Filomat 36:8 (2022), 2617–2636
https://doi.org/10.2298/FIL2208617L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Construction of Hom-Leibniz H-Pseudoalgebras

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Abstract. This paper is devoted to the construction of Hom-Leibniz H-pseudoalgebras, which unify Hom-Lie H-pseudoalgebras, Leibniz H-pseudoalgebras and Hom-Leibniz algebras. Firstly, we give the construction theorem and obtain a class of Hom-Leibniz H-pseudoalgebras. We also construct Hom-Leibniz H-pseudoalgebras from different perspectives, including Leibniz H-pseudoalgebras, Hom-Leibniz (resp. Hom-Lie, Hom-associative) H-pseudoalgebras and their representations, Hom-Leibniz (resp. Hom-associative) algebras. Then we give some properties of the representations of Hom-Leibniz H-pseudoalgebras. Finally, the annihilation algebras of Hom-Leibniz H-pseudoalgebras are investigated.

1. Introduction

The theory of conformal algebra was introduced by Kac as a normal language describing the singular part of the operator product expansion (OPE) in two-dimensional conformal field theory ([2, 10]), and it came to be useful for investigation of vertex algebras ([9]). As a multivariable generalization of (Lie) conformal algebra, Lie H-pseudoalgebra has close connections with the differential Lie algebras of Ritt and Hamiltonian formalism in the theory of nonlinear evolution equation ([3–5, 20]). The name of Lie H-pseudoalgebra is motivated by the fact that it is a Lie algebra in the pseudotensor category $\mathcal{M}^*(H)$ ([1]). More precisely, let H be a cocommutative Hopf algebra over a field k, then $\mathcal{M}^*(H)$ is a pseudotensor category with the same objects as ${}_H\mathcal{M}$ (the category of left H-module), but with a particular pseudotensor structure

$$Lin(\{L_i\}_{i\in I}, M) = Hom_{H^{\otimes I}}(\boxtimes_{i\in I}L_i, H^{\otimes I} \otimes_H M),$$

where I is a finite non-empty set and $\boxtimes_{i\in I}$ is the tensor product functor ${}_H\mathcal{M}^I \longrightarrow_{H^{\otimes I}} \mathcal{M}$. The algebra in this category is usually called an H-pseudoalgebra. Associative algebras and Lie algebras in this pseudotensor category $\mathcal{M}^*(H)$ were studied in [1]. Left symmetric algebras in this category were introduced in [18]. In 2015, Wu [19] defined Leibniz algebras in the pseudotensor category $\mathcal{M}^*(H)$ (called Leibniz H-pseudoalgebras), which are the "non skew-commutative" generalization of Lie H-pseudoalgebras.

Hom-algebras were firstly studied in [6], where the authors introduced the structures of Hom-Lie algebras to describe some q-deformations of Witt and Virasoro algebras. Hom-algebras of different types

2020 *Mathematics Subject Classification*. 16T05, 17B61, 17A32, 18C40.
Keywords. Hom-Leibniz H-pseudoalgebra, Hom-Lie H-pseudoalgebra, Hom-Leibniz algebra, annihilation algebra.
Received: 12 July 2021; Revised: 25 December 2021; Accepted: 29 December 2021
Communicated by Dijana Mosić
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Research supported by the Doctoral Fund of Henan Institute of Technology (No. KQ2101) and the National Natural Science Foundation of China (Nos. 11871144, 12101188).
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have been investigated by many researchers. It is well-known that the commutator-algebra of a Hom-associative algebra is a Hom Lie-algebra ([14]). Conversely, from a given Hom-Lie algebra, there is a universal enveloping Hom-associative algebra. Another Hom-type algebra closely related to Hom-Lie algebra is the Hom-Leibniz algebra ([8, 14]), which is a "nonantisymmetric" generalization of Hom-Lie algebra and a Hom-analogue of Leibniz algebra ([11, 12]). Many more properties and structures of Hom-Lie and Hom-Leibniz algebras have been developed (see [8, 15, 21] and reference therein).

In 2012, Sun ([17]) generalized Lie (resp. associative) H-pseudoalgebras to Hom-type, called Hom-Lie (resp. Hom-associative) H-pseudoalgebras. Different from Hom-type algebras, Hom-type H-pseudoalgebras are obtained by twisting the identities using morphisms of H-modules. In particular, for the one-dimensional Hopf algebra $H = k$, a Hom-Lie H-pseudoalgebra is just an ordinary Hom-Lie algebra.

Motivated by this, we define Hom-Leibniz H-pseudoalgebra, as the generalization of Hom-Lie H-pseudoalgebra, Leibniz H-pseudoalgebra and Hom-Leibniz algebra. In addition, we are committed to the construction of this new structure.

This paper is organized as follows. In Section 2, we introduce the definitions of Hom-Leibniz H-pseudoalgebras and their representations. Then we obtain the method of constructing a Hom-Leibniz H-pseudoalgebra via a Leibniz H-pseudoalgebra. More generally, we get a class of Hom-Leibniz H-pseudoalgebras.

In Section 3, we construct Hom-Leibniz H-pseudoalgebras from related algebras and their representations, including Hom-associative (resp. Hom-Leibniz) algebras, associative algebras, (the representations of) Hom-Lie (resp. Hom-Leibniz) H-pseudoalgebras and Hom-associative H-pseudoalgebras.

In Section 4, we construct the representations of Hom-Leibniz H-pseudoalgebras. Especially, we show that every irreducible representation of a Hom-Leibniz H-pseudoalgebra is either skew-symmetric or right trivial. Thus a Hom-Leibniz H-pseudoalgebra (L, α) is a Hom-Lie H-pseudoalgebra if (L, α) is an irreducible representation of (L, α).

In Section 5, we mainly discuss the annihilation algebra of a Hom-Leibniz H-pseudoalgebra, which is the main tool in the study of Hom-Leibniz H-pseudoalgebras and their representations.

Throughout this paper, all vector spaces, linear maps and tensor products are considered over an algebraically closed field k, H is a cocommutative Hopf algebra. We use Sweedler's notations [16], for a coalgebra C, we write its comultiplication as $\Delta(c) = c_1 \otimes c_2$, for all $c \in C$. For any vector space V, we will define $\sigma(f \otimes g) = g \otimes f$, $(12)(f \otimes g \otimes h) = g \otimes f \otimes h$ and $(123)(f \otimes g \otimes h) = h \otimes f \otimes g$, for all $f, g, h \in V$. Similarly, we have symbols (13), (23), (132) and so on.

2. Hom-Leibniz *H*-pseudoalgebras

We begin with several basic definitions which will be used later.

Definition 2.1. ([17]) A Hom-associative H-pseudoalgebra is a triple $(A, \mu = *, \alpha)$ (abbr. (A, α)), where A is a left H-module, $\mu \in Hom_{H\otimes H}(A \otimes A, (H \otimes H) \otimes_H A)$(denote $\mu(x \otimes y) = x * y$, called the pseudoproduct), $\alpha \in Hom_H(A, A)$, such that

$$\alpha(x) * (y * z) = (x * y) * \alpha(z)$$

in $H^{\otimes 3} \otimes_H A$ for all $x, y, z \in A$.

A Hom-associative H-pseudoalgebra (A, α) is commutative if $x * y = (\sigma \otimes_H id)(y * x)$ for all $x, y \in A$.

Remark 2.2. (1) The condition $\mu \in Hom_{H\otimes H}(A\otimes A, (H\otimes H)\otimes_H A)$ means that the pseudoproduct μ satisfies H-bilinearity, that is, for all $x, y \in A$ and $f, g \in H$,

$$fx * gy = (f \otimes g \otimes_H 1)(x * y) = \sum_i ff_i \otimes gg_i \otimes_H e_i,$$

if $x * y = \sum_i f_i \otimes g_i \otimes_H e_i$

(2) We can extend the pseudoproduct μ from $A\otimes A \longrightarrow (H\otimes H)\otimes_H A$ to maps $(H^{\otimes 2}\otimes_H A)\otimes A \longrightarrow H^{\otimes 3}\otimes_H A$ and $A\otimes(H^{\otimes 2}\otimes_H A) \longrightarrow H^{\otimes 3}\otimes_H A$ by letting

$$(f\otimes_H x)*y=\sum_i (f\otimes 1)(\Delta\otimes id)(f_i\otimes g_i)\otimes_H e_i,$$

$$x*(f\otimes_H y)=\sum_i (1\otimes f)(id\otimes\Delta)(f_i\otimes g_i)\otimes_H e_i,$$

where $f\in H\otimes H$ and $x*y=\sum_i f_i\otimes g_i\otimes_H e_i$.

Definition 2.3. ([17]) A Hom-Lie H-pseudoalgebra is a triple $(L,\mu=[*],\beta)$ (abbr. (L,β)), where L is a left H-module, $\mu\in Hom_{H\otimes H}(L\otimes L,(H\otimes H)\otimes_H L)$(denote $\mu(a\otimes b)=[a*b]$, called the pseudobracket), $\beta\in Hom_H(L,L)$, satisfying

• skew-commutativity,

$$[a*b]=-(\sigma\otimes_H id)[b*a].$$

• Hom-Jacobi identity,

$$[[a*b]*\beta(c)]=[\beta(a)*[b*c]]-((12)\otimes_H id)[\beta(b)*[a*c]]$$

in $H^{\otimes 3}\otimes_H L$ for all $a,b,c\in L$.

Definition 2.4. ([17]) Let $(L,[*],\beta)$ be a Hom-Lie H-pseudoalgebra. A representation of (L,β) is a triple (M,ρ,γ), where M is a left H-module, $\rho\in Hom_{H\otimes H}(L\otimes M,(H\otimes H)\otimes_H M)$(denote $\rho(m\otimes n)=[m*n]$), $\gamma\in Hom_H(M,M)$, satisfying

$$[[a*b]*\gamma(m)]=[\beta(a)*[b*m]]-((12)\otimes_H id)[\beta(b)*[a*m]]$$

in $H^{\otimes 3}\otimes_H M$, for all $a,b\in L$ and $m\in M$.

Definition 2.5. ([19]) A Leibniz H-pseudoalgebra is a pair $(L,\mu=[,])$, where L is a left H-module, $\mu\in Hom_{H^{\otimes 2}}(L\otimes L,(H\otimes H)\otimes_H L)$ (denote $\mu(a\otimes b)=[a,b]$, called the pseudobracket), satisfying

$$[[a,b],c]=[a,[b,c]]-((12)\otimes_H id)[b,[a,c]]$$

in $H^{\otimes 3}\otimes_H L$ for all $a,b,c\in L$.

Now we introduce the definitions of Hom-Leibniz H-pseudoalgebras and their representations.

Definition 2.6. A (left) Hom-Leibniz H-pseudoalgebra is a triple $(L,\mu=[,],\alpha)$ (abbr. (L,α)), where L is a left H-module, $\mu\in Hom_{H\otimes H}(L\otimes L,(H\otimes H)\otimes_H L)$, $\alpha\in Hom_H(L,L)$ satisfying

$$[[a,b],\alpha(c)]=[\alpha(a),[b,c]]-((12)\otimes_H id)[\alpha(b),[a,c]] \tag{1}$$

in $H^{\otimes 3}\otimes_H L$ for all $a,b,c\in L$. Sometimes, the pseudoproduct μ in a Hom-Leibniz H-pseudoalgebra is called the pseudobracket.

In the sequel, by a Hom-Leibniz H-pseudoalgebra we will always mean a left Hom-Leibniz H-pseudoalgebra.

