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Nearest Southeast Submatrix that Makes Two Prescribed Eigenvalues

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Abstract. Given four complex matrices A, B, C and D where $A \in \mathbb{C}^{n\times n}$ and $D \in \mathbb{C}^{m\times m}$ and given two distinct arbitrary complex numbers λ_1 and λ_2, so that they are not eigenvalues of the matrix A, we find a nearest matrix from the set of matrices $X \in \mathbb{C}^{m\times m}$ to matrix D (with respect to spectral norm) such that the matrix $\begin{pmatrix} A & B \\ C & X \end{pmatrix}$ has two prescribed eigenvalues λ_1 and λ_2.

1. Introduction

The spectral distance from an $n \times n$ matrix A to the set of matrices of rank at most r is equal to $\sigma_r(A)$, and $\sigma_r(A)$ denotes the rth singular value of the matrix A.

Let Φ be a complex $n \times n$ matrix, and let $\mathbb{L}$ be a set of $n \times n$ matrices with a multiple zero eigenvalue. In the paper [5], A.N. Malyshev obtained the following formula for 2-norm distance from Φ to $\mathbb{L}$:

$$\rho_2(\Phi, \mathbb{L}) = \min_{L\in\mathbb{L}} \|\Phi - L\|_2 = \max_{\phi\geq 0} \sigma_{2n-1}(P(\phi)), \tag{1}$$

in which

$$P(\phi) = \begin{pmatrix} \Phi & \phi I_n \\ 0 & \Phi \end{pmatrix}, \tag{2}$$

and $\sigma_i(\cdot)$ denotes the ith singular value of the corresponding matrix. It is assumed that the singular values of any matrix are arranged in decreasing order.

The spectral norm distance of an $n \times n$ matrix Φ to the set of matrices with two prescribed eigenvalues was computed by J. M. Gracia [2] for $\phi_\star \neq 0$ (where $P(\phi)$ gets its maximum at the point $\phi_\star$) and for other cases by Ross A. Lippert [4]. Let $A \in \mathbb{C}^{n\times n}$ be an invertible matrix and $D \in \mathbb{C}^{m\times m}$, J.M. Gracia and F.E. Velasco in their recent paper [3] found the spectral distance from a set of matrices $X \in \mathbb{C}^{m\times m}$ to matrix D, such that, the matrix

$$\Gamma_X = \begin{pmatrix} A & B \\ C & X \end{pmatrix}, \tag{3}$$

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has a multiple eigenvalue zero, i.e.

$$\min_{\substack{X\in\mathbb{C}^{m\times m}\\ m(0,\Gamma_X)\geq 2}} \|X-D\| = \sup_{\gamma\in\mathbb{R}} \sigma_{2m-1}(P(\gamma,D)),$$

where

$$P(\gamma,D) = \begin{pmatrix} \mathcal{M} & \gamma\mathcal{N} \\ 0 & \mathcal{M} \end{pmatrix},$$

$$\mathcal{M} := D - CA^{-1}B,$$

$$\mathcal{N} := I_m + CA^{-2}B,$$

and $m(\lambda_0,\Gamma_X)$ denotes the algebraic multiplicity of λ_0 as an eigenvalue of Γ_X.

Nazari and Nezami in [6] introduced a correction for Gracia and Velasco's formula, when the matrix Γ_D is a block normal matrix.

In this paper, for the given four complex matrices $A \in \mathbb{C}^{n\times n}$, B, C and $D \in \mathbb{C}^{m\times m}$ and for two given distinct complex numbers λ_1 and λ_2 which are not eigenvalues of matrix A, we find the nearest matrix to matrix D, from the set of matrices $X \in \mathbb{C}^{m\times m}$ such that matrix Γ_X has two prescribed eigenvalues λ_1 and λ_2.

Using the notations in [3], let us denote the Cartesian product $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{m\times m}$ by $L_{n,m}$. Given $\Gamma_D \in \mathbb{C}^{(m+n)\times(m+n)}$ the spectrum of Γ_D will be denoted by $\Lambda(\Gamma_D)$.

Two unitary vectors u, v are a pair of singular vectors of matrix Γ_X for the singular value σ if $\Gamma_X v = \sigma u$ and $(\Gamma_X)^H u = \sigma v$.

2. Function $P(\gamma)$

Assume that $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{N} \in \mathbb{C}^{m\times m}$ that

$$\mathcal{M}_1 = (D-\lambda_1 I_m) - C(A-\lambda_1 I_n)^{-1}B, \tag{4}$$

$$\mathcal{M}_2 = (D-\lambda_2 I_m) - C(A-\lambda_2 I_n)^{-1}B, \tag{5}$$

$$\mathcal{N} = I_m + C(A-\lambda_1 I_n)^{-1}(A-\lambda_2 I_n)^{-1}B, \tag{6}$$

and $\gamma \in \mathbb{R}$ and

$$P(\gamma) = \begin{pmatrix} \mathcal{M}_1 & \gamma\mathcal{N} \\ 0 & \mathcal{M}_2 \end{pmatrix}, \qquad p(\gamma) = \sigma_{2m-1}(P(\gamma)).$$

From Lemma 26 of [3] we have the Lemmas 2.1 to 2.4.

Lemma 2.1. *For each $\gamma \in \mathbb{R}$, $\sigma_{2m-1}(P(\gamma))$ is an even function.*

Lemma 2.2. *If $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{N} \in \mathbb{C}^{m\times m}$ and* rank$(\mathcal{N}) \geq 2$ *for $m \geq 2$, then*

$$\lim_{\gamma\to\infty} \sigma_{2m-1}\begin{pmatrix} \mathcal{M}_1 & \gamma\mathcal{N} \\ 0 & \mathcal{M}_2 \end{pmatrix} = 0.$$

Lemma 2.3. *The function $p(\gamma)$ is bounded on $\mathbb{R}$.*

Lemma 2.4. *If for some $\gamma \neq 0$, $p(\gamma) = 0$, then for each $\gamma \in \mathbb{R}$, $p(\gamma) = 0$.*

Now we bring, Lemma 5 of [5].

