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Long Time Behavior of an Two Diffusion Stochastic SIR Epidemic Model with Nonlinear Incidence and Treatment

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Abstract. In this paper we propose a stochastic SIR epidemic model to evaluate effect of the randomness on treatment and nonlinear incidence rate. More precisely, we perturb both nonlinear incidence and treatment rates in deterministic SIR model with Gaussian white noise and obtain two diffusion stochastic model. For the model, we theoretically prove that it's solution is positive and global, and then, we obtain the conditions under which we can claim the existence of the stationary distribution. Also, by constructing suitable Lyapunov functions, we establish sufficient conditions for p-th moment and almost sure exponential stability of disease-free equilibrium. Conditions for disease extinction are obtained, as well. We close the paper by presenting numerical simulations to verify our theoretical results. For that purpose we use real-life data for spread of cholera in the Department of Artibonite in Haiti, as well as for influenza A H1N1 in Guangdong Province, China.

1. Introduction and Motivation

Infectious diseases are diseases caused by a certain type of bacteria, parasites, viruses or fungi. Most of these organisms do not pose a significant threat to health, especially when the immune system of the host is fully functional. However, those that can impair health can even endanger the life of the sufferer. Disease-causing organisms can be transmitted from person to person through: skin contact, the transfer of bodily fluids, touching an object that a person carrying the pathogen has also touched, as well as through animal stings and bites, or the consumption of contaminated food or water. Infectious diseases can affect anyone, although people with compromised immune systems are at higher risk of acquiring the infection.

As it was pointed out in [10] it was expected that infectious diseases, in particular bacterial infections, would no longer be a major public health problem when some new drugs, especially penicillin, were discovered. However, with the development of drug-resistant bacteria, fatal viral diseases such as AIDS, and severe acute respiratory syndrome (SARS) caused by variants of Corona viruses, this early optimism appears unwarranted. We are currently witnessing a global pandemic that has changed our lives, claimed many lives, and whose consequences for the global economy are yet to be discussed in the years ahead.

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Mathematical models have long been used for understanding the dynamics of epidemics in populations and predicting outcomes of effective control strategies. Different types of the diseases are modeled by different epidemiological models. In the classic SIR model, which is one of the most commonly implemented and the basis for other models, the total population is divided into three compartments: "susceptible", "infected" and "recovered" as the pathogen spreads from person to person. The SIR model is one of the most important models in epidemiological patterns and disease control which was initially proposed and studied by Kermack and McKendrick in a set of three articles from 1927, 1932, and 1933. Because of their importance to the field of theoretical epidemiology, these articles were republished ([5]–[7]). Since then, various of SIR models have been investigated by many researchers due to their theoretical and practical significance. Namely, while in the classical SIR model, the incidence rate of new infections is linear, i.e. $g(I) = \beta I$, where β is the number of adequate contacts per unit time, it indicates that the number of new infections is proportional to the number of existing ones. However, after examining the cholera epidemic in Bari in 1973, Capasso et al. [2] introduced a nonlinear saturated incidence rate $g(I) = \frac{\beta I}{1+kI}$, which prevents the unlimited number of contacts and when $I \to \infty$ we have that $g(I) \to \frac{\beta}{k}$. A therm $\frac{1}{1+kI}$ represents precautionary measures for persons who have a disturbed immune system and are therefore more susceptible to infection, which is completely understandable to consider.

Besides precautionary measures, another method to prevent and control the spread of infectious diseases is a treatment. In classical epidemic models, the treatment rate of the infective persons is assumed to be proportional to the number of the infective individuals [1]. Meanwhile, during the SARS outbreaks in 2003 when number of infected persons was dramatically increasing in Beijing, researchers started to consider the capacity of the health-care system from both modeling and analyzing points of view. Nowadays, we are also witnessing the need for new Covid hospitals, bearing in mind that the existing capacities were not sufficient to take care of all infected people and to provide them appropriate care. In order to investigate the effect of limited medical treatment on the spread of infectious disease Wang et al. [19] introduced a constant treatment function in the SIR model. Zhang and Liu [21] replace the constant function by continuous-differentiable treatment function $h(I) = \frac{rI}{1+aI}$, where $\frac{r}{a}$ represents the maximum supply medical resources per unit time. Since the limitation of medical resources $\frac{r}{a}$ and supply efficiency $\frac{1}{1+aI}$ are dependent, it is better to modify the treatment function by

$$h(I) = \frac{\alpha I}{\omega + I},$$

where α represents the saturation constant (maximum medical resources per unit time) and $\omega > 0$ is the half-saturation constant (when $I = \omega$, then $h(I) = \frac{\alpha}{2}$), which measures the efficiency in the supply of medical resources in the sense that less ω indicates the greater efficiency. In [23] the authors consider SIR epidemiological model in order to understand how the limited medical resources and their supply efficiency affect the transmission of infectious diseases. The model is given by the system of ordinary differential equations of the form

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \frac{\beta IS}{1+kI} - \mu S, \\ \frac{dI}{dt} &= \frac{\beta IS}{1+kI} - (\mu + \epsilon + \gamma)I - \frac{\alpha I}{\omega + I}, \\ \frac{dR}{dt} &= \gamma I + \frac{\alpha I}{\omega + I} - \mu R, \end{aligned} \tag{1}$$

with the initial conditions $S(0) = S_0$, $I(0) = I_0$, $R(0) = R_0$. Besides the parameters which are already explained, let us give biological interpretation of the other parameters of model (1). The recruitment rate of the population is denoted by Λ, μ is the natural death rate of the population, γ represents the natural recovery rate, and ϵ is mortality rate related to the disease.

All previously mentioned models are deterministic. However, for diseases that spread mainly in the human population, the nature of the growth and spread of the epidemic is accidental due to the unpredictability of mutual contacts. That is the main reason why the stochastic models could be the more appropriate way of modeling epidemics. In the literature, there are many papers which consider dynamics

([3, 14]), stationary distribution (see [12, 13, 15], for instance) or stability (see [4, 9, 17, 18], among the others) of the stochastic epidemiological models. In majority of these papers the authors introduce a usual stochastic perturbation (i.e. linear noise) to represent uncertainty. In the literature, four approaches to introduce stochastic perturbations to the deterministic model can be found: time Markov chain model, parameter perturbation, perturbations proportional to the variables, and centering around the equilibria of deterministic models. We consider model (1) and use second approach, i.e. we assume that accidental nature of mutual contacts among the people will manifest trough the number of the adequate contacts β, and that accidental growth of epidemic may affect the saturation constant α. Thus, we perturb $\beta \to \beta + \sigma_1 \dot{B}_1(t)$ and $\alpha \to \alpha + \sigma_2 \dot{B}_2(t)$, where $B_1(t)$ and $B_2(t)$ are two independent standard Brownian motions defined on complete probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying the usual conditions (it is right continuous and increasing, while $\mathcal{F}_0$ contains all P-null sets) and σ_1 and σ_2 are real constants, and obtain two diffusion stochastic model with nonlinear incidence and treatment of the form

$$\begin{aligned} dS &= \left(\Lambda - \frac{\beta IS}{1+kI} - \mu S\right) dt - \frac{\sigma_1 IS}{1+kI} dB_1(t), \\ dI &= \left(\frac{\beta IS}{1+kI} - (\mu + \epsilon + \gamma)I - \frac{\alpha I}{\omega + I}\right) dt + \frac{\sigma_1 IS}{1+kI} dB_1(t) - \frac{\sigma_2 I}{\omega + I} dB_2(t), \\ dR &= \left(\gamma I + \frac{\alpha I}{\omega + I} - \mu R\right) dt + \frac{\sigma_2 I}{\omega + I} dB_2(t), \end{aligned} \tag{2}$$

with the initial conditions $S(0) = S_0$, $I(0) = I_0$, $R(0) = R_0$. In [22] the authors consider stationary distribution of the stochastic SIR model which is similar to model (2). However, they include random effects by adding linear noise into deterministic system, which is assumed to be proportional to $S(t)$, $I(t)$ and $R(t)$. The fact that model (2) has two nonlinear diffusions makes the study of the considered model more complex then in previously mentioned models.

