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Closedness and Self-Adjointness Criteria for Block Operator Matrices Involving Fully-Subordinate Entries

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Abstract.
In the present paper, we consider a 2×2 block operator matrices with unbounded entry operators acting on Banach spaces. Under some conditions, we develop criteria for its self-adjointness and closedness. The obtained results are applied to an Hamiltonian operator matrix.

1. Introduction

In mathematical physics, block operator matrices arise in various areas of engineering, physics, applied mathematics and transport theory [3–7, 9, 10]. Hence, the spectral properties of block operator matrices play a crucial role as they govern for instance the solvability and stability of the underlying physical systems. The criteria for the closedness and self-adjointness of block operator matrices with unbounded entries have attracted considerable attention and have been well investigated in literature, (see for example [1, 8, 14, 15, 17]).

Recently, in [16], A. A. Shkalikov and K. Trunk considered in the Banach spaces X_1 and X_2 the linear operators A, B, C, and D with domains $\mathcal{D}(A)$, $\mathcal{D}(B)$, $\mathcal{D}(C)$ and $\mathcal{D}(D)$, respectively. These operators are assumed that they act as follows:

$$A : \mathcal{D}(A) \subset X_1 \to X_1, \ \ C : \mathcal{D}(C) \subset X_1 \to X_2,$$
$$B : \mathcal{D}(B) \subset X_2 \to X_1, \ \ D : \mathcal{D}(D) \subset X_2 \to X_2.$$

Then, the linear operator

$$L := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with domain $\mathcal{D}(L) := (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(B) \cap \mathcal{D}(D))$ is defined in the space $X = X_1 \times X_2$.

In general, the operators occurring in L are unbounded and L doesn't need to be closed or to be a closable operator, even if its entries are closed. However, under some suitable conditions, L is closable and its closure $\overline{L}$ can be determined.

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More precisely, in [16], A. A. Shkalikov and K. Trunk studied the stability of closedness or closability, as well as the stability of self-adjointness, for the operator matrix L in the diagonal dominant case. This case is characterized under the condition assuming that the operators C and B are relatively bounded (or subordinated) with respect to the operators A and D, respectively. This situation implies that $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\mathcal{D}(D) \subset \mathcal{D}(B)$ and that there exist constants ν_1, ν_2, M_{ν_1} and M_{ν_2} satisfying the following inequalities:

$$\|Cx\| \leq \nu_1\|Ax\| + M_{\nu_1}\|x\|, \ \text{ for all } \ x \in \mathcal{D}(A),$$

$$\|By\| \leq \nu_2\|Dy\| + M_{\nu_2}\|y\|, \ \text{ for all } \ y \in \mathcal{D}(D).$$

The lower bounds of ν_1 and ν_2 such that the above inequalities are fulfilled with some constants M_{ν_1} and M_{ν_2} are denoted respectively by ν_1^* and ν_2^* and called the A-bound and the D-bound of the operators C and B. If $\nu_1^*\nu_2^* < 1$, the authors proved in [16] the closedness or closability, as well as the stability of self-adjointness, for the operator matrix L.

However, in some applications the above condition $\nu_1^*\nu_2^* < 1$ is very restrictive. Hence, the purpose of this paper is to build a common framework for this problem. To this interest, we try to use the concept of fully subordination introduced in [12] as a natural generalization of relative boundedness or subordination and we consider, in this case, that for every sufficiently small α and β the following estimations hold:

$$\|Cx\| \leq \alpha\|Ax\| + \phi_\alpha(x) \ \text{ for all } \ x \in \mathcal{D}(A),$$

and

$$\|By\| \leq \beta\|Dy\| + \phi_\beta(y) \ \text{ for all } \ y \in \mathcal{D}(D),$$

where ϕ_α and ϕ_β are continuous convex functionals of x and y.

This concept of fully subordination enables us to ameliorate the conditions of closedness, closability or self-adjointness for the operator matrix L and we give an application to an Hamiltonian operator matrix which plays a fundamental role in physics as it is closely related to anharmonic oscillators.