Lemma 2.5. *Let Ω be an open subset of $\mathbb{R}$ and $F : \Omega \longrightarrow \mathbb{C}^{m\times n}$ be an analytic function on Ω. If the function $\sigma_i(F(t))$ has a positive local maximum (or minimum) at $t_\star \in \Omega$, then there exists a pair of singular vectors $u \in \mathbb{C}^{m\times 1}$, $v \in \mathbb{C}^{n\times 1}$ of $F(t_\star)$ corresponding to $\sigma_i(F(t_\star))$ such that*

$$\mathrm{Re}\left(u^H \frac{dF}{dt}(\gamma_\star)v\right) = 0.$$

Let $0 \neq \gamma_\star \in \mathbb{R}$, and the function $p(\gamma)$ has a local extremum at $\gamma_\star$, then $\sigma_{2m-1}\begin{pmatrix} \mathcal{M}_1 & \gamma_\star\mathcal{N} \\ 0 & \mathcal{M}_2 \end{pmatrix} = \sigma_\star > 0$. If $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^{2m\times 1}$ are the right and left singular vectors associated to $\sigma_\star$ respectively, where $u_1, u_2, v_1, v_2 \in \mathbb{C}^{m\times 1}$, then

$$P(\gamma_*)v = \sigma_\star u, \tag{7}$$

$$P(\gamma_*)^H u = \sigma_\star v, \tag{8}$$

$$\begin{aligned} u_1^H u_1 + u_2^H u_2 &= 1, \\ v_1^H v_1 + v_2^H v_2 &= 1. \end{aligned} \tag{9}$$

By Lemma 2.5,

$$\mathrm{Re}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^H \frac{dP}{d\gamma}(\gamma_\star)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = 0.$$

Also by the definition of $P(\gamma)$ we have

$$\frac{dP}{d\gamma}(\gamma_\star) = \begin{pmatrix} 0 & \mathcal{N} \\ 0 & 0 \end{pmatrix},$$

thus, from two above relations we obtain

$$\mathrm{Re}(u_1^H \mathcal{N} v_2) = 0. \tag{10}$$

Now, by multiplying both sides of (7) from left by $(u_1^H, -u_2^H)$, we can write

$$(u_1^H, -u_2^H)\begin{pmatrix} \mathcal{M}_1 & \gamma_\star\mathcal{N} \\ 0 & \mathcal{M}_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_\star(u_1^H u_1 - u_2^H u_2),$$

therefore

$$\left(u_1^H \mathcal{M}_1, \gamma_\star u_1^H \mathcal{N} - u_2^H \mathcal{M}_2\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_\star(u_1^H u_1 - u_2^H u_2),$$

so

$$u_1^H \mathcal{M}_1 v_1 + \gamma_\star u_1^H \mathcal{N} v_2 - u_2^H \mathcal{M}_2 v_2 = \sigma_\star(u_1^H u_1 - u_2^H u_2). \tag{11}$$

By multiplying (8) from left by $(v_1^H, -v_2^H)$, we have the same relation as

$$v_1^H \mathcal{M}_1^H u_1 - \gamma_\star v_2^H \mathcal{N}^H u_1 - v_2^H \mathcal{M}_2^H u_2 = \sigma_\star(v_1^H v_1 - v_2^H v_2). \tag{12}$$

By taking conjugate transpose from both side (11), we have

$$v_1^H \mathcal{M}_1^H u_1 + \gamma_\star v_2^H \mathcal{N}^H u_1 - v_2^H \mathcal{M}_2^H u_2 = \sigma_\star(u_1^H u_1 - u_2^H u_2). \tag{13}$$

By multiplying relation (12) by −1 and add to relation (13) we have the following relation

$$2\gamma_\star v_2^H \mathcal{N}^H u_1 = -\sigma_\star (v_1^H v_1 - v_2^H v_2) + \sigma_\star (u_1^H u_1 - u_2^H u_2). \tag{14}$$

The right hand side of the above relation is real, and since $\gamma_\star \neq 0$, then $v_2^H \mathcal{N}^H u_1$ is real, so the conjugate of it, $u_1^H \mathcal{N} v_2$ is also real. Thus from (10) we get

$$u_1^H \mathcal{N} v_2 = 0. \tag{15}$$

Now we can provide the following lemmas similar to [2].

Lemma 2.6. *If $\gamma_\star > 0$ is the local extremum of $p(\gamma)$ and $\sigma_\star = p(\gamma_*) > 0$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^{2m\times 1}$, where $u_1, u_2, v_1, v_2 \in \mathbb{C}^{m\times 1}$ are the right and left singular vectors corresponding to $\sigma_\star = p(\gamma_*)$ respectively, then*

$$u_1^H \mathcal{N} v_2 = 0.$$

Lemma 2.7. *If u_1, u_2, v_1 and v_2 are the vectors in the previous Lemma and $U = (u_1, u_2)$ and $V = (v_1, v_2)$ are two matrices in $\mathbb{C}^{m\times 2}$, then*

$$U^H U = V^H V.$$

Proof. We construct the proof similar to the [3]. From relations (14) and (15), we have

$$\sigma_\star (v_1^H v_1 - v_2^\star v_2) = \sigma_\star (u_1^H u_1 - u_2^H u_2).$$

Since $\sigma_\star > 0$, then

$$v_1^H v_1 - v_2^\star v_2 = u_1^H u_1 - u_2^H u_2.$$

If we assume that $\alpha := v_1^H v_1 - v_2^H v_2$, then $\alpha = u_1^H u_1 - u_2^H u_2$. Then by (9) we get

$$2v_1^H v_1 = 1 + \alpha, \quad 2u_1^H u_1 = 1 + \alpha, \quad 2v_2^H v_2 = 1 - \alpha, \quad 2u_2^H u_2 = 1 - \alpha,$$

and so

$$v_1^H v_1 = \frac{1+\alpha}{2} = u_1^H u_1, \tag{16}$$

$$v_2^H v_2 = \frac{1-\alpha}{2} = u_2^H u_2. \tag{17}$$

By multiplying both sides of (7) from left by $(0, u_1^H)$ and both sides of (8) from left by $(v_2^H, 0)$ we have the following equations.

$$(0, u_1^H \mathcal{M}_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_\star u_1^H u_2,$$

and

$$(v_2^H \mathcal{M}_1^H, 0) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sigma_\star v_2^H v_1$$

so that

$$u_1^H \mathcal{M}_2 v_2 = \sigma_\star u_1^H u_2, \tag{18}$$

$$v_2^H \mathcal{M}_1^H u_1 = \sigma_\star v_2^H v_1. \tag{19}$$

By taking conjugate transpose of both sides of (19) and reduce from (18), we obtain

$$u_1^H\mathcal{M}_2v_2-\sigma_\star u_1^Hu_2=u_1^H\mathcal{M}_1v_2-\sigma_\star v_1^Hv_2.$$

By definition of $\mathcal{M}_1$ and $\mathcal{M}_2$ in the relation (4) and (5), we deduce that

$$u_1^H((D-\lambda_2I_m)-C(A-\lambda_2I_n)^{-1}B)v_2-\sigma_\star u_1^Hu_2=u_1^H((D-\lambda_1I_m)-C(A-\lambda_1I_n)^{-1}B)v_2-\sigma_\star v_1^Hv_2.$$

By some computations and by lemma 2.6 we have

$$\sigma_\star u_1^Hu_2=\sigma_\star v_1^Hv_2.$$

Since $\sigma_\star>0$,

$$u_1^Hu_2=v_1^Hv_2, \tag{20}$$

then

$$U^HU=\begin{pmatrix}u_1^H\\u_2^H\end{pmatrix}(u_1,u_2)=\begin{pmatrix}u_1^Hu_1 & u_1^Hu_2\\u_2^Hu_1 & u_2^Hu_2\end{pmatrix},$$

and

$$V^HV=\begin{pmatrix}v_1^H\\v_2^H\end{pmatrix}(v_1,v_2)=\begin{pmatrix}v_1^Hv_1 & v_1^Hv_2\\v_2^Hv_1 & v_2^Hv_2\end{pmatrix},$$

and by (16) and (17) and (20), we have $U^HU=V^HV$. □

The following lemma can be seen in [4].