Remark 2.7. (i) In particular, for the one-dimensional Hopf algebra $H=k$, a Hom-Leibniz H-pseudoalgebra is just an ordinary Hom-Leibniz algebra. When $\alpha=id$, a Hom-Leibniz H-pseudoalgebra is an ordinary Leibniz H-pseudoalgebra.

(ii) A Hom-Leibniz H-pseudoalgebra (L,α) is a Hom-Lie H-pseudoalgebra if and only if the pseudoproduct $[,]$ is skew-commutative (i.e., $[a,b]=-(\sigma\otimes_H id)[b,a]$, for all $a,b\in L$). Hence any Hom-Lie H-pseudoalgebra is obviously a Hom-Leibniz H-pseudoalgebra.

Similar to the usual Hom-Leibniz algebra, we can define a right Hom-Leibniz H-pseudoalgebra via replacing condition (2.1) by the following

$$[\alpha(a), [b, c]] = [[a, b], \alpha(c)] - ((23) \otimes_H id)[[a, c], \alpha(b)]$$

for all $a, b, c \in L$. It is easy to prove that a Hom-Leibniz H-pseudoalgebra $(L, [,], \alpha)$ is a right Hom-Leibniz H-pseudoalgebra if and only if $[a, [b, c]] = -((123) \otimes_H id)[[b, c], a]$ for all $a, b, c \in L$.

A Hom-Leibniz H-pseudoalgebra $(L, \mu = [,], \alpha)$ is called multiplicative if it satisfies the additional condition $[\alpha(a), \alpha(b)] = (id \otimes_H \alpha)[a, b]$ for all $a, b \in L$. For example, a Leibniz H-pseudoalgebra is a multiplicative Hom-Leibniz H-pseudoalgebra with $\alpha = id \in Hom_H(L, L)$.

Clearly, if $(A, *, \alpha)$ is a (multiplicative) Hom-associative H-pseudoalgebra, then $(A, [,], \alpha)$ is a (multiplicative) Hom-Leibniz H-pseudoalgebra with

$$[x, y] = x * y - (\sigma \otimes_H id)(y * x)$$

for all $x, y \in A$.

Let $(L, [,], \alpha)$ and $(L', [,]', \alpha')$ be two (multiplicative) Hom-Leibniz H-pseudoalgebras. An H-linear map $f : (L, [,], \alpha) \longrightarrow (L', [,]', \alpha')$ is called a morphism of (multiplicative) Hom-Leibniz H-pseudoalgebras if f satisfies $f \circ \alpha = \alpha' \circ f$ and $(id \otimes_H f)[a, b] = [f(a), f(b)]'$ for all $a, b \in L$.

Definition 2.8. Let (L, α) be a Hom-Leibniz H-pseudoalgebra. A representation of (L, α) or a (L, α)-module is a quadruple $(M, \rho_l, \rho_r, \beta)$(abbr. (M, β)), where M is a left H-module, $\beta \in Hom_H(M, M)$, $\rho_l \in Hom_{H\otimes H}(L \otimes M, H^{\otimes 2} \otimes_H M)$ and $\rho_r \in Hom_{H\otimes H}(M \otimes L, H^{\otimes 2} \otimes_H M)$ (we use $[a, m]$ and $[m, a]$ to denote $\rho_l(a \otimes m)$ and $\rho_r(m \otimes a)$), which satisfies

$(HLM1)$ $(id_{H^{\otimes 2}} \otimes_H \beta)[a, m] = [\alpha(a), \beta(m)]$,

$(HLM2)$ $(id_{H^{\otimes 2}} \otimes_H \beta)[m, a] = [\beta(m), \alpha(a)]$,

$(HLM3)$ $[[m, a], \alpha(b)] = [\beta(m), [a, b]] - ((12) \otimes_H id)[\alpha(a), [m, b]]$,

$(HLM4)$ $[[a, m], \alpha(b)] = [\alpha(a), [m, b]] - ((12) \otimes_H id)[\beta(m), [a, b]]$,

$(HLM5)$ $[[a, b], \beta(m)] = [\alpha(a), [b, m]] - ((12) \otimes_H id)[\alpha(b), [a, m]]$,

for all $a, b \in L$ and $m \in M$.

Moreover, (N, β) is called a submodule of (M, β) if N is an H-submodule of M, $\beta(n) \in N$, $[a, n] \in H^{\otimes 2} \otimes_H N$ and $[n, a] \in H^{\otimes 2} \otimes_H N$ for all $a \in L$ and $n \in N$.

Remark 2.9. (i) For any Hom-Leibniz H-pseudoalgebra (L, α), $M = L$ is a representation of (L, α) with the pseudobracket of L.

(ii) By using conditions $(HLM3)$ and $(HLM4)$, we have $[[m, a], \alpha(b)] = -((12) \otimes_H id)[[a, m], \alpha(b)]$ for all $a, b \in L, m \in M$.

Similar to the case of Hom-Lie H-pseudoalgebras [17], the following theorem provides a construction method of a Hom-Leibniz H-pseudoalgebra from a given Leibniz H-pseudoalgebra.

Theorem 2.10. Let $(L, [,])$ be a Leibniz H-pseudoalgebra and α an endomorphism of L (i.e., $[\alpha(a), \alpha(b)] = (id \otimes_H \alpha)([a, b])$). Define $[,]_\alpha \in Hom_{H\otimes H}(L \otimes L, (H \otimes H) \otimes_H L)$ by

$$[a, b]_\alpha = [\alpha(a), \alpha(b)] = (id \otimes_H \alpha)[a, b], \quad \forall\, a, b \in L,$$

then $(L, [,]_\alpha, \alpha)$ is a multiplicative Hom-Leibniz H-pseudoalgebra.

Moreover, suppose that $(L', [,]')$ is another Leibniz H-pseudoalgebra and $\alpha' : L' \longrightarrow L'$ is an endomorphism of L'. If $f : L \longrightarrow L'$ is a morphism of Leibniz H-pseudoalgebras that satisfies $f \circ \alpha = \alpha' \circ f$, then

$$f : (L, [,]_\alpha, \alpha) \longrightarrow (L', [,]'_{\alpha'}, \alpha')$$

is a morphism of multiplicative Hom-Leibniz H-pseudoalgebras.

Proof. Since α is an endomorphism of Leibniz H-pseudoalgebra L, we have

$$[\alpha(a), [b, c]_\alpha]_\alpha = [\alpha(a), (id \otimes_H \alpha)([b, c])]_\alpha = (id \otimes_H \alpha^2)([a, [b, c]]),$$

and

$$[[a, b]_\alpha, \alpha(c)]_\alpha = [(id \otimes_H \alpha)([a, b]), \alpha(c)]_\alpha = (id \otimes_H \alpha^2)([[a, b], c]),$$

for all $a, b, c \in L$. It follows that

$$\begin{aligned} & [\alpha(a), [b, c]_\alpha]_\alpha - ((12) \otimes_H id)([\alpha(b), [a, c]_\alpha]_\alpha) - [[a, b]_\alpha, \alpha(c)]_\alpha \\ = \ & (id \otimes_H \alpha^2)([a, [b, c]] - ((12) \otimes_H id)([b, [a, c]]) - [[a, b], c]) \\ = \ & 0. \end{aligned}$$

Observe that $[\alpha(a), \alpha(b)]_\alpha = (id \otimes_H \alpha)([\alpha(a), \alpha(b)]) = (id \otimes_H \alpha)([a, b]_\alpha)$, therefore $(L, [,]_\alpha, \alpha)$ is a multiplicative Hom-Leibniz H-pseudoalgebra. Also, by using $f \circ \alpha = \alpha' \circ f$, we have

$$\begin{aligned} (id \otimes_H f)[a, b]_\alpha &= (id \otimes_H f)[\alpha(a), \alpha(b)] \\ &= [f\alpha(a), f\alpha(b)]' \\ &= [\beta f(a), \alpha' f(b)]' \\ &= [f(a), f(b)]'_{\alpha'}, \end{aligned}$$

as needed. □

Example 2.11. Let H be a commutative Hopf algebra, $G(H)$ the set of all grouplike elements of H. Suppose that $H\{e\}$ is a free Leibniz H-pseudoalgebra with the pseudobracket given by $[e, e] = a \otimes_H e$ for any $a \in H \otimes H$. We define $\alpha : H\{e\} \longrightarrow H\{e\}$ as follows

$$\alpha(e) = ge \quad \text{with} \quad g \in G(H).$$

Then α is an endomorphism of Leibniz H-pseudoalgebra $H\{e\}$ because of the commutativity of H. By Theorem 2.10, $(L = H\{e\}, [,]_\alpha, \alpha)$ is a multiplicative Hom-Leibniz H-pseudoalgebra with the pseudobracket

$$[e, e]_\alpha = [ge, ge] = (g \otimes g)a \otimes_H e.$$

Example 2.12. Let $L = H\{e_1, e_2\}$ be a free H-module with basis $\{e_1, e_2\}$, and $G(H)$ the set of all grouplike elements of H. Then L is a Leibniz H-pseudoalgebra with the pseudobracket

$$[e_1, e_1] = [e_1, e_2] = 0, \ \ [e_2, e_1] = 1 \otimes 1 \otimes_H e_1, \ \ [e_2, e_2] = 1 \otimes 1 \otimes_H e_1.$$

We define $\alpha : H\{e_1, e_2\} \longrightarrow H\{e_1, e_2\}$ as follows

$$\alpha(e_1) = ge_1, \quad \alpha(e_2) = ge_2,$$

where $g \in G(H)$. Then α is an endomorphism of Leibniz H-pseudoalgebra L. By Theorem 2.10, $(L = H\{e_1, e_2\}, [,]_\alpha, \alpha)$ is a multiplicative Hom-Leibniz H-pseudoalgebra with the pseudobracket

$$\begin{aligned} &[e_1, e_2]_\alpha = [e_1, e_1]_\alpha = 0, \\ &[e_2, e_1]_\alpha = [e_2, e_2]_\alpha = g \otimes g \otimes_H e_1. \end{aligned}$$

More generally, we can obtain a class of multiplicative Hom-Leibniz H-pseudoalgebras by the following theorem.

Theorem 2.13 Let $(L, [,], \alpha)$ be a multiplicative Hom-Leibniz H-pseudoalgebra. Then so is $L^n = (L, [,]^n = (id \otimes_H \alpha^{2^n - 1})[,], \beta = \alpha^{2^n})$ for each $n \geq 0$.

