The Moore-Penrose Inverse of Accretive Operators with Application to Quadratic Operator Pencils

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Abstract. We establish some relationships between an m-accretive operator and its Moore-Penorse inverse. We derive some perturbation result the Moore-Penorse inverse of a maximal accretive operator. As an application we give a factorization theorem for a quadratic pencil of accretive operators. Also, we study a result of existence, uniqueness, and maximal regularity of the strict solution for complete abstract second order differential equation. Illustrative examples are also given.

1. introduction

The Moore-Penrose inverse of a linear operator in Hilbert space is a useful generalization of the ordinary inverse. This generalized inverse is an important theoretical and practical tool in algebra and analysis (Markov chains, singular differential and difference equations, iterative methods...), see [4, 31]. In particular, in [11, 12] (and the references therein) the perturbation analysis for the Moore-Penrose inverse of closed operators has been considered. Also, the expression of the generalized inverse of the perturbed operator has been investigated. In the paper [19] necessary and sufficient conditions for the cone nonnegativity of Moore–Penrose inverses of unbounded Gram operators are derived. These conditions include statements on acuteness of certain closed convex cones in infinite-dimensional real Hilbert spaces. In [3] a complete description of the left quotient and the right quotient of two bounded operators operators is given via the Moore-Penrose inverse. The objectives of this paper are to derive the properties of m-accretive operators via the Moore-Penrose inverse and establish some interesting results, especially for the perturbation analysis of Moore-Penrose inverses as well as of maximal accretive operators. Recall that a linear operator T with domain $\mathcal{D}(T)$ in a complex Hilbert space $\mathcal{H}$, is called accretive if its numerical range $W(T)$ is contained in the closed right half-plane, and if further has no proper accretive extensions in $\mathcal{H}$, it called maximal accretive, m-accretive for short. In particular, every m-accretive operator is accretive and closed densely defined, its adjoint is also m-accretive (cf. [16], p. 279). This class is of particular interest and related to the semi-group theory in follwing sens: an operator T is m-accretive if and only if $-T$ generates a strongly continuous contraction semigroup (Theorem of Lumer-Phillips).

In this paper, we explore the following two questions, the first what can be said about the m-accretivity of the Moore-Penrose of an m-accretive operator and conversely? the second concern the perturbation problem: Let T be m-accretive operator with a bounded Moore-Penrose inverse, what condition on the operator S can guarantee that $T + S$ is m-accretive and its Moore-Penrose inverse exists and it has the simplest expression?

In this work, we give a certain answers to the mentioned problems. This paper is organized as follows: In section 2, we establish some relationships between an m-accretive operator and its Moore-Penorse inverse. In section 3, we consider the perturbation for the m-accretive operator and its Moore-Penrose inverse. We prove that under weaker conditions that considered perturbation does not change the null space and the range space, consequently the perturbed operator is a closed EP operator. Utilizing this result, we study a class of a quadratic operator pencil $Q(\lambda) = \lambda^2 I - 2\lambda T - S, (\lambda \in \mathbb{C})$, where the coefficients of which are accretive operators. Our aim, in Section 4, is to investigate a canonical factorization like $(\lambda I - Z_1)(\lambda I - Z_2)$ for of such pencils based on the perturbation theory of accretive operators. We also obtain a criterion in order that the linear factors, into which the pencil splits, generates an holomorphic semi-group of contraction operators. As an illustration, in section 5, we establish a theorem of existence, uniqueness, and maximal regularity of the strict solution of an abstract second order evolutionary equations generated by such pencils.

2. Accretive operator and the Moore-Penrose inverse

Throughout this paper $\mathcal{H}$ is a complex Hilbert space with inner product $< \cdot, \cdot >$ and norm $\| \cdot \|$. For a closed linear operator T on $\mathcal{H}$ we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\mathcal{N}(T)$, $\sigma(T)$ and $\rho(T)$ the domain, the range, the kernel, the spectrum and the resolvent set of T, respectively. The space of bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. For two possibly unbounded linear operators T, S on $\mathcal{H}$ their product TS is defined on its natural domain $\mathcal{D}(TS) := \{x \in \mathcal{D}(S) : Sx \in \mathcal{D}(T)\}$ and their sum $T + S$ is defined in $\mathcal{D}(T+S) = \mathcal{D}(T) \cap \mathcal{D}(S)$. An inclusion $T \subseteq S$ denotes inclusion of graphs, i.e., it means that S extends T. A possibly unbounded operator T on $\mathcal{H}$ commutes with a bounded operator $S \in \mathcal{B}(\mathcal{H})$ if the graph of T is $S \times S$-invariant, or equivalently if $ST \subseteq TS$.

Recall that a linear operator T with domain $\mathcal{D}(T)$ in $\mathcal{H}$ is said to be accretive if

$$\text{Re} < Tx, x > \geq 0 \qquad \text{for all } x \in \mathcal{D}(T)$$

or, equivalently if

$$\|(\lambda + T)x\| \geq \lambda \|x\| \qquad \text{for all } x \in \mathcal{D}(T) \text{ and } \lambda > 0.$$

An accretive operator T is called *maximal accretive*, or *m-accretive* for short, if T has no proper accretive extensions in $\mathcal{H}$. The following conditions are equivalent:

1. T is *m*-accretive.
2.
$$(\lambda + T)^{-1} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad \left\|(\lambda + T)^{-1}\right\| \leq \frac{1}{\lambda} \text{ for } \lambda > 0.$$

3. T is accretive densely defined and $\mathcal{R}(\lambda + T) = \mathcal{H}$ for some (and hence for every) $\lambda > 0$;
4. T is accretive densely defined and closed, and T^* is accretive;
5. $-T$ generates contractive one-parameter semigroup $\mathcal{T}(t) = \exp(-tT), t \geq 0$.

In particular, a bounded accretive operator is m-accretive.

The numerical range of a linear operator $T : \mathcal{D}(T) \to \mathcal{H}$ it is defined by

$$W(T) := \{< Tx, x >: \quad x \in \mathcal{D}(T), \quad \text{with } \|x\| = 1\}, \tag{1}$$

It is well-known that $W(T)$ is a convex set of the complex plane (the Toeplitz-Hausdorff theorem), and in general is neither open nor closed, even for a closed operator T. Clearly, an operator T is accretive when $W(T)$ is contained in the closed right half-plane

$$W(T) \subset \overline{\mathbb{C}_+} := \{z \in \mathbb{C} : \text{Re}(z) \geq 0\}.$$

Further, if T is m-accretive operator then $W(T)$ has the so-called spectral inclusion property

$$\sigma(T) \subset \overline{W(T)}. \tag{2}$$

Recall that a linear operator T in $\mathcal{H}$ is called *sectorial* with vertex $z = 0$ and semi-angle $\omega \in [0, \pi/2)$, or *ω-accretive* for short, if its numerical range is contained in a closed sector with semi-angle ω,

$$W(T) \subset \overline{\mathcal{S}(\omega)} := \{z \in \mathbb{C} : |\arg z| \le \omega\} \tag{3}$$

or, equivalently,

$$|\mathrm{Im} < Tx, x >| \le \tan \omega \, \mathrm{Re} < Tx, x > \qquad \text{for all } x \in \mathcal{D}(T).$$

An ω-accretive operator T is called m-ω-accretive, if it is *m*-accretive. We have T is m-ω-accretive if and only if the operators $e^{\pm i\theta}T$ is m-accretive for $\theta = \frac{\pi}{2} - \omega, 0 < \omega \le \pi/2$. The resolvent set of an m-ω-accretive operator T contains the set $\mathbb{C} \setminus \overline{\mathcal{S}(\omega)}$ and

$$\|(T - \lambda I)^{-1}\| \le \frac{1}{\mathrm{dist}\,(\lambda, \mathcal{S}(\omega))}, \quad \lambda \in \mathbb{C} \setminus \overline{\mathcal{S}(\omega)}.$$

In particular, m-$\pi/2$-accretivity means m-accretivity. A 0-accretive operator is symmetric. An operator is positive if and only if it is m-0-accretive.

It is known that the C_0-semigroup $\mathcal{T}(t) = \exp(-tT), t \ge 0$, has contractive and holomorphic continuation into the sector $\overline{\mathcal{S}(\pi/2\omega)}$ if and only if the generator T is m-ω-accretive, see [16, Theorem V-3.35].