This paper is organized as follows: Section 2 is devoted to study the closure of the matrix operator L. In the third section, we investigate under sufficient conditions the self-adjointness as well as the boundedness of the operator L. In the last section, an application to an Hamiltonian matrix operator is presented.

2. The closability of the matrix operator L

Let X_1, X_2 be Banach spaces. In the product space $X_1 \times X_2$ equipped with the norm

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \|x\|_{X_1} + \|y\|_{X_2}, \tag{1}$$

we consider the following matrix operator

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the linear operators A, B, C and D act as

$$A : \mathcal{D}(A) \subset X_1 \to X_1, \ \ C : \mathcal{D}(C) \subset X_1 \to X_2,$$
$$B : \mathcal{D}(B) \subset X_2 \to X_1, \ \ D : \mathcal{D}(D) \subset X_2 \to X_2.$$

In the next, we will assume that the entries satisfy the following conditions:

($H1$) A (respectively, B) is a densely defined closable operator in X_1 (respectively, X_2).

($H2$) The operator C is A-fully subordinate, i.e., $\mathcal{D}(C) \supset \mathcal{D}(A)$ and for every sufficiently small α, we have

$$\|Cx\| \leq \alpha\|Ax\| + \phi_\alpha(x) \text{ for all } x \in \mathcal{D}(A),$$

where ϕ_α is a continuous convex functional of x.

($H3$) The operator B is D-fully subordinate, i.e.,$\mathcal{D}(B) \supset \mathcal{D}(D)$ and for every sufficiently small β, we have

$$\|By\| \leq \beta\|Dy\| + \phi_\beta(y) \text{ for all } y \in \mathcal{D}(D),$$

where ϕ_β is a continuous convex functional of y.

Lemma 2.1. *Let A and B be two linear operators from a Banach space X into a Banach space Y, having the same domain $\mathcal{D} \subset X$. If A is a closed operator and for every sufficiently small η, we have*

$$\|(A - B)\varphi\| \leq \eta\|A\varphi\| + \phi_\eta(\varphi) \text{ for all } \varphi \in \mathcal{D}, \tag{2}$$

where ϕ_η is a continuous convex functional of φ, then B is also closed.

Proof. Let $x \in \mathcal{D}$. In view of Eq. (2), we have

$$\begin{aligned} \|Bx\| &\leq \|Ax\| + \|(B - A)x\| \\ &\leq \|Ax\| + \eta\|Ax\| + \phi_\eta(x) \\ &\leq (1 + \eta)\|Ax\| + \phi_\eta(x), \end{aligned} \tag{3}$$

and

$$\begin{aligned} \|Ax\| &\leq \|Bx\| + \|(A - B)x\| \\ &\leq \|Bx\| + \eta\|Ax\| + \phi_\eta(x). \end{aligned} \tag{4}$$

Hence, Eq. (4) entails

$$(1 - \eta)\|Ax\| \leq \|Bx\| + \phi_\eta(x). \tag{5}$$

Taking $\eta < 1$, Eq. (5) implies that

$$\|Ax\| \leq \frac{1}{1-\eta}\|Bx\| + \frac{1}{1-\eta}\phi_\eta(x). \tag{6}$$

Now, let $(x_n)_n$ be a sequence in $\mathcal{D}$ such that $x_n \to x$ and $Bx_n \to g$ as $n \to \infty$. We will prove that $x \in \mathcal{D}$ and $g = Bx$.

In fact, since $\phi_\eta(0) = 0$, it follows from Eq. (6) that

$$\|A(x_n - x_m)\| \leq \frac{1}{1-\eta}\|B(x_n - x_m)\| + \frac{1}{1-\eta}\phi_\eta(x_n - x_m) \to 0 \text{ as } m, n \to \infty.$$

So, the sequence $(Ax_n)_n$ is a Cauchy sequence in Y, and hence it is convergent. However, A is closed, thus, $x \in \mathcal{D}$ and $Ax_n \to Ax$. Now, using Eq. (3), we get

$$\|B(x_n - x)\| \leq (1 + \eta)\|A(x_n - x)\| + \phi_\eta(x_n - x) \to 0 \text{ as } n \to \infty.$$

Consequently, $Bx_n \to Bx$ and $g = Bx$. Therefore, B is closed.