Proof. For all $a, b \in L$, we denote $[a, b] = f^{a,b} \otimes g^{a,b} \otimes_H e_{a,b}$ for convenience. Since $(L, [,], \alpha)$ is a multiplicative Hom-Leibniz H-pseudoalgebra, we have

$$[[a, b], \alpha(c)] = [\alpha(a), [b, c]] - ((12) \otimes_H id)[\alpha(b), [a, c]],$$

which is equivalent to

$$\begin{aligned} & f^{a,b} f_1^{e_{a,b},\alpha(c)} \otimes g^{a,b} f_2^{e_{a,b},\alpha(c)} \otimes g^{e_{a,b},\alpha(c)} \otimes_H e_{e_{a,b},\alpha(c)} \\ = \quad & f^{\alpha(a),e_{b,c}} \otimes f^{b,c} g_1^{\alpha(a),e_{b,c}} \otimes g^{b,c} g_2^{\alpha(a),e_{b,c}} \otimes_H e_{\alpha(a),e_{b,c}} \\ & -f^{a,c} g_1^{\alpha(b),e_{a,c}} \otimes f^{\alpha(b),e_{a,c}} \otimes g^{a,c} g_2^{\alpha(b),e_{a,c}} \otimes_H e_{\alpha(b),e_{a,c}}. \end{aligned} \tag{2}$$

Now we prove that $L^1 = (L, [,]^1 = (id \otimes_H \alpha)[,], \beta = \alpha^2)$ is a multiplicative Hom-Leibniz H-pseudoalgebra. We only verify that the pseudobracket $[,]^1$ satisfies condition (2.1). The rest is easy to prove and we omit the details. By assumption, we have $[\alpha(a), \alpha(b)] = (id \otimes_H \alpha)[a, b]$, i.e.,

$$f^{\alpha(a),\alpha(b)} \otimes g^{\alpha(a),\alpha(b)} \otimes_H e_{\alpha(a),\alpha(b)} = f^{a,b} \otimes g^{a,b} \otimes_H \alpha(e_{a,b}).$$

Using the above equation, we compute:

$$\begin{aligned} & [\beta(a), [b, c]^1]^1 = (1 \otimes f^{b,c} \otimes g^{b,c})(id \otimes \Delta)(f^{\beta(a),\alpha(e_{b,c})} \otimes g^{\beta(a),\alpha(e_{b,c})}) \otimes_H \alpha(e_{\beta(a),\alpha(e_{b,c})}) \\ = \quad & f^{\beta(a),\alpha(e_{b,c})} \otimes f^{b,c} g_1^{\beta(a),\alpha(e_{b,c})} \otimes g^{b,c} g_2^{\beta(a),\alpha(e_{b,c})} \otimes_H \alpha(e_{\beta(a),\alpha(e_{b,c})}) \\ = \quad & f^{\alpha(a),e_{b,c}} \otimes f^{b,c} g_1^{\alpha(a),e_{b,c}} \otimes g^{b,c} g_2^{\alpha(a),e_{b,c}} \otimes_H \alpha^2(e_{\alpha(a),e_{b,c}}). \end{aligned}$$

Interchanging the roles of a and b in the above equation, we get

$$((12) \otimes_H id)[\beta(b), [a, c]^1]^1 = f^{a,c} g_1^{\alpha(b),e_{a,c}} \otimes f^{\alpha(b),e_{a,c}} \otimes g^{a,c} g_2^{\alpha(b),e_{a,c}} \otimes_H \alpha^2(e_{\alpha(b),e_{a,c}}).$$

Similarly, we can obtain

$$[[a, b]^1, \beta(c)]^1 = f^{a,b} f_1^{e_{a,b},\alpha(c)} \otimes g^{a,b} f_2^{e_{a,b},\alpha(c)} \otimes g^{e_{a,b},\alpha(c)} \otimes_H \alpha^2(e_{e_{a,b},\alpha(c)}).$$

According to equation (2.2), we get $[[a, b]^1, \beta(c)]^1 = [\beta(a), [b, c]^1]^1 - ((12) \otimes_H id)[\beta(b), [a, c]^1]^1$. Hence L^1 is a multiplicative Hom-Leibniz H-pseudoalgebra.

Note that $L^0 = (L, [,], \alpha), L^1 = (L, [,]^1 = (id \otimes_H \alpha)[,], \beta = \alpha^2)$ and $L^{n+1} = (L^n)^1$. By an induction argument, the conclusion holds. □

3. The construction of Hom-Leibniz *H*-pseudoalgebras

In this section, we construct Hom-Leibniz H-pseudoalgebras from Hom-Lie (resp. Hom-Leibniz, Hom-associative) H-pseudoalgebras, Hom-associative algebras, Hom-Leibniz algebras, respectively.

First we construct Hom-Leibniz H-pseudoalgebras from Hom-type H-pseudoalgebras.

Proposition 3.1. Let $\varphi : H' \longrightarrow H$ be a homomorphism of Hopf algebras and $(L, [,]', \alpha)$ a (multiplicative) Hom-Leibniz H'-pseudoalgebra. Then $(H \otimes_{H'} L, [,], \beta = id \otimes \alpha)$ is a (multiplicative) Hom-Leibniz H-pseudoalgebra with the pseudobracket

$$[x \otimes_{H'} a, y \otimes_{H'} b] = \sum_i (x\varphi(f_i) \otimes y\varphi(g_i)) \otimes_H (1 \otimes_{H'} e_i),$$

where $x, y \in H, a, b \in L$ and $[a, b]' = \sum_i f_i \otimes g_i \otimes_{H'} e_i$.

Proof. For all $x \otimes a, y \otimes b, z \otimes c \in H \otimes_{H'} L$, we check that

$$[[x \otimes a, y \otimes b], \beta(z \otimes c)] = [\beta(x \otimes a), [y \otimes b, z \otimes c]] - ((12) \otimes_H id)[\beta(y \otimes b), [x \otimes a, z \otimes c]].$$

Denote $[a,b] = f^{a,b} \otimes g^{a,b} \otimes_H e_{a,b}$ for all $a, b \in L$, then we have

$$\begin{aligned} & [\beta(x\otimes a), [y\otimes b, z\otimes c]] \\ = \; & (1\otimes y\varphi(f^{b,c})\otimes z\varphi(g^{b,c}))(id\otimes\Delta)(x\varphi(f^{\alpha(a),e_{b,c}})\otimes\varphi(g^{\alpha(a),e_{b,c}}))\otimes_H(1\otimes e_{\alpha(a),e_{b,c}}) \\ = \; & x\varphi(f^{\alpha(a),e_{b,c}})\otimes y\varphi(f^{b,c}g_1^{\alpha(a),e_{b,c}})\otimes z\varphi(g^{b,c}g_2^{\alpha(a),e_{b,c}})\otimes_H(1\otimes e_{\alpha(a),e_{b,c}}). \end{aligned}$$

Interchanging the roles of $x \otimes a$ and $y \otimes b$ in the above equation, we get

$$\begin{aligned} & -((12)\otimes_H id)[\beta(y\otimes b), [x\otimes a, z\otimes c]] \\ = \; & -x\varphi(f^{a,c}g_1^{\alpha(b),e_{a,c}})\otimes y\varphi(f^{\alpha(b),e_{a,c}})\otimes z\varphi(g^{a,c}g_2^{\alpha(b),e_{a,c}})\otimes_H(1\otimes e_{\alpha(b),e_{a,c}}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} & [[x\otimes a, y\otimes b], \beta(z\otimes c)] \\ = \; & x\varphi(f^{a,b}f_1^{e_{a,b},\alpha(c)})\otimes y\varphi(g^{a,b}f_2^{e_{a,b},\alpha(c)})\otimes z\varphi(g^{e_{a,b},\alpha(c)})\otimes_H(1\otimes e_{e_{a,b},\alpha(c)}). \end{aligned}$$

Since $(L, [,]', \alpha)$ is a Hom-Leibniz H'-pseudoalgebra, we have

$$[[a,b]', \alpha(c)]' = [\alpha(a), [b,c]']' - ((12)\otimes_H id)[\alpha(b), [a,c]']'$$

which is equivalent to

$$\begin{aligned} & f^{a,b}f_1^{e_{a,b},\alpha(c)}\otimes g^{a,b}f_2^{e_{a,b},\alpha(c)}\otimes g^{e_{a,b},\alpha(c)}\otimes_{H'} e_{e_{a,b},\alpha(c)} \\ & = f^{\alpha(a),e_{b,c}}\otimes f^{b,c}g_1^{\alpha(a),e_{b,c}}\otimes g^{b,c}g_2^{\alpha(a),e_{b,c}}\otimes_{H'} e_{\alpha(a),e_{b,c}} \\ & -f^{a,c}g_1^{\alpha(b),e_{a,c}}\otimes f^{\alpha(b),e_{a,c}}\otimes g^{a,c}g_2^{\alpha(b),e_{a,c}}\otimes_{H'} e_{\alpha(b),e_{a,c}}. \end{aligned} \tag{3. 1}$$

By (3.1), we can get $[[x\otimes a, y\otimes b], \beta(z\otimes c)] = [\beta(x\otimes a), [y\otimes b, z\otimes c]] - ((12)\otimes_H id)[\beta(y\otimes b), [x\otimes a, z\otimes c]]$. This completes the proof. □

The following corollaries are obvious.

Corollary 3.2. Let H' be a Hopf subalgebra of H and $(L, [,]', \alpha)$ a (multiplicative) Hom-Leibniz H'-pseudoalgebra. Then $(H \otimes_{H'} L, [,], \beta = id_H \otimes \alpha)$ is a (multiplicative) Hom-Leibniz H-pseudoalgebra with the pseudobracket

$$[x\otimes_{H'} a, y\otimes_{H'} b] = (x\otimes y\otimes_H 1)[a,b] = \sum_i xf_i\otimes yg_i\otimes_H(1\otimes_{H'} e_i),$$

where $x, y \in H, a, b \in L$ and $[a,b]' = \sum_i f_i \otimes g_i \otimes_{H'} e_i$.

Corollary 3.3. Let H be a cocommutative Hopf algebra and $(L, [,]', \alpha)$ a (multiplicative) Hom-Leibniz algebra. Then $(H \otimes L, [,], \beta = id_H \otimes \alpha)$ is a (multiplicative) Hom-Leibniz H-pseudoalgebra with the pseudobracket

$$[x\otimes_{H'} a, y\otimes_{H'} b] = x\otimes y\otimes_H(1\otimes_{H'}[a,b]'),$$

for all $x, y \in H, a, b \in L$.

In general, the tensor product of two Hom-Leibniz H-pseudoalgebras is not a Hom-Leibniz H-pseudoalgebra. However, we have the following result.

Theorem 3.4. Let $(L_1, \mu = *, \alpha)$ be a commutative Hom-associative H_1-pseudoalgebra, and $(L_2, \mu = [,], \beta)$ a Hom-Leibniz H_2-pseudoalgebra. Then the tensor product $(L = L_1 \otimes L_2, \mu = [,]', \alpha \otimes \beta)$ is a Hom-Leibniz $H := H_1 \otimes H_2$-pseudoalgebra with the pseudoproduct given by

$$[x\otimes a, y\otimes b]' = \sum_{i,j}(f_i\otimes m_j)\otimes(g_i\otimes n_j)\otimes_H(e_i\otimes t_j),$$

where $x * y = \sum_i f_i \otimes g_i \otimes_{H_1} e_i$ and $[a,b] = \sum_j m_j \otimes n_j \otimes_{H_2} t_j$.

Moreover, $(L, \alpha\otimes\beta)$ is a Hom-Lie H-pseudoalgebra if and only if (L_2,β) is a Hom-Lie H_2-pseudoalgebra.

