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Existence and Uniqueness of Solutions to Some Classes of Nonlocal Semilinear Conformable Fractional Differential or Integrodifferential Equations

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Abstract. The purpose of this paper is to give and prove the fundamental theorem of conformable fractional calculus which is given only in previous associated papers for differentiable functions, and then we scrutinize the existence and uniqueness of solutions to some semilinear Cauchy problems for nonlocal conformable fractional integrodifferential or differential equations which their nonlinear terms include fractional derivative or fractional integral. Two examples are investigated to elucidate the main results.

1. Introduction

Fractional calculus plays a pivotal role in applied mathematics and other scientific fields such as chemistry, physics, engineering, biology, economics as well as signal processing and telecommunication. It describes the most diminutive details of natural phenomena, which is better than using the integer calculus. For the history and more details about applications and significant results on fractional calculus, we refer to [4, 6, 9, 12, 21, 23–26, 28].

A lot of definitions for fractional derivative have been given over the years [17], such as Riemann-Liouville, Caputo, Grunwald-Letnikov, Hadamard derivative, etc. Conformable fractional derivative (CFD) is similar to the directional derivative. It satisfies the classical formulas of the product and quotient of two functions and it has a simple chain rule. For other properties of the CFD, see [1, 16]. The physical interpretation of the CFD is given by D. Zhao and M. Luo in [30] as a special velocity in a specific direction. Many authors used the CFD in modeling physical problems by using their properties to obtain exact and approximate solutions, see for instance [2, 3, 7, 8, 18, 20, 22].

Many authors are interested in studying nonlinear Cauchy problems which have fractional derivatives or fractional integral in its nonlinear term, see [13, 15, 19, 29].

Nonlocal conditions have a main role in describing some peculiarities of chemical, biological, physical or other processes that occur at assorted positions inside the domain, which is clearly not possible with the end-point (initial or boundary) conditions. In fact, nonlocal conditions give accurate results, more precise measurements, and better effects than the conventional conditions.

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M. Herzallah and A. Radwan [14] studied the existence and uniqueness of solutions to the nonlocal-impulsive semilinear conformable fractional differential equation

$$\begin{cases} D^{\alpha}u(t) = A(t)u(t) + f(t,u(t)),\ t \in J^{*} = J \setminus \{t_1,t_2,...,t_m\},\ J = [0,k]; \\ u(0) + g(u) = u_0; \\ u(t_i^{+}) = u(t_i^{-}) + y_i,\ i = 1,2,...,m. \end{cases}$$

where $A(t)$ is a bounded linear operator on a Banach space X, t_i satisfy $0 < t_1 < t_2 < ... < t_m < k$, and $u(t_i^{+}) = lim_{\varepsilon\to 0^{+}} u(t_i+\varepsilon)$ and $u(t_i^{-}) = lim_{\varepsilon\to 0^{-}} u(t_i+\varepsilon)$ represent the right and lift limits of $u(t)$ at $t = t_i$.

Motivated by [14], this paper aims to discuss the existence and uniqueness of solutions of the following two nonlocal problems:

$$\begin{cases} D^{\alpha}u(t) = A(t)u(t) + f(t,u(t),I^{\beta}u(t)),\ \ 0 < \beta \leq \alpha < 1,\ \ t \in [0,T]; \\ \sum_{k=1}^{m} a_k\ u(t_k) = u_0,\ \sum_{k=1}^{m} a_k \neq 0. \end{cases} \tag{1}$$

and

$$\begin{cases} D^{\alpha}u(t) = A(t)u(t) + f(t,u(t),D^{\beta}u(t)),\ \ 0 < \beta \leq \alpha < 1,\ \ t \in [0,T]; \\ \sum_{k=1}^{m} a_k\ u(t_k) = u_0,\ \sum_{k=1}^{m} a_k \neq 0. \end{cases} \tag{2}$$

where D^{α}, D^{β} refer to the CFD's of order α, β sequentially, I^{β} denotes the CFI of order β, $A(t)$ is a bounded linear operator on a Banach space X with constant domain $D(A) \subset X$, $u_0 \in D(A)$ and t_k satisfies $0 < t_1 < t_2 < ... < t_m < T$, $k = 1,2,...,m$.

This paper is organized as follows: In section 2, we demonstrate some notations, definitions, and theorems with giving proof of the fundamental theorem of conformable fractional calculus. In section 3, we discuss the existence and uniqueness of solutions to Problem (1). Section 4 deals with studying the existence and uniqueness of solutions to Problem (2). Finally, section 5 has some examples to make clear the validity and importance of the main results.

2. Preliminaries

Let $J := [0,T]$, $T > 0$, and $C(J,X)$ be the set of all continuous functions $u : J \to X$ with the norm $\|u\|_C = \max\{\|u(t)\| : t \in J\}$.

Definition 2.1 (CFD [1, 16, 30]). *Given a function $u : [0,\infty) \to \mathbb{R}$. The CFD of u of order $\alpha \in (0,1)$ is given by*

$$D^{\alpha}u(t) = \lim_{\varepsilon\to 0} \frac{u(t+\varepsilon t^{1-\alpha}) - u(t)}{\varepsilon},\ \ t > 0.$$

If $\lim_{t\to 0^{+}} D^{\alpha}u(t)$ exists, we set $D^{\alpha}u(0) = \lim_{t\to 0^{+}} D^{\alpha}u(t)$.