Lemma 2.8. *Let $q\geq 2$ and $\Gamma_X\in\mathbb{C}^{q\times q}$ and $\lambda_1,\ \lambda_2\in\Lambda(\Gamma_X)$, then*

$$\operatorname{rank}\begin{pmatrix}\Gamma_X-\lambda_1I_q & \gamma I_q\\0 & \Gamma_X-\lambda_2I_q\end{pmatrix}\leq 2q-2,\qquad\forall\gamma\in\mathbb{R}.$$

By Theorem 1.1 from [1] and Theorem 5 from [3] we have the next Theorem.

Theorem 2.9. *Given a matrix partitioned in the following form*

$$\begin{pmatrix}A & B\\C & D\end{pmatrix},$$

with $A\in\mathbb{C}^{n\times n}$ and $D\in\mathbb{C}^{m\times m}$. For each matrix $X\in\mathbb{C}^{m\times m}$, let

$$\Gamma_X=\begin{pmatrix}A & B\\C & X\end{pmatrix},$$

and let us call

$$\rho:=\operatorname{rank}[A,B]+\operatorname{rank}\begin{bmatrix}A\\C\end{bmatrix}-\operatorname{rank}A,$$

$$M:=(I_n-AA^\dagger)B,\qquad N:=C(I_n-A^\dagger A),$$

and

$$P(X):=(I-NN^\dagger)(X-CA^\dagger B)(I-M^\dagger M).$$

Then for each $X\in\mathbb{C}^{m\times m}$, we have

$$\operatorname{rank}\Gamma_X=\rho+\operatorname{rank}P(X).$$

Moreover, for each integer r such that $\rho\leq r<\operatorname{rank}\Gamma_D$,

$$\min\{\|X-D\|:X\in\mathbb{C}^{m\times m},\operatorname{rank}\Gamma_X\leq r\}=\sigma_{s+1}(P(D)),$$

where $s=r-\rho$.

3. Lower bound for minimum of problem

Assume that $\alpha = (A, B, C) \in L_{n\times m}$ and $X \in \mathbb{C}^{m\times m}$ and

$$m(\lambda_i, \Gamma_X) \geq 1, \qquad i = 1, 2.$$

by Lemma 2.8 we have

$$\mathrm{rank}\begin{pmatrix} A-\lambda_1 I_n & B & \gamma I_n & 0 \\ C & X-\lambda_1 I_m & 0 & \gamma I_m \\ 0 & 0 & A-\lambda_2 I_n & B \\ 0 & 0 & C & X-\lambda_2 I_m \end{pmatrix} =$$

$$\mathrm{rank}\begin{pmatrix} A-\lambda_1 I_n & \gamma I_n & B & 0 \\ 0 & A-\lambda_2 I_n & 0 & B \\ C & 0 & X-\lambda_1 I_m & \gamma I_m \\ 0 & C & 0 & X-\lambda_2 I_m \end{pmatrix} \leq 2(n+m)-2.$$

Then we call

$$\mathcal{A}(\gamma) := \begin{pmatrix} A-\lambda_1 I_n & \gamma I_n \\ 0 & A-\lambda_2 I_n \end{pmatrix},$$

$$\mathcal{B} := \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix},$$

$$\mathcal{C} := \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix},$$

$$\mathcal{X}(\gamma) := \begin{pmatrix} X-\lambda_1 I_m & \gamma I_n \\ 0 & X-\lambda_2 I_m \end{pmatrix}.$$

From Theorem 2.9, we have

$$\rho(\gamma) = \mathrm{rank}\begin{bmatrix} \mathcal{A}(\gamma) & \mathcal{B} \end{bmatrix} + \mathrm{rank}\begin{bmatrix} \mathcal{A}(\gamma) \\ \mathcal{C} \end{bmatrix} - \mathrm{rank}\mathcal{A}(\gamma),$$

$$\begin{aligned} &s(\gamma) = 2m + 2n - 2 - \rho(\gamma), \\ &M(\gamma) = \left(I_{2n} - \mathcal{A}(\gamma)\mathcal{A}(\gamma)^{\dagger}\right)\mathcal{B}, \\ &N(\gamma) = \mathcal{C}(I_{2n} - \mathcal{A}(\gamma)^{\dagger}\mathcal{A}(\gamma)), \\ &P(\gamma, X-\lambda_1 I_m, X-\lambda_2 I_m) := (I_{2m} - N(\gamma)N(\gamma)^{\dagger})\left(\mathcal{X}(\gamma) - \mathcal{C}\mathcal{A}(\gamma)^{\dagger}\mathcal{B}\right)(I_{2m} - M(\gamma)^{\dagger}M(\gamma)), \end{aligned} \tag{21}$$

and so

$$\mathrm{rank}\begin{pmatrix} \mathcal{A}(\gamma) & \mathcal{B} \\ \mathcal{C} & \mathcal{X}(\gamma) \end{pmatrix} = \rho(\gamma) + \mathrm{rank}(P(\gamma, X-\lambda_1 I_m, X-\lambda_2 I_m)).$$

Since

$$m(\lambda_i, \Gamma_X) \geq 1, \qquad i = 1, 2,$$

for any $\gamma \in \mathbb{R}$, we have

$$\rho(\gamma) + \mathrm{rank}(P(\gamma, X-\lambda_1 I_m, X-\lambda_2 I_m) \leq 2n + 2m - 2 \Longleftrightarrow$$

$$\begin{aligned} &\mathrm{rank}(P(\gamma, X-\lambda_1 I_m, X-\lambda_2 I_m) \leq 2n + 2m - 2 - \rho(\gamma) = s(\gamma) \\ &\Longrightarrow \sigma_{s(\gamma)+1}(P(\gamma, X-\lambda_1 I_m, X-\lambda_2 I_m) = 0. \end{aligned} \tag{22}$$

Lemma 3.1. *If $\gamma \in \mathbb{R}$ and $X \in \mathbb{C}^{m\times m}$, then*

$$|\sigma_i(P(\gamma, X-\lambda_1 I_m, X-\lambda_2 I_m)) - \sigma_i(P(\gamma, D-\lambda_1 I_m, D-\lambda_2 I_m))| \leq \|X-D\|, \quad (23)$$

for $i = 1, 2, \cdots, 2m$.