Proof. Here, we use the following notation for convenience: for all $a_i, b_i \in L_i, i = 1,2$, we denote

$$\mu(a_i \otimes b_i) = (a_i)_{b_i} \otimes 1 \otimes_{H_i} (b_i)^{a_i}.$$

Since $(L_1, *, \alpha)$ is a commutative Hom-associative H_1-pseudoalgebra and $(L_2, [,], \beta)$ is a Hom-Leibniz H_2-pseudoalgebra, we have $(x*y)*\alpha(z) = \alpha(x)*(y*z) = ((12)\otimes_{H_1} id)(\alpha(y)*(x*z))$ and $[[a,b],\beta(c)] = [\beta(a),[b,c]] - ((12)\otimes_{H_2} id)[\beta(b),[a,c]]$ for all $x,y,z \in L_1, a,b,c \in L_2$. That is

$$\begin{aligned} & x_y(y^x)_{\alpha(z)1} \otimes (y^x)_{\alpha(z)2} \otimes 1 \otimes_{H_1} \alpha(z)^{y^x} \\ = \;& \alpha(x)_{z^y} \otimes y_z \otimes 1 \otimes_{H_1} (z^y)^{\alpha(x)} \\ = \;& x_z \otimes \alpha(y)_{z^x} \otimes 1 \otimes_{H_1} (z^x)^{\alpha(y)}, \end{aligned} \tag{3. 2}$$

and

$$\begin{aligned} & a^b(b^a)_{\beta(c)1} \otimes (b^a)_{\beta(c)2} \otimes 1 \otimes_{H_2} \beta(c)^{b^a} \\ = \;& \beta(a)_{c^b} \otimes b_c \otimes 1 \otimes_{H_2} (c^b)^{\beta(a)} - a^c \otimes \beta(b)_{c^a} \otimes 1 \otimes_{H_2} (c^a)^{\beta(b)}. \end{aligned} \tag{3. 3}$$

Now we check the condition (2.1). By using (3.2) and (3.3), we have

$$\begin{aligned} & [\alpha(x)\otimes\beta(a), [y\otimes b, z\otimes c]']' - ((12)\otimes_H id)[\alpha(y)\otimes\beta(b), [x\otimes a, z\otimes c]']' \\ = \;& (\alpha(x)_{z^y} \otimes \beta(a)_{c^b}) \otimes (y_z \otimes b_c) \otimes (1\otimes 1) \otimes_H ((z^y)^{\alpha(x)} \otimes (c^b)^{\beta(a)}) \\ & -(x_z \otimes a_c) \otimes (\alpha(y)_{z^x} \otimes \beta(b)_{c^a}) \otimes (1\otimes 1) \otimes_H ((z^x)^{\alpha(y)} \otimes (c^a)^{\beta(b)}) \\ = \;& (x^y(y^x)_{\alpha(z)1} \otimes a^b(b^a)_{\beta(c)1}) \otimes ((y^x)_{\alpha(z)2} \otimes (b^a)_{\beta(c)2}) \otimes (1\otimes 1) \\ & \otimes_H (\alpha(z)^{y^x} \otimes \beta(c)^{b^a}). \\ = \;& [[x\otimes a, y\otimes b]', \alpha(z)\otimes\beta(c)]', \end{aligned}$$

as desired. The last assertion of the theorem follows from a routine computation and we omit the details. This completes the proof. □

Example 3.5. Let $H = H_1$ and $L_1 = H\{e\}$ be a commutative associative H_1-pseudoalgebra with the pseudoproduct $e * e = 1 \otimes 1 \otimes_{H_1} e$. We define $\alpha : H\{e\} \longrightarrow H\{e\}$ as follows

$$\alpha(e) = ge, \ \forall\, g \in G(H).$$

Then α is an endomorphism of H_1-pseudoalgebra $H\{e\}$. By Theorem 2.9 in [17], $(H\{e\}, *_\alpha, \alpha)$ is a commutative Hom-associative H_1-pseudoalgebra with the pseudoproduct

$$e *_\alpha e = ge * ge = g \otimes g \otimes_H e.$$

Furthermore, let $H_2 = k$ and (L_2, β) a usual Hom-Leibniz algebra. Then $(L = L_1 \otimes L_2 = He \otimes L_2, \alpha\otimes\beta)$ is a Hom-Leibniz H-pseudoalgebra.

Let $\phi : H_1' \longrightarrow H_1$ and $\psi : H_2' \longrightarrow H_2$ be homomorphisms of Hopf algebras. Suppose that (L_1, α) is a commutative Hom-associative H_1'-pseudoalgebra and (L_2, β) is a Hom-Leibniz H_2'-pseudoalgebra. By Remark 2.17 in [17], $(H_1 \otimes_{H_1'} L_1, *, id\otimes\alpha)$ is a cocommutative Hom-associative H_1-pseudoalgebra with the pseudoproduct

$$(h \otimes_{H_1'} a) * (g \otimes_{H_1'} b) = \sum_i h\phi(f_i) \otimes (g\phi(g_i) \otimes_H ((1 \otimes_{H_1'} e_i)$$

if $a * b = \sum_i f_i \otimes g_i \otimes_{H_1'} e_i$. By Proposition 3.1, $(H_2 \otimes_{H_2'} L_2, [,], id \otimes \beta)$ is a Hom-Leibniz H_2-pseudoalgebra defined as above. Then by Theorem 3.4, $((H_1 \otimes_{H_1'} L_1) \otimes (H_2 \otimes_{H_2'} L_2), \widetilde{*}, id \otimes \alpha \otimes id \otimes \beta)$ is a Hom-Leibniz $H := H_1 \otimes H_2$-pseudoalgebra with the pseudobracket

$$\begin{aligned} & ((h \otimes_{H_1'} a) \otimes (h' \otimes_{H_2'} a')) \widetilde{*} ((g \otimes_{H_1'} b) \otimes (g' \otimes_{H_2'} b')) \\ = & \sum_{i,j} (h\phi(f_i) \otimes h'\psi(m_j)) \otimes (g\phi(g_i) \otimes g'\psi(n_j)) \otimes_H ((1 \otimes_{H_1'} e_i) \otimes (1 \otimes_{H_2'} t_j)), \end{aligned}$$

where $a * b = \sum_i f_i \otimes g_i \otimes_{H_1'} e_i, [a', b'] = \sum_j m_j \otimes n_j \otimes_{H_2'} t_j$ for all $a, b \in L_1$ and $a', b' \in L_2$.

Unlike Theorem 2.13, we show another method of constructing multiplicative Hom-Leibniz H-pseudoalgebra from a given multiplicative Hom-Leibniz H-pseudoalgebra in the following.

Proposition 3.6. Let $(L, [,], \alpha)$ be a (multiplicative) Hom-Leibniz H-pseudoalgebra, $\lambda \in k$ a fixed scalar. Suppose that $R : L \longrightarrow L$ is an endomorphism of H-module satisfying

$$R \circ \alpha = \alpha \circ R, \tag{3. 4}$$

$$[R(a), R(b)] = (id \otimes_H R)([R(a), b] + [a, R(b)] + \lambda[a, b]), \tag{3. 5}$$

for all $a, b \in L$. Then $(L, \{,\}, \alpha)$ is a (multiplicative) Hom-Leibniz H-pseudoalgebra with the pseudobracket

$$\{a, b\} = [R(a), b] + [a, R(b)] + \lambda[a, b].$$

Proof. For all $a, b \in L$ and $f, g \in H$, we have

$$\begin{aligned} & \{fa, gb\} = [R(fa), gb] + [fa, R(gb)] + \lambda[fa, gb] \\ = & [f(R(a)), gb] + [fa, g(R(b))] + \lambda[fa, gb] \\ = & (f \otimes g \otimes 1)([R(a), b] + [a, R(b)] + \lambda[a, b]) \\ = & (f \otimes g \otimes 1)\{a, b\}. \end{aligned}$$

Hence $\{,\}$ is H-bilinear. Now we verify the condition (2.1). For all $a, b, c \in L$, we have

$$\begin{aligned} \{\{a, b\}, \alpha(c)\} = & [(id \otimes_H R)[R(a), b], \alpha(c)] + [[R(a), b], R(\alpha(c))] + \lambda[[R(a), b], \alpha(c)] \\ & + [(id \otimes_H R)[a, R(b)], \alpha(c)] + [[a, R(b)], R(\alpha(c))] + \lambda[[a, R(b)], \alpha(c)] \\ & + \lambda[(id \otimes_H R)[a, b], \alpha(c)] + \lambda[[a, b], R(\alpha(c))] + \lambda^2[[a, b], \alpha(c)] \\ = & [[R(a), R(b)], \alpha(c)] + [[R(a), b], R(\alpha(c))] + \lambda[[R(a), b], \alpha(c)] \\ & + [[a, R(b)], R(\alpha(c))] + \lambda[[a, R(b)], \alpha(c)] + \lambda[[a, b], R(\alpha(c))] + \lambda^2[[a, b], \alpha(c)]. \end{aligned}$$

Similarly,

$$\begin{aligned} \{\alpha(a), \{b, c\}\} = & [\alpha(a), [R(b), R(c)]] + [R(\alpha(a)), [R(b), c]] + \lambda[\alpha(a), [R(b), c]] \\ & + [R(\alpha(a)), [b, R(c)]] + \lambda[\alpha(a), [b, R(c)]] + \lambda[R(\alpha(a)), [b, c]] + \lambda^2[\alpha(a), [b, c]]. \end{aligned}$$

Interchanging the roles of a and b in the above equation, we obtain

$$\begin{aligned} \{\alpha(b), \{a, c\}\} = & [\alpha(b), [R(a), R(c)]] + [R(\alpha(b)), [R(a), c]] + \lambda[\alpha(b), [R(a), c]] \\ & + [R(\alpha(b)), [a, R(c)]] + \lambda[\alpha(b), [a, R(c)]] + \lambda[R(\alpha(b)), [a, c]] + \lambda^2[\alpha(b), [a, c]]. \end{aligned}$$

Then we compute:

$$\begin{aligned}
& \{\alpha(a),\{b,c\}\}-((12)\otimes_H id)\{\alpha(b),\{a,c\}\} \\
= \; & [\alpha(R(a)),[R(b),c]]-((12)\otimes_H id)[\alpha(R(b)),[R(a),c]] \\
& +[\alpha(a),[R(b),R(c)]]-((12)\otimes_H id)[\alpha(R(b)),[a,R(c)]] \\
& +\lambda[\alpha(a),[R(b),c]]-((12)\otimes_H id)\lambda[\alpha(R(b)),[a,c]] \\
& +[\alpha(R(a)),[b,R(c)]]-((12)\otimes_H id)[\alpha(b),[R(a),R(c)]] \\
& +\lambda[\alpha(a),[b,R(c)]]-((12)\otimes_H id)\lambda[\alpha(b),[a,R(c)]] \\
& +\lambda[\alpha(R(a)),[b,c]]-((12)\otimes_H id)\lambda[\alpha(b),[R(a),c]] \\
& +\lambda^2[\alpha(a),[b,c]]-((12)\otimes_H id)\lambda^2[\alpha(b),[a,c]] \\
= \; & [[R(a),R(b)],\alpha(c)]+[[a,R(b)],\alpha(R(c))]+\lambda[[a,R(b)],\alpha(c)]+[[R(a),b],\alpha(R(c))] \\
& +\lambda[[a,b],\alpha(R(c))]+\lambda[[R(a),b],\alpha(c)]+\lambda^2[[a,b],\alpha(c)] \\
= \; & \{\{a,b\},\alpha(c)\}.
\end{aligned}$$

This completes the proof. □

Remark 3.7. By Example 2.12, $(L=H\{e_1,e_2\},[,]_\alpha,\alpha)$ is Hom-Leibniz H-pseudoalgebra. Define $R:L\longrightarrow L$ by $R(e_1)=R(e_2)=-\lambda e_1$ or $R(e_1)=-\lambda e_1, R(e_2)=-\lambda e_2$, where $0\neq\lambda\in k$. Then the map satisfies (3.4) and (3.5).

As in Proposition 3.6, we construct Hom-Leibniz H-pseudoalgebra from a given Hom-type H-pseudoalgebra via an morphism of H-modules.

