Filomat 36:8 (2022), 2847–2856
https://doi.org/10.2298/FIL2208847K

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Interpolative Hardy-Rogers Type Multivalued Contractions via a Simulation Function

Erdal Karapınar[a,b,c], Ahsan Ali[d], Azhar Hussain[d,e], Hassen Aydi[c,f,g]

[a] *Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam*
[b] *Department of Mathematics, Çankaya University, Etimesgut, Ankara, Turkey*
[c] *Department Of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[d] *Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan*
[e] *Department of Mathematics, University of Chakwal, Chakwal 48800, Pakistan*
[f] *Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia*
[g] *Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa*

Abstract. In this paper, the notion of multivalued interpolative Hardy-Rogers-contractions using generalized simulation functions is introduced. We establish some related fixed point results and we provide some examples. We also prove data dependence of the fixed point sets. Moreover, we present strict fixed point set, well-posedness and homotopy results.

1. Introduction and basic definitions

From now on, all considered sets and subsets are assumed to be non-empty. For a set M, the symbol $P(M)$ indicates all subsets of M. Further, the symbol $P_c(M)$ ($P_{cb}(M)$, respectively) denotes the class of all closed (all closed and bounded, respectively) subsets of M. The function $d : M \times M \to \mathbb{R}_0^+$ forms a standard metric. Throughout this paper, the pairs (M, d) and (M^*, d) denote a metric space and a complete metric space, respectively.

Let A and B be closed subsets of M. Consider the functional $\Delta, H : P_c(M) \times P_c(M) \to \mathbb{R}_0^+$ that are defined as

$$\begin{aligned} \Delta(A, B) &= \sup\{D(\upsilon, B);\ \upsilon \in A\}, \\ H(A, B) &= \max\{\Delta(A, B), \Delta(B, A)\} \end{aligned}$$

where $D(\upsilon, B) = \inf\{d(\upsilon, \tau);\ \tau \in B\}$, $A, B \in P_c(M)$. Here, H is known as the Pompeiu-Hausdorff distance, and Δ is called a gap distance. Here, (M, d) is complete if and only if $(P_{cb}(M), H)$ is complete, see e.g. [5], [6], [17] and [19].

An auxiliary function, announced as a simulation function, defined by Khojasteh *et al.* [11] to merge several results in fixed point theory. A function $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is called a simulation function if

2020 *Mathematics Subject Classification*. Primary 54E35; 26D15
Keywords. Multivalued interpolative Hardy-Rogers $\mathcal{Z}_{\mathcal{G}}$-contraction; Data dependence; Strict fixed points; Well-posedness; Homotopy
Received: 03 December 2020; Revised: 08 September 2021; Accepted: 15 September 2021
Communicated by Adrian Petrusel
Corresponding Author: Hassen Aydi
Email addresses: karapinar@mail.cmuh.org.tw (Erdal Karapınar), ali_ahsan45@yahoo.com (Ahsan Ali), hafiziqbal30@yahoo.com (Azhar Hussain), hassen.aydi@isima.rnu.tn (Hassen Aydi)

(ζ_1) $\zeta(t,r) < r - t$ for all $t, r > 0$;

(ζ_2) for $\{t_n\}, \{r_n\} \subset \mathbb{R}_0^+$, if $\lim_{n\to\infty} t_n = \lim_{n\to\infty} r_n > 0$ then

$$\lim_{n\to\infty} \sup \ \zeta(t_n, r_n) < 0.$$

We emphasize that in the original definition, there was a superfluous condition: $\zeta(0,0) = 0$.

Definition 1.1. *[11] A self-mapping T on (M, d) is named as a $\mathcal{Z}$-contraction with respect to ζ if*

$$\zeta(d(T\upsilon, T\tau), d(\upsilon, \tau)) \geq 0 \ \text{for all } \upsilon, \tau \in M.$$

Example 1.2. *[11] Suppose that $\kappa \in (0,1)$, and $\varphi : \mathbb{R}_0^+ \to [0,1)$ is a mapping such that $\lim_{t\to r^+} sup\ \psi(t) < 1$ for all $r > 0$; and $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function such that $\psi(r) = 0 \Leftrightarrow r = 0$. Let $\zeta_i : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, $i = 1, 2, 3$ be defined by*

(i) $\zeta_1(t,r) = \kappa r - t$,
(ii) $\zeta_2(t,r) = r\varphi(r) - t$,
(iii) $\zeta_3 = r - \psi(r) - t$.