Recall that for bounded operator T, we have

$$\mathrm{Re}(T) = \frac{1}{2}(T + T^*) \qquad \text{and} \qquad \mathrm{Im}(T) = \frac{1}{2i}(T - T^*),$$

where $\mathrm{Re}(T)$ and $\mathrm{Im}(T)$ are self-adjoint operators and called it the real and imaginary parts of T, with

$$T = \mathrm{Re}(T) + i\mathrm{Im}(T),$$

Such decomposition is unique and called the cartesian decomposition of T. In this case, T is accretive if $\mathrm{Re}(T)$ is a nonnegative operator.

The spectral radius and the numerical radius of a bounded operator T are defined, respectively, by

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

and

$$w(T) = \sup_{\|x\|=1} |< Tx, x >|.$$

The next results is a generalization of [1, Lemma 2. and Theorem 2.] from matrices to bounded operators.

Lemma 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ such that $\mathrm{Re}(T) \ge \delta I$, for some $\delta > 0$. Denote by $\mathcal{S} = \mathrm{Im}(T)(\mathrm{Re}(T))^{-1}$. Then there exists $\lambda_0 \in \sigma(\mathcal{S}), |\lambda_0| = r(\mathcal{S}) = w(\mathcal{S}) = \|\mathcal{S}\|$ such that T is a sectorial operator with semiangle $\omega = \arctan |\lambda_0|$ and*

$$|\lambda_0| \le \sqrt{\frac{\|T\|^2}{\delta^2} - 1}. \tag{4}$$

Proof. By assumption,

$$< \mathrm{Re}(T)x, x > \ge \delta \|x\|^2$$

for all $x \in \mathcal{H}$. Hence we get

$$|< \mathrm{Im}(T)x, x >| \le \|\mathrm{Im}(T)\| \, \|x\|^2 \le \frac{1}{\delta} \|\mathrm{Im}(T)\| < \mathrm{Re}(T)x, x > .$$

Thus, T is sectorial operator with a vertex at the origin with a semiangle ω. In particular, for $\mathrm{Re}(T)^{-\frac{1}{2}}x$, we have

$$\left| < \mathrm{Im}(T)\mathrm{Re}(T)^{-\frac{1}{2}}x, \mathrm{Re}(T)^{-\frac{1}{2}}x > \right| \le \left\| \mathrm{Re}(T)^{-\frac{1}{2}}\mathrm{Im}(T)\mathrm{Re}(T)^{-\frac{1}{2}} \right\| \|x\|^2$$

this implies that $tan(\omega) = \|\mathcal{T}\|$, where $\mathcal{T} = \mathrm{Re}(T)^{-\frac{1}{2}}\mathrm{Im}(T)\mathrm{Re}(T)^{-\frac{1}{2}}$. Since $\mathcal{T}$ is a self-adjoint operator we have $r(\mathcal{T}) = w(\mathcal{T}) = \|\mathcal{T}\|$. Thus assert the existence of a $\lambda_0 \in \sigma(\mathcal{T})$ such that $|\lambda_0| = r(\mathcal{T}) = w(\mathcal{T}) = \|\mathcal{T}\|$. Since

$$\mathcal{T} = \mathrm{Re}(T)^{-\frac{1}{2}}\mathcal{S}\mathrm{Re}(T)^{\frac{1}{2}} = \mathrm{Re}(T)^{\frac{1}{2}}\mathcal{S}^*\mathrm{Re}(T)^{-\frac{1}{2}},$$

the self-adjoint operators $\mathcal{T}$ and $\mathcal{S}$ have the same spectrum (which is real) and hence have the same closure of the numerical range. This shows that $\lambda_0 \in \sigma(\mathcal{S})$, with $|\lambda_0| = w(\mathcal{T}) = w(\mathcal{S})$ and hence $\omega = arctan |\lambda_0|$.

Now, assume that λ_0 is an eigenvalue of $\mathcal{T}$ with $|\lambda_0| = \|\mathcal{T}\|$, there exists $u \in \mathcal{H}$ with $\|u\| = 1$ and $\mathcal{T}u = \lambda_0 u$. We have

$$\mathrm{Re}(T)^{-\frac{1}{2}}T\mathrm{Re}(T)^{-\frac{1}{2}} = (I + i\mathcal{T})$$

and

$$\left\| \mathrm{Re}(T)^{-\frac{1}{2}}T\mathrm{Re}(T)^{-\frac{1}{2}} \right\|^2 \ge \|(I + i\mathcal{T})u\|^2 = 1 + \|\mathcal{T}\|^2 = 1 + |\lambda_0|^2,$$

which implies that

$$|\lambda_0|^2 \le \frac{\|T\|^2}{\delta^2} - 1.$$

Now we consider the general case. Let $\varepsilon > 0$. It follows from the spectral theorem that there exists a self-adjoint bounded operator P such that $\|P\| \le \varepsilon$ and the operator $\mathcal{T} + P$ has an eigenvalue such that the modulus equals $\|\mathcal{T} + P\|$. As above, we take the operator $I + i(\mathcal{T} + P)$ instead of $(I + i\mathcal{T})$, we get

$$\frac{\|T\|^2}{\delta^2} + \varepsilon^2 \ge 1 + |\lambda_0|^2.$$

Letting $\varepsilon \longrightarrow 0$, we obtain (4). $\quad\square$

If we assume, in Lemma 2.1, that the numerical range of T is closed, then $\lambda_0 \in W(T)$, but the extreme points of the numerical range are in the point spectrum, so λ_0 must be an eigenvalue of $\mathcal{T}$. So, we have the following

Corollary 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ with closed numerical range such that $\mathrm{Re}(T) \ge \delta I$, for some $\delta > 0$. Denote by $\mathcal{S} = \mathrm{Im}(T)(\mathrm{Re}(T))^{-1}$. Then there exists $\lambda_0 \in \sigma_p(\mathcal{S})$, $|\lambda_0| = w(\mathcal{S})$ such that T is a sectorial operator with semiangle $\omega = arctan |\lambda_0|$ and λ_0 verifies (4).*

Remark 2.3. 1. *Since $\mathrm{Re}(T)$ is strongly nonnegative, we know that*

$$\sigma(\mathcal{S}) \subseteq \{\frac{\beta}{\alpha} : \quad \alpha \in \overline{W(\mathrm{Re}(T))}, \beta \in \overline{W(\mathrm{Im}(T))}\}.$$

So $\lambda_0 = \dfrac{\beta_0}{\alpha_0}$ for $\alpha_0 \in \overline{W(\mathrm{Re}(T))}$ and $\beta_0 \in \overline{W(\mathrm{Im}(T))}$.

2. *As we can see from the proof above that T can be represented as*

$$T = \mathrm{Re}(T)^{\frac{1}{2}}(I + i\mathcal{T})\mathrm{Re}(T)^{\frac{1}{2}}$$

with $\mathcal{T} = \mathrm{Re}(T)^{-\frac{1}{2}}\mathrm{Im}(T)\mathrm{Re}(T)^{-\frac{1}{2}}$ and $\tan(\omega) = \|\mathcal{T}\|$. This is exactly the representation given in [16, Theorem VI-3.2]. In our case the selfadjoint operator is uniquely determined.

Next, in order to give some new results about accretive operator by using the Moore-Penrose inverse, let recall the definition of this generalized inverse for a closed densely defined operator.

Definition 2.4. *[4] Let T be a closed densely defined on $\mathcal{H}$. Then there exists a unique closed densely defined operator $T^\dagger$, with domain $\mathcal{D}(T^\dagger) = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$ such that*

$$TT^\dagger T = T \quad on \ \mathcal{D}(T), \qquad T^\dagger TT^\dagger = T^\dagger \quad on \ \mathcal{D}(T^\dagger),$$

$$TT^\dagger = P_{\overline{\mathcal{R}(T)}} \quad on \ \mathcal{D}(T^\dagger), \qquad T^\dagger T = P_{\mathcal{N}(T)^\perp} \quad on \ \mathcal{D}(T),$$

with $P_{\mathcal{M}}$ denotes the orthogonal projection onto a closed subspace $\mathcal{M}$.