Proposition 3.8. Let $(A,*,\alpha)$ be a (multiplicative) Hom-associative H-pseudoalgebra, $D:A\longrightarrow A$ an endomorphism of H-module such that

$$D\circ\alpha=\alpha\circ D,$$
$$(id\otimes_H D)(a*D(b))=D(a)*D(b)=(id\otimes_H D)(D(a)*b),$$

for all $a,b\in A$. Then $(A,[,],\alpha)$ is a (multiplicative) Hom-Leibniz H-pseudoalgebra with the pseudobracket

$$[a,b]=D(a)*b-(\sigma\otimes_H id)(b*D(a)).$$

Proof. Firstly, since $D:A\longrightarrow A$ is an endomorphism of H-module, we have

$$\begin{aligned}
[fa,gb] &= D(fa)*gb-(\sigma\otimes_H id_A)(gb*D(fa)) \\
&= fD(a)*gb-(\sigma\otimes_H id)(gb*fD(a)) \\
&= (f\otimes g\otimes 1)((D(a)*b)-(\sigma\otimes_H id_A)(b*D(a))) \\
&= (f\otimes g\otimes 1)[a,b],
\end{aligned}$$

for all $a,b\in A$ and $f,g\in H$. Then the pseudobracket $[,]$ is H-bilinear.

Secondly, assume that $(A,*,\alpha)$ is multiplicative, then

$$\begin{aligned}
(id\otimes\alpha)[a,b] &= (id\otimes\alpha)(D(a)*b)-(id\otimes\alpha)((\sigma\otimes_H id)(b*D(a))) \\
&= \alpha(D(a))*\alpha(b)-(\sigma\otimes_H id)(\alpha(b)*\alpha(D(a))) \\
&= D(\alpha(a))*\alpha(b)-(\sigma\otimes_H id)(\alpha(b)*D(\alpha(a))) \\
&= [\alpha(a),\alpha(b)].
\end{aligned}$$

Hence $(A,[,],\alpha)$ is multiplicative.

Finally, we have

$$\begin{aligned}
[[a,b],\alpha(c)] &= (D(a)*D(b))*\alpha(c)-((132)\otimes_H id)(\alpha(c)*(id\otimes_H D)(D(a)*b)) \\
&\quad +((132)\otimes_H id)(\alpha(c)*(\sigma\otimes_H id)(D(b)*D(a)))-((12)\otimes_H id) \\
&\quad ((D(b)*D(a))*\alpha(c)) \\
&= (D(a)*D(b))*\alpha(c)-((132)\otimes_H id)(\alpha(c)*(D(a)*D(b))) \\
&\quad +((13)\otimes_H id)(\alpha(c)*(D(b)*D(a)))-((12)\otimes_H id)((D(b)*D(a))*\alpha(c)),
\end{aligned}$$

and similarly,

$$\begin{aligned}[\alpha(a),[b,c]] &= D(\alpha(a))*(D(b)*c)-((123)\otimes_H id)((D(b)*c)*D(\alpha(a)))\\ &\quad +((13)\otimes_H id)((c*D(b))*D(\alpha(a)))-((23)\otimes_H id)(D(\alpha(a))*(c*D(b))),\end{aligned}$$

$$\begin{aligned}[\alpha(b),[a,c]] &= D(\alpha(b))*(D(a)*c)-((123)\otimes_H id)((D(a)*c)*D(\alpha(b)))\\ &\quad +((13)\otimes_H id)((c*D(a))*D(\alpha(b)))-((23)\otimes_H id)(D(\alpha(b))*(c*D(a))).\end{aligned}$$

It follows that

$$\begin{aligned}&[\alpha(a),[b,c]]-((12)\otimes_H id)[\alpha(b),[a,c]]\\ =\ & D(\alpha(a))*(D(b)*c)-((123)\otimes_H id)((D(b)*c)*D(\alpha(a)))\\ &+((13)\otimes_H id)((c*D(b))*D(\alpha(a)))-((23)\otimes_H id)(D(\alpha(a))*(c*D(b)))\\ &-((12)\otimes_H id)(D(\alpha(b))*(D(a)*c))+((23)\otimes_H id)((D(a)*c)*D(\alpha(b)))\\ &-((132)\otimes_H id)((c*D(a))*D(\alpha(b)))+((123)\otimes_H id)(D(\alpha(b))*(c*D(a)))\\ =\ & (D(a)*D(b))*\alpha(c)-((132)\otimes_H id)(\alpha(c)*(D(a)*D(b)))\\ &+((13)\otimes_H id)(\alpha(c)*(D(b)*D(a)))-((12)\otimes_H id)((D(b)*D(a))*\alpha(c))\\ =\ & [[a,b],\alpha(c)].\end{aligned}$$

This completes the proof. □

Next, we construct Hom-Leibniz H-pseudoalgebras from (Hom-)associative algebras.

Proposition 3.9. Let (A,α) be a Hom-associative algebra equipped with a k-linear map $D: A \longrightarrow A$ such that

$$D\circ\alpha = \alpha\circ D, \tag{3. 6}$$

$$D(aD(b)) = D(a)D(b) = D(D(a)b), \tag{3. 7}$$

for all $a,b\in A$. Then $(H\otimes H\otimes A, [,], id\otimes id\otimes\alpha)$ is a Hom-Leibniz H-pseudoalgebra with the pseudobracket

$$[f\otimes x\otimes a, g\otimes y\otimes b] = (f\otimes gx_1)\otimes_H(1\otimes yx_2\otimes D(a)b)-(fy_1\otimes g)\otimes_H(1\otimes xy_2\otimes bD(a))$$

for all $a,b\in A$ and $f,g,x,y\in H$.

Proof. It is easy to show that L is a left H-module with the action

$$h(f\otimes x\otimes a) = hf\otimes x\otimes a$$

and the pseudobracket $[,]$ is H-bilinear. Here we just verify condition (2.1). For all $a,b,c\in A$ and $f,g,h,x,y,z\in H$, we have

$$\begin{aligned}&[[f\otimes x\otimes a, g\otimes y\otimes b], h\otimes z\otimes\alpha(c)]\\ =\ & (f\otimes gx_1\otimes 1)(\Delta\otimes id)(1\otimes hy_1x_{21})\otimes_H 1\otimes zy_2x_{22}\otimes D(D(a)b)\alpha(c)\\ &-(f\otimes gx_1\otimes 1)(\Delta\otimes id)(z_1\otimes h)\otimes_H 1\otimes yx_2z_2\otimes\alpha(c)D(D(a)b)\\ &-(fy_1\otimes g\otimes 1)(\Delta\otimes id)(1\otimes hx_1y_{21})\otimes_H 1\otimes zx_2y_{22}\otimes D(bD(a))\alpha(c)\\ &+(fy_1\otimes g\otimes 1)(\Delta\otimes id)(z_1\otimes h)\otimes_H 1\otimes xy_2z_2\otimes\alpha(c)D(bD(a))\\ =\ & (f\otimes gx_1\otimes hy_1x_{21})\otimes_H 1\otimes zy_2x_{22}\otimes D(D(a)b)\alpha(c)\\ &-(fz_{11}\otimes gx_1z_{12}\otimes h)\otimes_H 1\otimes yx_2z_2\otimes\alpha(c)D(D(a)b)\\ &-(fy_1\otimes g\otimes hx_1y_{21})\otimes_H 1\otimes zx_2y_{22}\otimes D(bD(a))\alpha(c)\\ &+(fy_1z_{11}\otimes gz_{12}\otimes h)\otimes_H 1\otimes xy_2z_2\otimes\alpha(c)D(bD(a)).\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
& [f\otimes x\otimes \alpha(a),[g\otimes y\otimes b,h\otimes z\otimes c]] \\
= & (f\otimes gx_{11}\otimes hy_1x_{12})\otimes_H 1\otimes zy_2x_2\otimes D(\alpha(a))(D(b)c) \\
& -(fz_1y_{21}\otimes g\otimes hy_1)\otimes_H 1\otimes xz_2y_{22}\otimes (D(b)c)D(\alpha(a)) \\
& -(f\otimes gz_1x_{11}\otimes hx_{12})\otimes_H 1\otimes yz_2x_2\otimes D(\alpha(a))(cD(b)) \\
& +(fy_1z_{21}\otimes gz_1\otimes h)\otimes_H 1\otimes xy_2z_{22}\otimes (cD(b))D(\alpha(a)) \\
= & (f\otimes gx_1\otimes hy_1x_2)\otimes_H 1\otimes zy_2x_3\otimes D(D(a)b)\alpha(c) \\
& -(fz_1y_2\otimes g\otimes hy_1)\otimes_H 1\otimes xz_2y_3\otimes D(\alpha(b))(cD(a)) \\
& -(f\otimes gz_1x_1\otimes hx_2)\otimes_H 1\otimes yz_2x_3\otimes (D(a)c)D(\alpha(b)) \\
& +(fy_1z_2\otimes gz_1\otimes h)\otimes_H 1\otimes xy_2z_3\otimes \alpha(c)D(bD(a))
\end{aligned}
$$

and

$$
\begin{aligned}
& ((12)\otimes_H id)[g\otimes y\otimes \alpha(b),[f\otimes x\otimes a,h\otimes z\otimes c]] \\
= & (fy_{11}\otimes g\otimes hx_1y_{12})\otimes_H 1\otimes zx_2y_2\otimes D(\alpha(b))(D(a)c) \\
& -(f\otimes gz_1x_{21}\otimes hx_1)\otimes_H 1\otimes yz_2x_{22}\otimes (D(a)c)D(\alpha(b)) \\
& -(fz_1y_{11}\otimes g\otimes hy_{12})\otimes_H 1\otimes xz_2y_2\otimes D(\alpha(b))(cD(a)) \\
& +(fz_1\otimes gx_1z_{21}\otimes h)\otimes_H 1\otimes yx_2z_{22}\otimes (cD(a))D(\alpha(b)) \\
= & (fy_1\otimes g\otimes hx_1y_2)\otimes_H 1\otimes zx_2y_3\otimes D(bD(a))\alpha(c) \\
& -(f\otimes gz_1x_2\otimes hx_1)\otimes_H 1\otimes yz_2x_3\otimes (D(a)c)D(\alpha(b)) \\
& -(fz_1y_1\otimes g\otimes hy_2)\otimes_H 1\otimes xz_2y_3\otimes D(\alpha(b))(cD(a)) \\
& +(fz_1\otimes gx_1z_2\otimes h)\otimes_H 1\otimes yx_2z_3\otimes \alpha(c)D(D(a)b).
\end{aligned}
$$

Using the above equations, we have

$$
\begin{aligned}
& [f\otimes x\otimes \alpha(a),[g\otimes y\otimes b,h\otimes z\otimes c]]-((12)\otimes_H id)[g\otimes y\otimes \alpha(b),[f\otimes x\otimes a,h\otimes z\otimes c]] \\
= & [[f\otimes x\otimes a,g\otimes y\otimes b],h\otimes z\otimes \alpha(c)].
\end{aligned}
$$

This completes the proof. □

Remark 3.10. Let $A = sp\{x,y\}$ and define $\alpha : A \longrightarrow A$ by $\alpha(x) = -x, \alpha(y) = y$. Then (A,α) is a Hom-algebra with the multiplication $xy = yx = -x,\ x^2 = 0,\ y^2 = y$. Define a map $D : A \longrightarrow A$ by $D(x) = -x, D(y) = -y$. Then D satisfies (3.6) and (3.7).

Proposition 3.11. Let Y be a commutative associative algebra and a right H-module satisfying $(xy)h = (xh_1)(yh_2))$. Suppose that (L,α) is a Hom-Leibniz H-pseudoalgebra, then $(Y\otimes L, id\otimes\alpha)$ is a Hom-Leibniz H-pseudoalgebra with the left H-module action

$$h(x\otimes a) = xS(h_1)\otimes h_2a$$

and the pseudobracket

$$[x\otimes a,y\otimes b] = \sum_i f_{i1}\otimes g_{i1}\otimes_H (xf_{i2})(yg_{i2})\otimes c_i,$$

where $[a,b] = \sum_i f_i\otimes g_i\otimes_H c_i$ for all $x\otimes a, y\otimes b\in Y\otimes L, h\in H$.