The rest of this paper is arranged as follows. In Section 2 we give auxiliary lemmas which will be used in the sequel. Section 3 contains the result which gives us the existence and uniqueness of global positive solution to system (2). Then, in Section 4 the sufficient conditions for the unique stationary distribution are found. Extinction of the disease is subject of Section 5. More precisely, in this section we obtain conditions under which the disease-free equilibrium of system (2) is p-th moment and almost surely exponentially stable, as well as the conditions under which extinction of the disease occurs. Section 6 introduces some simulations to verify our analytical results. For that purpose we use real life data of the cholera outbreak in the Department of Artibonite in Haiti, and also realistic parameters of influenza A H1N1 in Guangdong Province, China.

2. Preliminaries

Let us present some auxiliary results that we will use in the proofs of the main results of the paper, and which may be found in [16]. As it was mentioned in the previous section, $(\Omega, \mathcal{F}, P)$ is a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying the usual conditions (it is right continuous and increasing, while $\mathcal{F}_0$ contains all P-null sets). Let $B(t)$ denotes a m-dimensional standard Brownian motion defined on the probability space. Consider the d-dimensional stochastic differential equation

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB(t), \quad t \geq 0, \tag{3}$$

with initial value $X(0) = X_0 \in \mathbb{R}^d$.

Denote by $C^{1,2}\left([0,\infty) \times \mathbb{R}^d; [0,\infty)\right)$ the family of all nonnegative functions $V(t, X)$ such that they are continuously differentiable with respect to t and twice with respect to X. If $V \in C^{1,2}\left([0,\infty) \times \mathbb{R}^d; [0,\infty)\right)$, we set

$$V_t = \frac{\partial V}{\partial t}, \quad V_X = \begin{pmatrix} \frac{\partial V}{\partial X_1} \\ \vdots \\ \frac{\partial V}{\partial X_d} \end{pmatrix}, \quad V_{XX} = \begin{pmatrix} \frac{\partial^2 V}{\partial X_1 \partial X_1} & \cdots & \frac{\partial^2 V}{\partial X_1 \partial X_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial X_d \partial X_1} & \cdots & \frac{\partial^2 V}{\partial X_d \partial X_d} \end{pmatrix}.$$

Define the differential operator L associated with Eq. (3) by

$$LV(t,X)=V_t(t,X)+V_X^T(t,X)f(t,X)+\frac{1}{2}trace[g^T(t,X)V_{XX}(t,X)g(t,X)].$$

In view of the Itô formula, we have

$$dV(t,X) = LV(t,X)dt + V_X^T(t,X)g(t,X)dB(t).$$

Theorem 2.1. *($\mathbb{D}$-invariance. [8]) Let $\mathbb{D}$ and $\mathbb{D}_n$ be open sets in $\mathbb{R}^d$ with*

$$\mathbb{D}_n \subseteq \mathbb{D}_{n+1}, \quad \overline{\mathbb{D}}_n \subseteq \mathbb{D}, \quad \text{and} \quad \mathbb{D} = \bigcup_n \mathbb{D}_n$$

and suppose f and g satisfy the existence and uniqueness conditions for solutions of Eq. (3) on each set $\{(t,X) : t > 0, X \in \mathbb{D}_n\}$. Suppose there is a nonnegative continuous function $V : [0,\infty) \times \mathbb{D} \to [0,\infty)$ with continuous partial derivatives and satisfying $LV \le cV$ for some positive constant c and $t > 0$, $X \in \mathbb{D}$. If also,

$$\inf_{t>0,\ X\in\mathbb{D}\setminus\mathbb{D}_n} V(t,X) \to \infty \ \text{ as } \ n \to \infty,$$

then, for any $X_0 \in \mathbb{D}$ there is a unique Markovian continuous time solution $X(t)$ of (3) with $X(0) = X_0$, and $X(t) \in \mathbb{D}$ for all $t > 0$ a.s.

Definition 2.2. *Let $X(t)$ be a regular time-homogenous Markov process in $\mathbb{R}^d$ described by the stochastic differential equation*

$$dX(t) = f(X(t))dt + \sum_{r=1}^{m} g_r(X(t))dB_r(t). \tag{4}$$

The diffusion matrix of the process $X(t)$ is defined as follows

$$A(x) = (a_{ij}(x)), \qquad a_{ij}(x) = \sum_{r=1}^{m} g_r^i(x)g_r^j(x).$$

Lemma 2.3. *[24] The system (4) is positive recurrent if there is a bounded open subset D of $\mathbb{R}^d$ with a regular (i.e., smooth) boundary, and*

(i) *there exist some $i = 1, 2, \ldots, d$ and a positive constant κ such that*

$$a_{ii}(x) \ge \kappa \quad \text{for any} \quad x \in D,$$

(ii) *there exists a nonnegative function $V : D^c \to [0;\infty)$ such that V is twice continuously differentiable and that for some $\theta > 0$,*

$$LV(x) \le -\theta, \quad \text{for any} \quad x \in D^c.$$

Moreover, the positive recurrent process $X(t)$ has a unique stationary distribution $\bar{\mu}(\cdot)$ with density in $\mathbb{R}^d$ such that for any Borel set $B \in \mathbb{R}^d$

$$\lim_{t\to\infty} \mathbb{P}(t,x,B) = \bar{\mu}(B),$$

and

$$\mathbb{P}_x\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T h\left(X(t)\right)dt = \int_{\mathbb{R}^d} h(x)\bar{\mu}(dx)\right\} = 1$$

for all $x \in \mathbb{R}^d$ and $h : \mathbb{R}^d \to \mathbb{R}$ be a function integrable with respect to the measure $\bar{\mu}$.

Definition 2.4. *The trivial equilibrium of Eq. (3) is said to be:*
(i) almost surely exponentially stable if for all $X_0 \in \mathbb{R}^d$ it holds

$$\limsup_{t\to\infty} \frac{1}{t} \ln|X(t, X_0)| < 0 \ \ a.s.$$

(ii) p-th moment exponentially stable if there is a pair of positive constants C_1 and C_2 such that for all $X_0 \in \mathbb{R}^d$ it holds

$$E\left(|X(t, X_0)|^p\right) \le C_1|X_0|^p e^{-C_2 t}.$$

Theorem 2.5. *Suppose that there exists a $C^{1,2}$-function V satisfying inequalities*

$$K_1|X|^p \le V(t, X) \le K_2|X|^p, \tag{5}$$
$$LV(t, X) \le -K_3|X|^p \tag{6}$$

for some $K_1, K_2, K_3 > 0$ and $p > 0$. Then the trivial equilibrium of Eq. (3) *is p-th moment exponentially stable.*

3. Existence and Uniqueness of Global Positive Solution

To study the dynamical properties of model (2), we first need to consider whether the solution is global and positive. Consider the set

$$\Gamma = \left\{(S(t), I(t), R(t)) \in \mathbb{R}^3_+ : \ S(t) + I(t) + R(t) < \frac{\Lambda}{\mu}\right\}. \tag{7}$$

Theorem 3.1. *There is a unique continuous time, Markovian global solution $(S(t), I(t), R(t))$ of system* (2)*, on $t \ge 0$, for any initial condition $(S_0, I_0, R_0) \in \Gamma$. This solution is invariant with respect to Γ with probability 1.*

Proof. Since the coefficients of system (2) are locally Lipschitz continuous, then, for any initial value there exists a unique local solution $(S(t), I(t), R(t))$ on $t \in [0, \tau(\Gamma))$, where $\tau(\Gamma)$ represents explosion time. To show that this solution is global, we need to prove that $\tau(\Gamma) = \infty$ a.s. Let

$$\Gamma_n = \left\{(S(t), I(t), R(t)) : \ e^{-n} < S < \frac{\Lambda}{\mu} - e^{-n},\ e^{-n} < I < \frac{\Lambda}{\mu} - e^{-n},\ e^{-n} < R < \frac{\Lambda}{\mu} - e^{-n},\ S + I + R < \frac{\Lambda}{\mu}\right\},$$

for $n \in \mathbb{N}$. The system (2) has a unique solution up to stopping time $\tau(\Gamma_n)$.