This unique operator $T^\dagger$ is called the Moore-Penrose inverse of T. (or the Maximal Tseng generalized Inverse in the terminology of [4]). Clearly,

1. $\mathcal{N}(T^\dagger) = \mathcal{R}(T)^\perp$,
2. $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp \cap \mathcal{D}(T)$.

As a consequence of the closed graph theorem $T^\dagger$ is bounded if and only if $\mathcal{R}(T)$ is closed in $\mathcal{H}$, see [4].

Now, if we assume that T is an m-accretive operator, then

$$\mathcal{N}(T) = \mathcal{N}(T^*) \quad \text{and} \quad \mathcal{N}(T) \subseteq \mathcal{D}(T) \cap \mathcal{D}(T^*). \tag{5}$$

Thus $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^*)}$ and $\mathcal{H} = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T)$. Consequently, the operator T is written in a matrix form with respect to mutually orthogonal subspaces decomposition as follows

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(T)} \\ \mathcal{N}(T) \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\mathcal{R}(T)} \\ \mathcal{N}(T) \end{bmatrix};$$

with T_1 is an operator on $\overline{\mathcal{R}(T)} \cap \mathcal{D}(T)$ is injective with dense range in $\overline{\mathcal{R}(T)}$. Also, its Moore-Penrose inverse is given by

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\mathcal{R}(T)} \\ \mathcal{N}(T) \end{bmatrix},$$

with T_1^{-1} from $\mathcal{R}(T)$ to $\overline{\mathcal{R}(T)} \cap \mathcal{D}(T)$ is closed operator densely defined on $\overline{\mathcal{R}(T)}$ and $\mathcal{N}(T^\dagger) = \mathcal{N}(T) = \mathcal{N}(T^*)$. Further, $\mathcal{R}(T)$ is closed if and only if T_1^{-1} is bounded from $\mathcal{R}(T)$ to $\mathcal{R}(T) \cap \mathcal{D}(T)$.

In the next result, we redefine a bounded sectorial operator with semiangle $< \pi/2$ via the Moore-Penrose inverse.

Lemma 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$. T is a sectorial operator with semiangle ω, $0 \leq \omega < \pi/2$, if and only if the following two conditions are fulfilled:*

1. $\text{Re}(T) \geq 0$.
2. $\overline{\mathcal{R}(T)} \subset \overline{\mathcal{R}(\text{Re}(T))}$.

In this case, $\omega = arctan |\lambda_0|$ for some $\lambda_0 \in \sigma(\mathcal{S})$, $|\lambda_0| = w(\mathcal{S})$ where $\mathcal{S} = \text{Im}(T)(\text{Re}(T))^\dagger$.

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be a sectorial operator with semiangle ω. Therefore $\text{Re}(T)$ is a nonnegative and $\mathcal{N}(T) = \mathcal{N}(T^*) = \mathcal{N}(\text{Re}(T))$. It follows that $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(\text{Re}(T))}$.

Conversely, let the two conditions of the theorem hold. Then $\mathcal{N}(T) = \mathcal{N}(T^*) \subset \mathcal{N}(\text{Re}(T))$, indeed, if $Tx = 0$ for some $x \neq 0$, then $\text{Re} < Tx, x >=< \text{Re}(T)x, x >= 0$. Consequently, $\text{Re}(T)x = 0$ and $T^*x = 2\text{Re}(T)x - Tx = 0$. Since $\mathcal{H} = \mathcal{N}(\text{Re}(T)) \oplus \overline{\mathcal{R}(\text{Re}(T))}$, we get

$$\overline{\mathcal{R}(\text{Re}(T))} \subset \overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^*)}$$

with $\overline{\mathcal{R}(T)}$ reduces T. By (2) we obtain $\overline{\mathcal{R}(\text{Re}(T))} = \overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^*)}$. It follows that the subspace $\overline{\mathcal{R}(T)}$ reduces also the operator $\text{Re}(T)$. Moreover, the restriction of $\text{Re}(T)$ to $\overline{\mathcal{R}(T)}$ is strongly nonnegative. So by Lemma 2.1, the restriction of T to $\overline{\mathcal{R}(T)}$ is sectorial operator with semiangle $\omega = arctan |\lambda_0|$ such that $\lambda_0 \in \sigma(\mathcal{S}_{|\overline{\mathcal{R}(T)}})$ and $|\lambda_0| = \sup_{\|x\|=1} \left| < \mathcal{S}_{|\overline{\mathcal{R}(T)}} x, x > \right|$, where $\mathcal{S}_{|\overline{\mathcal{R}(T)}} = \text{Im}(T)(\text{Re}(T)_{|\overline{\mathcal{R}(T)}})^{-1} P_{|\overline{\mathcal{R}(T)}}$. In this case $\mathcal{S} = \text{Im}(T)(\text{Re}(T))^\dagger$. $\square$

Now, we consider an unbounded operator T.

Proposition 2.6. *If T is m-accretive operator, then $T^\dagger$ is m-accretive.*

Proof. By assumption,
$$\text{Re} < Tx, x > \geq 0 \quad \text{for all } x \in \mathcal{D}(T) \cap \mathcal{N}(T)^\perp = \mathcal{R}(T^\dagger).$$

Hence
$$\text{Re} < y, T^\dagger y > \geq 0 \quad \text{for all } y \in \mathcal{R}(T).$$

Now let $x \in \mathcal{D}(T^\dagger) = \mathcal{R}(T) \oplus \mathcal{N}(T)$, then $x = x_1 + x_2$, with $x_1 \in \mathcal{R}(T)$ and $x_2 \in \mathcal{N}(T)$. Therefore,
$$\text{Re} < x, T^\dagger x > = \text{Re} < x_1, T^\dagger x_1 > \geq 0,$$

which implies
$$\text{Re} < x, T^\dagger x > \geq 0 \quad \text{for all } x \in \mathcal{D}(T^\dagger).$$

Since $T^\dagger$ is closed densely defined and $(T^\dagger)^*$ is accretive, it follows that $T^\dagger$ is m-accretive. □

It well known that by [4, Theorem 2; p 341], $T^{\dagger\dagger} = T$, this yields to

Corollary 2.7. *$T^\dagger$ is m-accretive operator if and only if T is m-accretive.*

Corollary 2.8. *T is m-accretive operator with closed range if and only if $T^\dagger$ is bounded and accretive.*

Corollary 2.9. *If T is m-accretive operator with closed range, then T is an EP (Equal Projections) operator, that is, $T^\dagger$ bounded and $TT^\dagger = T^\dagger T$ on $\mathcal{D}(T)$.*

Proposition 2.10. *Let T be an accretive bounded operator. If $W(T) \subseteq \overline{\mathbb{D}}$ and $W(T^\dagger) \subseteq \overline{\mathbb{D}}$, then T is unitary on $\mathcal{R}(T)$.*

Proof. It well known by [9, Theorem 1.3-1] that the numerical radius is equivalent to the usual operator norm;
$$w(T) \leq \|T\| \leq 2w(T).$$

Hence, the assumption that $w(T^\dagger) \leq 1$ implies that $T^\dagger$ is bounded. Thus $\mathcal{R}(T)$ is closed. Since T is m-accretive, then $\mathcal{R}(T) = \mathcal{R}(T^*) = \mathcal{N}(T)^\perp$. We consider the restriction of T from $\mathcal{R}(T)$ into itself. Since $T^\dagger_{|\mathcal{R}(T)} = (T_{|\mathcal{R}(T)})^{-1}$, $w((T_{\mathcal{R}(T)})^{-1}) = w(T^\dagger_{|\mathcal{R}(T)}) = w(T^\dagger) \leq 1$. Combining this with $w(T_{|\mathcal{R}(T)}) = w(T) \leq 1$ and applying [30, Corollary 1.] to $(T_{|\mathcal{R}(T)})^{-1}$ and $T_{|\mathcal{R}(T)}$, we conclude that $T_{|\mathcal{R}(T)}$ is unitary on $\mathcal{R}(T)$. □

3. A perturbation results

In the following, we shall consider the perturbation of Moore-Penrose inverse of m-accretive operators. This gives an example of EP-operators, see [14, Theorem 3.12]. The part about the Moore-Penrose invertibility of the operator $T + S$ appears to be new.