Proof. It is a routine computation and we omit the details. □

Finally, we construct Hom-Leibniz H-pseudoalgebras from the representations of Hom-type H-pseudoalgebras.

Proposition 3.12. Let $(M, *, \beta)$ be a representation of Hom-Lie H-pseudoalgebra $(L, [*], \alpha)$, and $L' = L \oplus M$ a direct sum of left H-modules. Then $(L', \alpha + \beta)$ is a Hom-Leibniz H-pseudoalgebra with the pseudobracket

$$[x + a, y + b] = [x * y] + x * b, \quad \forall\, x, y \in L, a, b \in M.$$

Proof. It is easy to show that L' is a left H-module with $h(x + a) = hx + ha$, for $h \in H, x + a \in L \oplus M$. Since $(\alpha + \beta)(h(x + a)) = \alpha(hx) + \beta(ha) = h\alpha(x) + h\beta(a) = h((\alpha + \beta)(x + a))$, and

$$\begin{aligned} & [h(x + a), g(y + b)] \\ = \;& [hx + ha, gy + gb] = [hx * gy] + (hx) * (gb) \\ = \;& (h \otimes g \otimes 1)([x * y] + x * b) = (h \otimes g \otimes 1)[x + a, y + b], \end{aligned}$$

for all $x, y, z \in L, a, b, c \in M$ and $h, g \in H$, we have $\alpha + \beta \in Hom_H(L', L')$ and $[,] \in Hom_{H^{\otimes 2}}(L' \otimes L', H \otimes H \otimes_H L')$. Now we check the condition (2.1) as follows

$$\begin{aligned} & [[x + a, y + b], (\alpha + \beta)(z + c)] \\ = \;& [[x * y] * \alpha(z)] + [x * y] * \beta(c) \\ = \;& [\alpha(x) * [y * z]] - ((12) \otimes_H id)[\alpha(y) * [x * z]] + \alpha(x) * (y * c) - ((12) \otimes_H id)(\alpha(y) * (x * c)) \\ = \;& [(\alpha + \beta)(x + a), [y + b, z + c]] - ((12) \otimes id)[(\alpha + \beta)(y + b), [x + a, z + c]]. \end{aligned}$$

This completes the proof. □

Proposition 3.13. Let (L_1, α) and (L_2, β) be two Hom-Leibniz H-pseudoalgebras, and (L_2, β) a representation of (L_1, α), such that

$$[[a, b], \alpha(x)] = [\beta(a), [b, x]] - ((12) \otimes_H id)[\beta(b), [a, x]], \tag{3. 8}$$
$$[[x, a], \beta(b)] = [\alpha(x), [a, b]] - ((12) \otimes_H id)[\beta(a), [x, b]], \tag{3. 9}$$
$$[[a, x], \beta(b)] = [\beta(a), [x, b]] - ((12) \otimes_H id)[\alpha(x), [a, b]], \tag{3. 10}$$

for all $x, y \in L_1$ and $a, b \in L_2$. Then $(L = L_1 \oplus L_2, \alpha + \beta)$ is a Hom-Leibniz H-pseudoalgebra with the pseudobracket

$$[x + a, y + b] = [x, y] + [x, b] + [a, y] + [a, b].$$

Proof. For all $h, g \in H, x, y \in L_1$ and $a, b \in L_2$, we have

$$\begin{aligned} h((\alpha + \beta)(x + a)) &= h(\alpha(x) + \beta(a)) = h\alpha(x) + h\beta(a) \\ &= \alpha(hx) + \beta(ha) = (\alpha + \beta)(h(x + a)), \end{aligned}$$

and

$$\begin{aligned} & (h \otimes g \otimes_H 1)[x + a, y + b] \\ = \;& (h \otimes g \otimes_H 1)[x, y] + (h \otimes g \otimes_H 1)[x, b] + (h \otimes g \otimes_H 1)[a, y] + (h \otimes g \otimes_H 1)[a, b] \\ = \;& [hx, gy] + [hx, gb] + [ha, gy] + [ha, gb] = [hx + ha, gy + gb] \\ = \;& [h(x + a), g(y + b)]. \end{aligned}$$

It follows that $\alpha + \beta$ is a homomorphism of L and $(L, \alpha + \beta)$ satisfies the H-bilinearity.

Now we check the condition (2.1). For all $x, y, z \in L_1$ and $a, b, c \in L_2$, we have

$$\begin{aligned} & [[x + a, y + b], \alpha(z) + \beta(c)] \\ = \;& [[x, y] + [x, b] + [a, y] + [a, b], \alpha(z) + \beta(c)] + [[x, y], \alpha(z)] + [[x, y], \beta(c)] + [[x, b], \alpha(z)] \\ & + [[a, y], \alpha(z)] + [[a, b], \alpha(z)] + [[a, b], \beta(c)] + [[x, b], \beta(c)] + [[a, y], \beta(c)]. \end{aligned}$$

Similarly, we get

$$\begin{aligned}&[\alpha(x)+\beta(a),[y+b,z+c]]\\=\;&[\alpha(x),[y,z]]+[\alpha(x),[y,c]]+[\alpha(x),[b,z]]+[\alpha(x),[b,c]]+[\beta(a),[y,z]]\\&+[\beta(a),[y,c]]+[\beta(a),[b,z]]+[\beta(a),[b,c]],\end{aligned}$$

and

$$\begin{aligned}&[\alpha(y)+\beta(b),[x+a,z+c]]\\=\;&[\alpha(y),[x,z]]+[\alpha(y),[x,c]]+[\alpha(y),[a,z]]+[\alpha(y),[a,c]]+[\beta(b),[x,z]]\\&+[\beta(b),[x,c]]+[\beta(b),[a,z]]+[\beta(b),[a,c]].\end{aligned}$$

Observe that, in general, using equations ($HLM1$)-($HLM4$) and (3.8)-(3.10), we can obtain

$$\begin{aligned}&[[x+a,y+b],\alpha(z)+\beta(c)]\\=\;&[\alpha(x)+\beta(a),[y+b,z+c]]-((12)\otimes_H id)([\alpha(y)+\beta(b),[x+a,z+c]]).\end{aligned}$$

This completes the proof. □

Example 3.14. Let (M,β) be a representation of Hom-Leibniz H-pseudoalgebra (L,α). Then (M,β) is a Hom-Leibniz H-pseudoalgebra with $[m,n]=0$ for all $m,n\in M$. By Proposition 3.13, $(L\oplus M,\alpha\otimes\beta)$ is a Hom-Leibniz H-pseudoalgebra.

4. Representations of Hom-Leibniz *H*-pseudoalgebras

For a Hom-Leibniz H-pseudoalgebra (L,α), the (L,α)-module (M,β) is said to be skew-symmetric (resp. right trivial) if $[a,m]=-(\sigma\otimes_H id)[m,a]$ (resp. $[m,a]=0$) for all $a\in L, m\in M$. It is called simple if every submodule N of M is equal to either 0 or M, and there is an element $a\in L$ and an element $m\in M$ such that either $[a,m]\neq 0$ or $[m,a]\neq 0$. A simple (L,α)-module is also called an irreducible representation of (L,α).

Proposition 4.1. Let (L,α) be a Hom-Leibniz H-pseudoalgebra and (M,β) a (L,α)-module. We denote $M_1:=\{m\in M|[m,a]=0 \text{ for all } a\in L\}$ and $M_2:=\{m\in M|[a,m]+(\sigma\otimes_H id)[m,a]=0 \text{ for all } a\in L\}$. Then (M_1,β) and (M_2,β) are submodules of (M,β). Therefore, every simple (L,α)-module is either right trivial or skew-symmetric.

Proof. For all $a\in L, m\in M_1$, we have $[a,hm]=(1\otimes h\otimes_H 1)[a,m]=0$, so $hm\in M_1$, i.e., M_1 is a H-submodule of M. Similarly, we can obtain that M_2 is a H-submodule of M. Since $[\beta(m),\alpha(a)]=(id_{H^{\otimes 2}}\otimes_H\alpha)[m,a]=0$ and $[[a,m],\alpha(b)]=[\alpha(a),[m,b]]-((12)\otimes_H id)[\beta(m),[a,b]]=0$ for all $m\in M_1$, we have $\beta(m)\in M_1$, $[a,m]\in H^{\otimes 2}\otimes_H M_1$ and $[m,a]\in H^{\otimes 2}\otimes_H M_1$. Hence (M_1,β) is a submodule of (M,β).

Now we prove that (M_2,β) is a submodule of (M,β). For all $a,b\in L$ and $n\in M_2$, we have

$$\begin{aligned}&[\alpha(b),[n,a]]+((123)\otimes_H id)[[n,a],\alpha(b)]\\=\;&[\alpha(b),[n,a]]+((123)\otimes_H id)([\beta(n),[a,b]]-((12)\otimes_H id)[\alpha(a),[n,b]])\\=\;&-((23)\otimes_H id)[\alpha(b),[a,n]]+((123)\otimes_H id)[\beta(n),[a,b]]-((13)\otimes_H id)[\alpha(a),[n,b]]\\=\;&-((23)\otimes_H id)(((12)\otimes_H id)[\alpha(a),[b,n]]-((12)\otimes_H id)[[a,b],\beta(n)])\\&+((123)\otimes_H id)[\beta(n),[a,b]]-((13)\otimes_H id)[\alpha(a),[n,b]]\\=\;&((132)\otimes_H id)[[a,b],\beta(n)]+((123)\otimes_H id)[\beta(n),[a,b]]-((132)\otimes_H id)[\alpha(a),[b,n]]\\&-((13)\otimes_H id)[\alpha(a),[n,b]]\\=\;&0,\end{aligned}$$

and

$$\begin{aligned}
& [\alpha(b),[a,n]]+((123)\otimes_H id)[[a,n],\alpha(b)] \\
= & -((12)\otimes_H id)[[a,b],\beta(n)]+((12)\otimes_H id)[\alpha(a),[b,n]]+((123)\otimes_H id)[[a,n],\alpha(b)]) \\
= & -((12)\otimes_H id)[[a,b],\beta(n)]-((123)\otimes_H id)[\alpha(a),[n,b]]+((123)\otimes_H id)[[a,n],\alpha(b)]) \\
= & -((12)\otimes_H id)[[a,b],\beta(n)]+((123)\otimes_H id)[[a,n],\alpha(b)] \\
& -((123)\otimes_H id)(((12)\otimes_H id)[\beta(n),[a,b]]-((12)\otimes_H id)[[n,a],\alpha(b)]) \\
= & -((12)\otimes_H id)[[a,b],\beta(n)]+((123)\otimes_H id)[[a,n],\alpha(b)]-((13)\otimes_H id)[\beta(n),[a,b]] \\
& +((13)\otimes_H id)[[n,a],\alpha(b)] \\
= & \ 0.
\end{aligned}$$

Then $[a,n]\in H^{\otimes 2}\otimes_H M_2$ and $[n,a]\in H^{\otimes 2}\otimes_H M_2$. It follows that (M_2,β) is a submodule of (M,β). Suppose that M is simple, then $M_1=0$ or $M_1=M$. If $M_1=M$, then M is right trivial. Next, we assume that $M_1=0$. Using ($HLM3$) and ($HLM4$), we have $[[m,a],\alpha(b)]+(\sigma\otimes_H id)[[a,m],\alpha(b)]=[[m,a]+(\sigma\otimes_H id)[a,m],\alpha(b)]=0$ for all $a,b\in L$ and $m\in M$. Since $M_1=0$, we have $[m,a]+(\sigma\otimes_H id)[a,m]=0$. Thus M is skew-symmetric. □

Remark 4.2. Let (L,α) be a Hom-Leibniz H-pseudoalgebra and (L,α) an irreducible representation of (L,α). Then by Proposition 4.1, the submodule (L,α) of (L,α) is skew-symmetric, i.e., (L,α) is a Hom-Lie H-pseudoalgebra.