Let us define a C^2-function

$$V(S, I, R) = S - \ln S + I - \ln I + R + \frac{\Lambda}{\mu} - S - \ln\left(\frac{\Lambda}{\mu} - S\right),$$

on set Γ and assume that $EV(S, I, R) < \infty$. Nonnegativity of this function can be seen from inequality $u - 1 - \ln u \ge 0$ for any $u > 0$ and we have $V(S, I, R) \ge 3$ for $(S, I, R) \in \Gamma$.
Application of differential operator L on V yields

$$\begin{aligned}
LV = {} & \left(\frac{1}{\frac{\Lambda}{\mu} - S} - \frac{1}{S}\right)\left(\Lambda - \frac{\beta IS}{1 + kI} - \mu S\right) + \left(1 - \frac{1}{I}\right)\left(\frac{\beta IS}{1 + kI} - (\mu + \epsilon + \gamma)I - \frac{\alpha I}{\omega + I}\right) + \gamma I + \frac{\alpha I}{\omega + I} - \mu R \\
& + \frac{\sigma_1^2 I^2}{2(1 + kI)^2} + \frac{\sigma_1^2 I^2 S^2}{2(\frac{\Lambda}{\mu} - S)^2(1 + kI)^2} + \frac{\sigma_1^2 S^2}{2(1 + kI)^2} + \frac{\sigma_2^2}{2(\omega + I)^2} \\
= {} & -\frac{\beta IS}{(\frac{\Lambda}{\mu} - S)(1 + kI)} - \frac{\Lambda}{S} + \frac{\beta I(1 + S)}{1 + kI} - (\mu + \epsilon)I - \frac{\beta S}{1 + kI} + 3\mu + \epsilon + \gamma + \frac{\alpha}{\omega + I} - \mu R \\
& + \frac{\sigma_1^2 I^2}{2(1 + kI)^2} + \frac{\sigma_1^2 I^2 S^2}{2(\frac{\Lambda}{\mu} - S)^2(1 + kI)^2} + \frac{\sigma_1^2 S^2}{2(1 + kI)^2} + \frac{\sigma_2^2}{2(\omega + I)^2}.
\end{aligned}$$

Removing some nonpositive terms and using fact $S + I + R < \frac{\Lambda}{\mu}$, we obtain

$$\begin{aligned} LV &\leq \frac{\beta}{k}\left(1+\frac{\Lambda}{\mu}\right)+3\mu+\epsilon+\gamma+\frac{\alpha}{\omega}+\frac{\sigma_1^2}{2k^2}+\frac{\sigma_1^2\Lambda^2}{\mu^2}+\frac{\sigma_2^2}{2\omega^2} \\ &:= 3c, \end{aligned}$$

where $c = \frac{1}{3}\left(\frac{\beta}{k}\left(1+\frac{\Lambda}{\mu}\right)+3\mu+\epsilon+\gamma+\frac{\alpha}{\omega}+\frac{\sigma_1^2}{2k^2}+\frac{\sigma_1^2\Lambda^2}{\mu^2}+\frac{\sigma_2^2}{2\omega^2}\right)$ is a positive constant. Therefore,

$$LV(S,I,R) \leq cV(S,I,R),$$

since $V(S,I,R) \geq 3$ for $(S,I,R) \in \Gamma$ and $\inf_{(S,I,R)\in\Gamma\setminus\Gamma_n} V(S,I,R) > n+2$ for $n \in \mathbb{N}$.
Now, we define $C^{1,2}$ function

$$W(t,S,I,R) = e^{-ct}V(S,I,R)$$

on $[0,\infty)\times\Gamma$. Applying operator L on W gives

$$\begin{aligned} LW(t,S,I,R) &= -ce^{-ct}V(S,I,R) + e^{-ct}LV(S,I,R) \\ &= e^{-ct}(-cV(S,I,R) + LV(S,I,R)) \\ &\leq 0. \end{aligned}$$

If we define $\tau_n := \min\{t, \tau(\Gamma_n)\}$ and apply Dynkin's formula, we get

$$\begin{aligned} EW(\tau_n, S(\tau_n), I(\tau_n), R(\tau_n)) &= EW(0,S(0),I(0),R(0)) + E\int_0^{\tau_n} LW(u,S(u),I(u),R(u))du \\ &\leq EW(0,S(0),I(0),R(0)) \\ &= EV(S(0),I(0),R(0)) = V(S_0,I_0,R_0). \end{aligned}$$

Next, to show that $P(\tau(\Gamma_n) < t) = 0$, we take the expected value

$$\begin{aligned} E\left(e^{c(t-\tau_n)}V(S(\tau_n),I(\tau_n),R(\tau_n))\right) &= E\left(e^{ct}e^{-c\tau_n}V(S(\tau_n),I(\tau_n),R(\tau_n))\right) \\ &= E\left(e^{ct}W(\tau_n,S(\tau_n),I(\tau_n),R(\tau_n))\right) \\ &\leq e^{ct}V(S_0,I_0,R_0). \end{aligned}$$

Bearing in mind that $\Gamma_n \subseteq \Gamma$, we obtain

$$\begin{aligned} 0 &\leq P(\tau(\Gamma) < t) \leq P(\tau(\Gamma_n) < t) \\ &= P(\tau_n < t) \\ &= E(\mathbb{I}_{\{\tau_n<t\}}) \\ &\leq E\left(e^{c(t-\tau_n)}\frac{V(S(\tau_n),I(\tau_n),R(\tau_n))}{\inf_{(S,I,R)\in\Gamma\setminus\Gamma_n}V(S,I,R)}\mathbb{I}_{\{\tau_n<t\}}\right) \\ &\leq e^{ct}\frac{V(S_0,I_0,R_0)}{\inf_{(S,I,R)\in\Gamma\setminus\Gamma_n}V(S,I,R)} \\ &\leq e^{ct}\frac{V(S_0,I_0,R_0)}{n+2} \quad \to 0 \quad \text{as} \quad n\to\infty, \end{aligned}$$

for all $(S_0,I_0,R_0) \in \Gamma_n$ and for all fixed $t \in [0,\infty)$.
Thus, $P(\tau(\Gamma) < t) = P(\tau(\Gamma_n) < t) = 0$ for all $(S_0,I_0,R_0) \in \Gamma$ and $t \geq 0$, that is, $P(\tau(\Gamma) = \infty) = 1$.

This proves the invariance property and the global existence of the solution $(S(t),I(t),R(t))$ on Γ. Uniqueness and continuity of the solution is obtained by Theorem 2.1 . □

Corollary 3.2. *The set* Γ *defined by* (7) *is almost surely positively invariant by system* (2)*, that is, if* $(S_0,I_0,R_0) \in \Gamma$*, then* $P\{(S(t),I(t),R(t)) \in \Gamma\} = 1$ *for all* $t \geq 0$.

From now on, we always assume that the initial value $(S_0,I_0,R_0) \in \Gamma$.

4. Existence of Stationary Distribution

In this section, we will establish sufficient conditions for the positive recurrence and the existence of a stationary distribution for system (2). The existence of a stationary distribution means that the system is stochastically weak stabile, i.e. the disease will prevail in the population in long time.