Theorem 3.1. *Let T is m-accretive operator and S is bounded and accretive. We have*

1. *$T + S$ is m-accretive.*
2. *If $\mathcal{R}(T)$ is closed, $\mathcal{R}(S) \subseteq \mathcal{R}(T)$ and $\left\|T^\dagger S\right\| < 1$. Then*
   - *$\mathcal{R}(T + S) = \mathcal{R}(T)$ is closed and $\mathcal{N}(T + S) = \mathcal{N}(T)$.*

- *T + S is an EP operator, and*

$$(T + S)^\dagger = (I + T^\dagger S)^{-1}T^\dagger = T^\dagger(I + ST^\dagger)^{-1}.$$

In particular,

$$T^\dagger = (T + S)^\dagger(I + ST^\dagger),$$

and

$$\left\|(T + S)^\dagger - T^\dagger\right\| \leq \frac{\|S\| \left\|T^\dagger\right\|^2}{1 - \left\|T^\dagger S\right\|}.$$

Proof. (1) Clearly, the operator $T + S$, with $\mathcal{D}(T + S) = \mathcal{D}(T)$, is densely defined, closed and accretive. Since also its adjoint operator $(T + S)^* = T^* + S^*$ is accretive, the operator $T + S$ is m-accretive.

(2) If $\mathcal{R}(S) \subseteq \mathcal{R}(T)$, then it is obvious that $\mathcal{R}(T+S) \subseteq \mathcal{R}(T)$ and $TT^\dagger S = P_{\mathcal{R}(T)}S = S$. Conversely, let $y \in \mathcal{R}(T)$, so $y = Tx$ for some $x \in \mathcal{D}(T)$. The condition $\left\|T^\dagger S\right\| < 1$ implies that $(I + T^\dagger S)^{-1}$ exists and bounded. Hence, there exists $u \in \mathcal{D}(T)$ such that $x = (I + T^\dagger S)u$. This shows that $y = T(I + T^\dagger S)u = Tu + Su \in \mathcal{R}(T + S)$. Hence $R(T) \subseteq \mathcal{R}(T + S)$. Consequently, $\mathcal{R}(T + S) = \mathcal{R}(T)$ is closed.

Since T and $T + S$ are m-accretive with closed ranges, then

$$\mathcal{N}(T + S) = \mathcal{R}(T + S)^\perp = \mathcal{R}(T)^\perp = \mathcal{N}(T).$$

Now we prove that $(T + S)^\dagger = (I + T^\dagger S)^{-1}T^\dagger$. Since, $\mathcal{R}(T + S)$ is closed and $\mathcal{N}(T + S) = \mathcal{R}(T + S)$, by Corollary 2.9, it follows that $T + S$ is an EP operator.

Put $\mathbf{T} = (I + T^\dagger S)^{-1}T^\dagger$. We show that $\mathbf{T}$ satisfies all the axioms of the Definition 2.4. First let us remark that, since $(I + T^\dagger S)^{-1}$ is invertible, $\mathcal{D}(\mathbf{T}) = \mathcal{D}(T^\dagger) = \mathcal{R}(T) \oplus \mathcal{N}(T)$, $\mathcal{N}(\mathbf{T}) = \mathcal{N}(T^\dagger) = \mathcal{R}(T)^\perp = \mathcal{N}(T + S)$.

Let $v \in \mathcal{R}(\mathbf{T})$, then there exists $u \in \mathcal{D}(\mathbf{T})$ such that $v = \mathbf{T}u = (I + T^\dagger S)^{-1}T^\dagger u$. Hence $T^\dagger u = v + T^\dagger Sv \in \mathcal{R}(T) \cap \mathcal{D}(T)$. So $v = T^\dagger u - T^\dagger Sv \in \mathcal{D}(T)$.

Now for $v \in \mathcal{D}(T)$,

$$\begin{aligned}
\mathbf{T}(T + S)v &= (I + T^\dagger S)^{-1}T^\dagger(T + S)v \\
&= (I + T^\dagger S)^{-1}T^\dagger(T + TT^\dagger S)v \quad \text{(since } S = TT^\dagger S\text{)} \\
&= (I + T^\dagger S)^{-1}T^\dagger T(I + T^\dagger S)v \\
&= (I + T^\dagger S)^{-1}P_{\mathcal{N}(T)^\perp}(I + T^\dagger S)v \\
&= (I + T^\dagger S)^{-1}P_{\mathcal{R}(T)}(I + T^\dagger S)v \\
&= (I + T^\dagger S)^{-1}(I + T^\dagger S)v \\
&= v = P_{\mathcal{R}(T)}v \\
&= P_{\mathcal{R}(T+S)}v \\
&= P_{\mathcal{N}(T+S)^\perp}v.
\end{aligned}$$

and for $u \in \mathcal{D}(\mathbf{T})$,

$$\begin{aligned}
(T + S)\mathbf{T}u &= (T + S)(I + T^\dagger S)^{-1}T^\dagger u \\
&= (T + TT^\dagger S)(I + T^\dagger S)^{-1}T^\dagger u \quad \text{(since } S = TT^\dagger S\text{)} \\
&= T(I + T^\dagger S)(I + T^\dagger S)^{-1}T^\dagger u \\
&= TT^\dagger u = P_{\mathcal{R}(T)}u \\
&= P_{\mathcal{R}(T+S)}u.
\end{aligned}$$

The uniqueness of $(T + S)^\dagger$ follows from Definition 2.4.

Since $\mathcal{R}(S) \subseteq \mathcal{R}(T)$, by Neumann series, we have

$$(I + T^\dagger S)^{-1} T^\dagger = \sum_{n=0}^{\infty} (-T^\dagger S)^n T^\dagger = \sum_{n=0}^{\infty} T^\dagger (-ST^\dagger)^n = T^\dagger (I + ST^\dagger)^{-1}. \tag{6}$$

For the last inequality, we can see that

$$\begin{aligned}
(T + S)^\dagger - T^\dagger &= (I + ST^\dagger)^{-1} T^\dagger - (I + ST^\dagger)(I + ST^\dagger)^{-1} T^\dagger \\
&= [I - (I + ST^\dagger)](I + ST^\dagger)^{-1} T^\dagger \\
&= (-ST^\dagger)(I + ST^\dagger)^{-1} T^\dagger.
\end{aligned}$$

Hence we get the desired inequality. $\quad\square$

Similarly, we have

Theorem 3.2. *Let T is m-accretive operator and S is bounded and accretive, then the Theorem 3.1 hold true, if $\mathcal{R}(T)$ is closed, $\mathcal{N}(T) \subseteq \mathcal{N}(S)$ and $\left\| ST^\dagger \right\| < 1$.*

Proof. (1) Since the operator $T + S$ is m-accretive, its adjoint operator $(T + S)^* = T^* + S^*$ is also m-accretive.
　(2) If $\mathcal{R}(T^*)$ is closed and $\mathcal{N}(T) \subseteq \mathcal{N}(S)$, then it is obvious that the last inclusion gives $\mathcal{R}(S^*) \subseteq \mathcal{R}(T^*)$. Also, the condition $\left\| ST^\dagger \right\| < 1$ implies $\left\| (ST^\dagger)^* \right\| = \left\| (T^*)^\dagger S^* \right\| < 1$, and conversely. Hence by Theorem 3.1, $\mathcal{R}(T + S) = \mathcal{R}(T^* + S^*) = \mathcal{R}(T^*) = \mathcal{R}(T)$ is closed and $\mathcal{N}(T + S) = \mathcal{N}(T^* + S^*) = \mathcal{N}(T^*) = \mathcal{N}(T)$. Now, we proceed as in the proof of Theorem 3.1. $\quad\square$

Remark 3.3. *Recall that the reduced minimum modulus of a non-zero operator T is defined by*

$$\gamma(T) = \inf\{\|Tx\| : x \in N(T)^\perp \cap \mathcal{D}(T), \|x\| = 1\}.$$

If $T = 0$ then we take $\gamma(T) = \infty$. Note that (see [16]), $\mathcal{R}(T)$ is closed if an only if $\gamma(T) > 0$. In that case, $\gamma(T) = \dfrac{1}{\left\| T^\dagger \right\|}$, where $T^\dagger$ is the Moore-Penrose inverse of T. Let us remark that if we assume that $\|S\| < \dfrac{1}{\gamma(T)}$ instead the condition $\left\| T^\dagger S \right\| < 1$, then the Theorem 3.1 hold true.