Definition 2.2 (CFI [1, 16, 30]). *Given a function $u : [0,\infty) \to \mathbb{R}$. Then for $t > 0$ and $\alpha \in (0,1)$, the CFI of u of order α is defined by*

$$I^{\alpha}u(t) = \int_0^t s^{\alpha-1}u(s)ds,$$

where the integral is the usual Riemann improper integral.

Proposition 2.3 ([1, 5, 16, 30]). *Let $\alpha \in (0,1]$ and $f,g : J \to \mathbb{R}$ be α-differentiable functions at a point $t > 0$. Then*

1. *The CFD and CFI are linear operators;*
2. *If f is α-differentiable at $t_0 > 0$, then f is continuous at t_0;*
3. *The function f could be α-differentiable at a point but not differentiable;*

4. *The CFD satisfies*

$$D^{\alpha}(fg)(t) = f(t)(D^{\alpha}g(t)) + (D^{\alpha}f(t))g(t), \quad \text{and} \quad D^{\alpha}\left(\frac{f}{g}\right)(t) = \frac{g(t)(D^{\alpha}f(t)) - f(t)(D^{\alpha}g(t))}{g^2(t)};$$

5. *If f, in addition, is differentiable, then* $D^{\alpha}f(t) = t^{1-\alpha}(\frac{df}{dt})$ *and* $I^{\alpha}D^{\alpha}f(t) = f(t) - f(0)$.

Theorem 2.4 ([1, 16])**.** *Let $a > 0$ and $u : [a, b] \to \mathbb{R}$ be a given function that satisfies*

1. *u is continuous on $[a, b]$;*
2. *u is α-differentiable for some $\alpha \in (0, 1)$. Then, there exists $c \in (a, b)$ such that*

$$D^{\alpha}u(c) = \frac{u(b) - u(a)}{\frac{b^{\alpha}}{\alpha} - \frac{a^{\alpha}}{\alpha}}.$$

Theorem 2.5 (Conformable Gronwall inequality [1])**.** *Let u be a continuous, nonnegative function on an interval $[0, b]$ and ψ, ϕ be nonnegative constants such that*

$$u(t) \leq \phi + \psi \int_0^t s^{\alpha-1}u(s)ds, \quad t \in J.$$

Then for all $t \in [0, b]$,

$$u(t) \leq \phi \exp\left(\frac{\psi t^{\alpha}}{\alpha}\right).$$

Theorem 2.6 (Fundamental theorem of conformable fractional calculus)**.** *Let $f : [0, b] \to \mathbb{R}$, then we have the following*

(*a*) *If f is an $\alpha-$differentiable function where $\alpha \in (0, 1]$. Then $I^{\alpha}D^{\alpha}f(t) = f(t) - f(0)$.*

(*b*) *If f is an Riemann integrable function on $[0, b]$ and f is continuous for $c \in (0, b)$ then $I^{\alpha}f(t)$ is $\alpha-$differentiable at c and $D^{\alpha}I^{\alpha}f(c) = f(c)$.*

Proof. Let $\{t_0, t_1, t_2, ..., t_n\}$ be an arbitrary partition of $[0, t]$, then we get from the conformable fractional mean value theorem that there is $u_i \in (t_{i-1}, t_i)$ where

$$f(t_i) - f(t_{i-1}) = D^{\alpha}f(u_i)\left(\frac{t_i^{\alpha}}{\alpha} - \frac{t_{i-1}^{\alpha}}{\alpha}\right).$$

Using the ordinary mean value theorem, we get $v_i \in (t_{i-1}, t_i)$ where

$$f(t_i) - f(t_{i-1}) = D^{\alpha}f(u_i)v_i^{\alpha-1}(t_i - t_{i-1})$$

Thus, we get

$$f(t) - f(0) = \sum_{i=1}^{n} D^{\alpha}f(u_i)v_i^{\alpha-1}(t_i - t_{i-1})$$

Let $n \to \infty$, we get (a).

Now, let $F(t) = I^{\alpha}f(t)$, where $f(t)$ is Riemann integrable for $t \in [0, b]$, and is continuous at $c \neq 0$, we get that the function $t^{\alpha-1}f(t)$ is Riemann integrable and is continuous at c. From the fundamental theorem of calculus, we get that $F(t)$ is differentiable at the point c and satisfies that $F'(c) = c^{\alpha-1}f(c)$ but we have if the function is differentiable, we get its fractional conformable derivative in the form

$$D^{\alpha}F(c) = c^{1-\alpha}F'(c) = c^{1-\alpha}c^{\alpha-1}f(c) = f(c).$$

□

Theorem 2.7 (Schaefer's fixed point theorem [10, 11, 27])**.** *Let X be a Banach space and $F : X \to X$ be a completely continuous operator. If the set $E(F) = \{x \in X : \eta Fx = x\}$ for some $\eta \in [0, 1]$ is bounded, then F has at least one fixed point.*

