ *represents a blurred, but noise-free, image. The vector* $g$ *in* (2) *represents the available blur- and noise-contaminated image associated with* $\hat{f}$. *The blurring matrix* $A$ *is determined by a point-spread function (PSF), which determines how each pixel is smeared out (blurred), and by the boundary conditions, which specify our assumptions on the scene just outside the available image; see [8, 20] for details.*

In this example, consider the original 'cameraman' image in Figure 5, which has an $256 \times 256$ matrix representation. The white lines have been applied to show the observed image domain which is restored in Figure 5. $PSF = psfDefocus([7,7],3)$ in [20] is applied to blur the true image, and a 'noisy' right-hand-side $g$ is generated using MATLAB code $g = \hat{g} + 0.001 \times rand(size(\hat{g}))$. The reflexive boundary condition is imposed. Furthermore, the PSNR of the degrade image is 53.9855.

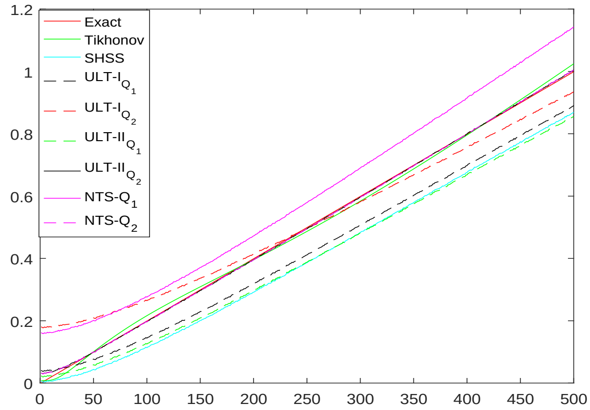


Figure 3: Comparison of the exact solution of *foxgood*(500) problem and its numerical solutions.

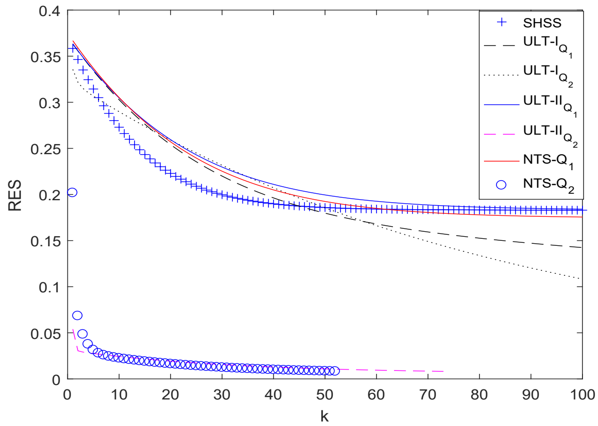


Figure 4: Residual errors versus iteration $k$ for *foxgood*(500).

Table 3: Numerical results of iteration methods of Example 4.3.

| Method | $s$ | $\alpha$ | IT | CPU | PNSR |
|---|---|---|---|---|---|
| SHSS | - | 0.3438 | 200 | 73.4675 | 60.7997 |
| $\mathrm{ULT}-\mathrm{I}_{Q_1}$ | 1.0735 | - | 200 | 3.1283 | 60.5340 |
| $\mathrm{ULT}-\mathrm{I}_{Q_2}$ | 0.6 | - | 200 | 29.6901 | 60.4434 |
| $\mathrm{ULT}-\mathrm{II}_{Q_1}$ | 1.0505 | - | 200 | 0.5126 | 60.5516 |
| $\mathrm{ULT}-\mathrm{II}_{Q_2}$ | 0.0540 | - | 179 | 26.6037 | 60.8202 |
| $\mathrm{NTS}-\mathrm{Q}_1$ | 2 | 0.7108 | 200 | 0.3188 | 60.5480 |
| $\mathrm{NTS}-\mathrm{Q}_2$ | 0.0001 | 28.6869 | 6 | 0.4846 | 60.8202 |

We use the blurred and noise image as an initial guess, and the stopping criterion is $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-7}$, where $r^{(k)}$ denotes the residual after the $k$th iteration. The maximum iteration is set to be 200. We use the GCV scheme to determine a suitable value for regularization parameter $\mu = 0.0527$ for both iterative methods. The matrix $A$ is approximated by $B_k$ and $C_k$ minimizing $\|A - \sum_k B_k \otimes C_k\|$. Due to the fact that the intercepted image is 64×64, we store the matrix $A$ sparsely and compute of the matrix-vector multiplications directly in Algorithms 3.1-3.2 without fast algorithm.