Proposition 3.4. *Let T be an accretive such that T^2 is m-accretive. Then*

(i) T is m-accretive. Further, if T is θ-accretive with $\theta < \pi/4$, then T^2 is m-2θ-accretive.

(ii) If $\mathcal{R}(T)$ is closed, then $\mathcal{R}(T^2)$ is closed and $\gamma(T^2) \geq \dfrac{\gamma(T)^2}{2}$.

Proof. (i) Since T is an accretive operator, by [13, Theorem 1.2], we have ,

$$\|Tx\|^2 \leq \nu \|x\|^2 + \frac{1}{\nu} \left\| T^2 x \right\|^2, \tag{7}$$

for all $x \in \mathcal{D}(T^2)$ and an arbitrary $\nu > 0$. Choosing $\nu > 0$ so large that $\dfrac{1}{\nu} < 1$, we obtain T is T^2-bounded with lower bound < 1. Then $T^2 + T$ with domain $\mathcal{D}(T^2)$ is m-accretive. Now, let us remark that

$$(\frac{1}{4}I + T^2 + T)x = (\frac{1}{2}I + T)^2 x$$

for all $x \in \mathcal{D}(T^2)$. Since the operator on the left-hand side is invertible, then $(\frac{1}{2}I + T)^2$ is invertible, so $\frac{1}{2}I + T$ is also invertible. It follows that T is m-accretive. Now, we applied [6, Theorem 4.].

(ii) By the Landau-Kolmogorov inequality, [18, Theorem.], applied to T, we have

$$\|Tx\|^2 \leq 2 \left\|T^2x\right\| \|x\|,$$

for all $x \in \mathcal{N}(T^2)^\perp \cap \mathcal{D}(T^2)$. It follows that

$$\|x\|^2 \gamma(T)^2 \leq \|Tx\|^2 \leq 2 \left\|T^2x\right\| \|x\|,$$

and hence

$$\left\|T^2x\right\| \geq \frac{\gamma(T)^2}{2} \|x\|,$$

for all $x \in \mathcal{N}(T^2)^\perp \cap \mathcal{D}(T^2)$. Now by the definition of $\gamma(T^2)$ we obtain $\gamma(T^2) \geq \frac{\gamma(T)^2}{2}$. $\quad\square$

By Proposition 3.4, Theorem 3.1 and Theorem 3.2,

Corollary 3.5. *Let T^2 be m-accretive, T accretive with closed range and S is bounded and accretive. If $\mathcal{R}(S) \subseteq \mathcal{R}(T)$ and $\left\|(T^\dagger)^2 S\right\| < 1$. (or $\mathcal{N}(T) \subseteq \mathcal{N}(S)$ and $\left\|S(T^2)^\dagger\right\| < 1$). Then*

- *$\mathcal{R}(T^2 + S) = \mathcal{R}(T^2) = \mathcal{R}(T)$ is closed and $\mathcal{N}(T^2 + S) = \mathcal{N}(T^2) = \mathcal{N}(T)$.*

- *$\mathcal{H} = \mathcal{R}(T^2 + S) \oplus \mathcal{N}(T^2 + S)$.*

- *$T^2 + S$ is an EP operator, and*

$$(T^2 + S)^\dagger = (I + (T^\dagger)^2 S)^{-1}(T^\dagger)^2 = (T^\dagger)^2(I + S(T^\dagger)^2)^{-1}.$$

If further, T is injective, then T, T^2 and $T^2 + S$ are invertible, and

$$(T^2 + S)^{-1} = (I + T^{-2}S)^{-1}T^{-2} = T^{-2}(I + ST^{-2})^{-1}.$$

Proof. Since T^2 is m-accretive, then T is also m-accretive by Proposition 3.4. Hence $\mathcal{N}(T^2) = \mathcal{N}(T)$ and $\mathcal{R}(T^2) = \mathcal{R}(T)$. By [3, Lemma 5.5], we have $(T^\dagger)^2 = (T^2)^\dagger$. Now the result is obtained by Theorem 3.1 and Theorem 3.2. $\quad\square$

Remark 3.6.
- *By Remark 3.3, the Corollary 4.2 hold true if we assume $\left\|T^\dagger S\right\| < \gamma(T)$ instead $\left\|(T^\dagger)^2 S\right\| < 1$.*

- *Ôta showed in [27, Theorem 2.1] that, if T is closed and an accretive such that there is a positive integer n with $\mathcal{D}(T^n)$ is dense in $\mathcal{H}$ and $\mathcal{R}(T^n) \subset \mathcal{D}(T)$, then T is bounded . In particular, for a closed and accretive operator T, if $\mathcal{R}(T)$ is contained in $\mathcal{D}(T)$, or in $\mathcal{D}(T^*)$, then T is automatically bounded, see also [27, Theorem 3.3].*

- *In general, if T^2 is m-accretive; then T fails to be accretive. Take $T = i\dfrac{d}{dx}$ on $L^2(\mathbb{R})$. The operator T has its spectrum on both sides of the origin. But $T^2 = -\dfrac{d^2}{dx^2}$ is a nonegative selfadjoint operator.*

4. An application to quadratic operator pencil

Consider in the Hilbert space $\mathcal{H}$ the following quadratic operator pencil

$$Q(\lambda) = \lambda^2 I - 2\lambda T - S, \tag{8}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter and the two operators T and S with domain $\mathcal{D}(T)$ and $\mathcal{D}(S)$, respectively.

One of the approaches to study the spectral properties of quadratic operator pencil (8) consists of the reducing them to a first order system in a suitable space, [7, 28]. However, as it was pointed out in [32],

this way may be unpractical in the situation when the space is difficult to construct or it is complicated for applications. In addition, as it was mentioned in [33–35], the direct treatment of higher order problems allows to get more general results. An useful approach to address the study and computation of the spectral structure of quadratic operators pencil is through the use of factorization. This method was developed by Krein and Langer [15], for a quadratic pencils of self-adjoint operators and by Langer [20], for self-adjoint polynomial operator pencils, see also [8, 10, 20]. The main idea of this approach consist to factoring them and studying the spectral properties of the factors. Of particular interest is the separation of spectral values of Q_n between the spectra of the roots. Such separation may be complicated, even in the case of eigenvalues, see [29] and references therein. The problem is of great importance in spectral theory of such general operators. Moreover, its understanding is crucial in the study of performance properties of many systems.

The purpose of this section is to extend some earlier factorization results essentially those given in [10, 15, 22, 24] for the self-adjoint quadratic operators case, to (8) based on the perturbation theory of accretive operators together with the uniquely determined fractional powers of the maximal accretive operators. We also obtain a criterion in order that the linear factors, into which the pencil splits, generates a holomorphic semi-group of contraction operators.

We mention that if T is m-accretive, then for each $\alpha \in (0, 1)$ the fractional powers T^α, $0 < \alpha < 1$, are defined by the following Balakrishnan formula, see [2],

$$T^\alpha x = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{\alpha-1} T(\lambda + T)^{-1} x dt,$$

for all $x \in \mathcal{D}(T)$. The operators T^α are m-$(\alpha\pi)/2$-accretive and, if $\alpha \in (0, 1/2)$, then $\mathcal{D}(T^\alpha) = \mathcal{D}(T^{*\alpha})$. It was proved in [17, Theorem 5.1] that, if T is m-accretive, then $\mathcal{D}(T^{1/2}) \cap \mathcal{D}(T^{*1/2})$ is a core of both $T^{1/2}$ and $T^{*1/2}$ and the real part $\mathrm{Re} T^{1/2} := (T^{1/2} + T^{*1/2})/2$ defined on $\mathcal{D}(T^{1/2}) \cap \mathcal{D}(T^{*1/2})$ is a selfadjoint operator. Further, by [17, Corollary 2],

$$\mathcal{D}(T) = \mathcal{D}(T^*) \implies \mathcal{D}(T^{1/2}) = \mathcal{D}(T^{*1/2}) = \mathcal{D}(T_R^{1/2}) = \mathcal{D}[\phi], \tag{9}$$

where ϕ is the closed form associated with the sectorial operator T via the first representation theorem [16, Sect. VI.2.1] and T_R is the non-negative selfadjoint operator associated with the real part of ϕ given by $\mathrm{Re}\,\phi := (\phi + \phi^*)/2$, see [2, 16, 17, 23, 25, 26].