3. Existence and uniqueness of solutions to Problem (1)

Consider the following assumptions:
(H_1) $A(t)$ is a bounded linear operator on a Banach space X where $t \to A(t)$ is continuous in the strong operator topology and $M = \max\{\|A(t)\|: t \in J\}$.
(H_2) $f : J \times X \times X \to X$ is continuous. $N = \max\{f(t, 0, 0) : t \in J\}$ and there exist positive constants μ, γ such that for all $t \in J$ and $u, v \in X$:

$$\|f(t, u_2(t), v_2(t)) - f(t, u_1(t), v_1(t))\| \leq \mu\|u_2(t) - u_1(t)\| + \gamma\|v_2(t) - v_1(t)\|.$$

Lemma 3.1. *Problem* (1) *is equivalent to the integral equation*

$$\begin{aligned} u(t) \;=\; & \frac{u_0}{\sum_{k=1}^m a_k} - \frac{1}{\sum_{k=1}^m a_k} \sum_{k=1}^m a_k \int_0^{t_k} s^{\alpha-1} A(s)u(s)ds \\ & - \frac{1}{\sum_{k=1}^m a_k} \sum_{k=1}^m a_k \int_0^{t_k} s^{\alpha-1} f(s, u(s), I^\beta u(s))ds \\ & + \int_0^t s^{\alpha-1} A(s)u(s)ds + \int_0^t s^{\alpha-1} f(s, u(s), I^\beta u(s))ds. \end{aligned} \tag{3}$$

Proof. Acting I^α on both sides of the fractional integrodifferential equation of (1) with applying part (a) of Theorem 2.4, we obtain

$$u(t) = u(0) + \int_0^t s^{\alpha-1} A(s)u(s)ds + \int_0^t s^{\alpha-1} f(s, u(s), I^\beta u(s))ds, \tag{4}$$

Putting $t = t_k$ in (4) with applying the nonlocal condition of (1), we have

$$\begin{aligned} u(0) \;=\; & \frac{u_0}{\sum_{k=1}^m a_k} - \frac{1}{\sum_{k=1}^m a_k} \sum_{k=1}^m a_k \int_0^{t_k} s^{\alpha-1} A(s)u(s)ds \\ & - \frac{1}{\sum_{k=1}^m a_k} \sum_{k=1}^m a_k \int_0^{t_k} s^{\alpha-1} f(s, u(s), I^\beta u(s))ds. \end{aligned} \tag{5}$$

From (5) into (4), we get (3).
Conversely, if we α-differentiate (3) with applying part (b) of Theorem 2.4, we get

$$\begin{aligned} D^\alpha u(t) &= D^\alpha I^\alpha A(t)u(t) + D^\alpha I^\alpha f(t, u(t), I^\beta u(t)) \\ &= A(t)u(t) + f(t, u(t), I^\beta u(t)), \end{aligned}$$

which is the fractional integrodifferential equation of (1) . Putting $t = t_k$ in (3),

$$\begin{aligned} \sum_{k=1}^m a_k u(t_k) \;&=\; u_0 - \sum_{k=1}^m a_k \int_0^{t_k} s^{\alpha-1} A(s)u(s)ds - \sum_{k=1}^m a_k \int_0^{t_k} s^{\alpha-1} f(s, u(s), I^\beta u(s))ds \\ & \quad + \sum_{k=1}^m a_k \int_0^{t_k} s^{\alpha-1} A(s)u(s)ds + \sum_{k=1}^m a_k \int_0^{t_k} s^{\alpha-1} f(s, u(s), I^\beta u(s))ds \\ &=\; u_0. \end{aligned}$$

which completes the proof. □

Definition 3.2. *By a solution of Problem* (1), *we mean a function* $u \in C(J, D(A))$ *which is* α*-differentiable and satisfies* (3).

Consider the nonempty, bounded, convex and closed set B_r:

$$B_r = \left\{ u \in C([0,T], X) : \|u\|_C \leq r,\ r = \frac{\rho\|u_0\| + \frac{2NT^\alpha}{\alpha}}{1 - \frac{2T^\alpha}{\alpha}\left(M + \mu + \frac{\gamma T^\beta}{\beta}\right)} \right\}. \tag{6}$$

where $\rho := \frac{1}{\sum_{k=1}^{m}|a_k|}$.

Theorem 3.3. *If the assumptions* (H_1)-(H_2) *are satisfied, and* $\frac{T^\alpha}{\alpha}\left(M + \mu + \frac{\gamma T^\beta}{\beta}\right) < \frac{1}{2}$, *then Problem* (1) *has a unique solution* $u \in B_r$.