In Table 3, we report the IT, CPU times and PSNR of the tested methods. From Table 3, it can be seen that the $\mathrm{NTS} - \mathrm{Q}_2$ method outperforms the other six ones as it needs the least IT, just 6 iterations, and less CPU times to achieve a highest PSNR value. The restored images by the tested iteration methods are illustrated in Figure 6. The obtained results reveal the superiority of Algorithms 3.1 and 3.2 over other examined iterative schemes. As the numerical results show, the presented versions of NTS method can be effectively applied to solve the image restoration problem.

The plots of PSNR with respect to the iterations $k$ are drawn in Figure 7 to illustrate the convergence behaviour of our methods. Figure 7 clearly shows that among these iteration methods, the convergence rate of the $\mathrm{NTS} - \mathrm{Q}_2$ method is the most effective as its PSNR increases the fastest and the $\mathrm{NTS} - \mathrm{Q}_2$ method requires the least iteration steps to give a better quality of the computed restoration than other ones.

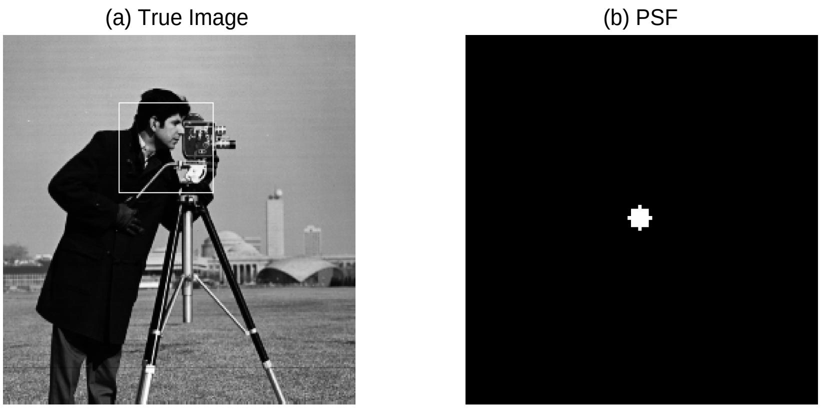


Figure 5: True image and PSF of Example 4.3.

**Example 4.4.** *(Image restoration) The original image is a brain image of dimension* $256 \times 256$ *from MATLAB. It is shown on the left-hand side of Figure 8. PSF and the 'noisy' right-hand-side g are the same as in Example 4.3. The blurred and noisy image is shown in the top left hand corner of Figure 9. Here the Periodic boundary condition is employed, and therefore the matrix A is block circulant with circulant blocks, which can perform matrix-vector multiplications via two-dimensional fast Fourier transformations (FFTs). Furthermore, the PSNR of the degrade image is 27.5769.*

The GCV scheme is used to determine a suitable value for regularization parameter $\mu = 0.042$. The stopping criterion is $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-4}$, where $r^{(k)}$ denotes the residual after the $k$th iteration. The maximum iteration is set to be 600. The numbers of iterations, CPU times and PSNR of the tested methods for solving the linear image restoration problem (6) are listed in Table 4. By comparing the results of Table 4, it can be clearly seen that the versions of NTS method needs the less CPU times, and the reason for this is that no linear system is sloved in the proposed methods. Moreover, the $\mathrm{NTS} - \mathrm{Q}_2$ iteration method performs the best since it uses the least IT and CPU times to achieve a highest PSNR, implying the highest quality of the restoration compared with the six ones. The image restoration results are shown in Figure 9 and demonstrate that a high visual quality of the restored images can be obtained.