Our result of this section read as follows

Theorem 4.1. *Let T^2 be m-accretive, T is accretive and S is accretive bounded operator. Then, we have*

1. *The operator $\Delta = T^2 + S$ with domain $\mathcal{D}(T^2)$ is m-accretive.*

2. *Δ admits a fractional powers Δ^α m-$(\alpha\pi/2)$-accretive for each $0 < \alpha < 1$.*

3. *$Z_1 = T + \Delta^{\frac{1}{2}}$ and $Z_2 = T - \Delta^{\frac{1}{2}}$ with domain $\mathcal{D}(T) \cap \mathcal{D}(\Delta^{\frac{1}{2}})$ are T^2-bounded with lower bound < 1 and closable operators. Further, Z_1 is accretive densely defined.*
   If further, $\mathcal{D}(\Delta^{\frac{1}{2}}) \subset \mathcal{D}(T)$, then

4. *Z_1 is m-$\pi/4$-accretive and for any $\varepsilon > 0$, there exists $r > 0$, such that $-Z_2 + r$ is m-$(\pi/4 + \varepsilon)$-accretive. In particular, $-Z_1$ and $Z_2 - r$ generates holomorphic C_0-semigroup of contraction operators $\mathcal{T}_1(z)$ and $\mathcal{T}_2(z)$ of angle $\frac{\pi}{4}$ and $\pi/4 - \varepsilon$, respectively.*

5. *The spectra of Z_1 and Z_2 are separated,*
   $$\sigma(Z_1) \cap \sigma(Z_2) = \emptyset.$$

6. *If $T(\mathcal{D}(T^2)) \subset \mathcal{D}(T^2)$ and $\Delta^{\frac{1}{2}}(\mathcal{D}(T^2)) \subset \mathcal{D}(T^2)$, then Q take the following form,*

$$Q(\lambda)x = \frac{1}{2}(\lambda I - Z_1)(\lambda I - Z_2)x + \frac{1}{2}(\lambda I - Z_2)(\lambda I - Z_1)x, \tag{10}$$

for all $x \in \mathcal{D}(T^2)$.

In particular, if $TS = ST$ on $\mathcal{D}(T^2)$, then Q admits the following canonical factorization

$$Q(\lambda)x = (\lambda I - Z_1)(\lambda I - Z_2)x = (\lambda I - Z_2)(\lambda I - Z_1)x, \tag{11}$$

for all $x \in \mathcal{D}(T^2)$.

Proof. (1) An immediate consequence of Theorem 3.1-(1).

(2) Δ admits fractional powers Δ^α m-$(\alpha\pi/2)$-accretive for each $0 < \alpha < 1$, see [2, 17].

(3) By (2); Δ admits unique root $\Delta^{\frac{1}{2}}$ m-$(\pi/4)$-accretive operator with $\mathcal{D}(T^2)$ is a core of $\Delta^{\frac{1}{2}}$. So we define the following operators

$$Z_1 = T + \Delta^{\frac{1}{2}}$$

and

$$Z_2 = T - \Delta^{\frac{1}{2}}$$

with domain $\mathcal{D}(T) \cap \mathcal{D}(\Delta^{\frac{1}{2}})$. Both of Z_1 and Z_2 are densely defined on $\mathcal{H}$ with numerical range is not the whole complex plane, it follows that Z_1 and Z_2 are closable operators. Now, we prove that Z_1 and Z_2 are T^2-bounded with lower bound < 1.

By Proposition 3.4, T is also m-accretive. Hence, by [13, Theorem 1.2], we have for an arbitrary $\rho_1 > 0$ and $\rho_2 > 0$,

$$\|Tx\|^2 \leq \rho_1 \|x\|^2 + \frac{1}{\rho_1} \|T^2x\|^2, \tag{12}$$

and

$$\left\|\Delta^{\frac{1}{2}}x\right\|^2 \leq \rho_2 \|x\|^2 + \frac{1}{\rho_2} \|\Delta x\|^2, \tag{13}$$

for all $x \in \mathcal{D}(T^2)$ (cf. by [25, Chap. 2, Theorem 6.10]).

By (12) and (13), it follows that

$$\|Z_i x\|^2 \leq 2\|Tx\|^2 + 2\left\|\Lambda^{\frac{1}{2}}x\right\|^2$$

$$\leq 2(\rho_1 \|x\|^2 + \frac{1}{\rho_1}\left\|T^2x\right\|^2) + 2(\rho_2 \|x\|^2 + \frac{1}{\rho_2}\|\Lambda x\|^2)$$

$$\leq 2(\rho_1 \|x\|^2 + \frac{1}{\rho_1}\left\|T^2x\right\|^2) + 2\rho_2 \|x\|^2 + \frac{4}{\rho_2}(\left\|T^2x\right\|^2 + \|Sx\|^2)$$

$$\leq 2(\rho_1 + \rho_2 + \frac{2\|S\|^2}{\rho_2})\|x\|^2 + 2(\frac{1}{\rho_1} + \frac{2}{\rho_2})\left\|T^2x\right\|^2$$

$$\leq \nu_1 \|x\|^2 + \nu_2 \left\|T^2x\right\|^2,$$

for some $\nu_1, \nu_2 > 0$, $i = 1, 2$ and all $x \in \mathcal{D}(A^2)$. Since ρ_1 and ρ_2 are arbitrary, we can choose $\nu_2 < 1$.

(4) Now assume that $\mathcal{D}(\Delta^{\frac{1}{2}}) \subset \mathcal{D}(T)$. It follows that

$$\|Tx\| \leq a\|x\| + b\left\|\Delta^{\frac{1}{2}}x\right\| \tag{14}$$

for all $x \in \mathcal{D}(\Delta^{\frac{1}{2}})$ and for some nonnegative constants a and b. By (13), we obtain

$$\|Tx\|^2 \leq 2a(1 + \rho_2)\|x\|^2 + \frac{2b}{\rho_2}\|\Delta x\|^2,$$

for all $x \in \mathcal{D}(\Delta)$ and an arbitrary $\rho_2 > 0$. Thus

$$\left\|T(t + \Delta^{\frac{1}{2}})^{-1}x\right\|^2 \leq 2a(1 + \rho_2)\left\|(t + \Delta^{\frac{1}{2}})^{-1}x\right\|^2 + \frac{2b}{\rho_2}\left\|\Delta(t + \Delta^{\frac{1}{2}})^{-1}x\right\|^2,$$

for all $x \in \mathcal{H}$.

Hence

$$\left\| T(t + \Delta^{\frac{1}{2}})^{-1} \right\|^2 \leq \frac{2a}{t^2}(1 + \rho_2) + \frac{2b}{\rho_2} \left\| \Delta(t + \Delta^{\frac{1}{2}})^{-1} \right\|^2.$$

Letting t to $+\infty$, we assert that

$$M = \sup_{t>0} \left\| T(t + \Delta^{\frac{1}{2}})^{-1} \right\| < \frac{2b}{\rho_2^2}.$$

(cf. [36, Proposition 2.12]). Since ρ_2 is arbitrary, we can choose it such that $\frac{2b}{\rho_2^2} < 1$. Since T is m-accretive and $\Delta^{\frac{1}{2}}$ is m-$(\pi/4)$-accretive, then Z_1 is m-accretive. By [16, Theorem IX-1.24], the factor $-Z_1$ generates holomorphic C_0-semigroup $\mathcal{T}_1(z)$ of angle $\frac{\pi}{4}$.

On the other hand, since $\Delta^{\frac{1}{2}}$ is m-$(\pi/4)$-accretive and $-T$ satisfy (14), by [16, Theorem IX-2.4], for any $\varepsilon > 0$, there exist nonnegative constants r and s such that $a, b < s$ and $(Z_2 - r)$ is the generator of holomorphic C_0-semigroup $\mathcal{T}_2(z)$ of angle $\frac{\pi}{2} - \frac{\pi}{4} - \varepsilon$. This implies that $-Z_2 + r$ is m-ψ-accretive with $\psi = \frac{\pi}{4} + \varepsilon$.

(5) It follows from the item (4), $W(Z_1)$ is contained in the right half complex plan and $W(Z_2)$ in the let side with a non zero distance between their closure.