Proof. Similar to Lemma 22 of [3] the proof is obtained directly. □

Now by relations (22) and (23) we have

$$\sigma_{s(\gamma)+1}(P(\gamma, D-\lambda_1 I_m, D-\lambda_2 I_m)) \leq \|X-D\|,$$

so

$$\sup_{\gamma\in\mathbb{R}} \sigma_{s(\gamma)+1}(P(\gamma, D-\lambda_1 I_m, D-\lambda_2 I_m)) \leq \min_{\substack{X\in\mathbb{C}^{m\times m}\\ m(\lambda_i,\Gamma_X)\geq 1\\ i=1,2}} \|X-D\|.$$

In continue we assume that λ_1 and λ_2 do not belong to $\Lambda(A)$ and we solve the problem (If one of these numbers be an eigenvalue of A, then $s(\gamma)$ is not equal to $2m-1$).

There are two following cases for m:

- $m > 1$,
- $m = 1$.

The proof of existence a matrix X such that λ_1 and λ_2 are eigenvalues of matrix Γ_X, is similar to the section 3 of [3]. For the cases $m > 1$ and $m = 1$ that $\mathcal{N} = 0$, we introduce a method for constructing matrix X such that λ_1 and λ_2 are eigenvalues of Γ_X and for the case $m = 1$ when $\mathcal{N} \neq 0$, we prove that there is no matrix X.

4. The cases that λ_1 and λ_2 do not belong to $\Lambda(A)$

Firstly we consider $m > 1$ and since the local maximum of $\sigma_{2m-1}(P(\gamma, D-\lambda_1 I_m, D-\lambda_2 I_m))$ happens in $\gamma_\star$, we also consider the following three cases:

- $\gamma_\star \neq 0$,
- $\gamma_\star = 0$,
- $\gamma_\star = \infty$.

4.1. The case $\gamma_\star \neq 0$

By relation (21) we have $s(\gamma) + 1 = 2m - 1$, then we prove the following Theorem.

Theorem 4.1. *If $\alpha = (A, B, C) \in L_{n,m}$ and $D \in \mathbb{C}^{m\times m}$, where $\lambda_1, \lambda_2 \notin \Lambda(A)$, then*

$$\min_{\substack{X\in\mathbb{C}^{m\times m}\\ m(\lambda_i,\Gamma_X)\geq 1\\ i=1,2}} \|X-D\| = \sup_{\gamma\in\mathbb{R}^{\neq 0}} \sigma_{2m-1}(P(\gamma, D-\lambda_1 I_m, D-\lambda_2 I_m)).$$

Proof. It is sufficient to show that

$$\sup_{\gamma\in\mathbb{R}} \sigma_{2m-1}(P(\gamma, D-\lambda_1 I_m, D-\lambda_2 I_m)) \geq \min_{\substack{X\in\mathbb{C}^{m\times m}\\ m(\lambda_i,\Gamma_X)\geq 1\\ i=1,2}} \|X-D\|.$$

Set

$$\mathcal{M}_1 = (D - \lambda_1 I_m) - C(A - \lambda_1 I_n)^{-1}B,$$

$$\mathcal{M}_2 = (D - \lambda_2 I_m) - C(A - \lambda_2 I_n)^{-1}B,$$

$$\mathcal{N} = I_m + C(A - \lambda_1 I_n)^{-1}(A - \lambda_2 I_n)^{-1}B,$$

$$P(\gamma, D - \lambda_1 I_m, D - \lambda_2 I_m) = \begin{pmatrix} \mathcal{M}_1 & \gamma\mathcal{N} \\ 0 & \mathcal{M}_2 \end{pmatrix},$$

$$p(\gamma) = \sigma_{2m-1}(P(\gamma, D - \lambda_1 I_m, D - \lambda_2 I_m)).$$

Let $D_\star$ be the matrix such that $\Gamma_{D_\star} = \begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}$ has two eigenvalues λ_1, λ_2 and

$$\|D - D_\star\| = \max_{\gamma\in\mathbb{R}} p(\gamma),$$

and let the local maximum of $p(\gamma)$ happens in $\gamma_\star > 0$ and $p(\gamma_*, D - \lambda_1 I_m, D - \lambda_2 I_m) = \sigma_\star > 0$. According to Lemma 2.6, we assume that $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^{2m\times 1}$, where $u_1, u_2, v_1, v_2 \in \mathbb{C}^{m\times 1}$ are the right and left singular vectors corresponding to $\sigma_\star = p(\gamma_*)$ respectively and

$$U = (u_1, u_2), V = (v_1, v_2) \in \mathbb{C}^{m\times 2}.$$

We define $\Delta = \sigma_\star UV^\dagger$ and prove that $\|\Delta\| = \sigma_\star$ and λ_1, λ_2 are the eigenvalues of

$$\begin{pmatrix} A & B \\ C & D^* \end{pmatrix},$$

where

$$D_\star = D - \Delta. \tag{24}$$

By Lemma 2.7 we have $V^H V = U^H U$. There is a unitary matrix $W \in \mathbb{C}^{m\times m}$ such that $U = WV$. Hence

$$\|D - D_\star\| = \sigma_\star\|UV^\dagger\| = \sigma_\star\|WVV^\dagger\| = \sigma_\star\|VV^\dagger\| = \sigma_\star.$$

Now we prove that λ_1 and λ_2 are eigenvalues of $\Gamma_{D_\star}$.
From [2], Page 287, equations (31) and (32) we have

$$\Delta v_2 = \sigma_\star u_2, \tag{25}$$

$$u_1^H \Delta = \sigma_\star v_1^H. \tag{26}$$

and from [2], Page 287, since we have $\Delta V = \sigma_\star U$, so

$$D_\star V = DV - \sigma_\star U, \tag{27}$$

so $\text{rank}V^H V = \text{rank}V$ and $\|(v_1, v_2)^H\| = 1$, thus we deduce that $\text{rank}V^H V \geq 1$. Therefore we have the following cases:

- $\text{rank}V = 2$,
- $\text{rank}V = 1, v_2 = 0$
- $\text{rank}V = 1, v_2 \neq 0$

4.1.1. rank$V = 2$

Since rank$V = 2$, so v_1, v_2 are linearly independent. Hence we should find the vectors $w_1, w_2 \in \mathbb{C}^{n\times 1}$ such that

$$\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}\begin{pmatrix} w_2 & w_1 \\ v_2 & v_1 \end{pmatrix} = \begin{pmatrix} w_2 & w_1 \\ v_2 & v_1 \end{pmatrix}\begin{pmatrix} \lambda_1 & -\gamma_\star \\ 0 & \lambda_2 \end{pmatrix}. \tag{28}$$

Because λ_1 and λ_2 are not eigenvalues of matrix A, let us assume that

$$w_2 = -(A - \lambda_1 I_n)^{-1} B v_2,$$
$$w_1 = -(A - \lambda_2 I_n)^{-1} B v_1 + \gamma_\star (A - \lambda_2 I_n)^{-1}(A - \lambda_1 I_n)^{-1} B v_2.$$