Let us denote

$$R_0^s := \frac{\beta}{\left(\frac{\beta}{k} + \mu + \frac{\sigma_1^2}{2k^2}\right)\left(\mu + \epsilon + \gamma + \frac{\alpha}{\omega} + \frac{\sigma_2^2}{2\omega^2}\right)}. \tag{8}$$

Theorem 4.1. *If $R_0^s > 1$, then the solution $(S(t), I(t), R(t))$ of system (2) with any initial value $(S_0, I_0, R_0) \in \Gamma$ is positive recurrent and admits a unique ergodic stationary distribution in Γ.*

Proof. The diffusion matrix of system (2) is

$$A = \begin{pmatrix} \frac{\sigma_1^2 I^2 S^2}{(1+kI)^2} & -\frac{\sigma_1^2 I^2 S^2}{(1+kI)^2} & 0 \\ -\frac{\sigma_1^2 I^2 S^2}{(1+kI)^2} & \frac{\sigma_1^2 I^2 S^2}{(1+kI)^2} + \frac{\sigma_2^2 I^2}{(\omega+I)^2} & -\frac{\sigma_2^2 I^2}{(\omega+I)^2} \\ 0 & -\frac{\sigma_2^2 I^2}{(\omega+I)^2} & \frac{\sigma_2^2 I^2}{(\omega+I)^2} \end{pmatrix}.$$

Define the following bounded open subset of Γ

$$D_\varepsilon = \left\{(S, I, R) \in \Gamma \ : \varepsilon < S < \frac{1}{\varepsilon},\ \varepsilon < I < \frac{1}{\varepsilon},\ \varepsilon^2 < R < \frac{1}{\varepsilon^2}\right\},$$

where $0 < \varepsilon < 1$ is a sufficiently small number. For any $(S, I, R) \in D_\varepsilon$ and, for example, $i = 3$, we have

$$a_{33}(S, I, R) = \frac{\sigma_2^2 I^2}{(\omega + I)^2} \geq \frac{\sigma_2^2 \varepsilon^2}{\left(\omega + \frac{1}{\varepsilon}\right)^2} = \frac{\sigma_2^2 \varepsilon^4}{(\omega\varepsilon + 1)^2}.$$

Thus the condition (*i*) in Lemma 2.3 is satisfied.

To verify the condition (*ii*) in Lemma 2.3, we only need to construct a nonnegative C^2-function $V : \Gamma \to [0, \infty)$ such that $LV(S, I, R) \leq -1$ for any $(S, I, R) \in \Gamma \setminus D_\varepsilon$.

Define

$$V_1(S, I, R) = S + \left(2 + \frac{k}{\mu}\right) I + 2R - c_1 \ln S - c_2 \ln I,$$

where c_1 and c_2 are positive constant which will be determined later. Application of the Itô formula to V_1, as well as the definition of set Γ in (7) and well known $AM - GM$ inequality, yields

$$\begin{aligned} LV_1 &= \Lambda + \frac{\mu + k}{\mu}\frac{\beta I S}{1 + kI} - \mu S - 2(\mu + \epsilon)I - \frac{k}{\mu}(\mu + \epsilon + \gamma)I - \frac{k}{\mu}\frac{\alpha I}{\omega + I} - 2\mu R \\ &\quad - \frac{c_1\Lambda}{S} + \frac{c_1\beta I}{1 + kI} + c_1\mu + \frac{c_1\sigma_1^2 I^2}{2(1 + kI)^2} - \frac{c_2\beta S}{1 + kI} + c_2(\mu + \epsilon + \gamma) + \frac{c_2\alpha}{\omega + I} + \frac{c_2\sigma_1^2 S^2}{2(1 + kI)^2} + \frac{c_2\sigma_2^2}{2(\omega + I)^2} \\ &\leq -kI - 1 - \frac{c_1\Lambda}{S} - \frac{c_2\beta S}{1 + kI} + 1 + \Lambda + c_1\left(\frac{\beta}{k} + \mu + \frac{\sigma_1^2}{2k^2}\right) + c_2\left(\mu + \epsilon + \gamma + \frac{\alpha}{\omega} + \frac{\sigma_2^2}{2\omega^2}\right) + \frac{(\mu + k)\beta}{\mu}S + \frac{c_2\sigma_1^2}{2}S^2 \\ &\leq -3\sqrt[3]{c_1 c_2 \beta \Lambda} + 1 + \Lambda + c_1\left(\frac{\beta}{k} + \mu + \frac{\sigma_1^2}{2k^2}\right) + c_2\left(\mu + \epsilon + \gamma + \frac{\alpha}{\omega} + \frac{\sigma_2^2}{2\omega^2}\right) + \frac{(\mu + k)\beta\Lambda}{\mu^2} + \frac{c_2\sigma_1^2\Lambda^2}{2\mu^2}. \end{aligned}$$

Let

$$c_1\left(\frac{\beta}{k} + \mu + \frac{\sigma_1^2}{2k^2}\right) = c_2\left(\mu + \epsilon + \gamma + \frac{\alpha}{\omega} + \frac{\sigma_2^2}{2\omega^2}\right) = \Lambda.$$

Then we have

$$c_1 = \frac{\Lambda}{\frac{\beta}{k} + \mu + \frac{\sigma_1^2}{2k^2}}, \quad c_2 = \frac{\Lambda}{\mu + \epsilon + \gamma + \frac{\alpha}{\omega} + \frac{\sigma_2^2}{2\omega^2}}$$

and $c_1, c_2 \leq \frac{\Lambda}{\mu}$. Thus,

$$\begin{aligned} LV_1 &\leq -3\Lambda \sqrt[3]{\frac{\beta}{\left(\frac{\beta}{k} + \mu + \frac{\sigma_1^2}{2k^2}\right)\left(\mu + \epsilon + \gamma + \frac{\alpha}{\omega} + \frac{\sigma_2^2}{2\omega^2}\right)}} + 3\Lambda + 1 + \frac{(\mu + k)\beta\Lambda}{\mu^2} + \frac{\sigma_1^2\Lambda^3}{2\mu^3} \\ &= -3\Lambda\left(\sqrt[3]{R_0^s} - 1\right) + 1 + \frac{(\mu + k)\beta\Lambda}{\mu^2} + \frac{\sigma_1^2\Lambda^3}{2\mu^3} \\ &= -Q + K, \end{aligned} \tag{9}$$

where

$$Q = 3\Lambda\left(\sqrt[3]{R_0^s} - 1\right), \qquad K = 1 + \frac{(\mu + k)\beta\Lambda}{\mu^2} + \frac{\sigma_1^2\Lambda^3}{2\mu^3}.$$

It is obvious that the both constants are positive, bearing in mind the condition of the theorem. Next, define

$$V_2(S) = -\ln S, \quad V_3(R) = R^2, \quad V_4(S, I, R) = \frac{1}{2}(S + I + R)^2, \quad V_5(S, I, R) = -\ln\left(\frac{\Lambda}{\mu} - (S + I + R)\right).$$

Applying the differential operator L to V_2, V_3, V_4, V_5, respectively, and taking into account definition (7) of set Γ, we obtain

$$\begin{aligned} LV_2 &= -\frac{\Lambda}{S} + \frac{\beta I}{1 + kI} + \mu + \frac{\sigma_1^2 I^2}{2(1 + kI)^2} \\ &\leq -\frac{\Lambda}{S} + \frac{\beta I}{1 + kI} + \mu + \frac{\sigma_1^2}{2k^2}, \end{aligned} \tag{10}$$

$$\begin{aligned} LV_3 &= 2\frac{\alpha RI}{\omega + I} + 2\gamma RI - 2\mu R^2 + \frac{\sigma_2^2 I^2}{(\omega + I)^2} \\ &\leq 2\alpha R + 2\gamma RI - 2\mu R^2 + \sigma_2^2 \\ &\leq \frac{2\alpha\Lambda}{\mu} + 2\gamma RI + \sigma_2^2, \end{aligned} \tag{11}$$

$$\begin{aligned} LV_4 &= (S + I + R)(\Lambda - \mu(S + I + R) - \epsilon I) \\ &\leq \Lambda(S + I + R) - \mu(S + I + R)^2 \\ &\leq \Lambda(S + I + R) - \frac{\mu}{2}(S + I + R)^2 - \frac{\mu}{2}(S^2 + I^2 + R^2) \\ &= -\frac{\mu}{2}\left(S + I + R - \frac{\Lambda}{\mu}\right)^2 + \frac{\Lambda^2}{2\mu} - \frac{\mu}{2}(S^2 + I^2 + R^2) \\ &\leq \frac{\Lambda^2}{2\mu} - \frac{\mu}{2}(S^2 + I^2 + R^2), \end{aligned} \tag{12}$$

$$\begin{aligned} LV_5 &= -\frac{1}{\frac{\Lambda}{\mu} - (S + I + R)}(-\Lambda + \mu(S + I + R) + \epsilon I) \\ &= \mu - \frac{\epsilon I}{\frac{\Lambda}{\mu} - (S + I + R)} \\ &\leq \mu \end{aligned} \tag{13}$$