(6) We have $\mathcal{D}(T^2) \subset \mathcal{D}(Z_1) = \mathcal{D}(Z_2) = \mathcal{D}(\Delta^{\frac{1}{2}}) \subset \mathcal{D}(T)$.

The fact that $T(\mathcal{D}(T^2)) \subset \mathcal{D}(T^2)$ and $\Delta^{\frac{1}{2}}(\mathcal{D}(T^2)) \subset \mathcal{D}(T^2)$, we have $\mathcal{D}(T^2) \subset \mathcal{D}(T\Delta^{\frac{1}{2}})$, $\mathcal{D}(T^2) \subset \mathcal{D}(\Delta^{\frac{1}{2}}T)$ and $\mathcal{D}(T^2) \subset \mathcal{D}(Z_1^2)$. Now, by items (1), (2) and (3), we can easily verify that

$$Z_1^2 x - TZ_1 x - Z_1 Tx - Sx = 0,$$

for all $x \in \mathcal{D}(T^2)$, hence on $\mathcal{D}(T^2)$, we have

$$\begin{aligned} Q(\lambda) &= Q(\lambda) - (Z_1^2 - TZ_1 - Z_1 T - S) \\ &= \lambda^2 I - 2\lambda T - S - Z_1^2 + TZ_1 + Z_1 T + S \\ &= \lambda^2 I - Z_1^2 - T(\lambda - Z_1) - (\lambda - Z_1)T \\ &= \frac{1}{2}(\lambda - Z_1)(\lambda I + Z_1 - 2T) + \frac{1}{2}(\lambda I + Z_1 - 2T)(\lambda - Z_1) \\ &= \frac{1}{2}(\lambda I - Z_1)(\lambda I - Z_2) + \frac{1}{2}(\lambda I - Z_2)(\lambda I - Z_1). \end{aligned}$$

This gives the form (10). Now, if $TS = ST$ on $\mathcal{D}(T^2)$, then $\Delta T = T\Delta$. Thus $\Delta^{\frac{1}{2}}$ commutes with T on $\mathcal{D}(T^2)$, wich implies that (11). $\square$

Now the fact that $\mathcal{N}(\Delta^{\frac{1}{2}}) = \mathcal{N}(\Delta)$ and $\overline{\mathcal{R}(\Delta^{\frac{1}{2}})} = \overline{\mathcal{R}(\Delta)}$, by Corollary 3.5 and Theorem 3.2, we have

Corollary 4.2. *Let T^2 be m-accretive, T accretive with closed range and S is bounded and accretive. If $\mathcal{R}(S) \subseteq \mathcal{R}(T)$ and $\left\| (T^\dagger)^2 S \right\| < 1$ (or $\mathcal{N}(T) \subseteq \mathcal{N}(S)$ and $\left\| S(T^2)^\dagger \right\| < 1$). Then*

- $\mathcal{R}(Z_1) = \mathcal{R}(Z_2) = \mathcal{R}(\Delta^{\frac{1}{2}}) = \mathcal{R}(T)$ *is closed and* $\mathcal{N}(Z_1) = \mathcal{N}(Z_2) = \mathcal{N}(\Delta^{\frac{1}{2}}) = \mathcal{N}(T)$.

- $\mathcal{H} = \mathcal{R}(\Delta^{\frac{1}{2}}) \oplus \mathcal{N}(\Delta^{\frac{1}{2}})$.

- Z_1, Z_2 *and* $\Delta^{\frac{1}{2}}$ *are EP operators.*

- *If further, T is injective, then Z_1, Z_2 and $\Delta^{\frac{1}{2}}$ are invertible.*

An immediate consequence of this corollary, the operator Z_1 and Z_1 are written in a matrices form with respect to mutually orthogonal subspaces decomposition as follows

$$Z_1 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix};$$

and

$$Z_2 = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix};$$

with A and B on $\mathcal{R}(T) \cap \mathcal{D}(T)$ are injective operators with closed range. In this case, if $AB = BA$, we have

$$Q(\lambda) = \begin{bmatrix} (\lambda - A)(\lambda - B) & 0 \\ 0 & \lambda^2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix}.$$

Also, $Q(0)$ is Moore-penrose invertible and $Q^\dagger(0) = Z_1^\dagger Z_2^\dagger = A^{-1}B^{-1}$. If T is injective, then $Q(0)$ is invertible and $Q^{-1}(0) = Z_1^{-1}Z_2^{-1}$. Consequently, S is also invertible. The block Vandermonde operator corresponding to Z_1, Z_2 is given by

$$\mathcal{V}(Z_1, Z_2) = \begin{bmatrix} I & I \\ Z_1 & Z_2 \end{bmatrix}.$$

We have,

$$\mathcal{V}(Z_1, Z_2) = \begin{bmatrix} I & 0 \\ Z_2 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & Z_1 - Z_2 \end{bmatrix}\begin{bmatrix} I & I \\ 0 & I \end{bmatrix},$$

where the left and right factors on the right-hand side are invertible. So $\mathcal{V}(Z_1, Z_2)$ is invertible if and only if $2\Delta^{\frac{1}{2}} = (Z_1 - Z_2)$ is invertible. Now, we apply [22, Corollary 29.12, Corollary 29.13 and Remark 29.14] taking in account that $\mathcal{N}(\Lambda^{\frac{1}{2}}) = \mathcal{N}((\Lambda^{\frac{1}{2}})^*)$, we obtain,

$$\sigma(Z_1) \cap \sigma(Z_2) = \emptyset \qquad \text{and} \qquad \sigma(Z_1) \cup \sigma(Z_2) = \sigma(Q(.)).$$

5. An application to a second order linear boundary value problem

Denote $[0, +\infty)$ by $\mathbb{R}_+$, and let $C^k(\mathbb{R}_+, \mathcal{D})$ be the set of all k-times (strongly) continuously differentiable functions mapping $\mathbb{R}_+$, into $\mathcal{D} \subseteq \mathcal{H}$. In this section we consider the following abstract second order linear boundary value problem,

$$u''(t) - 2Tu'(t) - Su(t) = 0, \qquad t \in (0, 1), \tag{15}$$

$$u(0) = u_0, \qquad u(1) = u_1 \tag{16}$$

where $u' = \dfrac{d}{dt}$.

Theorem 5.1. *Let T^2 be m-accretive, T is accretive and S is bounded and accretive. Assume that*

1. $\mathcal{D}(\Delta^{\frac{1}{2}}) \subset \mathcal{D}(T)$.
2. $T(\mathcal{D}(T^2)) \subset \mathcal{D}(T^2)$ and $\Delta^{\frac{1}{2}}(\mathcal{D}(T^2)) \subset \mathcal{D}(T^2)$.
3. Δ is invertible.
4. T commutes with $\Delta^{1/2}$ on $\mathcal{D}(T^2)$.

Then of any constant vectors $u_0, u_1 \in \mathcal{D}(T)$ the vector valued function,

$$u(t) = e^{-(1-t)Z_1}x_0 + e^{tZ_2}x_1, \qquad t \in (0, 1),$$

with

$$x_0 = (I - e^{-2\Delta^{\frac{1}{2}}})^{-1} \left[-e^{Z_2} u_0 + u_1 \right]$$

and

$$x_1 = (I - e^{-2\Delta^{\frac{1}{2}}})^{-1} \left[u_0 - e^{-Z_1} u_1 \right]$$

is uniquely determined solution of (15)-(16), *with* $u(.) \in C^\infty((0,1), \mathcal{H}) \cap C^1((0,1), \mathcal{D}(T))$.