Lemma 2.2. *Assume that hypotheses* $(H1)$-$(H3)$ *hold, then the operators* B *and* C *admit extensions to* $\mathcal{D}(\overline{D}) \cup \mathcal{D}(B)$ *and* $\mathcal{D}(\overline{A}) \cup \mathcal{D}(C)$ *denoted by* $\widetilde{B}$ *and* $\widetilde{C}$ *respectively. Moreover, we have*

$$\|\widetilde{C}x\| \leq \alpha\|\overline{A}x\| + \phi_\alpha(x), \quad x \in \mathcal{D}(\overline{A}) \tag{7}$$

and

$$\|\widetilde{B}y\| \leq \beta\|\overline{D}y\| + \phi_\beta(y), \quad y \in \mathcal{D}(\overline{D}). \tag{8}$$

Proof. Let $x_n \in \mathcal{D}(A)$ such that $x_n \to x$ and $Ax_n \to u$ as $n \to \infty$. Since A is closable then $x \in \mathcal{D}(\overline{A})$ and $u = \overline{A}x$. Applying hypothesis $(H2)$, we have

$$\|Cx_n\| \leq \alpha\|Ax_n\| + \phi_\alpha(x_n). \tag{9}$$

As $(x_n)_n$ and $(Ax_n)_n$ are convergent sequences, it follows from Eq. (9) that the sequence $(Cx_n)_n$ is also convergent. So, we define $\widetilde{C}$ as

$$\begin{cases} \widetilde{C}x = Cx, \quad x \in \mathcal{D}(C) \\ \widetilde{C}x = \lim\limits_n Cx_n, \quad x \in \mathcal{D}(\overline{A}). \end{cases}$$

Taking the limit in Eq. (9), we get

$$\|\widetilde{C}x\| \leq \alpha\|\overline{A}x\| + \phi_\alpha(x), \quad x \in \mathcal{D}(\overline{A}).$$

By an analogous reasoning as the above, we have

$$\|\widetilde{B}y\| \leq \beta\|\overline{D}y\| + \phi_\beta(y), \quad y \in \mathcal{D}(\overline{D}),$$

where $\widetilde{B}$ designates the extension of the operator B to the domain $\mathcal{D}(\overline{D}) \cup \mathcal{D}(B)$.

Theorem 2.3. *Assume that hypotheses* $(H1)$-$(H3)$ *are satisfied. Then, the operator* L *is closable and its closure has the form*

$$\overline{L} = \begin{pmatrix} \overline{A} & \widetilde{B} \\ \widetilde{C} & \overline{D} \end{pmatrix} \textit{ with domain } \mathcal{D}(\overline{L}) = \mathcal{D}(\overline{A}) \times \mathcal{D}(\overline{D}).$$

Proof. It is easy to check that under hypotheses $(H2)$ and $(H3)$, the domain of definition of the closure $\overline{L}$ (if it exists) has to contain $\mathcal{D}(\overline{A}) \times \mathcal{D}(\overline{D})$. To see this, it suffices to take elements of the space $X_1 \times X_2$ with the zero first or second coordinates.
Now, let us consider the matrix operator

$$T = \begin{pmatrix} \overline{A} & \widetilde{B} \\ 0 & \overline{D} \end{pmatrix} \text{ with domain } \mathcal{D}(T) = \mathcal{D}(\overline{A}) \times \mathcal{D}(\overline{D}),$$

and we will prove that it is closed. So, let $(x_n, y_n)^T \in \mathcal{D}(T)$ converging to $(x, y)^T \in X_1 \times X_2$ and let $(u_n, v_n)^T := T(x_n, y_n)^T = (\overline{A}x_n + \widetilde{B}y_n, \overline{D}y_n)^T$ converging to $(u, v)^T \in X_1 \times X_2$ as $n \to \infty$.