As in the previous example, the PSNR with respect to the iterations $k$ is shown in Figure 10. It can be seen that the convergence behaviour of the $\mathrm{NTS} - \mathrm{Q}_1$ method is almost same as $\mathrm{ULT} - \mathrm{I}_{\mathrm{Q}_1}$ and $\mathrm{ULT} - \mathrm{II}_{\mathrm{Q}_1}$

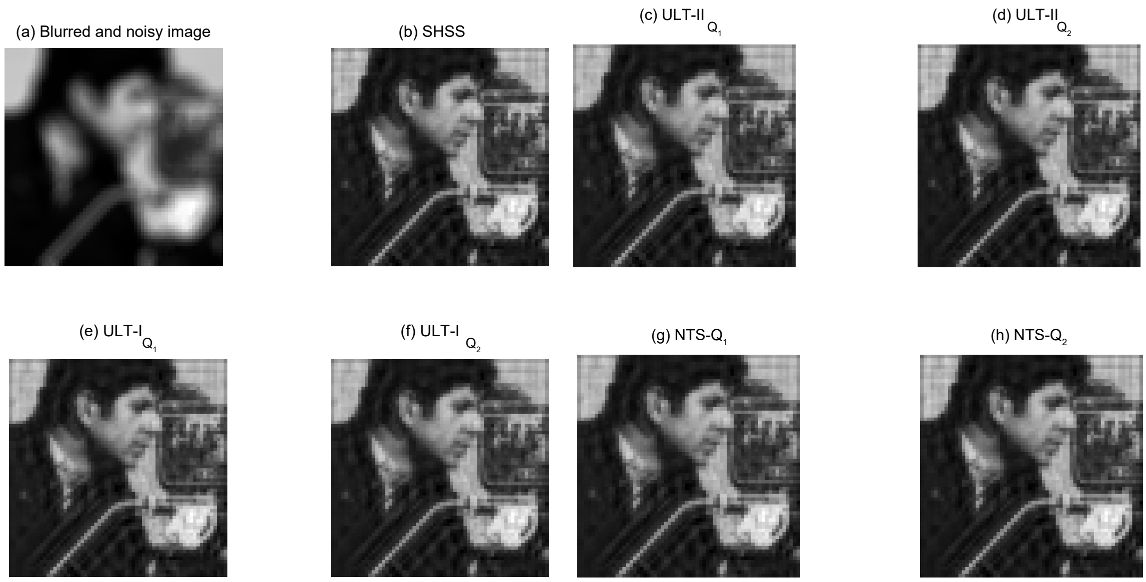


Figure 6: Restored images with various methods in Example 4.3.

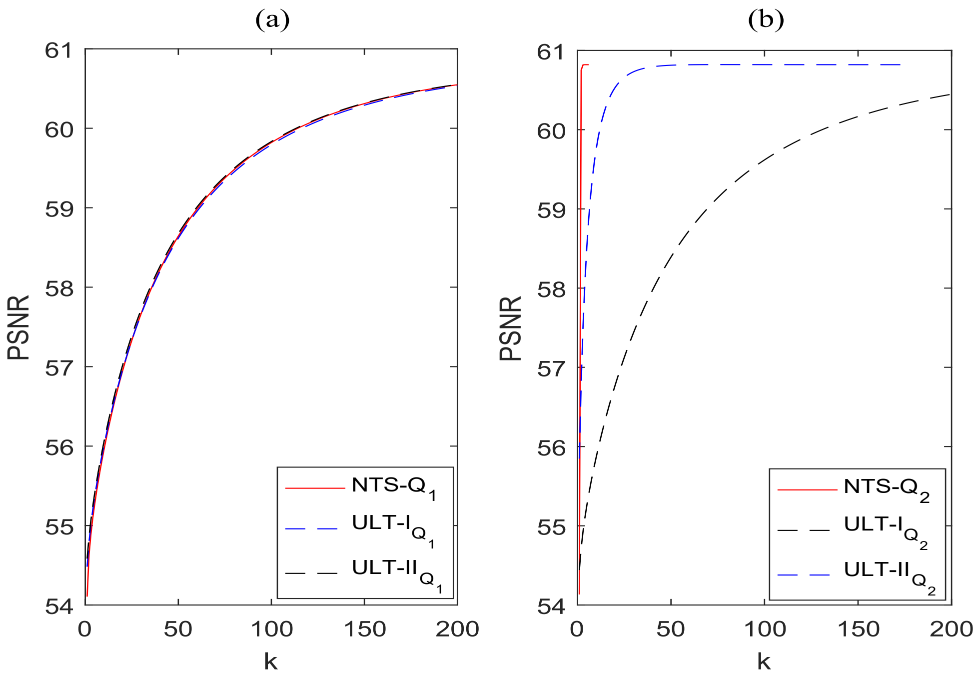


Figure 7: PSNR versus the iteration number $k$ for the restored images in Example 4.3.