Proof. Define the operator $L : C(J,X) \to C(J,X)$ such that

$$\begin{aligned} Lu(t) \ = \ & \frac{u_0}{\sum_{k=1}^m a_k} - \frac{1}{\sum_{k=1}^m a_k}\sum_{k=1}^m a_k \int_0^{t_k} s^{\alpha-1}A(s)u(s)ds \\ & - \frac{1}{\sum_{k=1}^m a_k}\sum_{k=1}^m a_k \int_0^{t_k} s^{\alpha-1} f(s, u(s), I^\beta u(s))ds \\ & + \int_0^t s^{\alpha-1}A(s)u(s)ds + \int_0^t s^{\alpha-1} f(s, u(s), I^\beta u(s))ds. \end{aligned} \tag{7}$$

Due to the continuity of u, f and $A(t)$, L is well defined. For $u, v \in B_r$, we have

$$\begin{aligned} \|Lu(t)\| \ \leq \ & \rho\|u_0\| + \rho\sum_{k=1}^m |a_k| \int_0^{t_k} s^{\alpha-1}\|A(s)\|\|u(s)\|ds \\ & + \rho\sum_{k=1}^m |a_k| \int_0^{t_k} s^{\alpha-1}\|f(s,u(s),I^\beta u(s)) - f(s,0,0)\|ds \\ & + \rho\sum_{k=1}^m |a_k| \int_0^{t_k} s^{\alpha-1}\|f(s,0,0)\|ds + \int_0^t s^{\alpha-1}\|A(s)\|\|u(s)\|ds \\ & + \int_0^t s^{\alpha-1}\|f(s,u(s),I^\beta u(s)) - f(s,0,0)\|ds + \int_0^t s^{\alpha-1}\|f(s,0,0)\|ds. \end{aligned}$$

Then,

$$\|Lu\| \ \leq \ \rho\|u_0\| + \frac{2T^\alpha}{\alpha}\left[N + r\left(M + \mu + \frac{\gamma T^\beta}{\beta}\right)\right] = \ r. \tag{8}$$

Thus, L maps B_r into itself. Furthermore,

$$\begin{aligned} \|Lu(t) - Lv(t)\| \ \leq \ & \rho\sum_{k=1}^m |a_k| \int_0^{t_k} s^{\alpha-1}\|A(s)\|\|u(s) - v(s)\|ds \\ & + \rho\sum_{k=1}^m |a_k| \int_0^{t_k} s^{\alpha-1}\|f(s,u(s),I^\beta u(s)) - f(s,v(s),I^\beta v(s))\|ds \\ & + \int_0^t s^{\alpha-1}\|A(s)\|\|u(s) - v(s)\|ds \\ & + \int_0^t s^{\alpha-1}\|f(s,u(s),I^\beta u(s)) - f(s,v(s),I^\beta v(s))\|ds. \end{aligned}$$

Then,

$$\|Lu - Lv\| \leq \frac{2T^\alpha}{\alpha}\left(M + \mu + \frac{\gamma T^\beta}{\beta}\right)\|u - v\|. \tag{9}$$

Since $\frac{2T^\alpha}{\alpha}\left(M + \mu + \frac{\gamma T^\beta}{\beta}\right) < 1$, then as a consequence of Banach's fixed point principle, L is a contraction mapping and has a unique fixed-point which is the unique solution of Problem (1). □

Theorem 3.4. *If the assumptions* (H_1) *and* (H_2) *are satisfied, then Problem* (1) *has at least one solution* $u \in B_r$.

Proof. Consider the operator L defined by (7). In view of Schaefer's fixed point theorem, the proof will be given in four steps.
Step 1. (L is continuous)
Let $\{u_n\}$ be a sequence in $C(J, X)$ which converges to $u \in C(J, X)$ as $n \to \infty$ for all $t \in J$.
As in proving (9), we get

$$\|Lu_n - Lu\| \leq \frac{2T^\alpha}{\alpha}\left(M + \mu + \frac{\gamma T^\beta}{\beta}\right)\|u_n - u\|. \tag{10}$$

The right hand side of (10) tends to zero as n tends to ∞. Therefore, L is continuous.
Step 2. (L maps bounded sets into bounded sets in $C(J, X)$)
It is enough to show that for any $r > 0$ there exists $l_1 > 0$ such that for each $u \in B_r$ we have $\|Lu\| \leq l_1$. Let $t \in J$ and $u \in B_r$. By using (7), we get

$$\|Lu\| \leq \rho\|u_0\| + \frac{2T^\alpha}{\alpha}\left[N + r\left(M + \mu + \frac{\gamma T^\beta}{\beta}\right)\right] \leq l_1, \quad l_1 > 0. \tag{11}$$

Step 3. (L maps bounded sets into equicontinuous sets of $C(J, X)$)
Let $t \in J$, $0 \leq t_1 < t_2 \leq T$ and $u \in B_r$. Consider then,

$$\begin{aligned} \|Lu(t_2) - Lu(t_1)\| &\leq \int_{t_1}^{t_2} s^{\alpha-1}\|A(s)\|\|u(s)\|ds + \int_{t_1}^{t_2} s^{\alpha-1}\|f(s, 0, 0)\|ds \\ &\quad + \int_{t_1}^{t_2} s^{\alpha-1}\|f(s, u(s), \int_0^s \tau^{\beta-1}u(\tau)d\tau) - f(s, 0, 0)\|ds \\ &\leq \frac{|t_2^\alpha - t_1^\alpha|}{\alpha}\left[N + r\left(M + \mu + \frac{|t_2^\beta - t_1^\beta|}{\beta}\right)\right]. \end{aligned} \tag{12}$$

Letting t_2 tends to t_1, the right hand side of (12) tends to zero. Therefore, the class of functions $\{Lu(t)\}$ is equicontinuous. As a consequence of Steps 1-3 together with Arzela-Ascoli theorem, we can conclude that the operator L is a completely continuous operator.
Step 4. (A priori bounds)
We have to show that $E(L) = \{u \in C([0, T], X) : \eta Lu = u\}$, for some $\eta \in [0, 1]$, is bounded. Let $u \in E(L)$ and there exist some $\eta \in [0, 1]$ such that $\eta Lu = u$.