We prove that w_1 and w_2 are holding in (28). By (27), we know

$$D_\star v_i = D v_i - \sigma_\star u_i, \qquad i = 1, 2; \tag{29}$$

from the second line (28), we have

$$-C(A - \lambda_1 I_n)^{-1} B v_2 + D_\star v_2 = \lambda_1 v_2,$$
$$-C(A - \lambda_2 I_n)^{-1} B v_1 + \gamma_\star C(A - \lambda_2 I_n)^{-1}(A - \lambda_1 I_n)^{-1} B v_2 + D_\star v_1 = -\gamma_\star v_2 + \lambda_2 v_1,$$

by (29), we find

$$-C(A - \lambda_1 I_n)^{-1} B v_2 + (D - \lambda_1 I_m) v_2 - \sigma_\star u_2 = 0,$$
$$-C(A - \lambda_2 I_n)^{-1} B v_1 + \gamma_\star C(A - \lambda_2 I_n)^{-1}(A - \lambda_1 I_n)^{-1} B v_2 + (D - \lambda_2 I_m) v_1 - \sigma_\star u_1$$
$$= -\gamma_\star v_2.$$

From these relation and by definitions $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{N}$, we have

$$\mathcal{M}_1 v_2 = \sigma_\star u_2, \qquad \mathcal{M}_2 v_1 + \gamma_\star \mathcal{N} v_2 = \sigma_\star u_1,$$

This equation is the section of equation (7) and above relations also is correct, so equation (28) is hold. So w_1 and w_2 satisfy the required conditions in (28). Therefore λ_1 and λ_2 are eigenvalues of $\Gamma_{D_\star}$.

4.1.2. rank$V = 1, v_2 = 0$

From Lemma 27 in [3], we know that $u_2 = 0$ and $u_1 \neq 0$, so it suffices to find vectors $w_1, w_2 \in \mathbb{C}^{1\times n}, w_2 \neq 0$ such that

$$\begin{pmatrix} w_1 & u_1^H \\ w_2 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix} = \begin{pmatrix} \bar{\lambda}_1 & 0 \\ 1 & \bar{\lambda}_2 \end{pmatrix}\begin{pmatrix} w_1 & u_1^H \\ w_2 & 0 \end{pmatrix}. \tag{30}$$

Let

$$w_1 = -u_1^H C(A - \lambda_1 I_n)^{-1}, \qquad w_2 = -u_1^H C(A - \lambda_1 I_n)^{-1}(A - \lambda_2 I_n)^{-1}.$$

Now we prove that w_1 and w_2 are hold in (30). By taking conjugate transpose from both sides of (30) and second line it and definition $\mathcal{N}$, we earn

$$-u_1^H C(A - \lambda_1 I_n)^{-1} B + u_1^H (D_\star - \lambda_1 I_m) = 0,$$
$$u_1^H \mathcal{N} = 0.$$

The second equation is right from Lemma 27 in [3] . In order to prove the first equation, since $v_2 = 0$ and by (26), and definition $\mathcal{M}_1$, we write

$$u_1^H \mathcal{M}_1 = \sigma_\star v_1^H \Longleftrightarrow \mathcal{M}_1^H u_1 = \sigma_\star v_1,$$

This equation is the section of equation (8) and the above relations also correct, so equation (30) is held. In this Case $w_2 \neq 0$, if $w_2 = 0$ then $-u_1^H C A^{-2} = 0$, so $-u_1^H C A^{-2} B = 0$ and $u_1^H [I_m + C A^{-2} B] = u_1^H$ and by definition $\mathcal{N}$ we have $u_1^H \mathcal{N} = u_1^H$, that, it is wrong by Lemma 27 in [3]. So w_1, w_2 satisfy in the condition (30) and λ_1 and λ_2 are eigenvalues of $\Gamma_{D_\star}$.

4.1.3. $\operatorname{rank} V = 1, v_2 \neq 0$

We prove that there are vectors $w_1, w_2 \in \mathbb{C}^{n\times 1}$, $w_1 \neq 0$ such that

$$\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}\begin{pmatrix} w_2 & w_1 \\ v_2 & 0 \end{pmatrix} = \begin{pmatrix} w_2 & w_1 \\ v_2 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}. \tag{31}$$

Let us assume that

$$w_2 = -(A - \lambda_1 I_n)^{-1} B v_2, \qquad w_1 = -(A - \lambda_2 I_n)^{-1}(A - \lambda_1 I_n)^{-1} B v_2.$$

We want to show that w_1 and w_2 are hold in (31). From (29) and the second line of (31), we have

$$-C(A - \lambda_1 I_n)^{-1} B v_2 + (D - \lambda_1 I_m) v_2 - \sigma_\star u_2 = 0,$$

$$-C(A - \lambda_2 I_n)^{-1}(A - \lambda_1 I_n)^{-1} B v_2 = v_2.$$

Now, by replacing $\mathcal{M}_1$ and $\mathcal{N}$ in both above formulas, we have

$$\mathcal{M}_1 v_2 = \sigma_\star u_2, \qquad \mathcal{N} v_2 = 0.$$

These relations are a combination of Lemma 28 in [3] and (8), then (31) is held. In this case $w_1 \neq 0$, if $w_1 = 0$ we have

$$-(A - \lambda_2 I_n)^{-1}(A - \lambda_1 I_n)^{-1} B v_2 = 0,$$

so

$$C(A - \lambda_2 I_n)^{-1}(A - \lambda_1 I_n)^{-1} B v_2 = 0,$$

and

$$[I_m + C(A - \lambda_2 I_n)^{-1}(A - \lambda_1 I_n)^{-1} B] v_2 = v_2.$$

By replacing the matrix $\mathcal{N}$ in the above relation we have

$$\mathcal{N} v_2 = v_2,$$

but this relation is wrong by Lemma 28 in [3]. So w_1, w_2 satisfy the condition (31) and λ_1 and λ_2 are eigenvalues of $\Gamma_{D_\star}$.

4.2. The case $\gamma_\star = 0$

Assume that $\sigma_{2m-1}\begin{pmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{pmatrix} = \sigma_\star > 0$, then two cases happens.

- Case 1: $\sigma_m(\mathcal{M}_1) \geq \sigma_m(\mathcal{M}_2) > 0$,
- Case 2: $\sigma_m(\mathcal{M}_2) \geq \sigma_m(\mathcal{M}_1) > 0$.