As $\overline{D}$ is closed, we obtain $y \in \mathcal{D}(\overline{D}) \subset \mathcal{D}(\widetilde{B})$ and $\overline{D}y = v$. Moreover, Eq. (8) yields

$$\|\widetilde{B}y_n - \widetilde{B}y\| \leq \beta\|\overline{D}y_n - \overline{D}y\| + \phi_\beta(y_n - y). \tag{10}$$

So, Eq. (10) entails that $\widetilde{B}y_n \to \widetilde{B}y$ and then $\overline{A}x_n \to u - \widetilde{B}y$ as $n \to \infty$. Since $\overline{A}$ is closed, we get $x \in \mathcal{D}(\overline{A})$ and $\overline{A}x = u - \widetilde{B}y$. Consequently, the matrix operator T is closed.

Now, we denote by G the following matrix operator

$$G = \begin{pmatrix} 0 & 0 \\ \widetilde{C} & 0 \end{pmatrix} \text{ with domain } \mathcal{D}(G) = \mathcal{D}(T).$$

To complete the proof of our result, we will show that the matrix operator G is T-fully subordinate. So, let $(x, y)^T \in \mathcal{D}(G)$, we observe that:

$$\begin{aligned}
\left\| G \begin{pmatrix} x \\ y \end{pmatrix} \right\| &= \|\widetilde{C}x\| \\
&\leq \alpha\|\overline{A}x\| + \phi_\alpha(x) \\
&\leq \alpha\left(\|\overline{A}x + \widetilde{B}y\| + \|\widetilde{B}y\|\right) + \phi_\alpha(x) \\
&\leq \alpha\left(\|\overline{A}x + \widetilde{B}y\| + \beta\|\overline{D}y\| + \phi_\beta(y)\right) + \phi_\alpha(x) \\
&\leq \max(\alpha, \alpha\beta)\left(\|\overline{A}x + \widetilde{B}y\| + \|\overline{D}y\|\right) + \phi_\alpha(x) + \alpha\phi_\beta(y) \\
&\leq \max(\alpha, \alpha\beta)\left(\|\overline{A}x + \widetilde{B}y\| + \|\overline{D}y\|\right) + \phi_{\alpha,\beta}\begin{pmatrix} x \\ y \end{pmatrix} \\
&\leq \max(\alpha, \alpha\beta)\left(\|\overline{A}x + \widetilde{B}y\| + \|\overline{D}y\|\right) + \phi_{\alpha,\beta}\begin{pmatrix} x \\ y \end{pmatrix} \\
&\leq \max(\alpha, \alpha\beta)\left\| \begin{pmatrix} \overline{A} & \widetilde{B} \\ 0 & \overline{D} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \right\| + \phi_{\alpha,\beta}\begin{pmatrix} x \\ y \end{pmatrix} \\
&\leq \max(\alpha, \alpha\beta)\left\| T\begin{pmatrix} x \\ y \end{pmatrix} \right\| + \phi_{\alpha,\beta}\begin{pmatrix} x \\ y \end{pmatrix},
\end{aligned}$$

where $\phi_{\alpha,\beta}(x, y) = \phi_\alpha x + \phi_\beta y$ is a continuous convex functional of (x, y). Hence, the matrix operator T is G-fully subordinate. Using the fact that T is closed, Lemma 2.1 implies that the operator $\overline{L} = T + G$ is closed.

Remark 2.4. *We note here that the closedness of the operator L is guaranteed whatever the constants α and β. Hence, Theorem 2.3 improves Theorem 1 in* [16].

3. Stability of self-adjointness of the matrix operator L

In this section, we consider X_1 and X_2 two Hilbert spaces and we concentrate ourselves on proving some results about stability of self-adjointness.

Theorem 3.1. *Suppose that hypotheses $(H1)$-$(H3)$ are satisfied with $C = B^*$. Assume further that the operators A and D are self-adjoint. Then, the matrix operator L is self-adjoint.*