Then ζ_i (for each $i = 1, 2, 3$) forms a simulation function.

By replacing (ζ_3) by (ζ_3'), the definition of a simulation function was extended by Roldán-López-de-Hierro *et al.* [20]:

(ζ_3') : if $\{t_n\}, \{r_n\} \subset \mathbb{R}_0^+$ so that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} r_n > 0$ and $t_n < r_n$, then

$$\lim_{n\to\infty} \sup \zeta(t_n, r_n) < 0.$$

We say that $\mathcal{G} : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is a C-class function [3] if:

(1) $\mathcal{G}(r,t) \leq r$;

(2) $\mathcal{G}(r,t) = r$ implies either $r = 0$, or $t = 0$, for all $r, t \in \mathbb{R}_0^+$.

Definition 1.3. *[12] A mapping $\mathcal{G} : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ has the property $C_{\mathcal{G}}$, if there is $C_{\mathcal{G}} \geq 0$ so that*

(1) *$\mathcal{G}(r,t) > C_{\mathcal{G}} \Rightarrow r > t$;*

(2) *$\mathcal{G}(t,t) \leq C_{\mathcal{G}}$, for all t.*

Immediate examples of C-class functions with $C_{\mathcal{G}}$-property are as follows:

(a) $\mathcal{G}(r,t) = r - t$, $C_{\mathcal{G}} = r, r \in [0, +\infty)$;

(b) $\mathcal{G}(r,t) = r - \frac{(2+t)t}{(1+t)}$ with $C_{\mathcal{G}} = 0$;

(c) $\mathcal{G}(r,t) = \frac{r}{1+kt}$ for $k \geq 1$ with $C_{\mathcal{G}} = \frac{r}{1+k}$, for $r \geq 2$.

Definition 1.4. *[4, 7, 12] A $C_{\mathcal{G}}$ simulation function is a mapping $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfying the following conditions:*

(1) *$\zeta(t,r) < \mathcal{G}(r,t)$ for all $t, r > 0$, where $\mathcal{G} : [0, +\infty)^2 \to \mathbb{R}$ is a C-class function;*

(2) *if $\{t_n\}, \{r_n\}$ are sequences in $(0, +\infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} r_n > 0$, and $t_n < r_n$, then $\lim\limits_{n\to\infty} \sup \zeta(t_n, r_n) < C_{\mathcal{G}}$;*

(3) $\zeta(0,0) = 0$.

In [4, 11, 12, 16, 20, 23], one can find a number of examples of simulation functions.

Example 1.5. **(1)** *$\zeta(t,r) = \frac{r}{r+1} - t$ for all $t, r > 0$;*

(2) *For a lower semi-continuous function $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, the function $\zeta(t,r) = r - \phi(r) - t$ for all $t, r > 0$, forms a $C_{\mathcal{G}}$ simulation function, whenever $\phi(r) = 0 \Leftrightarrow r = 0$.*

Very recently, the concept of interpolative contraction mappings has been introduced in [9] to enrich fixed point theory.

Theorem 1.6. *[9] Suppose that T is a self- mapping on (M^*, d). If*

$$d(T\upsilon, T\tau) \leq \kappa[d(\upsilon, T\upsilon)]^{\alpha}[d(\tau, T\tau)]^{1-\alpha} \tag{1}$$

for all $\upsilon, \tau \in M$ with $\upsilon \neq T\upsilon$, where $\alpha \in [0,1), \kappa \in [0,1)$, then T possesses a unique fixed point.