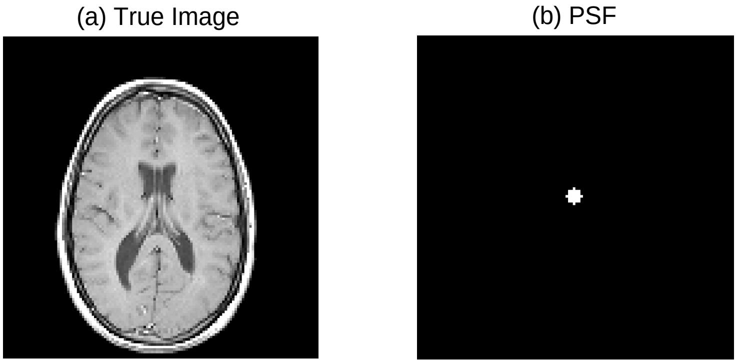


Figure 8: True image and PSF of Example 4.4.

Table 4: Numerical results of iteration methods of Example 4.4.

| Method | $s$ | $\alpha$ | IT | CPU | PNSR |
|---|---|---|---|---|---|
| SHSS | - | 0.3491 | 600 | 6.3296 | 39.6913 |
| $\mathrm{ULT}-\mathrm{I}_{Q_1}$ | 1.0197 | - | 600 | 6.0949 | 38.2191 |
| $\mathrm{ULT}-\mathrm{I}_{Q_2}$ | 12 | - | 600 | 7.2344 | 30.6109 |
| $\mathrm{ULT}-\mathrm{II}_{Q_1}$ | 1.0179 | - | 600 | 4.7235 | 38.2232 |
| $\mathrm{ULT}-\mathrm{II}_{Q_2}$ | 0.1338 | - | 429 | 5.4728 | 41.4648 |
| $\mathrm{NTS}-\mathrm{Q}_1$ | 0.51 | 246.1933 | 600 | 4.7153 | 38.2232 |
| $\mathrm{NTS}-\mathrm{Q}_2$ | 0.03 | 1.5259 | 190 | 2.0037 | 41.4715 |

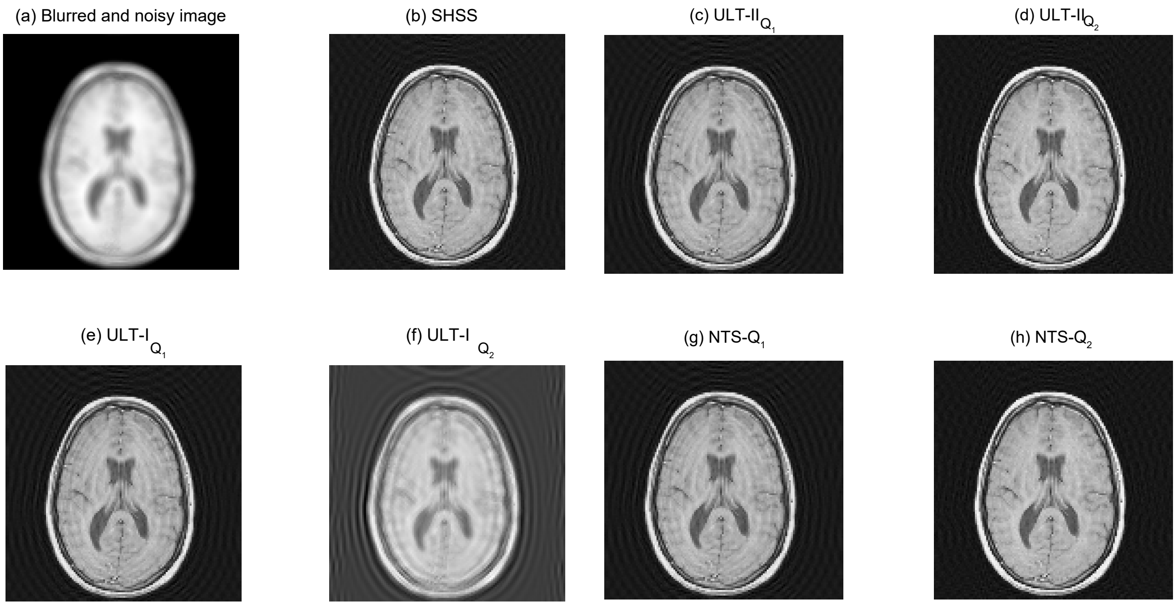


Figure 9: Restored images with various methods in Example 4.4.