Using (7) with applying (H_1) and (H_2), we have

$$\begin{aligned}
\|u(t)\| &= \|\eta Lu(t)\| \\
&\leq \eta\rho\|u_0\| + \eta\rho\sum_{k=1}^{m}|a_k|\int_0^{t_k} s^{\alpha-1}\|A(s)\|\|u(s)\|ds \\
&\quad + \eta\rho\sum_{k=1}^{m}|a_k|\int_0^{t_k} s^{\alpha-1}\|f(s,u(s),\int_0^s \tau^{\beta-1}u(\tau)d\tau) - f(s,0,0)\|ds \\
&\quad + \eta\rho\sum_{k=1}^{m}|a_k|\int_0^{t_k} s^{\alpha-1}\|f(s,0,0)\|ds + \eta\int_0^t s^{\alpha-1}\|A(s)\|\|u(s)\|ds \\
&\quad + \eta\int_0^t s^{\alpha-1}\|f(s,u(s),\int_0^s \tau^{\beta-1}u(\tau)d\tau) - f(s,0,0)\|ds + \eta\int_0^t s^{\alpha-1}\|f(s,0,0)\|ds \\
&\leq \eta\rho\|u_0\| + \eta M\|u\|\rho\sum_{k=1}^{m}|a_k|\int_0^{t_k} s^{\alpha-1}ds + \eta\mu\|u\|\rho\sum_{k=1}^{m}|a_k|\int_0^{t_k} s^{\alpha-1}ds \\
&\quad + \eta\gamma\|u\|\frac{T^\beta}{\beta}\rho\sum_{k=1}^{m}|a_k|\int_0^{t_k} s^{\alpha-1}ds + \eta N\rho\sum_{k=1}^{m}|a_k|\int_0^{t_k} s^{\alpha-1}ds + \eta N\int_0^t s^{\alpha-1}ds \\
&\quad + \eta\gamma\|u\|\frac{T^\beta}{\beta}\int_0^t s^{\alpha-1}ds + \eta(M+\mu)\int_0^t s^{\alpha-1}\|u(s)\|ds
\end{aligned}$$

then $\|u(t)\| \leq \phi_1 + \psi_1\int_0^t s^{\alpha-1}\|u(s)\|ds$ where $\phi_1 = \eta\left[\left(\rho\|u_0\| + \frac{2NT^\alpha}{\alpha}\right) + \frac{T^\alpha}{\alpha}\|u\|\left(M+\mu+\frac{2\gamma T^\beta}{\beta}\right)\right]$ and $\psi_1 = \eta(M+\mu)$. Applying Gronwall inequality (Theorem 2.3), we get $\|u(t)\| \leq \phi_1\exp\left(\frac{\psi_1 T^\alpha}{\alpha}\right)$. Therefore, $E(L)$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that the operator L has at least one fixed-point which is a solution of Problem (1). □

4. Existence and uniqueness of solutions to Problem (2)

Lemma 4.1. *For $0 < \beta \leq \alpha < 1$, every an α-differentiable function is a β-differentiable.*

Proof. Let $0 < \beta \leq \alpha < 1$ and u be an α-differentiable function. Since $D^\beta u(t) = \lim_{\varepsilon\to 0}\frac{u(t+\varepsilon t^{1-\beta})-u(t)}{\varepsilon}$, then

$$t^{\beta-\alpha}D^\beta u(t) = \lim_{\varepsilon\to 0}\frac{u(t+\varepsilon t^{1-\beta})-u(t)}{\varepsilon t^{\alpha-\beta}}.$$

Let $h = \varepsilon t^{\alpha-\beta}$. If $\varepsilon \to 0$, we get $h \to 0$. Hence, $t^{\beta-\alpha}D^\beta u(t) = \lim_{h\to 0}\frac{u(t+ht^{1-\alpha})-u(t)}{h} = D^\alpha u(t)$.
Then,

$$D^\beta u(t) = t^{\alpha-\beta}D^\alpha u(t). \tag{13}$$

Thus, the proof is completed. □

To facilitate our discussion. let

$$K := \frac{u_0}{\sum_{k=1}^m a_k} - \frac{1}{\sum_{k=1}^m a_k}\sum_{k=1}^{m} a_k\int_0^{t_k} s^{\beta-1}y(s)ds, \tag{14}$$

where $y(t)$ is the solution of the integral equation

$$y(t) = t^{\alpha-\beta}A(t)\left(K + \int_0^t s^{\beta-1}y(s)ds\right) + t^{\alpha-\beta}f(t,K+\int_0^t s^{\beta-1}y(s)ds, y(t)). \tag{15}$$

