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A New Variant of Hildebrandt's Theorem for the Weyl Spectrum in Banach Spaces

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Abstract. The main purpose of this paper is to establish a new variant of the Hildebrandt's theorem for the Weyl spectrum in a separable Banach space. This theorem asserts that the convex hull of the Weyl spectrum of an operator T is equal to the intersection of the Weyl numerical spectra of operators that are similar to T.

1. Introduction and Main Results

In 1918, O. Toeplitz [21] and F. Hausdorff [12] introduced a stronger invariant set for a linear operator T calling it the numerical range of T. This set is defined by

$$W(T) := \{\langle Tx, x\rangle, x \in X, \|x\| = 1\}$$

where $X = \mathbb{C}^n$ is a finite dimensional space endowed with Euclidean norm $\|.\|$.

The set $W(T)$ is invariant only under unitary equivalence and has very nice properties making it a stronger invariant than the spectrum of T denoted by $\sigma(T)$. These properties are:

(1) $W(T)$ contains $\sigma(T)$.

(2) $W(T)$ is closed.

(3) $W(T)$ is convex.

(4) $W(T) = \{0\}$ if and only if $T = 0$.

Later, the above formula of the numerical range was extended by M. H. Stone [20] to bounded operators acting on Hilbert spaces. This extension led to the construction of a unitary invariant subset satisfying properties (3) and (4), but lost properties (1) and (2).

In 1966 [13], S. Hildebrandt proved an important theorem which asserted that the convex hull of the spectrum of a bounded operator T on a complex Hilbert space was equal to the intersection of the numerical ranges of operators that were similar to T. This theorem reads as follows:

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Theorem 1.1. *[13]* For every bounded operator T in Hilbert space $\mathcal{H}$, we have

$$conv(\sigma(T)) = \bigcap \left\{ W(VTV^{-1}),\ V \text{ is a bounded invertible operator on } \mathcal{H} \right\}.$$

The notion of the numerical range was extensively studied during last five decades and was used in engineering as a rough estimate of eigenvalues of T. This set allows us to get more information about operators (see for example [4, 5, 10, 11]). Furthermore, a considerable attention has been devoted to extend the definition of the numerical range to operators acting on Banach spaces (see [3, 14, 17]), but all these interesting definitions still lose at least one property among the four properties already cited. For this reason, M. Adler et al. [1] have recently proposed a definition of the numerical spectrum, $\sigma_n(T)$, of a closed densely defined linear operator T on a Banach space X. Unlike the numerical range, the numerical spectrum is always an isometric invariant and satisfies all properties $(1)-(4)$.

In literature, spectral theory has witnessed an explosive development in the study of the Weyl spectrum of linear operators which opened up a new line to investigate some spectral properties of the underlying physical systems. The original definition of this spectrum goes back to H. Weyl [23] and later, which has considerably attracted the attention of many authors [7, 8, 15, 16, 18, 19].

In [19], it was shown that the Weyl spectrum of an operator T acting on a Banach space X, consists of those points of the spectrum which cannot be removed from the spectrum by the addition to T of a compact operator, that is:

$$\sigma_w(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma(T+K). \tag{1}$$

We draw the attention of the reader that if there exists a compact operator $K \in \mathcal{K}(X)$ such that $\sigma(T+K) = \sigma_w(T)$, then the operator T is said to satisfy the problem of Salinas. This property is valid for all bounded linear operators defined on a separable Hilbert space but not for all Banach spaces setting. (See [2] for more details).

Recently, J. Bračič et al. in [6] proved the following Hildebrandt theorem for the Weyl spectrum:

Theorem 1.2. *[6]* For every bounded operator T on a Hilbert space $\mathcal{H}$, we have:

$$conv(\sigma_w(T)) := \bigcap \left\{ W_w(VTV^{-1}),\ V \text{ is a bounded invertible operator on } \mathcal{H} \right\}$$

where the Weyl numerical range of T, denoted by $W_w(T)$, is defined by

$$W_w(T) := \bigcap_{K \in \mathcal{K}(X)} \overline{W(T+K)}.$$

A natural question can be asked about the extension of the Hildebrandt's theorem to bounded operators acting in Banach spaces. For this subject, analogously to the Weyl spectrum, we can define the Weyl numerical spectrum for a bounded operator T acting on a Banach space X, as follows:

$$\sigma_{w,n}(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma_n(T+K). \tag{2}$$