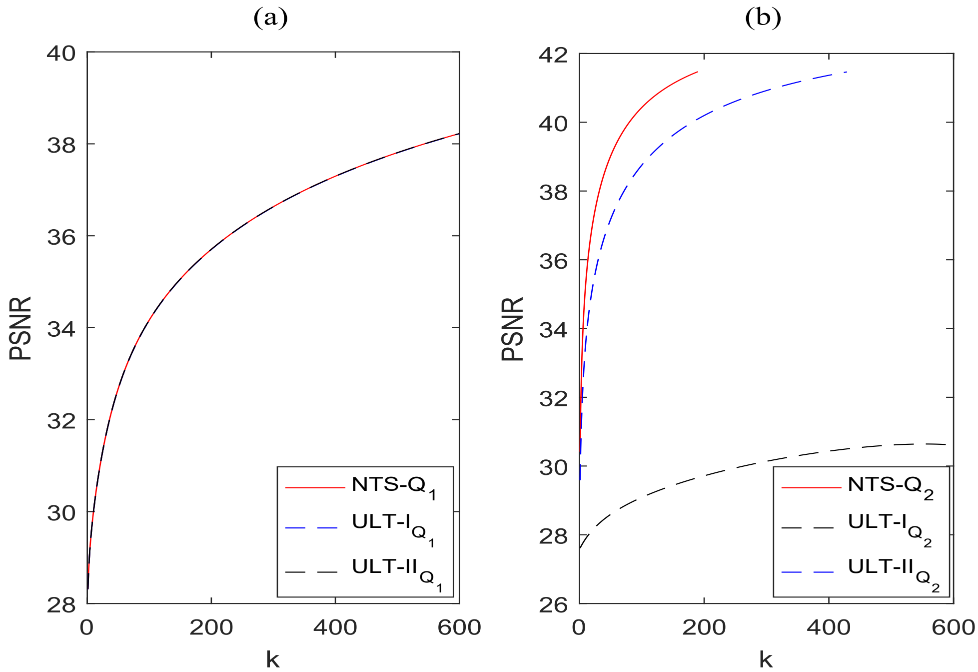


Figure 10: PSNR versus the iteration number $k$ for the restored images in Example 4.4.

ones, while the convergence rate of the $\mathrm{NTS}-\mathrm{Q}_2$ method is much faster than $\mathrm{ULT}-\mathrm{I}_{\mathrm{Q}_2}$ and $\mathrm{ULT}-\mathrm{II}_{\mathrm{Q}_2}$ ones, and it achieves the highest PSNR value. Comparison between these methods shows that the new methods are more reliable and effective than the SHSS and the versions of ULT methods in solving image restoration problems.

## 5. Conclusions

This paper studies the ill-posed inverse problems and proposes a new two-splitting (NTS) iteration method in which the parameter $\alpha$ and the parameter matrix $Q$ are incorporated. The NTS iteration method is based on the Hermitian and skew-Hermitian splitting (HSS) and the upper and lower triangular splitting (ULT) of the coefficient matrix $\mathcal{A}$ in (6). Besides, we obtain the convergence conditions and the optimal parameters minimizing the spectral radius of the iteration matrix of the versions of NTS method. The convergence rate of the NTS method can be speeded up by selecting appropriate parameters and parameter matrix. The presented numerical examples from the ill-posed problems and image restoration illustrate the efficiency of our method.

## Competing interests

The authors declare that they have no competing interests.