Now we construct the representations of Hom-Leibniz H-pseudoalgebras.

Proposition 4.3. Let (L,α) be a Hom-Lie H-pseudoalgebra, and (M,β) a representation of (L,α). Define $[m,a]=-(\sigma\otimes_H id)[a,m]$, or $[m,a]=0$ for all $a\in L, m\in M$. Then (M,β) is a representation of (L,α) as a Hom-Leibniz H-pseudoalgebra.

Proof. Obviously, (M,β) is a representation of (L,α) as a Hom-Leibniz H-pseudoalgebra when $[m,a]=0$. Suppose $[m,a]=-(\sigma\otimes_H id)[a,m]$, it suffices to show conditions ($HLM2$)-($HLM4$). For all $a,b\in L, m\in M$, it is easy to see that

$$[a,[m,b]]=-((23)\otimes_H id)[a,[b,m]],\quad [[a,m],b]=-((132)\otimes_H id)[b,[a,m]].$$

Using the above equations, we compute:

$$\begin{aligned}
(id\otimes_H\beta)[m,a] &= -(id\otimes_H\beta)(\sigma\otimes_H id)[a,m] \\
&= -(\sigma\otimes_H id)((id_{H^{\otimes 2}}\otimes_H\beta)[a,m]) \\
&= -(\sigma\otimes_H id)[\alpha(a),\beta(m)]=[\beta(m),\alpha(a)],
\end{aligned}$$

$$\begin{aligned}
[[m,a],\alpha(b)] &= ((13)\otimes_H id)[\alpha(b),[a,m]] \\
&= ((13)\otimes_H id)(-((12)\otimes_H id)[[a,b],\beta(m)]+((12)\otimes_H id)[\alpha(a),[b,m]]) \\
&= -((123)\otimes_H id)[[a,b],\beta(m)]+((123)\otimes_H id)[\alpha(a),[b,m]] \\
&= ((123)(132)\otimes_H id)[\beta(m),[a,b]]-((123)(23)\otimes_H id)[\alpha(a),[m,b]] \\
&= [\beta(m),[a,b]]-((12)\otimes_H id)[\alpha(a),[m,b]],
\end{aligned}$$

and

$$\begin{aligned}
[[a,m],\alpha(b)] &= -((132)\otimes_H id)[\alpha(b),[a,m]] \\
&= ((132)(12)\otimes_H id)[[a,b],\beta(m)]-((132)(12)\otimes_H id)[\alpha(a),[b,m]] \\
&= ((132)(12)(23)\otimes_H id)[\alpha(a),[m,b]]-((123)(12)(132)\otimes_H id)[\beta(m),[a,b]] \\
&= [\alpha(a),[m,b]]-((12)\otimes_H id)[\beta(m),[a,b]].
\end{aligned}$$

Hence we complete the proof. □

By Theorem 3.4, the tensor product of a commutative Hom-associative H_1-pseudoalgebra and a Hom-Leibniz H_2-pseudoalgebras is a Hom-Leibniz $H_1 \otimes H_2$-pseudoalgebra. This conclusion is also true for modules.

Proposition 4.4. Let (L_1, α_1) be a commutative Hom-associative H_1-pseudoalgebra, and (L_2, α_2) a Hom-Leibniz H_2-pseudoalgebra. Then $(M_1 \otimes M_2, \beta_1 \otimes \beta_2)$ is a representation of $(L = L_1 \otimes L_2, \alpha_1 \otimes \alpha_2)$, where (M_1, β_1) is a (L_1, α_1)-module and (M_2, β_2) is a (L_2, α_2)-module.

Proof. By Theorem 3.4, $(L = L_1 \otimes L_2, \alpha_1 \otimes \alpha_2)$ is a Hom-Leibniz $H = H_1 \otimes H_2$-pseudoalgebra. For all $m \otimes m' \in M_1 \otimes M_2$ and $a \otimes a' \in L_1 \otimes L_2$, we define

$$[a \otimes a', m \otimes m'] = (a * m) \otimes [a', m']$$

and

$$[m \otimes m', a \otimes a'] = (\sigma \otimes_{H_1} id)(a * m) \otimes [m', a'].$$

The proof completes by showing that $(HLM1)$-$(HLM5)$ hold, which is a routine computation and we omit the details. □

We also have the following conclusion about the tensor product of modules.

Proposition 4.5. Let (M, β) be a representation of Hom-Leibniz H-pseudoalgebra (L, α), and Y a commutative associative H-differential algebra with a right action of H. Suppose that N is a right Y-module and right H-module such that $(ny)h = (nh_1)(yh_2)$ for all $n \in N, y \in Y$ and $h \in H$. Then $(N \otimes M, id \otimes \beta)$ is a representation of Hom-Leibniz H-pseudoalgebra $(Y \otimes L, id \otimes \alpha)$.

Proof. Obviously, $N \otimes M$ is a left H-module with the action $h(n \otimes m) = nS(h_1) \otimes h_2 m$. Define

$$[n \otimes m, x \otimes a] = \sum_i h_{i1} \otimes l_{i1} \otimes_H (nh_{i2})(xl_{i2}) \otimes d_i,$$

$$[x \otimes a, n \otimes m] = \sum_j f_{j1} \otimes g_{j1} \otimes_H (ng_{j2})(xf_{j2}) \otimes e_j,$$

where $[m, a] = \sum_i h_i \otimes l_i \otimes_H d_i$ and $[a, m] = \sum_j f_j \otimes g_j \otimes_H e_j$.

For all $n \otimes m \in N \otimes M, x \otimes a \in Y \otimes L$ and $f, g \in H$, we have

$$\begin{aligned}
[f(n \otimes m), g(x \otimes a)] &= [nS(f_1) \otimes f_2 m, xS(g_1) \otimes g_2 a] \\
&= \sum_i (f_2 h_i)_1 \otimes (g_2 l_i)_1 \otimes_H (nS(f_1)(f_2 h_i)_2)(xS(g_1)(g_2 l_i)_2) \otimes d_i \\
&= \sum_i f h_{i1} \otimes g l_{i1} \otimes_H (nh_{i2})(xl_{i2}) \otimes d_i \\
&= (f \otimes g \otimes_H 1)[n \otimes m, x \otimes a].
\end{aligned}$$

Similarly, we can get $[g(x \otimes a), f(n \otimes m)] = (g \otimes f \otimes_H 1)[x \otimes a, n \otimes m]$.

Now we check that $(HLM1)$-$(HLM5)$ hold.

$$\begin{aligned}
(id \otimes_H (id \otimes \beta))[x \otimes a, n \otimes m] &= \sum_i f_{i1} \otimes g_{i1} \otimes_H (ng_{i2})(yf_{i2}) \otimes \beta(e_i) \\
&= [x \otimes \alpha(a), n \otimes \beta(m)] \\
&= [(id \otimes \alpha)(x \otimes a), (id \otimes \beta)(n \otimes m)].
\end{aligned}$$

Hence $(HLM1)$ holds. In the same way, we can obtain $(HLM2)$. Suppose

$$[a, m] = \sum_i f_i \otimes g_i \otimes_H e_i, \quad [e_i, \alpha(b)] = \sum_j f_{ij} \otimes g_{ij} \otimes_H e_{ij},$$

$$[m, b] = \sum_i s_i \otimes t_i \otimes_H c_i, \quad [\alpha(a), c_i] = \sum_j s_{ij} \otimes t_{ij} \otimes_H c_{ij},$$

$$[a, b] = \sum_i p_i \otimes q_i \otimes_H k_i, \quad [\beta(m), k_i] = \sum_j p_{ij} \otimes q_{ij} \otimes_H k_{ij}.$$

Then we have

$$
\begin{aligned}
& [[x\otimes a, n\otimes m], y\otimes \alpha(b)] \\
= & \sum_{i,j}(f_if_{ij1})_1\otimes (g_if_{ij2})_1\otimes g_{ij1}\otimes_H (n(g_if_{ij2})_2)(x(f_if_{ij1})_2)(yg_{ij2})\otimes e_{ij} \\
= & \sum_{i,j} s_{ij1}\otimes (s_it_{ij1})_1\otimes (t_it_{ij2})_1\otimes_H (n(s_it_{ij1})_2)(xs_{ij2})(y(t_it_{ij2})_2)\otimes c_{ij} \\
& -(p_iq_{ij1})_1\otimes p_{ij1}\otimes (q_iq_{ij2})_1\otimes_H (np_{ij2})(x(p_iq_{ij1})_2)(y(q_iq_{ij2})_2)\otimes k_{ij} \\
= & [x\otimes\alpha(a), [n\otimes m, y\otimes b]] - ((12)\otimes_H id)[n\otimes\beta(m), [x\otimes a, y\otimes b]].
\end{aligned}
$$

So $(HLM4)$ holds. The proofs of $(HLM3)$ and $(HLM5)$ are similar to the one of $(HLM4)$, so we omit the details. This completes the proof. □

5. Hom-annihilation algebras

The main tool in the study of H-pseudoalgebras and their representations is the annihilation algebra. In detail, a Leibniz algebra is usually associated to a Leibniz H-pseudoalgebra, that is the annihilation algebra. Moreover, the category of all Leibniz H-pseudoalgebras is a left module category over the category of all right H-differential commutative associative algebras (see Proposition 2.2 in [19]). In this section we consider the Hom-type generalization of annihilation algebras.

A Hom-associative algebra (A,α) is called a Hom-associative H-differential algebra ([17]) if it is also a left H-module such that $h(ab)=(h_1a)(h_2b)$, where $\Delta(h)=h_1\otimes h_2$ for all $a,b\in A$ and $h\in H$. Similarly, a Hom-Leibniz algebra $(L,[\cdot],\alpha)$ is called a Hom-Leibniz H-differential algebra if it is also a left H-module such that $h[a\cdot b]=[(h_1a)\cdot(h_2b)]$ for all $h\in H, a,b\in L$.

Let Y be an H-bimodule which is a commutative associative H-differential algebra both for the left and for the right action of H (i.e., $h(xy)=(h_1x)(h_2y), (xy)h=(xh_1)(yh_2)$, for all $x,y\in Y, h\in H$), for example, $Y=X:=H^*$. For a left H-module L, recall that $\mathcal{A}_YL=Y\otimes_H L$ is a left H-module via $h(x\otimes_H a)=hx\otimes_H a$ for all $h\in H, x\otimes_H a\in\mathcal{A}_YL$.

In addition, if $(L,[,],\alpha)$ is a Hom-Leibniz H-pseudoalgebra, then $(\mathcal{A}_YL, id\otimes\alpha)$ becomes a Hom-Leibniz H-differential algebra with bracket given by

$$[(x\otimes_H a)\cdot(y\otimes_H b)]=\sum_i (xf_i)(yg_i)\otimes_H e_i,$$

if $[a,b]=\sum_i f_i\otimes g_i\otimes_H e_i$. In fact, we have the following conclusion.