Proof. Clearly, the resolvent sets $\rho(A)$ and $\rho(D)$ of the operators A and D are not empty and contain the set $\mathbb{C}\backslash\mathbb{R}$ since the operators A and D are self-adjoint. Further, due to Theorem 2.3 the matrix operator L is closed and it is symmetric. Hence, in view of [11, *Theorem V.*3.16 , *p.* 271] it suffices to prove that the deficiency of $L - it$ is equal to zero, where $|t| > t_0$ and t_0 is sufficiently large. So, using the Frobenius-Schur factorization, we obtain

$$L - it = \begin{pmatrix} I & 0 \\ B^*(A - it)^{-1} & I \end{pmatrix}\begin{pmatrix} A - it & 0 \\ 0 & S_1(it) \end{pmatrix}\begin{pmatrix} I & (A - it)^{-1}B \\ 0 & I \end{pmatrix}, \tag{11}$$

where $S_1(it) = D - it - B^*(A - it)^{-1}B$.
It follows from hypothesis $(H2)$ that the operator $B^*(A - it)^{-1}$ is bounded on X_1 and its adjoint coincides with the operator $(A - it)^{-1}B$ on $\mathcal{D}(B)$. Thus, $(A - it)^{-1}B$ is bounded. As $\mathcal{D}(B)$ is dense on X_2, then this operator admits the bounded continuation to the whole space X_2. As a consequence, the first and the third factors

in the right hand side of Eq. (11) are bounded and boundedly invertible operators. To complete the proof, it is sufficient to reveal that the deficiency of $S_1(it)$ is equal to zero. Writing $S_1(it)$ as

$$\begin{aligned} S_1(it) &= D - it - B^*(A - it)^{-1}B \\ &= \left(I - B^*(A - it)^{-1}B(D - it)^{-1}\right)(D - it), \end{aligned} \tag{12}$$

we will prove that it is invertible with bounded inverse. Indeed, in view of hypothesis $(H2)$ we have

$$\|B^*(A - it)^{-1}x\| \leq \alpha\|A(A - it)^{-1}x\| + \phi_\alpha\left((A - it)^{-1}x\right) \text{ for all } x \in X_1. \tag{13}$$

As A is self-adjoint, it follows from [11, *Theorem V*.3.16 , *p*. 271] that the deficiency of the operator $A - it$ is equal to zero and

$$\|A(A - it)^{-1}x\| \leq \|x\| \text{ and } \|(A - it)^{-1}x\| \leq \frac{1}{|t|}\|x\|. \tag{14}$$

Although, for $|t| > t_0$ and t_0 is sufficiently large, Eq. (14) implies that $(A-it)^{-1}x \to 0$. Hence, $\phi_\alpha\left((A - it)^{-1}x\right) \to \phi_\alpha(0) = 0$. Consequently, Eqs (13) and (14) imply that

$$\|B^*(A - it)^{-1}x\| \leq \alpha\|x\|, \tag{15}$$

for $|t| > t_0$ and t_0 is sufficiently large. Similarly, we show that hypothesis $(H3)$ entails the estimate

$$\|B(D - it)^{-1}y\| \leq \beta\|y\|, \text{ for all } y \in X_2, \tag{16}$$

where $|t| > t_0$ and t_0 is sufficiently large. So, Eqs (15) and (16) yield

$$\|B^*(A - it)^{-1}B(D - it)^{-1}\| \leq \alpha\beta, \tag{17}$$

where $|t| > t_0$ and t_0 is sufficiently large. Since $\alpha\beta$ is sufficiently small, then $S_1(it)$ is invertible with bounded inverse and its deficiency is equal to zero.

Remark 3.2. *The result of Theorem 3.1 is very interesting since it improves Theorem* 2 *in* [16]*. In fact, if the operators A and D are self-adjoint, and $C = B^*$, the quadratic form of the operator matrix L takes real values only. Therefore, it is a symmetric operator. However, it is not necessarily self-adjoint and the authors in* [16] *proved that it is self-adjoint if the operators B^* and B are A- and D-bounded, and $\nu_1^*\nu_2^* < 1$ where ν_1^* and ν_2^* are respectively the A-bound and the D-bound of the operators B^* and B. By comparison to* [16, *Theorem* 2]*, Theorem 3.1 guarantees the self-adjointness with better hypotheses.*

In the next theorem, we will prove that the property of boundedness from below is preserved under the conditions of Theorem 3.1.

Theorem 3.3. *Assume that the conditions of Theorem 3.1 are satisfied. Furthermore, suppose that the operators A and D are bounded from below, i.e.,*

$$\langle Ax, x\rangle \geq \delta_A\langle x, x\rangle \text{ for all } x \in \mathcal{D}(A),$$
$$\langle Dy, y\rangle \geq \delta_D\langle y, y\rangle \text{ for all } y \in \mathcal{D}(D).$$

Then, the matrix operator L is self-adjoint and bounded from below.