This new concept allows us to provide an extension of Hildebrandt's theorem to the case of Banach space. Indeed, we will first generalize Theorem 1.1 by proving the following theorem:

Theorem 1.3. (Hildebrandt's Theorem in Banach space) Let X be a separable Banach space and T a bounded operator on X. Then

$$conv(\sigma(T)) = \bigcap \left\{ \sigma_n(VTV^{-1}),\ V \text{ is a bounded invertible operator on } X \right\}.$$

Next, we will extend Theorem 1.2 by proving the following result:

Theorem 1.4. Let X be a separable Banach space and T be a bounded operator on X satisfying the Salinas problem. Then, we have:

$$conv(\sigma_w(T)) = \bigcap \left\{\sigma_{w,n}(VTV^{-1}), \;\; V \text{ is a bounded invertible operator on } X\right\}.$$

Now, let us outline the content of this paper. In Section 2, we shall present some basic notations and properties connected to the main body of the paper. In Section 3, we will formulate straightforward auxiliary results to prove the Hildebrandt's theorem for the Weyl spectrum in a separable Banach space.

2. Basic Notations and Preliminary results

To outline the main topics of this paper, we need first to agree on some standard notations and introduce some terminology.

Let X be an infinite dimensional and separable Banach space and X' its dual. We denote by $\mathcal{L}(X)$ the set of all bounded linear operators on X and by $\mathcal{K}(X)$ the set of all compact operators which is a subspace of $\mathcal{L}(X)$. The range of a bounded operator T is denoted by $\mathcal{R}(T)$.

In the sequel, $\overline{conv}M$ denotes the closed convex hull of a given set M.

Definition 2.1. [1, Corollary 2.7] Let $T \in \mathcal{L}(X)$, the numerical spectrum of T is defined by

$$\sigma_n(T) := \overline{conv}\{< Tx, J(x) >, \;\; x \in X, \|x\| = 1, J(x) \in \mathfrak{J}(x)\}$$

where $\mathfrak{J}(x)$ denotes the (non empty) duality set defined by

$$\mathfrak{J}(x) := \{J(x) \in X' : \; < x, J(x) >= \|x\|^2 = \|J(x)\|^2\}.$$

Now, we recall some several basic properties about the numerical spectrum and we refer the reader to the paper of M. Adler et al. in [1] for more details.

Proposition 2.2. [1] Let $T \in \mathcal{L}(X)$. Then, we have

(i) $\sigma_n(T)$ is closed and convex.

(ii) $\sigma_n(\alpha T + \beta) = \alpha\sigma_n(T) + \beta$ for all complex numbers α and β.

(iii) $\sigma_n(T) = \sigma_n(U^{-1}TU)$ for all isometric isomorphisms U on X.

(iv) $\sigma(T) \subseteq \sigma_n(T)$.

(v) $\sigma_n(T) \subseteq \{\lambda \in \mathbb{C}$ such that $|\lambda| \leq \|T\|\}$.

Remark 2.3. (i) The role of the Weyl numerical spectrum in comparison with the Weyl spectrum mimics the role of the numerical spectrum in comparison with the spectrum.

(ii) From Proposition 2.2 (ii) and the definition of the Weyl numerical spectrum $\sigma_{w,n}(.)$ given by (2), we infer that this set is closed and convex being the intersection of closed convex sets.

One of the impediments to the development of a clear parallel theory for operators on Banach spaces compared to there for Hilbert spaces is the lack of a suitable notion of an adjoint operator. For this reason, T. L. Gill et al. in [9] gave the construction of the adjoint of bounded linear operators on a separable Banach space. This construction was established in order to generalize the well-known result of J. von Neumann given in [22] for bounded operators in Hilbert spaces. Let first recall the following result:

Theorem 2.4. [9, Theorem 5] Let X be a separable Banach space and let $T \in \mathcal{L}(X)$. Then, T has a well-defined adjoint T^* defined on X such that:

(*i*) the operator $T^*T \geq 0$ called maximal accretive,

(*ii*) $(T^*T)^* = T^*T$,

(*iii*) $I + T^*T$ has a bounded inverse.

3. Proof of Main Results

The proofs of theorems 1.3 and 1.4 of this paper require some preparatory results. For this purpose, we consider in this section a separable Banach space X. Due to [9, Theorem 1.2], there exists two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfying $\mathcal{H}_1 \subseteq X \subseteq \mathcal{H}_2$ as continuous dense embedding where we consider in the next that $X' \subset \mathcal{H}_2$. Now, let us start by proving the following lemma.