Lemma 4.2. *Problem* (2) *is equivalent to the integral equation*

$$u(t) = K + \int_0^t s^{\beta-1} y(s)ds, \tag{16}$$

Proof. Let u be a solution of Problem (2) and

$$y(t) = D^{\beta} u(t). \tag{17}$$

Since $u, A(t)$, and f are continuous, then $D^{\alpha} u$ is continuous and by Lemma 4.1, we have $D^{\beta} u$ is continuous. Therefore, y is also continuous. From (17), we get

$$u(t) = u(0) + \int_0^t s^{\beta-1} y(s)ds. \tag{18}$$

Putting $t = t_k$ in (18) with applying the nonlocal condition of (2), we obtain

$$u(0) = \frac{u_0}{\sum_{k=1}^m a_k} - \frac{1}{\sum_{k=1}^m a_k} \sum_{k=1}^m a_k \int_0^{t_k} s^{\beta-1} y(s)ds = K. \tag{19}$$

From (19) into (18), we get (16).
By using (17), Lemma 4.1 and (2), we have

$$\begin{aligned} y(t) &= D^{\beta} u(t) \\ &= t^{\alpha-\beta} D^{\alpha} u(t) \\ &= t^{\alpha-\beta} A(t) \left(K + \int_0^t s^{\beta-1} y(s)ds \right) + t^{\alpha-\beta} f(t, K + \int_0^t s^{\beta-1} y(s)ds, y(t)). \end{aligned}$$

Thus, u is a solution of (16) where $y(t)$ is the solution of (15).

Conversely, we have to show that if u is a solution of (16), u satisfies problem (2).
Putting $t = t_k$ in (16) and using (14), we have

$$\sum_{k=1}^m a_k u(t_k) = u_0 - \sum_{k=1}^m a_k \int_0^{t_k} s^{\beta-1} y(s)ds + \sum_{k=1}^m a_k \int_0^{t_k} s^{\beta-1} y(s)ds = u_0,$$

which is the nonlocal condition of Problem (2).
If we β-differentiate (16), we get $D^{\beta} u(t) = y(t)$. Using Lemma 4.1 and (15), we have

$$\begin{aligned} D^{\alpha} u(t) &= t^{\beta-\alpha} D^{\beta} u(t) \\ &= t^{\beta-\alpha} y(t) \\ &= A(t) \left(K + \int_0^t s^{\beta-1} y(s)ds \right) + f(t, K + \int_0^t s^{\beta-1} y(s)ds, y(t)) \\ &= A(t)u(t) + f(t, u(t), D^{\beta} u(t)). \end{aligned}$$

Therefore, u is a solution of Problem (2). □

Definition 4.3. *By a solution of Problem* (2), *we mean a function* $u \in C(J, D(A))$ *which is* α*-differentiable and satisfies* (16).

Consider the nonempty bounded set B_{σ}:

$$B_{\sigma} = \left\{ y \in C([0,T], X) : \|y\| \leq \sigma,\ \sigma = \frac{T^{\alpha-\beta} [\rho \|u_0\| (M + \mu) + N]}{1 - [\frac{2T^{\alpha}}{\beta}(M + \mu) + \gamma]} \right\}. \tag{20}$$

Theorem 4.4. *Let the assumptions* (H_1) *and* (H_2) *are satisfied. Then Problem* (2) *has a unique solution* $u \in B_\sigma$, *if*

$$\frac{2T^\alpha}{\beta}(M+\mu)+\gamma T^{\alpha-\beta}<1.$$

Proof. Consider the operator $P : C(J,X) \to C(J,X)$ such that

$$Py(t) = t^{\alpha-\beta}A(t)\left(K+\int_0^t s^{\beta-1}y(s)ds\right)+t^{\alpha-\beta}f(t,K+\int_0^t s^{\beta-1}y(s)ds, y(t)). \tag{21}$$

Due to the continuity of u, y, f and $A(t)$, P is well defined.

Let $t \in J$ and $y, z \in B_\sigma$. By using (13) and (21), we have

$$\begin{aligned} \|Py(t)\| \leq\ & \rho t^{\alpha-\beta}\|A(t)\|\|u_0\| + \rho t^{\alpha-\beta}\|A(t)\|\sum_{k=1}^m |a_k|\int_0^{t_k} s^{\beta-1}\|y(s)\|ds \\ & + t^{\alpha-\beta}\|A(t)\|\int_0^t s^{\beta-1}\|y(s)\|ds + t^{\alpha-\beta}\|f(t,0,0)\| \\ & + t^{\alpha-\beta}\|f(t,\ K+\int_0^t s^{\beta-1}y(s)ds,\ y(t)) - f(t,0,0)\|. \end{aligned}$$

Then,

$$\|Py\| \leq T^{\alpha-\beta}\left[\rho\|u_0\|(M+\mu)+N\right]+\sigma\left[\frac{2T^\alpha}{\beta}(M+\mu)+\gamma\right]=\sigma.$$