Proposition 5.1. Let (L,α) be a (multiplicative) Hom-Leibniz H-pseudoalgebra. Then $(\mathcal{A}_YL,[\cdot], id_Y\otimes\alpha)$ is a (multiplicative) Hom-Leibniz H-differential algebra with the action

$$h[(x\otimes_H a)\cdot(y\otimes_H b)]=[(h_1x\otimes_H a)\cdot(h_2y\otimes_H b)],$$

for all $h\in H, x\otimes_H a, y\otimes_H b\in\mathcal{A}_YL$.

A similar statement holds for modules as well: if (M,β) is a (L,α)-module, then $(\mathcal{A}_YM,$
$id_Y\otimes_H\beta)$ is a $(\mathcal{A}_YL, id\otimes\alpha)$-module with the compatible H-action $h(am)=(h_1a)(h_2m)$, $(am)h=(ah_1)(mh_2)$ and the module actions

$$
\begin{aligned}
[(x\otimes_H a)\cdot(y\otimes_H m)] &= \sum_i (xf_i)(yg_i)\otimes_H m_i, \\
[(y\otimes_H m)\cdot(x\otimes_H a)] &= \sum_j (yf'_j)(xg'_j)\otimes_H m'_j,
\end{aligned}
$$

where $[a,m]=\sum_i f_i\otimes g_i\otimes_H m_i, [m,a]=\sum_j f'_j\otimes g'_j\otimes_H m'_j$ for all $x\otimes_H a\in\mathcal{A}_YL, y\otimes_H m\in\mathcal{A}_YM$.

Proof. The proof in the case of Hom-Leibniz H-pseudoalgebra is similar to Proposition 3.1 in [17]. Here we just show that $(\mathcal{A}_Y M, id \otimes_H \beta)$ is a $(\mathcal{A}_Y L, id \otimes \alpha)$-module. For all $m \in M, a, b \in L, x, y, z \in Y$, suppose

$$[m,a] = \sum_i f_i \otimes g_i \otimes_H m_i, \quad [m_i, \alpha(b)] = \sum_j f_{ij} \otimes g_{ij} \otimes_H m_{ij},$$
$$[a,b] = \sum_i k_i \otimes l_i \otimes_H e_i, \qquad [\beta(m), e_i] = \sum_j k_{ij} \otimes l_{ij} \otimes_H e_{ij},$$
$$[m,b] = \sum_i p_i \otimes q_i \otimes_H n_i, \qquad [\alpha(a), n_i] = \sum_j p_{ij} \otimes q_{ij} \otimes_H n_{ij}.$$

Then we have

$$\begin{aligned} & [[(x \otimes_H m) \cdot (y \otimes_H a)] \cdot (z \otimes_H \alpha(b))] \\ = & \sum_i [((xf_i)(yg_i) \otimes_H m_i) \cdot (z \otimes_H \alpha(b))] \\ = & \sum_{i,j} (((xf_i)(yg_i))f_{ij})(zg_{ij}) \otimes_H m_{ij} \\ = & \sum_{i,j} (xf_i f_{ij1})(yg_i f_{ij2})(zg_{ij}) \otimes_H m_{ij}. \end{aligned}$$

Similarly,

$$[(x \otimes_H \beta(m)) \cdot [(y \otimes_H a) \cdot (z \otimes_H b)]] = \sum_{i,j} (xk_{ij})(yk_i l_{ij1})(zl_i l_{ij2}) \otimes_H e_{ij},$$

$$[(y \otimes_H \alpha(a)) \cdot [(x \otimes_H m) \cdot (z \otimes_H b)]] = \sum_{i,j} (xp_i q_{ij1})(yp_{ij})(zq_i q_{ij2}) \otimes_H n_{ij}.$$

Since $[[m,a], \alpha(b)] = [\beta(m), [a,b]] - ((12) \otimes_H 1)[\alpha(a), [m,b]]$, which is equivalent to

$$\begin{aligned} & \sum_{i,j} f_i f_{ij1} \otimes g_i f_{ij2} \otimes g_{ij} \otimes_H m_{ij} \\ = & \sum_{i,j} k_{ij} \otimes k_i l_{ij1} \otimes l_i l_{ij2} \otimes_H e_{ij} - \sum_{i,j} p_i q_{ij1} \otimes p_{ij} \otimes q_i q_{ij2} \otimes_H n_{ij}, \end{aligned}$$

it follows that

$$\begin{aligned} & [[(x \otimes_H m) \cdot (y \otimes_H a)] \cdot (z \otimes_H \alpha(b))] \\ = & [(x \otimes_H \beta(m)) \cdot [(y \otimes_H a) \cdot (z \otimes_H b)]] - [(y \otimes_H \alpha(a)) \cdot [(x \otimes_H m) \cdot (z \otimes_H b)]]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & [[(y \otimes_H a) \cdot (x \otimes_H m)] \cdot (z \otimes_H \alpha(b))] \\ = & [(y \otimes_H \alpha(a)) \cdot [(x \otimes_H m) \cdot (z \otimes_H b)]] - [(x \otimes_H \beta(m)) \cdot [(y \otimes_H a) \cdot (z \otimes_H b)]], \end{aligned}$$

and

$$\begin{aligned} & [[(y \otimes_H a) \cdot (z \otimes_H b)] \cdot (x \otimes_H \beta(m))] \\ = & [(y \otimes_H \alpha(a)) \cdot [(z \otimes_H b) \cdot (x \otimes_H m)]] - [(z \otimes_H \alpha(b)) \cdot [(y \otimes_H a) \cdot (x \otimes_H m)]]. \end{aligned}$$

This completes the proof. □

In particular, when $Y = X$, then $\mathcal{A}L = \mathcal{A}_X L = X \otimes_H L$ is a Hom-Leibniz H-differential algebra.

Definition 5.2. The Hom-Leibniz H-differential algebra $(\mathcal{A}L, [\cdot], id \otimes_H \alpha)$ is called a Hom-annihilation algebra of the Hom-Leibniz H-pseudoalgebra $(L, [,], \alpha)$.

Let $H_i (i = 1, 2)$ be cocommutative Hopf algebras, and Y_i commutative associative H_i-differential algebras both for the left and for the right action of H_i for $i = 1, 2$. Define the actions of $H = H_1 \otimes H_2$ on $Y = Y_1 \otimes Y_2$ as follows

$$(h_1 \otimes h_2)(y_1 \otimes y_2) = h_1 y_1 \otimes h_2 y_2, \ (y_1 \otimes y_2)(h_1 \otimes h_2) = y_1 h_1 \otimes y_2 h_2,$$

for all $h_1 \otimes h_2 \in H_1 \otimes H_2$ and $y_1 \otimes y_2 \in Y_1 \otimes Y_2$. Then Y is a commutative associative H-differential algebra. Under this assumption, we have the following result.

Theorem 5.3. Let (L_1, α_1) be a commutative Hom-associative H_1-pseudoalgebra, and (L_2, α_2) a Hom-Leibniz H_2-pseudoalgebra. Suppose that Y_i are H_i-bimodules, which are the commutative associative H_i-differential algebras. Then $(\mathcal{A}_{Y_1 \otimes Y_2}(L_1 \otimes L_2), \alpha) \simeq (\mathcal{A}_{Y_1} L_1 \otimes \mathcal{A}_{Y_2} L_2, \beta)$ as Hom-Leibniz $H := H_1 \otimes H_2$-differential algebras, where $\alpha = id \otimes id \otimes \alpha_1 \otimes \alpha_2$, $\beta = id \otimes \alpha_1 \otimes id \otimes \alpha_2$.

Proof. Let $H_1 = H_2 = k$ in Theorem 3.4, then $(\mathcal{A}_{Y_1} L_1 \otimes \mathcal{A}_{Y_2} L_2, \beta)$ is a Hom-Leibniz algebra with product

$$\begin{aligned} & \gamma(((x \otimes_{H_1} a) \otimes (x' \otimes_{H_2} a')) \otimes ((y \otimes_{H_1} b) \otimes (y' \otimes_{H_2} b'))) \\ = \ & (x \otimes_{H_1} a)(y \otimes_{H_1} b) \otimes (x' \otimes_{H_2} a')(y' \otimes_{H_2} b') \\ = \ & \sum_{i,j} ((x f_i)(y g_i) \otimes_{H_1} e_i) \otimes ((x' m_j)(y' n_j) \otimes_{H_2} l_j), \end{aligned}$$

where $a * b = \sum_i f_i \otimes g_i \otimes_{H_1} e_i$, $a' * b' = \sum_j m_j \otimes n_j \otimes_{H_2} l_j$, for all $x, y \in Y_1, x', y' \in Y_2, a, b \in L_1, a', b' \in L_2$. Furthermore, it is a Hom-Leibniz H-differential algebra by a routine computation. By Theorem 3.4, $(L_1 \otimes L_2, \alpha_1 \otimes \alpha_2)$ is a Hom-Leibniz H-pseudoalgebra. By Proposition 5.1, $(\mathcal{A}_{Y_1 \otimes Y_2}(L_1 \otimes L_2), \alpha)$ is a Hom-Leibniz H-differential algebra with product

$$\mu(((x \otimes x') \otimes_H (a \otimes a')) \otimes ((y \otimes y') \otimes_H (b \otimes b'))) = \sum_{i,j} (x f_i)(y g_i) \otimes (x' m_j)(y' n_j) \otimes_H (e_i \otimes l_j).$$

Now we define two maps as follows

$$\begin{aligned} \phi : (Y_1 \otimes Y_2) \otimes_H (L_1 \otimes L_2) & \longrightarrow (Y_1 \otimes_{H_1} L_1) \otimes (Y_2 \otimes_{H_2} L_2), \\ (x \otimes x') \otimes_H (a \otimes a') & \mapsto (x \otimes_{H_1} a) \otimes (x' \otimes_{H_2} a'), \end{aligned}$$

and

$$\begin{aligned} \varphi : (Y_1 \otimes_{H_1} L_1) \otimes (Y_2 \otimes_{H_2} L_2) & \longrightarrow (Y_1 \otimes Y_2) \otimes_H (L_1 \otimes L_2) \\ (x \otimes_{H_1} a) \otimes (x' \otimes_{H_2} a') & \mapsto (x \otimes x') \otimes_H (a \otimes a'). \end{aligned}$$

It is easy to check that ϕ is a left H-module isomorphism with the inverse φ. It remains to show that ϕ is a morphism of Hom-Leibniz algebra. Note that

$$\begin{aligned} \phi \circ \alpha((x \otimes x') \otimes_H (a \otimes a')) &= \phi((x \otimes x') \otimes_H (\alpha_1(a) \otimes \alpha_2(a'))) \\ &= (x \otimes_{H_1} \alpha_1(a)) \otimes (x' \otimes_{H_2} \alpha_2(a')) \\ &= \beta \circ \phi((x \otimes x') \otimes_H (a \otimes a')), \end{aligned}$$

and

$$\begin{aligned} & \phi \circ \mu(((x \otimes x') \otimes_H (a \otimes a')) \otimes ((y \otimes y') \otimes_H (b \otimes b'))) \\ = \ & \sum_{i,j} ((x f_i)(y g_i) \otimes_{H_1} e_i) \otimes ((x' m_j)(y' n_j) \otimes_{H_2} l_j) \\ = \ & \gamma(((x \otimes_{H_1} a) \otimes (x' \otimes_{H_2} a')) \otimes ((y \otimes_{H_1} b) \otimes (y' \otimes_{H_2} b'))) \\ = \ & \gamma(\phi((x \otimes x') \otimes_H (a \otimes a')) \otimes \phi((y \otimes y') \otimes_H (b \otimes b'))). \end{aligned}$$

So we complete the proof. □

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