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## References


[1] N. Aghazadeh, M. Bastani, and D. K. Salkuyeh. Generalized Hermitian and skew-Hermitian splitting iterative method for image restoration. Appl. Math. Model., 39:6126–6138, 2015.
[2] H. Aminikhah and M. Yousefi. A special generalized HSS method for discrete ill-posed problems. Comp. Appl. Math., 37:1507–1523, 2018.
[3] Z. Z. Bai, G. H. Golub, and M. K. Ng. Hermitian and skew-Hermitian splitting methods for non- Hermitian positive definite linear systems. SIAM J. Matrix Anal. Appl., 24:603–626, 2003.
[4] Z. Z. Bai, G. H. Golub, and J. Y. Pan. Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems. Numer. Math., 98:1–32, 2004.
[5] M. Benzi. A generalization of the Hermitian and skew-Hermitian splitting iteration. SIAM J. Matrix Anal. Appl., 31:360–374, 2009.
[6] C. Brezinski, G. Rodriguez, and S. Seatzu. Error estimates for linear systems with application to regularization. Numer. Algorithms, 49:85–104, 2008.
[7] T. M. Buzug. Computed Tomography. Springer, Berlin, 2008.
[8] B. Chalmond. Modeling and Inverse Problems in Image Analysis. Springer, New York, 2003.
[9] G. H. Cheng, X. Rao, and X. G. Lv. The comparisons of two special Hermitian and skew-Hermitian splitting methods for image restoration. Appl. Math. Model., 39:1275–1280, 2015.
[10] J. Chung, S. Knepper, and J. G. Nagy. Large-scale inverse problems in imaging. In Scherzer, O. (Ed.): Handbook of Mathematical Methods in Imaging. Springer, Heidelberg, 2011.
[11] H. Diao, Y. Wei, and S. Qiao. Structured condition numbers of structured Tikhonov regularization problem and their estimations. J. Comput. Appl. Math., 308:276–300, 2016.
[12] H. W. Engl, M. Hanke, and A. Neubauer. Regularization of Inverse Problems. Kluwer, Dordrecht, 1996.
[13] H. T. Fan, M. Bastani, B. Zheng, and X. Y. Zhu. A class of upper and lower triangular splitting iteration methods for image restoration. Appl. Math. Model., 54:82–111, 2018.
[14] G. H. Golub, M. Heath, and G. Wahba. Generalized cross-validation as a method for choosing a good ridge parameter. Technometrics, 21:215–223, 1979.
[15] C. W. Groetsch. The theory of Tikhonov regularization for Fredholm equations of the first kind. Pitman, Boston, 1984.
[16] P. C. Hansen. Analysis of discrete ill-posed problems by means of the L-curve. SIAM Rev., 34:561–580, 1992.
[17] P. C. Hansen. Regularization tools: A MATLAB package for analysis and solution of discrete ill-posed problems. Numer. Algorithms, 6:1–35, 1994.
[18] P. C. Hansen. Rank-Deficient and Discrete Ill-Posed Problems. SIAM, Philadelphia, 1998.
[19] P. C. Hansen. Regularization Tools version 4.0 for Matlab 7.3. Numer. Algorithms, 46:189–194, 2007.
[20] P. C. Hansen, J. G. Nagy, and D. P. OLeary. Deblurring Images: Matrices, Spectra and Filtering. SIAM, Philadelphia, 2006.
[21] S. Kindermann. Convergence analysis of minimization-based noise level-free parameter choice rules for linear ill-posed problems. Electron. Trans. Numer. Anal., 38:233–257, 2011.
[22] X. G. Lv, T. Z. Huang, Z. B. Xu, and X. L. Zhao. A special Hermitian and skew-Hermitian splitting method for image restoration. Appl. Math. Model., 37:1069–1082, 2013.
[23] V. Morozov. On the solution of functional equations by the method of regularization. Soviet Math. Dokl., 7:414–417, 1966.
[24] D. L. Phillips. A Technique for the Numerical Solution of Certain Integral Equations of the First Kind. J Assoc. Comput. Mach., 9:84–97, 1962.
[25] L. Reichel and A. Shyshkov. A new zero-finder for Tikhonov regularization. BIT Numer. Math., 48:627–643, 2008.
[26] A. N. Tikhonov. English translation of Dokl Akad Nauk. SSSR, 151:501–504, 1963.
[27] A. N. Tikhonov and V. Y. Arsenin. Solutions of ill-posed problems. Winston, DC, 1977.
[28] Y. M. Wei, P. P. Xie, and L. P. Zhang. Tikhonov regularization and randomized GSVD. SIAM J. Matrix Anal. Appl., 37:649–675, 2016.
[29] L. Zhang and Y. Wei. Randomized core reduction for discrete ill-posed problem. J. Comput. Appl. Math., 368:112797, 2020.
[30] N. M. Zhang, T. T. Lub, and Y. M. Wei. Semi-convergence analysis of Uzawa methods for singular saddle point problems. J. Comput. Appl. Math., 255:334–345, 2014.
[31] M. Zhdanov. Geophysical Inverse Theory and Regularization Problems. Elsevier, Amsterdam, 2002.