From Theorem 3.7 of [4], we have:

Theorem 4.2. *Let $M \in \mathbb{C}^{m\times m}$ and ΔM be a perturbation such that $M - \Delta M$ has two eigenvalues λ_1, λ_2 (or a multiple eigenvalue, $\lambda = \lambda_1 = \lambda_2$). Then we have*

$$\max\{\sigma_m(\mathcal{M}_1), \sigma_m(\mathcal{M}_2), p(\gamma_\star)\} \leqslant \|\Delta M\|_2,$$

and in a more precise way

$$\max\{\sigma_m(\mathcal{M}_1), \sigma_m(\mathcal{M}_2), p(\gamma_\star)\} = \min_{\Delta M} \|\Delta M\|_2.$$

4.2.1. Case 1

Let $\mathcal{M}_1 = U\Sigma V^H$ be the singular value decomposition of $\mathcal{M}_1$ with the smallest singular value σ_m and let u_m and v_m be the corresponding right and left singular vectors of σ_m respectively. It is known that

$$u_m^H \mathcal{N} v_m \neq 0.$$

If $u_m^H \mathcal{N} v_m = 0$, according to the definition of $\mathcal{N}$, we have

$$\begin{aligned} u_m^H \mathcal{N} v_m &= u_m^H (I_m + C(A - \lambda_1 I_n)^{-1}(A - \lambda_2 I_n)^{-1} B) v_m \\ &= u_m^H v_m + u_m^H C(A - \lambda_1 I_n)^{-1}(A - \lambda_2 I_n)^{-1} B v_m \\ &= 0, \end{aligned}$$

so

$$u_m^H I_m v_m = -u_m^H C(A - \lambda_1 I_n)^{-1}(A - \lambda_2 I_n)^{-1} B v_m,$$

since u_m and v_m are right and left singular vectors of $\mathcal{M}_1$ corresponding to the σ_m respectively, since $m > 1$, consequently

$$I_m = -(u_m^H)^{\dagger} u_m^H C(A - \lambda_1 I_n)^{-1}(A - \lambda_2 I_n)^{-1} B v_m (v_m)^{\dagger}$$

and finally

$$I_m = -C(A - \lambda_1 I_n)^{-1}(A - \lambda_2 I_n)^{-1} B,$$

therefore

$$\mathcal{N} = 0,$$

and this is impossible. Thus

$$u_m^H \mathcal{N} v_m \neq 0.$$

Assume that

$$\tilde{U}\tilde{\Sigma}\tilde{V}^H = \tilde{\mathcal{M}}_2 = \mathcal{M}_2 + \frac{(\lambda_2 - \lambda_1)\mathcal{N}}{u_m^H \mathcal{N} v_m} v_m u_m^H \mathcal{N}, \tag{32}$$

is the SVD of $\tilde{\mathcal{M}}_2$. We prove that σ_m is the singular value of $\tilde{\mathcal{M}}_2$.

From $\mathcal{M}_1 v_m = \sigma_m u_m$, we have

$$\begin{aligned} \tilde{\mathcal{M}}_2 v_m &= \mathcal{M}_2 v_m + \tfrac{(\lambda_2 - \lambda_1)\mathcal{N}}{u_m^H \mathcal{N} v_m} v_m u_m^H \mathcal{N} v_m \\ &= \left[(D - \lambda_2 I_m) - C(A - \lambda_2 I_n)^{-1} B\right] v_m + \tfrac{(\lambda_2 - \lambda_1)\mathcal{N}}{u_m^H \mathcal{N} v_m} v_m u_m^H \mathcal{N} v_m \\ &= (D - \lambda_1 I_m) v_m + (\lambda_1 - \lambda_2) I_m v_m - C(A - \lambda_1 I_n)^{-1} B v_m \\ &+ C(A - \lambda_1 I_n)^{-1} B v_m - C(A - \lambda_2 I_n)^{-1} B v_m + (\lambda_2 - \lambda_1)\mathcal{N} v_m \\ &= \mathcal{M}_1 v_m + (\lambda_1 - \lambda_2) v_m + C\left((A - \lambda_1 I_n)^{-1} - (A - \lambda_2 I_n)^{-1}\right) B v_m \\ &+ (\lambda_2 - \lambda_1)\mathcal{N} v_m \\ &= \mathcal{M}_1 v_m + (\lambda_1 - \lambda_2) v_m \\ &+ C[(A - \lambda_1 I_n)^{-1}(A - \lambda_2 I_n)^{-1}(A - \lambda_2 I_n) \\ &- (A - \lambda_2 I_n)^{-1}(A - \lambda_1 I_n)^{-1}(A - \lambda_1 I_n)] B v_m + (\lambda_2 - \lambda_1)\mathcal{N} v_m \\ &= \mathcal{M}_1 v_m + (\lambda_1 - \lambda_2) v_m \\ &+ C\left[(A - \lambda_1 I_n)^{-1}(A - \lambda_2 I_n)^{-1}(A - \lambda_2 I_n - A + \lambda_1 I_n)\right] B v_m \\ &+ (\lambda_2 - \lambda_1)\mathcal{N} v_m. \end{aligned} \tag{33}$$

Since we can permute the product of two matrices $(A-\lambda_2 I_n)^{-1}$ and $(A-\lambda_1 I_n)^{-1}$, so the relation (33) is equal to $\sigma_m u_m$. Similarly, we can also prove that

$$\tilde{\mathcal{M}}_2^H u_m = \sigma_m v_m,$$

and this shows that σ_m is the singular value of $\tilde{\mathcal{M}}_2$. If $\tilde{\sigma}_m$ is the smallest singular value of $\tilde{\mathcal{M}}_2$, then $\sigma_m \geq \tilde{\sigma}_m$. Now we define the matrix $D - D_\star$ as

$$D - D_\star = (u_m, \tilde{u}_m)\begin{pmatrix} \sigma_m & 0 \\ 0 & \tilde{\sigma}_m \end{pmatrix}(v_m, \tilde{v}_m)^H, \tag{34}$$

where $\tilde{u}_m$ and $\tilde{v}_m$ are the right and left singular vectors corresponding to $\tilde{\sigma}_m$.

We prove that $\|D - D_\star\| = \sigma_m$ and λ_1 and λ_2 are eigenvalues of $\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}$. For proving $\|D - D_\star\| = \sigma_m$, we know that σ_m is one of the singular values of $\tilde{\mathcal{M}}_2$ and u_m, v_m are the corresponding singular vectors. Assume that $\tilde{u}_m$ and $\tilde{v}_m$ are the corresponding singular vectors of $\tilde{\sigma}_m$ for $\tilde{\mathcal{M}}_2$, then we have $u_m^H \tilde{u}_m = v_m^H \tilde{v}_m$ (since U and V are unitary matrices, there are $\tilde{u}_m$ and $\tilde{v}_m$). Therefore, from the definition of matrix $D - D_\star$, we have

$$\|D - D_\star\| = \max(\sigma_m, \tilde{\sigma}_m) = \sigma_m.$$

Now we prove that λ_1 and λ_2 are eigenvalues of $\Gamma_{D_\star}$.