Define a C^2-function $\hat{V} : \Gamma \to \mathbb{R}$ as follows

$$\hat{V}(S,I,R) = MV_1(S,I,R) + V_2(S) + V_3(R) + V_4(S,I,R) + V_5(S,I,R),$$

where $M > 0$ is a sufficiently large number satisfying the following condition

$$-MQ + F \leq -2, \tag{14}$$

and

$$F := MK + \frac{\beta}{k} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + \frac{2\gamma\Lambda^2}{\mu^2} + \sigma_2^2 + \frac{\Lambda^2}{2\mu}.$$

Moreover, note that $\hat{V}(S,I,R)$ is not only continuous, but also tends to ∞ as (S,I,R) approaches the boundary of Γ. Therefore, it must be lower bounded and it achieves this lower bound at a point (S^0,I^0,R^0) in the interior of Γ. Then, we define a C^2-function $V : \Gamma \to [0,\infty)$ as follows

$$V(S,I,R) = MV_1(S,I,R) + V_2(S) + V_3(R) + V_4(S,I,R) + V_5(S,I,R) - \hat{V}(S^0,I^0,R^0).$$

In view of (9)-(13), we have

$$LV \leq -MQ + MK - \frac{\Lambda}{S} + \frac{\beta I}{1+kI} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + 2\gamma RI + \sigma_2^2 + \frac{\Lambda^2}{2\mu} - \frac{\mu}{2}(S^2+I^2+R^2). \tag{15}$$

In the set $\Gamma \setminus D_\varepsilon$, we can choose ε sufficiently small such that the following conditions hold

$$-\frac{\Lambda}{\varepsilon} + F \leq -1, \tag{16}$$
$$\beta\varepsilon \leq 1, \tag{17}$$
$$2\gamma\varepsilon \leq 1, \tag{18}$$
$$-\frac{\mu}{2\varepsilon^2} + F \leq -1, \tag{19}$$
$$-\frac{\mu}{2\varepsilon^4} + F \leq -1. \tag{20}$$

For convenience, we can divide $\Gamma \setminus D_\varepsilon$ into six domains

$$D_1 = \{(S,I,R) \in \Gamma \ : S \leq \varepsilon\},\ \ D_2 = \{(S,I,R) \in \Gamma \ : I \leq \varepsilon\},\ \ D_3 = \left\{(S,I,R) \in \Gamma \ : I \leq \frac{1}{\varepsilon},\ R \leq \varepsilon^2\right\},$$
$$D_4 = \left\{(S,I,R) \in \Gamma \ : S \geq \frac{1}{\varepsilon}\right\},\ \ D_5 = \left\{(S,I,R) \in \Gamma \ : I \geq \frac{1}{\varepsilon}\right\},\ \ D_6 = \left\{(S,I,R) \in \Gamma \ : R \geq \frac{1}{\varepsilon^2}\right\}.$$

Clearly, $D_\varepsilon^c = \Gamma \setminus D_\varepsilon = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6$. Next, we will prove that $LV(S,I,R) \leq -1$ for any $(S,I,R) \in D_\varepsilon^c$, which is equivalent to proving it on the above six domains, respectively.

Case 1. For any $(S,I,R) \in D_1$, by (15), we have

$$\begin{aligned} LV &\leq -\frac{\Lambda}{S} - MQ + MK + \frac{\beta I}{1+kI} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + 2\gamma RI + \sigma_2^2 + \frac{\Lambda^2}{2\mu} - \frac{\mu}{2}(S^2+I^2+R^2) \\ &\leq -\frac{\Lambda}{S} + MK + \frac{\beta}{k} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + \frac{2\gamma\Lambda^2}{\mu^2} + \sigma_2^2 + \frac{\Lambda^2}{2\mu} \\ &\leq -\frac{\Lambda}{\varepsilon} + F \\ &\leq -1, \end{aligned}$$

which follows from (16). Therefore

$$LV \leq -1 \ \text{ for any } (S,I,R) \in D_1.$$

Case 2. For any $(S, I, R) \in D_2$, by (15), we obtain

$$\begin{aligned} LV &\leq \frac{\beta I}{1+kI} - MQ + MK - \frac{\Lambda}{S} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + 2\gamma RI + \sigma_2^2 + \frac{\Lambda^2}{2\mu} - \frac{\mu}{2}(S^2 + I^2 + R^2) \\ &\leq \beta I - MQ + MK + \frac{\beta}{k} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + \frac{2\gamma\Lambda^2}{\mu^2} + \sigma_2^2 + \frac{\Lambda^2}{2\mu} \\ &\leq \beta\varepsilon - MQ + F \\ &\leq 1 - 2 = -1, \end{aligned}$$

which follows from (14) and (17). So

$$LV \leq -1 \text{ for any } (S, I, R) \in D_2.$$

Case 3. For any $(S, I, R) \in D_3$, according to (15), we derive

$$\begin{aligned} LV &\leq 2\gamma RI - MQ + MK - \frac{\Lambda}{S} + \frac{\beta I}{1+kI} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + \sigma_2^2 + \frac{\Lambda^2}{2\mu} - \frac{\mu}{2}(S^2 + I^2 + R^2) \\ &\leq 2\gamma\varepsilon^2\frac{1}{\varepsilon} - MQ + MK + \frac{\beta}{k} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + \frac{2\gamma\Lambda^2}{\mu^2} + \sigma_2^2 + \frac{\Lambda^2}{2\mu} \\ &\leq 2\gamma\varepsilon - MQ + F \\ &\leq 1 - 2 = -1, \end{aligned}$$

which follows from (14) and (18). Thus

$$LV \leq -1 \text{ for any } (S, I, R) \in D_3.$$

Case 4. For any $(S, I, R) \in D_4$, it follows from (15) that

$$\begin{aligned} LV &\leq -\frac{\mu}{2}S^2 - MQ + MK - \frac{\Lambda}{S} + \frac{\beta I}{1+kI} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + 2\gamma RI + \sigma_2^2 + \frac{\Lambda^2}{2\mu} - \frac{\mu}{2}(I^2 + R^2) \\ &\leq -\frac{\mu}{2}S^2 + MK + \frac{\beta}{k} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + \frac{2\gamma\Lambda^2}{\mu^2} + \sigma_2^2 + \frac{\Lambda^2}{2\mu} \\ &\leq -\frac{\mu}{2\varepsilon^2} + F \\ &\leq -1, \end{aligned}$$

which follows from (19). Hence

$$LV \leq -1 \text{ for any } (S, I, R) \in D_4.$$

Case 5. For any $(S, I, R) \in D_5$, in view of (15), we get

$$\begin{aligned} LV &\leq -\frac{\mu}{2}I^2 - MQ + MK - \frac{\Lambda}{S} + \frac{\beta I}{1+kI} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + 2\gamma RI + \sigma_2^2 + \frac{\Lambda^2}{2\mu} - \frac{\mu}{2}(S^2 + R^2) \\ &\leq -\frac{\mu}{2}I^2 + MK + \frac{\beta}{k} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + \frac{2\gamma\Lambda^2}{\mu^2} + \sigma_2^2 + \frac{\Lambda^2}{2\mu} \\ &\leq -\frac{\mu}{2\varepsilon^2} + F \\ &\leq -1, \end{aligned}$$

which follows from (19). So

$$LV \leq -1 \text{ for any } (S, I, R) \in D_5.$$

Case 6. For any $(S, I, R) \in D_6$, by (15), we have

$$\begin{aligned} LV &\leq -\frac{\mu}{2}R^2 - MQ + MK - \frac{\Lambda}{S} + \frac{\beta I}{1+kI} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + 2\gamma RI + \sigma_2^2 + \frac{\Lambda^2}{2\mu} - \frac{\mu}{2}(S^2 + I^2) \\ &\leq -\frac{\mu}{2}R^2 + MK + \frac{\beta}{k} + 2\mu + \frac{\sigma_1^2}{2k^2} + \frac{2\alpha\Lambda}{\mu} + \frac{2\gamma\Lambda^2}{\mu^2} + \sigma_2^2 + \frac{\Lambda^2}{2\mu} \\ &\leq -\frac{\mu}{2\varepsilon^4} + F \\ &\leq -1, \end{aligned}$$

which follows from (20). Therefore

$$LV \leq -1 \text{ for any } (S, I, R) \in D_6.$$

Based on the previous discussion we can obtain that for a sufficiently small ε

$$LV \leq -1 \text{ for any } (S, I, R) \in \Gamma \setminus D_\varepsilon,$$

so the condition (*ii*) of Lemma 2.3 is met. This completes the proof of Theorem 4.1. □