Lemma 3.1. Let $T \in \mathcal{L}(X)$, then T has a bounded extension $\overline{T}$ to $\mathcal{H}_2$ and $\rho(T) = \rho(\overline{T})$.

Proof. Clearly, since T is bounded, then by Theorem 1.4 in [9], it can be extended to a bounded linear operator $\overline{T}$ on $\mathcal{H}_2$.

First, we will prove that $\rho(\overline{T}) \subset \rho(T)$.

Indeed, if $\lambda \in \rho(\overline{T})$, then $\lambda I - \overline{T}$ has an inverse which entails that $\lambda I - T$ also has one. So, $\rho(\overline{T}) \subset \rho(T)$.

Second, we can show that $\rho(T) \subset \rho(\overline{T})$. That is, since $T \in \mathcal{L}(X)$, then $\rho(T) \neq \emptyset$. Let's $\lambda \in \rho(T)$, then $(\lambda I - T)^{-1}$ is a continuous mapping on X.

On the other hand, let's $f \in \mathcal{H}_2$. Since $\overline{T}$ is continuous, then there exists a sequence $\{f_n\} \subset X$, such that $\lim\limits_{n\to\infty} f_n = f$ and $\lim\limits_{n\to\infty} \overline{T} f_n = \overline{T} f$ in $\mathcal{H}_2$. Obviously, $\overline{T} f_n = T f_n$. This entails $(\lambda I - \overline{T})f = \lim\limits_{n\to\infty} (\lambda I - T) f_n$. However, by the boundedness fact of the operator $(\lambda I - T)^{-1}$ on X, we have

$$\|(\lambda I - \overline{T})f\|_{\mathcal{H}_2} = \lim_{n\to\infty} \|(\lambda I - T) f_n\|_{\mathcal{H}_2} \geq \lim_{n\to\infty} \delta \|f_n\|_{\mathcal{H}_2} = \delta \|f\|_{\mathcal{H}_2}, \text{ for some } \delta > 0.$$

Thus, $\lambda I - \overline{T}$ has a bounded inverse and we conclude that $\rho(T) \subset \rho(\overline{T})$. □

Now, we need to prove the following lemma which is useful in the proof of our main results.

Lemma 3.2. Let $T \in \mathcal{L}(X)$. If $\sigma(T)$ is contained in the open unit disk, then

$$V := \sum_{k=0}^{\infty} (\overline{T}^*)^k \overline{T}^k$$

is an invertible and accretive operator with

$$\|V^{\frac{1}{2}} \overline{T} V^{-\frac{1}{2}}\|_{\mathcal{H}_2} < 1,$$

where $\overline{T}$ and $\overline{T}^*$ are respectively the extensions of T and its adjoint T^* on $\mathcal{H}_2$.

Proof. Since T is bounded, Theorem 1.4 in [9] asserts that it has an adjoint T^*. Thus, these both operators can be extended to bounded linear operators $\overline{T}$ and $\overline{T}^*$ on $\mathcal{H}_2$. Furthermore, following Lemma 3.1, we have $\sigma(T) = \sigma(\overline{T})$.

We notice that

$$\limsup_{k\to\infty} \|(\overline{T}^*)^k \overline{T}^k\|_{\mathcal{H}_2}^{\frac{1}{k}} \leq \limsup_{k\to\infty} \|\overline{T}^k\|_{\mathcal{H}_2}^{\frac{2}{k}} = r(\overline{T})^2,$$

where $r(\overline{T})$ is the spectral radius of $\overline{T}$. Since $\sigma(T) = \sigma(\overline{T})$ is a subset of the open unit disk , then $r(\overline{T}) < 1$. Consequently, $\limsup_{k\to\infty} \|(\overline{T}^*)^k\overline{T}^k\|_{\mathcal{H}_2}^{\frac{1}{k}} < 1$. Hence, the sum converges absolutely. Clearly, V is an accretive operator being a sum of accretive operators.

One has

$$\langle \overline{T}^* V\overline{T}x, x\rangle_{\mathcal{H}_2} \geq 0, \quad \forall x \in \mathcal{H}_2,$$

it follows that $\overline{T}^* V\overline{T} = V - I_{\mathcal{H}_2}$. So, we infer that $V \geq I$. Thus, we claim that V is an invertible operator.