By the definition of the matrix $D - D_\star$ in (34), we have

$$(D - D_\star)(v_m, \tilde{v}_m) = (u_m, \tilde{u}_m)\begin{pmatrix} \sigma_m & 0 \\ 0 & \tilde{\sigma}_m \end{pmatrix}. \tag{35}$$

If we apply SVD for the matrix $\mathcal{M}_1$, we see that

$$Dv_m = \sigma_m u_m + C(A - \lambda_1 I_n)^{-1} B v_m + \lambda_1 v_m \tag{36}$$

and from (32)

$$D\tilde{v}_m = \tilde{\sigma}_m \tilde{u}_m - \frac{(\lambda_2 - \lambda_1)\mathcal{N}}{u_m^H \mathcal{N} v_m} v_m u_m^H \mathcal{N} \tilde{v}_m + C(A - \lambda_2 I_n)^{-1} B \tilde{v}_m + \lambda_2 \tilde{v}_m. \tag{37}$$

Considering the relations (35), (36) and (37) we obtain the following equations:

$$D_\star v_m = C(A - \lambda_1 I_n)^{-1} B v_m + \lambda_1 v_m, \tag{38}$$

$$D_\star \tilde{v}_m = \lambda_2 \tilde{v}_m + C(A - \lambda_2 I_n)^{-1} B \tilde{v}_m - \frac{(\lambda_2 - \lambda_1)\mathcal{N}}{u_m^H \mathcal{N} v_m} v_m u_m^H \mathcal{N} \tilde{v}_m. \tag{39}$$

To prove that λ_1 and λ_2 are eigenvalues of $\Gamma_{D_\star}$, we need to find the vectors $w_1, w_2 \in \mathbb{C}^{n\times 1}$ such that:

$$\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}\begin{pmatrix} w_1 & w_2 \\ v_m & \tilde{v}_m \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \\ v_m & \tilde{v}_m \end{pmatrix}\begin{pmatrix} \lambda_1 & -\frac{(\lambda_2-\lambda_1)}{u_m^H \mathcal{N} v_m} u_m^H \mathcal{N} \tilde{v}_m \\ 0 & \lambda_2 \end{pmatrix},$$

then from the above equation and relations (38) and (39), if we define two vectors w_1 and w_2 as follows

$$w_1 = (\lambda_1 I_n - A)^{-1} B v_m,$$
$$w_2 = (\lambda_2 I_n - A)^{-1} B \tilde{v}_m + \frac{(\lambda_2-\lambda_1)(\lambda_2 I_n - A)^{-1}(\lambda_1 I_n - A)^{-1}}{u_m^H \mathcal{N} v_m} v_m u_m^H \mathcal{N} \tilde{v}_m,$$

so λ_1 and λ_2 are eigenvalues of $\Gamma_{D_\star}$.

4.2.2. Case 2

As the Case 1 we have the following results.

Let $\mathcal{M}_2 = U\Sigma V^H$ be the singular value decomposition of $\mathcal{M}_2$ with the smallest singular value σ_m and let u_m and v_m be the corresponding right and left singular vectors of σ_m respectively. Because

$$u_m^H \mathcal{N} v_m \neq 0,$$

assume that

$$\tilde{U}\tilde{\Sigma}\tilde{V}^H = \tilde{\mathcal{M}}_1 = \mathcal{M}_1 + \frac{(\lambda_1 - \lambda_2)\mathcal{N}}{u_m^H \mathcal{N} v_m} v_m u_m^H \mathcal{N},$$

is the SVD of $\tilde{\mathcal{M}}_1$. If $\tilde{\sigma}_m$ is the smallest singular value of $\tilde{\mathcal{M}}_1$, then $\sigma_m \geq \tilde{\sigma}_m$. Now we define the matrix $D - D_\star$ as

$$D - D_\star = (u_m, \tilde{u}_m)\begin{pmatrix} \sigma_m & 0 \\ 0 & \tilde{\sigma}_m \end{pmatrix}(v_m, \tilde{v}_m)^H,$$

where $\tilde{u}_m$ and $\tilde{v}_m$ are the right and left singular vectors corresponding to the $\tilde{\sigma}_m$. Then $\|D - D_\star\| = \sigma_m$ and λ_1 and λ_2 are eigenvalues of $\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}$.

4.3. The case $\gamma_\star = \infty$

The case $\gamma_\star = \infty$ is very similar to the case of $\gamma_\star = \infty$ in [3], especially all of $\mathcal{M}$ must be replaced by $\mathcal{M}_1$ or $\mathcal{M}_2$. By definition $\Delta = \sigma_\star UV^\dagger$ and $D_\star = D - \Delta$, for proving that λ_1 and λ_2 are eigenvalues of the matrix $\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}$, we will separate two cases: $v_2 \neq 0$ and $v_2 = 0$. By sections 4.1.2 and 4.1.3 of this paper and section 5.3 in [3], it is very obvious that λ_1 and λ_2 are eigenvalues of matrix $\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}$, and $\|D - D_\star\| = \sigma_\star$.

4.4. The case $m = 1$

When $m = 1$, then from Theorem 4.2 we have

$$\min_{\substack{X \in \mathbb{C}^{1\times 1} \\ m(\lambda_i, \Gamma_X) \geq 1 \\ i=1,2}} \|X - D\| = \begin{cases} \infty, & if \quad \mathcal{N} \neq 0, \\ \max(|\mathcal{M}_1|, |\mathcal{M}_2|), & if \quad \mathcal{N} = 0. \end{cases}$$

So, when $\mathcal{N} \neq 0$,

$$\min_{\substack{X \in \mathbb{C}^{1\times 1} \\ m(\lambda_i, \Gamma_X) \geq 1 \\ i=1,2}} \|X - D\| = \infty,$$

i.e. there is no matrix X such that λ_1 and λ_2 be eigenvalues of matrix Γ_X.

When $\mathcal{N} = 0$, it suffices finding matrix $D_\star$ so that $|D - D_\star| = \max(|\mathcal{M}_1|, |\mathcal{M}_2|)$.

If $|\mathcal{M}_1| > |\mathcal{M}_2|$, we assume that $D_\star = \lambda_1 + C(A - \lambda_1 I_n)^{-1}B$, so

$$\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}\begin{pmatrix} -(A-\lambda_1 I_n)^{-1}B & -(A-\lambda_1 I_n)^{-1}B - (A-\lambda_2 I_n)^{-1}(A-\lambda_1 I_n)^{-1}B \\ 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} -(A-\lambda_1 I_n)^{-1}B & -(A-\lambda_1 I_n)^{-1}B - (A-\lambda_2 I_n)^{-1}(A-\lambda_1 I_n)^{-1}B \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_1 - \lambda_2 + 1 \\ 0 & \lambda_2 \end{pmatrix}.$$

Since $\mathcal{N} = 0$, the above relation is hold and the two vectors

$$\begin{pmatrix} -(A-\lambda_1 I_n)^{-1}B \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -(A-\lambda_1 I_n)^{-1}B - (A-\lambda_2 I_n)^{-1}(A-\lambda_1 I_n)^{-1}B \\ 1 \end{pmatrix}$$

are linearly independent. Therefore λ_1 and λ_2 are eigenvalues of $\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}$ and $|D - D_\star| = \max(|\mathcal{M}_1|, |\mathcal{M}_2|) = |\mathcal{M}_1|$.