Proof. In view of Theorem 3.1 the matrix operator L is self-adjoint. So, to prove that L is bound from below it is sufficient to show that the spectrum of L is bounded from below (see [11, *p*. 278]). Since the operator A is self-adjoint and bounded from below then it follows from [11, *p*. 278] that the spectrum of A is bounded from below and we have $A - \lambda$ is invertible for $\lambda < \delta_A$. Similarly, the operator $D - \lambda$ is invertible for $\lambda < \delta_D$. We note here that the proof of this result is technically similar to that of Theorem 3.1. In fact, it suffices to show the existence of a number $\delta < \min(\delta_A, \delta_D)$ such that Eq. (17) holds for all $\lambda < \delta$. As a matter of fact, it follows from [11, p. 272-273] that for $\lambda < \delta_A$ we have

$$\|(A-\lambda)^{-1}\| \leq \frac{1}{\delta_A - \lambda} \text{ and } \|A(A-\lambda)^{-1}\| \leq \sup_{\mu\in\sigma_A}\left|\frac{\mu}{\mu-\lambda}\right| \leq \max\left(1, \frac{|\delta_A|}{\delta_A-\lambda}\right). \tag{18}$$

Further, for $\lambda < \delta_D$ we obtain

$$\|(D-\lambda)^{-1}\| \leq \frac{1}{\delta_D - \lambda} \text{ and } \|D(D-\lambda)^{-1}\| \leq \sup_{\mu\in\sigma_D}\left|\frac{\mu}{\mu-\lambda}\right| \leq \max\left(1, \frac{|\delta_D|}{\delta_D-\lambda}\right). \tag{19}$$

Hence, in view of hypothesis $(H2)$ and Eq. (18) we have

$$\|B^*(A-\lambda)^{-1}x\| \leq \alpha \max\left(1, \frac{|\delta_A|}{\delta_A-\lambda}\right)\|x\| + \phi_\alpha\left((A-\lambda)^{-1}x\right). \tag{20}$$

Clearly, Eq. (18) implies that $(A-\lambda)^{-1}x \to 0$ as $\lambda \to -\infty$. Then, $\phi_\alpha((A-\lambda)^{-1}x) \to \phi_\alpha(0) = 0$ as $\lambda \to -\infty$. Consequently, it follows from Eq. (20) that

$$\|B^*(A-\lambda)^{-1}\| \leq \alpha, \text{ as } \lambda \to -\infty. \tag{21}$$

Analogously to Eq. (21), hypothesis $(H3)$ and Eq. (19) entail the estimate

$$\|B(D-\lambda)^{-1}\| \leq \beta, \text{ as } \lambda \to -\infty. \tag{22}$$

So, Eqs (21) and (22) yield

$$\|B^*(A-\lambda)^{-1}B(D-\lambda)^{-1}\| \leq \alpha\beta, \text{ as } \lambda \to -\infty.$$

However, $\alpha\beta$ is sufficiently small. Then, there exists $\delta \in \mathbb{R}$ such that Eq. (17) is satisfied for all $\lambda < \delta < \min(\delta_A, \delta_D)$.

4. Application

In this part, we consider the following Hamiltonian operator

$$L := \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix},$$

where $A = \frac{-d^2}{dx^2} + x^2$ is the harmonic oscillator in dimension one given with domain

$$\mathcal{D}(A) = \{f \in H^2(\mathbb{R}) \text{ such that } x^2 f \in L^2(\mathbb{R})\},$$

whereas the off-diagonal self-adjoint potentials $B = C = 2ia\frac{d}{dx}$ are given with domains

$$\mathcal{D}(B) = \mathcal{D}(C) = \{f \in L^2(\mathbb{R}) \text{ such that } f' \in L^2(\mathbb{R})\}.$$

It follows from [13] that the operator A is self-adjoint. Then, the operator L can be written as

$$\begin{pmatrix} A & B \\ C & -A \end{pmatrix}.$$

Lemma 4.1. *The operator C (respectively, B) is A-fully subordinate.*

Proof. The result of the above lemma is particularly inspired from [2, *Example* 7.2.4] and [13, *Lemma* 2.1]. For the reader's convenience, we will give this proof.