Now, since V is an accretive operator, we define the bounded operator $V^{\frac{1}{2}}\overline{T}V^{-\frac{1}{2}}$ satisfying

$$\|V^{\frac{1}{2}}\overline{T}V^{-\frac{1}{2}}\|^2_{\mathcal{H}_2} \leq \|V^{-\frac{1}{2}}\overline{T}^* V\overline{T}V^{-\frac{1}{2}}\|_{\mathcal{H}_2} = \|V^{-\frac{1}{2}}(V - I)V^{-\frac{1}{2}}\|_{\mathcal{H}_2} = \|I - V^{-1}\|_{\mathcal{H}_2}.$$

On the other hand, since V and $V - I$ are accretive operators, then $\sigma(V) \subseteq]1, +\infty[$.
Hence, we conclude that $\sigma(V^{-1}) \subseteq (0, 1]$. That is,

$$\sigma(I - V^{-1}) \subseteq [0, 1),$$

and thus

$$\|I - V^{-1}\|_{\mathcal{H}_2} < 1.$$

Which yields $\|V^{\frac{1}{2}}\overline{T}V^{-\frac{1}{2}}\|_{\mathcal{H}_2} < 1$ as desired.□

In the next, we will prove Theorem 1.3 by double inclusion.

Proof of Theorem 1.3 First, let us prove the first inclusion, that is:

$$conv(\sigma(\overline{T})) \subseteq \bigcap \left\{\sigma_n(V\overline{T}V^{-1}),\ V \text{is a bounded invertible operator on } \mathcal{H}_2\right\}.$$

Clearly, for every invertible operator $V \in \mathcal{L}(\mathcal{H}_2)$, we have $\sigma(V\overline{T}V^{-1}) = \sigma(\overline{T})$.
Therefore, following Proposition 2.2 (*iv*), we obtain

$$\sigma(\overline{T}) = \sigma(V\overline{T}V^{-1}) \subseteq \sigma_n(V\overline{T}V^{-1}).$$

According to the convexity of the numerical spectrum (see Proposition 2.2 (*i*)), we infer that

$$conv(\sigma(\overline{T})) \subseteq \sigma_n(V\overline{T}V^{-1}).$$

Consequently, we get the desired inclusion

$$conv(\sigma(\overline{T})) \subseteq \bigcap \left\{\sigma_n(V\overline{T}V^{-1}),\ V \text{is a bounded invertible operator on } \mathcal{H}_2\right\}.$$

In order to prove the second inclusion, we suppose that there exists $\lambda \in \mathbb{C}$, which satisfies that $\lambda \in \sigma_n(V\overline{T}V^{-1})$, for all invertible operator. Yet $\lambda \notin conv(\sigma(\overline{T}))$.

Now, by translating, scaling and thanking into account Proposition 2.2 (*ii*), we may assume that $conv(\sigma(\overline{T}))$ is a subset of the open unit disk of radius and $|\lambda| \geq 1$.

Since $\sigma(T) \subseteq conv(\sigma(\overline{T}))$ is a subset of the open disk of radius, then Lemma 3.2 implies that there exists an invertible element $V \in \mathcal{L}(\mathcal{H}_2)$ such that $\|V\overline{T}V^{-1}\|_{\mathcal{H}_2} < 1$. From Proposition 2.2 (*vi*), one has $\sigma_n(V\overline{T}V^{-1})$ is a subset of the open unit disk which contradicts the fact that $\lambda \in \sigma_n(V\overline{T}V^{-1})$ and $|\lambda| \geq 1$. Hence, we get the second inclusion and we conclude that

$$conv(\sigma(\overline{T})) = \bigcap \left\{\sigma_n(V\overline{T}V^{-1}),\ V \text{is a bounded invertible operator on } \mathcal{H}_2\right\}.$$

From here, taking the restriction T of $\overline{T}$ to X, the result follows and we get

$$conv(\sigma(T)) = \bigcap \left\{\sigma_n(VTV^{-1}),\ V\text{is a bounded invertible operator on } X\right\}.\square$$

Remark 3.3. Note that, Theorem 1.3 can be considered as a generalization and an extension of Theorem 2.2 in [6] to the Banach space case.

Now, we are able to prove Theorem 1.4 which gives an improvement of [6, Theorem 2.3].