If $|\mathcal{M}_2| > |\mathcal{M}_1|$, the argument is similar.

5. Numerical examples

In this section for the given four complex matrices $A \in \mathbb{C}^{n\times n}$, B, C and $D \in \mathbb{C}^{m\times m}$ and for the given two complex numbers λ_1 and λ_2, we find the nearest matrix $D_\star$ to matrix D from the set of matrices $X \in \mathbb{C}^{m\times m}$, such that the matrix

$$\Gamma_{D_\star} = \begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}$$

has two prescribed eigenvalues λ_1 and λ_2.

Example 5.1. *Let*

$$\Gamma_D = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = \begin{pmatrix} 5 & 1 & 3 & 9 & 5 \\ 7 & 1 & 7 & 1 & 5 \\ 5 & 0 & 6 & 1 & 8 \\ 9 & 4 & 7 & 6 & 4 \\ 2 & 4 & 9 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 6 & 5 & 4 \\ 7 & 2 & 6 \\ 5 & 0 & 3 \\ 3 & 7 & 7 \\ 1 & 2 & 3 \end{pmatrix},$$

$$C = \begin{pmatrix} 6 & 0 & 2 & 4 & 7 \\ 7 & 3 & 1 & 8 & 3 \\ 4 & 4 & 8 & 3 & 8 \end{pmatrix}, \qquad D = \begin{pmatrix} 7 & 7 & 6 \\ 3 & 9 & 4 \\ 2 & 3 & 8 \end{pmatrix}.$$

The set of eigenvalues of the matrix Γ_D *is equal to*

$$\{35.636798, 10.011182, -3.102620, -3.102620,$$

$$-0.101225, -0.101225, 3.664355, 2.095356\}.$$

We find the nearest submatrix $D_\star$ *to the matrix* D *such that the matrix* $\Gamma_{D_\star}$ *have two eigenvalues* 7 *and* 13.

The following results can be obtained for the problem. By subsection 4.1 we have

$$\gamma_\star = 5.1888125, \qquad \sigma_\star = 5.022005.$$

So by (24) *we have*

$$D - D_\star = \begin{pmatrix} -1.210412 & 2.781868 & 2.815794 \\ -0.318987 & -4.095686 & 1.307919 \\ 1.315925 & 0.813897 & -3.511032 \end{pmatrix},$$

$$\|D - D_\star\| = 5.022007,$$

and the set of eigenvalues of the matrix $\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}$ *is equal to*

$$\{35.7292282, 13.0000000, -2.6318610 + 3.04709591i,$$

$$-2.6318610 - 3.04709591i, -2.1080252, 7.0000000,$$

$$2.7298252 + 0.5791467i, 2.7298252 - 0.5791467i\}.$$

The behavior of $\sigma_{2m-1}(P(\gamma, D - \lambda_1 I_m, D - \lambda_2 I_m))$ *is shown in Figure 1.*

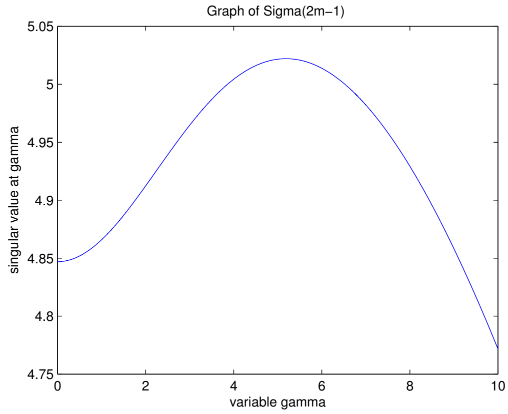

Figure 1:

Example 5.2. *For the matrix Γ_D in the previous example we find the nearest submatrix $D_\star$ to matrix D such that the matrix $\Gamma_{D_\star}$ have two eigenvalues 17 and 25.*

The following results can be obtained for the problem: By subsection 4.2 we have

$$\gamma_\star = 0, \qquad p(\gamma_\star) = 15.57954326,$$

Then by Case1 of subsection 4.2.1 and (32) respectively we have

$$\mathcal{M}_2 = \begin{pmatrix} 7.46107768 & 25.93855446 & 31.92616887 \\ 36.57155300 & 9.44580424 & 38.35263256 \\ 35.77308604 & 25.88561251 & 16.39206578 \end{pmatrix},$$

$$\tilde{\mathcal{M}}_1 = \begin{pmatrix} -27.54916572 & -3.10094508 & -10.77183205 \\ -22.62452901 & -26.98246631 & -17.21071049 \\ -15.63095423 & -4.70638090 & -30.49094956 \end{pmatrix},$$

by (34) we compute

$$D - D_\star = \begin{pmatrix} -9.61734431 & 7.57837149 & 5.59282542 \\ 2.24397165 & -12.17200253 & 5.484390350 \\ 5.51621342 & 7.88783877 & -11.19195324 \end{pmatrix},$$

so

$$\|D - D_\star\| = 18.6543550,$$

and the set of eigenvalue of the matrix $\begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}$ is equal to

$$\{34.97824316, 25.00000000, 17.00000000, -3.30416657 + 3.26137482i,$$

$$-3.30416657 - 3.26137482i, -1.95815033, 2.68333536, 6.88620506\}.$$

The behavior of $\sigma_{2m-1}(P(\gamma, D - \lambda_1 I_m, D - \lambda_2 I_m))$ is shown in Figure 2.

In Figure 2 we can see that the value $\max(\sigma_{2m-1}(P(\gamma, D - \lambda_1 I_m, D - \lambda_2 I_m)))$ must be 15.57954326. It is shown that $\|D - D_\star\| = \sigma_m(\mathcal{M}_2) > \max(\sigma_{2m-1}(P(\gamma, D - \lambda_1 I_m, D - \lambda_2 I_m)))$, that is right by Theorem 4.2.

Remark 5.3. *If λ_1, λ_2 are eigenvalues of matrix A, then in the similar method we can provide some the proofs, and instead of $(A - \lambda_1 I)^{-1}, (A - \lambda_2 I)^{-1}$ we must replace $(A - \lambda_1 I)^\dagger, (A - \lambda_2 I)^\dagger$.*

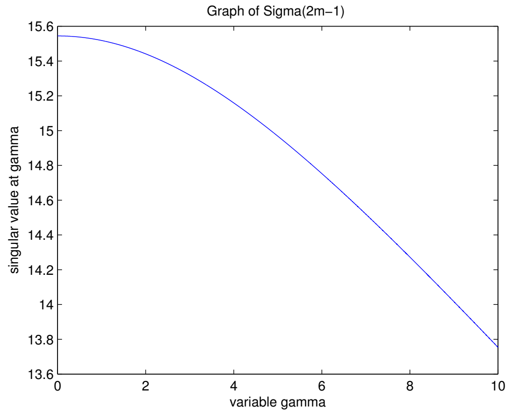

Figure 2:

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