5. Stochastic Disease-free Dynamics

In this section we consider long-time behavior of model (2). A very important quantity for all epidemiological models is basic reproduction number which describes the dynamics of the disease. More preciously, it represents the average number of secondary infections that single infected person may generate in a totally susceptible population. For model (1), basic reproduction number is computed in [23] by the next generation method, and it is given by

$$\mathcal{R}_0 = \frac{\beta\Lambda}{\mu\left(\mu + \gamma + \epsilon + \frac{\alpha}{\omega}\right)}.$$

If $\alpha = 0$, basic reproduction number $\mathcal{R}_0$ reduces to the basic reproduction number without medical treatment $\mathcal{R}^*$, which is given by

$$\mathcal{R}^* = \frac{\beta\Lambda}{\mu\left(\mu + \gamma + \epsilon\right)}. \tag{21}$$

For model (1) basic reproduction numbers $\mathcal{R}_0$ and $\mathcal{R}^*$, as well as values P_1, P^* and P^{**} given by

$$\begin{aligned} P^* &= 1 - \frac{\left[\sqrt{(\beta+\mu k)(\mu+\gamma+\epsilon)\omega} - \sqrt{\alpha\left(\frac{\mu}{\omega} - \beta - \mu k\right)}\right]^2}{\mu\left(\mu+\gamma+\epsilon+\frac{\alpha}{\omega}\right)}, \\ P^{**} &= 1 - \frac{\left[\sqrt{(\beta+\mu k)(\mu+\gamma+\epsilon)\omega} + \sqrt{\alpha\left(\frac{\mu}{\omega} - \beta - \mu k\right)}\right]^2}{\mu\left(\mu+\gamma+\epsilon+\frac{\alpha}{\omega}\right)}, \\ P_1 &= 1 + \frac{(\beta+\mu k)(\mu+\gamma+\epsilon)\omega^2 + \alpha\omega(\beta+\mu k) - \alpha\mu}{\omega\mu\left(\mu+\gamma+\epsilon+\frac{\alpha}{\omega}\right)}, \end{aligned}$$

control number of equilibria which is shown in Theorem 2.1 in [23]. For our stochastic model, we are interested in disease-free equilibrium $E_0 = (\Lambda/\mu, 0, 0)$.

5.1. Stability analysis

In this subsection, we investigate p-th moment and almost sure exponential stability of disease-free equilibrium E_0 by using the suitable Lyapunov function and other techniques of stochastic analysis.

Theorem 5.1. *Let $p \geq 2$. If the condition*

$$\mathcal{R}^* + \frac{(p-1)(\Lambda^2\sigma_1^2\omega^2 + \sigma_2^2\mu^2)}{2\mu^2\omega^2(\mu+\epsilon+\gamma)} < 1 \tag{22}$$

is satisfied, where $\mathcal{R}^*$ is given by (21), the disease-free equilibrium E_0 of (2) is p-th moment exponentially stable in Γ.

Proof. Let $p \geq 2$ and $(S_0, I_0, R_0) \in \Gamma$. In view of Corollary 3.2 the solution of system (2) remains in Γ. We define the Lyapunov function V as follows:

$$V(S,I,R) = m_1\left(\frac{\Lambda}{\mu} - S\right)^p + \frac{1}{p}I^p + m_2R^p. \tag{23}$$

where m_1 and m_2 are positive constants that are to be chosen later. It is easy to check that inequality (5) is satisfied (choosing $K_1 = \min\{m_1, 1/p, m_2\}$ and $K_2 = \max\{m_1, 1/p, m_2\}$).
Furthermore, by the Itô formula, it follows from $(S,I,R) \in \Gamma$ that

$$\begin{aligned}
LV &= -m_1p\left(\frac{\Lambda}{\mu} - S\right)^{p-1}\left(\Lambda - \frac{\beta IS}{1+kI} - \mu S\right) + \frac{m_1p(p-1)}{2}\left(\frac{\Lambda}{\mu} - S\right)^{p-2}\frac{\sigma_1^2I^2S^2}{(1+kI)^2} \\
&\quad + I^{p-1}\left(\frac{\beta IS}{1+kI} - (\mu+\epsilon+\gamma)I - \frac{\alpha I}{\omega+I}\right) + \frac{p-1}{2}I^{p-2}\left(\frac{\sigma_1^2I^2S^2}{(1+kI)^2} + \frac{\sigma_2^2I^2}{(\omega+I)^2}\right) \\
&\quad + m_2pR^{p-1}\left(\gamma I + \frac{\alpha I}{\omega+I} - \mu R\right) + \frac{m_2p(p-1)}{2}R^{p-2}\frac{\sigma_2^2I^2}{(\omega+I)^2} \\
&\leq -m_1p\mu\left(\frac{\Lambda}{\mu} - S\right)^p + \frac{m_1p\beta\Lambda}{\mu}\left(\frac{\Lambda}{\mu} - S\right)^{p-1}I + \frac{m_1p(p-1)\sigma_1^2\Lambda^2}{2\mu^2}\left(\frac{\Lambda}{\mu} - S\right)^{p-2}I^2 \\
&\quad + \frac{\beta\Lambda}{\mu}I^p - (\mu+\epsilon+\gamma)I^p + \frac{(p-1)\sigma_1^2\Lambda^2}{2\mu^2}I^p + \frac{(p-1)\sigma_2^2}{2\omega^2}I^p \\
&\quad + m_2p\frac{\alpha}{\omega}R^{p-1}I + m_2p\gamma R^{p-1}I - c_2p\mu R^p + \frac{c_2p(p-1)\sigma_2^2}{2\omega^2}R^{p-2}I^2.
\end{aligned}$$

In view of Young inequality, for $p \geq 2$ and $S, I, R, \varepsilon > 0$, we have that

$$\begin{aligned}
\left(\frac{\Lambda}{\mu} - S\right)^{p-1}I &\leq \frac{(p-1)\varepsilon}{p}\left(\frac{\Lambda}{\mu} - S\right)^p + \frac{1}{p}\varepsilon^{1-p}I^p, \\
\left(\frac{\Lambda}{\mu} - S\right)^{p-2}I^2 &\leq \frac{(p-2)\varepsilon}{p}\left(\frac{\Lambda}{\mu} - S\right)^p + \frac{2}{p}\varepsilon^{\frac{2-p}{p}}I^p, \\
R^{p-1}I &\leq \frac{(p-1)\varepsilon}{p}R^p + \frac{1}{p}\varepsilon^{1-p}I^p, \\
R^{p-2}I^2 &\leq \frac{(p-2)\varepsilon}{p}R^p + \frac{2}{p}\varepsilon^{\frac{2-p}{p}}I^p.
\end{aligned}$$