Let $f \in \mathcal{D}(A)$, we have

$$\|Af\|^2 = \|f''\|^2 + \|x^2 f\|^2 - 2\mathrm{Re}\langle f'', x^2 f\rangle.$$

Using integration by parts, we get

$$\|Af\|^2 = \|f''\|^2 + \|x^2 f\|^2 + 2\|xf'\|^2 + 4\mathrm{Re}\langle f', xf\rangle. \tag{23}$$

As

$$2\mathrm{Re}\langle f', xf\rangle \geq -\|f'\|^2 - \|xf\|^2,$$

Eq. (23) yields

$$\|Af\|^2 \geq \|f''\|^2 + \|x^2 f\|^2 - 2\|f'\|^2 - 2\|xf\|^2.$$

Completing the square, we obtain

$$\|x^2 f\|^2 - 4\|xf\|^2 = \|(x^2 - 2)f\|^2 - 4\|f\|^2 \geq -4\|f\|^2.$$

Now, using the Fourier transform, the following estimate holds

$$\|f''\|^2 - 4\|f'\|^2 = \|x^2\widehat{f}\|^2 - 4\|x\widehat{f}\|^2 \geq -4\|f\|^2.$$

Hence, we get

$$\begin{aligned}
\|Af\|^2 + \|f\|^2 &\geq \frac{2}{9}\|Af\|^2 + \|f\|^2 \\
&\geq \frac{1}{9}\left(\|f''\|^2 + \|x^2 f\|^2\right) + \frac{1}{9}\left(\|x^2 f\|^2 - 4\|xf\|^2 + \|f''\|^2 - 4\|f'\|^2\right) + \|f\|^2 \\
&\geq \frac{1}{9}\left(\|f''\|^2 + \|x^2 f\|^2 + \|f\|^2\right).
\end{aligned} \tag{24}$$

On the other hand, we have

$$\|(A+I)f\|^2 = \|Af\|^2 + \|f\|^2 + 2\langle Af, f\rangle \geq \|Af\|^2 + \|f\|^2. \tag{25}$$

So, Eqs (24) and (25) imply that

$$\begin{aligned}
\|f'\| &\leq \|f''\|^{\frac{1}{2}}\|f\|^{\frac{1}{2}} \\
&\leq (\|f''\|^2 + \|x^2 f\|^2 + \|f\|^2)^{\frac{1}{4}}\|f\|^{\frac{1}{2}} \\
&\leq c\|(A+I)f\|^{\frac{1}{2}}\|f\|^{\frac{1}{2}}, \quad c \geq 3.
\end{aligned} \tag{26}$$

Now, let α sufficiently small. Using Young inequality, Eq. (26) yields

$$\begin{aligned}
\|f'\| &\leq \left(c^2(2\alpha)^{-1}\|f\|\right)^{\frac{1}{2}}(2\alpha\|(T+I)f\|)^{\frac{1}{2}} \\
&\leq c^2(4\alpha)^{-1}\|f\| + \alpha\|(A+I)f\| \\
&\leq \left(c^2(4\alpha)^{-1} + 1\right)\|f\| + \alpha\|Af\| \\
&\leq \alpha\|Af\| + \phi_\alpha(f),
\end{aligned}$$

where $\phi_\alpha(f) = \left(c^2(4\alpha)^{-1} + 1\right)\|f\|$.

Theorem 4.2. *The hamiltonian operator L is self-adjoint.*

Proof. The result follows immediately from Theorem 3.1 and Lemma 4.1.

Remark 4.3. *Theorem 4.2 guarantees the self-adjointness of the hamiltonian operator L under less restrictive conditions than the one considered in* [16, *Theorem* 2]. *Indeed, from Theorem* 2 *in* [16], *it is required that* α^2 *is strictly smaller than* 1 *to obtain the self-adjointness of the hamiltonian operator L. Due to our result, this condition is usually satisfied for every sufficiently small* α.

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