Furthermore,

$$\begin{aligned} \|Py(t)-Pz(t)\| \leq\ & \rho t^{\alpha-\beta}\|A(t)\|\sum_{k=1}^m |a_k|\int_0^{t_k} s^{\beta-1}\|y(s)-z(s)\|ds \\ & + t^{\alpha-\beta}\|A(t)\|\int_0^t s^{\beta-1}\|y(s)-z(s)\|ds \\ & + \rho\mu t^{\alpha-\beta}\sum_{k=1}^m |a_k|\int_0^{t_k} s^{\beta-1}\|y(s)-z(s)\|ds \\ & + \mu t^{\alpha-\beta}\int_0^t s^{\beta-1}\|y(s)-z(s)\|ds + \gamma t^{\alpha-\beta}\|y(s)-z(s)\|. \end{aligned}$$

Then,

$$\|Py-Pz\| \leq \left(\frac{2T^\alpha}{\beta}(M+\mu)+\gamma T^{\alpha-\beta}\right)\|y-z\|. \tag{22}$$

Since $\frac{2T^\alpha}{\beta}(M+\mu)+\gamma T^{\alpha-\beta}<1$, the operator P is a contraction. P has a unique fixed-point which is the unique solution of Problem (2). Thus, the proof is completed. □

Theorem 4.5. *If the assumptions* (H_1) *and* (H_2) *are satisfied, then Problem* (2) *has at least one solution* $u \in B_\sigma$.

Proof. Consider the operator P defined by (21). Let $\{y_n\}_{n=1}^\infty$ be a sequence in $C(J,X)$ which converges to $y \in C(J,X)$ as $n \to \infty$ for all $t \in J$. As in proving (22), we can easily get

$$\|Py_n - Py\| \leq \left(\frac{2T^\alpha}{\beta}(M+\mu)+\gamma T^{\alpha-\beta}\right)\|y_n-y\|.$$

Letting n tends to ∞, we get Py_n tends to Py . Thus P is continuous.

Now, we show that for any $\sigma > 0$ there exists $l_2 > 0$ such that for each $y \in B_\sigma$ we have $\|Py\| \le l_2$. Let $y \in B_\sigma$. By using (14) and (21), we get

$$\|Py\| \;\le\; T^{\alpha-\beta}\left[\rho\|u_0\|(M+\mu)+N\right]+\left[\frac{2T^\alpha}{\beta}(M+\mu)+\gamma\right]\sigma \;\le\; l_2,\;\; l_2>0.$$

That is, P maps bounded sets into bounded sets in $C(J,X)$.

Let $t \in J,\ 0 \le t_1 < t_2 \le T$ and $y \in B_\sigma$. Since $t_2^{\alpha-\beta} \ge t_1^{\alpha-\beta}$, then

$$\begin{aligned}\|Py(t_2)-Py(t_1)\| \le\; & M\left[\left(\rho\|u_0\|+\frac{\sigma T^\beta}{\beta}\right)|t_2^{\alpha-\beta}-t_1^{\alpha-\beta}|+\frac{\sigma t_2^{\alpha-\beta}}{\beta}\,|t_2^\beta-t_1^\beta|\right]\\ & + t_2^{\alpha-\beta}\|f(t_2,\ K+\int_0^{t_2} s^{\beta-1}y(s)ds,\ y(t_2))-f(t_1,\ K+\int_0^{t_1} s^{\beta-1}y(s)ds,\ y(t_1))\|.\end{aligned}$$

Since f is continuous, the right hand side of the above inequality tends to zero as t_2 tends to t_1. Therefore, the class of functions $\{Py(t)\}$ is equicontinuous. As a consequence of steps 1-3 together with Arzela-Ascoli theorem, we can conclude that the operator P is a completely continuous operator.

Finally, for some $\eta \in [0,1]$, We have to show that $E(P) = \{y \in C([0,T],X) : \eta Py = y\}$ is bounded. Let $y \in E(P)$ and there exist some $\eta \in [0,1]$ such that $\eta Py = y$. Consider then,

$$\begin{aligned}\|y(t)\| &= \|\eta Py(t)\|\\ &\le \eta\rho t^{\alpha-\beta}\|A(t)\|\|u_0\|+\eta\rho t^{\alpha-\beta}\|A(t)\|\sum_{k=1}^m|a_k|\int_0^{t_k} s^{\beta-1}\|y(s)\|ds\\ &\quad+\eta t^{\alpha-\beta}\|A(t)\|\int_0^t s^{\beta-1}\|y(s)\|ds+\eta t^{\alpha-\beta}\|f(t,0,0)\|\\ &\quad+\eta t^{\alpha-\beta}\|f(t,\ K+\int_0^t s^{\beta-1}y(s)ds,\ y(t))-f(t,0,0)\|\\ &\le \eta\rho T^{\alpha-\beta}(M+\mu)\|u_0\|+\eta\rho T^{\alpha-\beta}(M+\mu)\|y\|\sum_{k=1}^m|a_k|\int_0^{t_k} s^{\beta-1}ds\\ &\quad+\eta T^{\alpha-\beta}(\gamma\|y\|+N)+\eta T^{\alpha-\beta}(M+\mu)\int_0^t s^{\beta-1}\|y(s)\|ds.\end{aligned}$$