Hence, we get

$$LV \leq -A_1\left(\frac{\Lambda}{\mu} - S\right)^p - A_2R^p - A_3I^p,$$

where

$$A_1 = \left[p\mu - \left(\frac{\beta\Lambda(p-1)}{\mu} + \frac{(p-1)(p-2)\sigma_1^2\Lambda^2}{2\mu^2}\right)\varepsilon\right]m_1,$$
$$A_2 = \left[p\mu - \left(\frac{\alpha(p-1)}{\omega} + (p-1)\gamma + \frac{(p-1)(p-2)\sigma_2^2}{2\omega^2}\right)\varepsilon\right]m_2,$$
$$A_3 = \mu + \epsilon + \gamma - \frac{\beta\Lambda}{\mu} - \frac{(p-1)\sigma_1^2\Lambda^2}{2\mu^2} - \frac{(p-1)\sigma_2^2}{2\omega^2}$$
$$- \left(\frac{\beta\Lambda}{\mu}\varepsilon^{1-p} + \frac{(p-1)\sigma_1^2\Lambda^2}{2\mu^2}\varepsilon^{\frac{2-p}{p}}\right)m_1 - \left(\left(\frac{\alpha}{\omega} + \gamma\right)\varepsilon^{1-p} + \frac{(p-1)\sigma_2^2}{2\omega^2}\varepsilon^{\frac{2-p}{p}}\right)m_2.$$

Now, we can choose ε, independently of m_1 and m_2, to be sufficiently small such that A_1 and A_2 are positive. In view of conditions of Theorem 5.1 we obtain

$$\mu + \epsilon + \gamma - \frac{\beta\Lambda}{\mu} - \frac{(p-1)\sigma_1^2\Lambda^2}{2\mu^2} - \frac{(p-1)\sigma_2^2}{2\omega^2} > 0.$$

Thus, by choosing m_1 and m_2 sufficiently small such A_3 is also positive, we obtain that the condition (6) is fulfilled with $K_3 = \min\{A_1, A_2, A_3\}$. In view of Theorem 2.5 the proof is completed. □

Theorem 5.2. *Let the condition*

$$\sigma_1^2 > \frac{\beta^2}{2\mu} \tag{24}$$

hold. Then the disease-free equilibrium E_0 of (2) *is almost surely exponentially stable in* Γ.

Proof. In order of proving the theorem, let $(S_0, I_0, R_0) \in \Gamma$, which ensures that the solution of system (2) remains in Γ, in view of Corollary 3.2. Also, let us define:

$$V(S, I, R) = \ln\left[\frac{\Lambda}{\mu} - S + I + R\right]. \tag{25}$$

Application of the Itô formula to function V yields

$$dV(S, I, R) = LV(S, I, R)dt + \frac{2}{\left[\frac{\Lambda}{\mu} - S + I + R\right]}\frac{\sigma_1 IS}{1 + kI}dB_1(t), \tag{26}$$

where

$$\begin{aligned} LV(S, I, R) &= \frac{1}{\left[\frac{\Lambda}{\mu} - S + I + R\right]}\left[-\left(\frac{\Lambda}{\mu} - S\right)\mu + 2\frac{\beta IS}{1 + kI} - (\mu + \epsilon)I - \mu R\right] - \frac{2}{\left[\frac{\Lambda}{\mu} - S + I + R\right]^2}\left(\frac{\sigma_1 IS}{1 + kI}\right)^2 \\ &= -\frac{\mu\left[\frac{\Lambda}{\mu} - S + I + R\right] - \epsilon I}{\left[\frac{\Lambda}{\mu} - S + I + R\right]} + 2\frac{\beta IS}{\left[\frac{\Lambda}{\mu} - S + I + R\right](1 + kI)} - 2\left(\frac{\sigma_1 IS}{\left[\frac{\Lambda}{\mu} - S + I + R\right](1 + kI)}\right)^2 \\ &\leq -\mu - \frac{\epsilon I}{\left[\frac{\Lambda}{\mu} - S + I + R\right]} + \frac{\beta^2}{2\sigma_1^2} \\ &\leq -\left(\mu - \frac{\beta^2}{2\sigma_1^2}\right). \end{aligned}$$

Hence, from (26), we obtain

$$\ln\left[\frac{\Lambda}{\mu} - S(t) + I(t) + R(t)\right] \leq \ln\left[\frac{\Lambda}{\mu} - S_0 + I_0 + R_0\right] - \left(\mu - \frac{\beta^2}{2\sigma_1^2}\right)t + M(t),$$

where $M(t) = \int_0^t \frac{2}{\frac{\Lambda}{\mu} - S(s) + I(s) + R(s)} \frac{\sigma_1 I(s) S(s)}{1 + kI(s)} dB_1(s)$ is a continuous local martingale and $M(0) = 0$. Furthermore,

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} \leq \frac{4\sigma_1^2\Lambda^2}{\mu^2} < \infty.$$

Thus, in view of the strong law of large numbers for local martingales, we have

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \qquad \text{a.s.}$$

which means, due to condition (24), that

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left[\frac{\Lambda}{\mu} - S(t) + I(t) + R(t)\right] \leq -\left(\mu - \frac{\beta^2}{2\sigma_1^2}\right) < 0,$$

which is desired result. □

5.2. Extinction of the Disease

In Theorem 5.2 we obtain that in long period all the trajectories of system (2) will approach to the disease-free equilibrium $E_0 = (\Lambda/\mu, 0, 0)$, or, in other words, the disease will become extinct exponentially with probability 1. By the next theorem we also provide extinction result, but in this case both intensities of the noise have an impact in extinction, unlike in Theorem 5.2 where extinction depends only on σ_1.

Theorem 5.3. *If the condition*

$$\frac{\beta^2}{2\sigma_1^2} + \frac{\alpha^2}{2\sigma_2^2} < \mu + \epsilon + \gamma \tag{27}$$

hold, then the disease will die out exponentially with probability 1, i.e.

$$\lim_{t\to\infty} I(t) = 0 \qquad a.s.$$

Proof. With the same reasoning as in the previous proofs, we suppose that $(S_0, I_0, R_0) \in \Gamma$. Let us define a function V in the following way:

$$V(S, I, R) = \ln I.$$

If we apply the Itô formula, we obtain

$$dV(S, I, R) = LV(S, I, R)dt + \frac{\sigma_1 S}{1 + kI} dB_1(t) - \frac{\sigma_2}{\omega + I} dB_2(t),$$

where

$$\begin{aligned} LV(S, I, R) &= \frac{\beta S}{1 + kI} - (\mu + \epsilon + \gamma) - \frac{\alpha}{\omega + I} - \frac{1}{2}\left[\left(\frac{\sigma_1 S}{1 + kI}\right)^2 + \left(\frac{\sigma_2}{\omega + I}\right)^2\right] \\ &\leq -\left(\mu + \epsilon + \gamma - \frac{\beta^2}{2\sigma_1^2} - \frac{\alpha^2}{2\sigma_2^2}\right). \end{aligned}$$

Thus,

$$\ln I(t) \leq \ln I_0 - \left(\mu + \epsilon + \gamma - \frac{\beta^2}{2\sigma_1^2} - \frac{\alpha^2}{2\sigma_2^2}\right)t + M_1(t) - M_2(t),$$

where $M_1(t) = \int_0^t \frac{\sigma_1 S(s)}{1+kI(s)} dB_1(s)$, and $M_2(t) = \int_0^t \frac{\sigma_2}{\omega+I(s)} dB_2(s)$ are the continuous local martingales and $M_1(0) = M_2(0) = 0$. By the same arguments as in the proof of Theorem 5.2 we obtain that

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = \lim_{t\to\infty} \frac{M_2(t)}{t} = 0 \qquad \text{a.s.}$$

which implies that

$$\limsup_{t\to\infty} \frac{1}{t} \ln I(t) \leq -\left(\mu + \epsilon + \gamma - \frac{\beta^2}{2\sigma_1^2} - \frac{\alpha^2}{2\sigma_2^2}\right).$$

The quantity in the bracket in the last inequality is positive, due to condition (27), and thus, the proof is complete. □

6. Numerical Simulations

6.1. Example 1

Cholera is an acute infection caused by bacterium *Vibrio cholerae* and affects the entire small intestine. The symptoms are watery diarrhea, vomiting, muscle cramps, dehydration. Incubation of cholera lasts from one to three days. Cholera can have a mild or potentially fatal form. As it was already mentioned, the initial symptoms are painless watery diarrhea and vomiting. Excessive fluid loss can lead to severe thirst, oliguria, muscle cramps, weakness, and significantly weakened skin turgor with inflamed eyes and wrinkled skin on the fingers. Uncomplicated cholera stops spontaneously and recovery occurs in three to 6 days. In untreated, severe cases, mortality can be grater than 50% (usually due to dehydration) but with prompt and adequate fluid and electrolyte infusion therapy, mortality is less then 1%. Most patients stop secreting *Vibrio cholerae* within 2 weeks, and a small proportion of them become chronic biliary carriers.