Proof of Theorem 1.4. The proof of Theorem 1.4 requires two steps:
Step 1: Let $T \in \mathcal{L}(X)$, we claim that

$$conv(\sigma_w(T)) \subseteq \bigcap \left\{\sigma_{w,n}(VTV^{-1}),\ V\text{is a bounded invertible operator on } X\right\}.$$

Indeed, since V is a bounded invertible operator, we have $\sigma(VTV^{-1}) = \sigma(T)$. Following Proposition 2.2 (*iv*), one has

$$\sigma(T) = \sigma(VTV^{-1}) \subseteq \sigma_n(VTV^{-1}).$$

Therefore

$$\sigma(T+K) = \sigma(V(T+K)V^{-1}) \subseteq \sigma_n(V(T+K)V^{-1}),\ \ \forall K \in \mathcal{K}(X).$$

Let $\widetilde{K}$ be the compact operator defined by $\widetilde{K} := VKV^{-1}$. Clearly, we have

$$\sigma(VTV^{-1} + \widetilde{K}) \subseteq \sigma_n(VTV^{-1} + \widetilde{K}).$$

So,

$$\bigcap_{K\in\mathcal{K}(X)} \sigma(T+K) \subseteq \bigcap_{K\in\mathcal{K}(X)} \left\{\sigma_n(VTV^{-1} + \widetilde{K}),\ V\text{is a bounded invertible operator on } X\right\}.$$

In other terms, we have

$$\sigma_w(T) \subseteq \bigcap \left\{\sigma_{w,n}(VTV^{-1}),\ V\text{is a bounded invertible operator on } X\right\}.$$

Therefore, taking the convex hulls, we deduce that

$$conv(\sigma_w(T)) \subseteq \bigcap \left\{\sigma_{w,n}(VTV^{-1}),\ V\text{is a bounded invertible operator on } X\right\}.$$

Step 2: At this step, we will prove the opposite inclusion, that is:

$$\bigcap \left\{\sigma_{w,n}(VTV^{-1}),\ V\text{is a bounded invertible operator on } X\right\} \subseteq conv(\sigma_w(T)).$$

Indeed, Since the operator T satisfy the Salinas problem, we infer that there exists a compact operator $K \in \mathcal{K}(X)$, such that

$$\sigma_w(T) = \sigma(T+K).$$

Therefore, by the convexity hulls of the Weyl spectrum of T and the spectrum of $T+K$, we deduce that

$$conv(\sigma_w(T)) = conv(\sigma(T+K)). \tag{3}$$

Using Theorem 1.3, it follows that:

$$\begin{aligned} conv(\sigma(T+K)) &= \bigcap \left\{\sigma_n(V(T+K)V^{-1}),\ V\text{is a bounded invertible operator on } X\right\} \\ &= \bigcap \left\{\sigma_n(VTV^{-1} + VKV^{-1}),\ V\text{is a bounded invertible operator on } X\right\} \\ &= \bigcap \left\{\sigma_n(VTV^{-1} + \widetilde{K}),\ V\text{is a bounded invertible operator on } X\right\}, \end{aligned} \tag{4}$$

where $\widetilde{K} := VKV^{-1}$ is a compact operator as the product of the compact operator K and the bounded invertible operator V.

On the other hand, we have:

$$\bigcap_{K\in\mathcal{K}(X)} \sigma_n(VTV^{-1}+K) \subseteq \sigma_n(VTV^{-1}+\widetilde{K}),$$

which allows us to conclude that:

$$\begin{aligned} &\bigcap \left\{ \bigcap_{K\in\mathcal{K}(X)} \sigma_n(VTV^{-1}+K),\ V\text{is a bounded invertible operator on } X \right\} \\ \subset\ &\bigcap \left\{ \sigma_n(VTV^{-1}+\widetilde{K}),\ V\text{is a bounded invertible operator on } X \right\}. \end{aligned}$$

According to Eqs. (4) and (3), we assert that:

$$\bigcap \left\{ \sigma_{w,n}(VTV^{-1}),\ V\text{is a bounded invertible operator on } X \right\} \subset conv(\sigma_w(T)).$$

Now, combining the results of the two steps, we conclude that

$$conv(\sigma_w(T)) = \bigcap \left\{ \sigma_{w,n}(VTV^{-1}),\ V\text{is a bounded invertible operator on } X \right\},$$

which ends the proof.□

Remark 3.4. Our results improve and extend those established by J. Bračič et al. given in [6] from Hilbert to Banach spaces by using the concept of numerical spectrum on the Banach space.

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