Then, $\|y(t)\| \le \phi_2+\psi_2\int_0^t s^{\beta-1}\|y(s)\|ds$ where $\phi_2 = \eta T^{\alpha-\beta}\left[(M+\mu)(\rho\|u_0\|+\frac{T^\beta}{\beta}\|y\|)+\gamma\|y\|+N\right]$ and $\psi_2 = \eta T^{\alpha-\beta}(M+\mu)$. By applying Gronwall inequality, we get $\|y(t)\| \le \phi_2\exp\left(\frac{\psi_2 T^\beta}{\beta}\right)$. Thus, $E(P)$ is bounded. Now we have the operator P is completely continuous and the set $E(P)$ is bounded which prove, by using Schaefer's fixed-point theorem, that P has at least one fixed-point $u \in B_\sigma$ which is a solution of Problem (2). □

5. Examples

Example 5.1. *Consider the nonlocal problem:*

$$\begin{cases}D^{0.8}u(t)=\frac{1}{15}tu(t)+\frac{e^{-\lambda t}}{1+e^t}\left[0.5u(t)+0.25v(t)\right],\;\; t\in[0,1];\\ u(0.2)+u(0.8)=1.\end{cases}\tag{23}$$

where $\lambda > 0$ is a constant.

We deduce that $\alpha = \frac{4}{5}$, $M = \frac{1}{15}$, $f(t,u(t),v(t)) = \frac{e^{-\lambda t}}{1+e^t}\left[0.5u(t) + 0.25v(t)\right]$, $N = 0$, $u_0 = 1$, $k \in \{1,2\}$, $t_1 = 0.2$, $t_2 = 0.8$, $\sum_{k=1}^{2} a_k = 2$ *and* $J = [0,1]$. *Let* $u, v \in C([0,1],\mathbb{R})$, *we have*

$$\begin{aligned}|f(t,u_1(t),v_1(t)) - f(t,u_2(t),v_2(t))| &\le \frac{e^{-\lambda t}}{1+e^t}\left[\frac{1}{2}|u_1(t)-u_2(t)| + \frac{1}{4}|v_1(t)-v_2(t)|\right] \\ &\le \frac{e^{-\lambda t}}{2}\left[\frac{1}{2}|u_1(t)-u_2(t)| + \frac{1}{4}|v_1(t)-v_2(t)|\right].\end{aligned}$$

Then, $\mu = \frac{e^{-\lambda t}}{4}$ *and* $\gamma = \frac{e^{-\lambda t}}{8}$. *Letting, for example,* $v(t) = I^{0.6}u(t)$ *and choosing some* $\lambda > 0$ *large enough, we get (in view of Theorem 3.2) that Problem* (1) *has a unique solution.*

Example 5.2. *Consider the nonlocal problem:*

$$\begin{cases} D^{0.75}u(t) = \int_0^{0.2} u(t)dt + \frac{e^{-\lambda t}}{1+e^t}\left[\frac{|u(t)|}{1+|u(t)|} + \frac{|v(t)|}{1+|v(t)|}\right], & t \in [0,1]; \\ u(0.3) + u(0.7) = 1. \end{cases} \tag{24}$$

where $\lambda > 0$ *is a constant.*

We deduce that $\alpha = \frac{3}{4}, M = 0.2$, $f(t,u(t),v(t)) = \frac{e^{-\lambda t}}{1+e^t}\left[\frac{|u(t)|}{1+|u(t)|} + \frac{|v(t)|}{1+|v(t)|}\right], u_0 = 1$, $k \in \{1,2\}$, $t_1 = 0.3$, $t_2 = 0.7$, $\sum_{k=1}^{2} a_k = 2$, $N = 0$ *and* $J = [0,1]$. *Let* $u, v \in C([0,1],\mathbb{R})$, *we have*

$$\begin{aligned}&|f(t,u_1(t),v_1(t)) - f(t,u_2(t),v_2(t))| \\ &\quad\le \frac{e^{-\lambda t}}{1+e^t}\left|\left(\frac{|u_1(t)|}{1+|u_1(t)|} - \frac{|u_2(t)|}{1+|u_2(t)|}\right) + \left(\frac{|v_1(t)|}{1+|v_1(t)|} - \frac{|v_2(t)|}{1+|v_2(t)|}\right)\right| \\ &\quad\le \frac{e^{-\lambda t}}{1+e^t}\left[\frac{||u_1(t)|-|u_2(t)||}{(1+|u_1(t)|)(1+|u_2(t)|)} + \frac{||v_1(t)|-|v_2(t)||}{(1+|v_1(t)|)(1+|v_2(t)|)}\right] \\ &\quad\le \frac{e^{-\lambda t}}{1+e^t}\left[|u_1(t)-u_2(t)| + |v_1(t)-v_2(t)|\right] \\ &\quad\le \frac{e^{-\lambda t}}{2}\left[|u_1(t)-u_2(t)| + |v_1(t)-v_2(t)|\right].\end{aligned}$$

Then, $\mu = \gamma = \frac{e^{-\lambda t}}{2}$. *Let* $v(t) = D^{0.5}u(t)$ *and choosing some* $\lambda > 0$ *large enough, we get (in view of Theorem 4.3) that Problem* (2) *has a unique solution.*

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