In [11] the authors provide numerical simulations for the cholera outbreak in the Department of Artibonite, Haiti, from 1st November 2010 until 1st May 2011. Namely, from 2007 until 2011 several cholera outbreaks occurred. They were register in Angola, Haiti and Zimbabwe. In Haiti, the first cases of cholera happened in Artibonite Department on 14th October 2010, and then the disease progressed along the Artibonite river and reached several departments. Only within one month, all departments had reported cases of cholera infection in rural areas and places without good conditions of public health. We use data presented there to confirm our theoretical results, and for more details we refer the readers to [11] and references cited therein.

Let us set daily values for model parameters of system (2) in the following way

$$\Lambda = 45.72, \quad \mu = 0.084, \quad \beta = 0.002, \quad k = 0.5, \quad \gamma = 0.2, \quad \epsilon = 0.0151, \quad \omega = 35, \quad \alpha = 7. \tag{28}$$

For such choice of parameters, and initial condition $(S_0, I_0, R_0) = (220, 5, 0)$, if we choose intensities of the noise $\sigma_1^2 = 0.001$ and $\sigma_2^2 = 0.00001$, we have that the condition of Theorem 5.2 is fulfilled.

We can observe that disease-free equilibrium $E_0 = (\Lambda/\mu, 0, 0)$ of model (2) is almost surely exponentially stable in Figures 1 and 2. We plotted deterministic trajectory (black line) of model (1) and 25 stochastic trajectories of model (2) for each compartment of the population. On the other hand, bearing in mind conditions of Theorem 3.1 from [23], we conclude that disease-free equilibrium of deterministic model (1) is unstable, which is expected since for such choice of the model parameters we may compute $\mathcal{R}_0 = 2.27 > 1$. Therefore, we can conclude that the noise in this case has a stabilizing effect on the system, even when the intensity of the noise is small, like it is the case with σ_2^2.

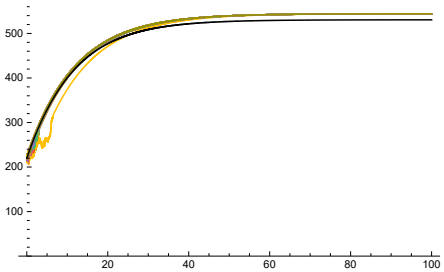

Figure 1: Deterministic and stochastic trajectories for susceptible individuals $S(t)$ of models (1) and (2) with initial condition $(S_0, I_0, R_0) = (220, 5, 0)$ and model parameters (28), and $\sigma_1^2 = 0.001$ and $\sigma_2^2 = 0.00001$.

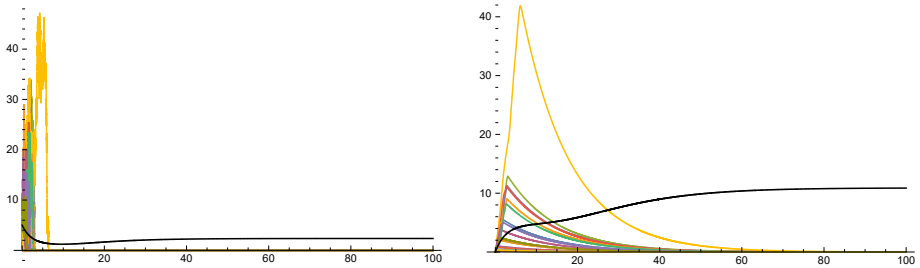

Figure 2: Deterministic and stochastic trajectories for infected $I(t)$ (left) and recovered individuals $R(t)$ (right) of models (1) and (2) with initial condition $(S_0, I_0, R_0) = (220, 5, 0)$ and model parameters (28), and $\sigma_1^2 = 0.001$ and $\sigma_2^2 = 0.00001$.

6.2. Example 2

Another disease we mention here is influenza. Influenza, commonly called "flu", is an infectious disease caused by flu viruses. Symptoms range from mild to severe and often include fever, runny nose, sore throat, muscle aches, headache, cough and fatigue. These symptoms usually begin 1-4 days after exposure to the virus and last for about 2-8 days. Diarrhea and vomiting may occur, especially in children. The flu can progress to pneumonia, which can be caused by a virus or a subsequent bacterial infection. Other complications of the infection include acute respiratory distress syndrome, meningitis, encephalitis, and worsening of pre-existing health problems such as asthma and cardiovascular disease.

Influenza A H1N1 is a subtype of influenza A virus. The well-known pandemic of this virus in the 20th and 21st centuries are the swine flu pandemic in 2009, the Russian flu pandemic in 1977, as well as the flu pandemic in 1918. The influenza A H1N1 virus, a new influenza virus, was first identified in April 2009 in Mexico and the USA, and by the June 2009, it was declared like pandemic. In [12] the authors consider spread of the influenza A H1N1 in Guangdong Province, where first case was confirmed on May 18, 2009 in Guangzhou. It was traveler who arrived from USA, but 2 weeks after that there appeared new local cases. The virus is highly infectious, and it was essential that the government put in place strategies to mitigate and control the disease in the face of uncertainty.

In order to show that our theoretical results may be applied to real life data, we use data from [12], but while the authors there assume that the previously immune hosts may become susceptible again, mainly due to the progressive antigenic drift of influenza virus, here we assume that people who have suffered from this disease acquire permanent immunity to that subtype of the virus and cannot become infected again. All the parameters are weekly data which may be found in [12] and references cited therein. They choose mortality rate to be $\frac{1}{70*52}$ which suggests the average life expectancy is 70 years. For simplicity, it is assumed that there is no death due to the disease, and that the mean recovery rate of infected individuals is 1. The total population size is $N = 10\,000$, which means that $\Lambda = \frac{10\,000}{70*52}$. Thus,

$$\Lambda = 2.75, \quad \mu = 0.00027, \quad \beta = 1.45 * 10^{-4}, \quad k = 0.7, \quad \gamma = 1, \quad \epsilon = 0, \quad \omega = 500, \quad \alpha = 0.005. \tag{29}$$

Now, for initial data $(S_0, I_0, R_0) = (500, 5, 0)$, if we choose $\sigma_1^2 = 0.00003$, $\sigma_2^2 = 0.01$, we may check that

condition (24) of Theorem 5.2 is not satisfied, but condition (27) of Theorem 5.3 is, and hence, extinction of the disease occurs (Figure 3).

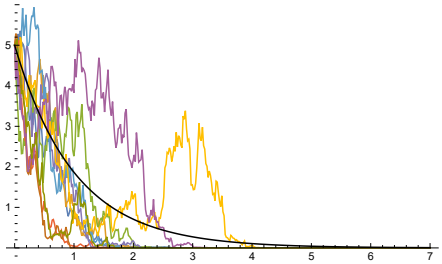

Figure 3: Deterministic and stochastic trajectories for infected individuals $I(t)$ of models (1) and (2) with initial condition $(S_0, I_0, R_0) = (500, 5, 0)$ and model parameters (29), and $\sigma_1^2 = 0.00003$ and $\sigma_2^2 = 0.01$.

For parameters (29) basic reproduction number is $\mathcal{R}_0 = 1.47 > 1$. Hence, it is expected for disease-free equilibrium of deterministic model (1) to be unstable, on the basis of Theorem 3.1 from [23]. However, model (1) has two more equilibrium states than the stochastic model (2). From Figure 3 we may conclude that, if coordinate I is small enough for such choice of parameters, deterministic trajectory approaches one of the interior equilibrium states in some sense. Thus, it would be interesting for future investigations to consider behavior of model (2) around the endemic equilibrium states of model (1), among the other things.

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