Proof. Under the assumptions, by Theorem 4.1, the factors $-Z_1$ and Z_2 generates bounded holomorphic C_0-semigroups. By [5, Lemma 2.38],

$$x(t) = e^{-(1-t)Z_1} x_0 \in \mathcal{D}(Z_1)^k)$$

and

$$y(t) = e^{tZ_2} x_1 \in \mathcal{D}(Z_2)^k),$$

for all $k \in \mathbb{N}$, $x_0, x_1 \in \mathcal{H}$ and $t \in (0,1)$. This implies that

$$u(t) = x(t) + y(t) \in \mathcal{H},$$

$$u'(t) = Z_1 x(t) + Z_2 y(t) \in \mathcal{D}(T)$$

and

$$u^{(2)}(t) = Z_1^2 x(t) + Z_2^2 y(t)$$

for all $t \in (0,1)$. We can easily see that u verifies (15). Since $(I - e^{-2\Delta^{\frac{1}{2}}})^{-1}$ and $(I - e^{-2\Delta^{\frac{1}{2}}})^{-1}$ exist and bounded on $\mathcal{H}$. Thus, we have

$$u(0) = e^{-Z_1} x_0 + x_1 = u_0,$$

and

$$u(1) = x_0 + e^{-Z_2} x_1 = u_1,$$

This completes the proof. $\square$

Example 5.2. *Let Ω is a smooth bounded domain of $\mathbb{R}^n$, Γ be the boundary of Ω, and $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$, $\eta_1 \geq 0$, $\eta \in \mathbb{R}$. We consider the following initial-boundary value problem in $L^2(\Omega)$,*

$$u''(t,x) - 2i(\eta\Delta - i\eta_1\Delta^2)u'(t,x) - \xi u(t,x) = 0, \qquad (t,x) \in (0,1) \times \Omega, \tag{17}$$

$$u(0,x) = u_0, \qquad u(1,x) = u_1, \qquad x \in \Omega, \tag{18}$$

$$u_{|\Gamma} = \Delta u_{|\Gamma} = 0, \tag{19}$$

where Δ denotes the Laplacian. It is known that $-\Delta$ with domain $H_0^1(\Omega) \cap H^2(\Omega) = \{u \in H^2(\Omega); u_{|\Gamma} = 0\}$ is a positive, self-adjoint operator on $L^2(\Omega)$. Its $L^2(\Omega)$ - normalized eigenfunctions are denoted w_j, and its eigenvalues counted with their multiplicities are denoted λ_j:

$$-\Delta w_j = \lambda_j w_j. \tag{20}$$

It is well known that $0 < \lambda_1 \leq ..., \leq \lambda_j \longrightarrow \infty$. Functional calculus can be defined using the eigenfunction expansion. In particular; if we denote by $(-\Delta)^\alpha$ the fractional powers of the Dirichlet Laplacian, with $0 \leq \alpha \leq 1$, then

$$(-\Delta)^\alpha u = \sum_{j=1}^\infty \lambda_j^\alpha u_j w_j$$

with

$$u_j = \int_\Omega u(x) w_j(y) dx$$

for $u \in \mathcal{D}((-\Delta)^\alpha) = \{u : (\lambda_j^\alpha u_j) \in \ell^2(\mathbb{N})\}$. Also, if $m\alpha$ is an integer, then

$$\mathcal{D}((-\Delta)^{m\alpha}) = H_0^{m\alpha}(\Omega) \cap H^{2m\alpha}(\Omega), \qquad m \geq 1. \tag{21}$$

See [21], for more details.

Set

$$T = i\eta\Delta + \eta_1\Delta^2, \qquad \mathcal{D}(T) = \mathcal{D}(\Delta^2) = \{u \in H^4(\Omega); u_{|\Gamma} = \Delta u_{|\Gamma} = 0\}.$$

and

$$Su = \xi u, \qquad \mathcal{D}(S) = L^2(\Omega)$$

Then the abstract version of problem (17)-(18) takes the form (15)-(16). We have,

- S is bounded and accretive.

- T is m-accretive.

- We have

$$T^2 = -\eta^2\Delta^2 + \eta_1^2\Delta^4 - 2i\eta\eta_1\Delta^3$$

with

$$\mathcal{D}(T^2) = \mathcal{D}(\Delta^4) = \{u \in H^8(\Omega); u_{|\Gamma} = \Delta u_{|\Gamma} = \Delta^2 u_{|\Gamma} = \Delta^3 u_{|\Gamma} = 0\}.$$

If we assume that $\lambda_1 \geq \dfrac{\eta^2}{\eta_1^2}$, then T is m-accretive.

- The operator

$$\Delta = T^2 + S = -\eta^2\Delta^2 + \eta_1^2\Delta^4 - 2i\eta\eta_1\Delta^3 + \xi,$$

with domain $\mathcal{D}(\Delta^4)$, is m-accretive.

- Δ admits a square root $\Delta^{1/2}$ m-$(\pi/4)$-accretive with $\mathcal{D}(\Delta^4)$ is a core of $\Delta^{1/2}$.

- The operators factors

$$Z_1 = T + \Delta^{\frac{1}{2}} = -\eta\Delta + \eta_1\Delta^2 + (-\eta^2\Delta^2 + \eta_1^2\Delta^4 - 2i\eta\eta_1\Delta^3 + \xi)^{\frac{1}{2}}$$

and

$$Z_2 = T - \Delta^{\frac{1}{2}} = -i\eta\Delta + \eta_1\Delta^2 - (-\eta^2\Delta^2 + \eta_1^2\Delta^4 - 2i\eta\eta_1\Delta^3 + \xi)^{\frac{1}{2}}$$

with domain $\mathcal{D}(\Delta^2)$ are closed operators.

- By an argument of functional calculus, we obtain

$$Tu = -\sum_{j=1}^{\infty}(i\eta - \eta_1\lambda_j)\lambda_j u_j w_j$$

for $u \in \mathcal{D}(\Delta^2)$. If $(\eta, \eta_1) \neq (0,0)$ then T is injective. Thus, is invertible with

$$T^{-1}u = -\sum_{j=1}^{\infty}\frac{1}{(\eta - i\eta_1\lambda_j)\lambda_j}u_j w_j$$

for $u \in \mathcal{D}(\Delta^2)$.

- *Assume that $(\eta, \eta_1) \neq (0,0)$ and $\xi \neq 0$. If $\sum_{j=1}^{\infty} \frac{1}{(\eta - i\eta_1\lambda_j)^2\lambda_j^2} < \frac{1}{|\xi|}$ then $\left\|ST^{-2}\right\| < 1$. In fact,*

$$
\begin{aligned}
\left\|T^{-1}u\right\|^2 &= \left\langle T^{-2}u, u \right\rangle \\
&= \sum_{j=1}^{\infty} \frac{1}{(\eta - i\eta_1\lambda_j)^2\lambda_j^2} \left|\left\langle u, w_j \right\rangle\right|^2 \\
&\leq \sum_{j=1}^{\infty} \frac{1}{(\eta - i\eta_1\lambda_j)^2\lambda_j^2} \|u\|^2 \\
&< \frac{1}{|\xi|} \|u\|^2.
\end{aligned}
$$

This implies that

$$
\left\|ST^{-2}\right\| \leq \|S\| \left\|T^{-1}\right\|^2 = |\xi| \left\|T^{-1}\right\|^2 < 1.
$$

- *If we assume that $(\eta, \eta_1) \neq (0,0)$, $\xi \neq 0$, $\lambda_1 \geq \frac{\eta^2}{\eta_1^2}$ and $\sum_{j=1}^{\infty} \frac{1}{(\eta - i\eta_1\lambda_j)^2\lambda_j^2} < \frac{1}{|\xi|}$; then by Theorem 3.2 and*

  Corollary 4.2, we conclude that $\Delta^{\frac{1}{2}}$, Z_1 and $-Z_2$ are m-$(\pi/4)$-accretive invertible operators. In particular, Z_1 and $-Z_2$ generates holomorphic C_0-semigroup of contraction operators $\mathcal{T}_1(z)$ and $\mathcal{T}_2(z)$ of angle $\frac{\pi}{4}$. If we assume further T commutes with $\Delta^{\frac{1}{2}}$ on $\mathcal{D}(T^2)$. Then all the statements of Theorem 5.1 hold. Consequently, for any pair of vectors $u_0, u_1 \in \mathcal{D}(T)$ the vector valued function,

  $$
  u(t,x) = e^{-(1-t)Z_1}v_0(x) + e^{tZ_2}v_1(x), \qquad t \in (0,1), \quad x \in \Omega
  $$

  with

  $$
  v_0(x) = (I - e^{-2\Delta^{\frac{1}{2}}})^{-1}\left[-e^{Z_2}u_0(x) + u_1(x)\right] \quad x \in \Omega
  $$

  and

  $$
  v_1(x) = (I - e^{-2\Delta^{\frac{1}{2}}})^{-1}\left[u_0(x) - e^{-Z_1}u_1(x)\right] \quad x \in \Omega
  $$

  is the unique solution of (17)-(19).

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