Interpolation theory is very deep and has been used widely in several research fields, see e.g. [1, 2, 10, 13, 14]. Karapınar, Agarwal and Aydi [8] indicated the gap on the proof of the uniqueness in Theorem 1.6.

Theorem 1.7. *[8] Suppose that T is a self- mapping on (M^*, d). If*

$$d(T\upsilon, T\tau) \leq c_1[d(\upsilon,\tau)] + c_2[d(\upsilon, T\upsilon)] + c_3[d(\tau, T\tau)] + c_4[\frac{1}{2}(d(\upsilon, T\tau) + d(\tau, T\upsilon)],$$

for all $\upsilon, \tau \in M^*$, where c_1, c_2, c_3, c_4 are non-negative reals such that $c_1 + c_2 + c_3 + c_4 < 1$, then T possesses a fixed point in M^*.

In this study, we define the notion of multivalued interpolative H-R-contractions via a simulation function $\mathcal{Z}_{\mathcal{G}}$ and prove some related fixed point results. Here, the notation H-R is the abbreviation of Hardy-Rogers. Basically, we revisit the renowned interpolate H-R- contraction for multivalued mappings via a generalized simulation function.

2. Main results

Definition 2.1. *Let (M, d) be a metric space. We say that $T : M \to P_{cb}(M)$ is a multivalued interpolative H-R-contraction via a simulation function $\mathcal{Z}_{\mathcal{G}}$, if there exist $\kappa \in [0,1)$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ such that*

$$\zeta(H(T\upsilon, T\tau), R(\upsilon, \tau)) \geq C_{\mathcal{G}} \tag{2}$$

where

$$R(\upsilon,\tau) = \kappa\ [d(\upsilon,\tau)]^{\alpha} \cdot [D(\upsilon, T\upsilon)]^{\beta} \cdot [D(\tau, T\tau)]^{\gamma} \cdot [\frac{1}{2}(D(\upsilon, T\tau) + D(\tau, T\upsilon))]^{1-\alpha-\beta-\gamma},$$

for all $\upsilon, \tau \in M \setminus Fix(T)$.

The following example supports Definition 2.1.

Example 2.2. *Let $M = \{0, 1, 2, 3, 4\}$ and $d(\upsilon, \tau) = |\upsilon - \tau|$. Define*

$$T\upsilon = \begin{cases} \{0,2\} & \text{if } \upsilon = 4 \\ \{1,2\} & \text{if } \upsilon = 0 \\ \{1,4\} & \text{if } \upsilon = 3 \\ \{1,3\} & \text{otherwise.} \end{cases}$$

Take $\upsilon, \tau \in M \setminus Fix(T)$ such that $\upsilon \neq \tau$. Let $\zeta(t,r) = \frac{12}{13}r - t$ and $\mathcal{G}(r,t) = r - t$ for all $r, t \in \mathbb{R}_0^+$, $C_(\mathcal{G}) = 0$. Then at $\upsilon = 4, \tau = 0$ and $\kappa = 0.9$, we have

$$\zeta(H(T\upsilon, T\tau), R(\upsilon, \tau)) = \frac{12}{13}R(\upsilon, \tau) - H(T\upsilon, T\tau) = 1.5184 \tag{3}$$

where $\alpha = 0.7$, $\gamma = 0$ *and* $\beta = 0.2$. *Note that*

$$\mathcal{G}(R(\upsilon,\tau), H(T\upsilon, T\tau)) = R(\upsilon,\tau) - H(T\upsilon, T\tau) = 1.728\ . \tag{4}$$

So, from (3) *and* (4)*, one writes*

$$0 \leq \zeta(H(T\upsilon, T\tau), R(\upsilon,\tau)) < \mathcal{G}(R(\upsilon,\tau), H(T\upsilon, T\tau)). \tag{5}$$

Hence, from (5)*, at* $\upsilon = 4$ *and* $\tau = 0$ *the operator* T *is a multivalued interpolative H-R-contraction via the simulation function* $\mathcal{Z}_{\mathcal{G}}$.

Theorem 2.3. *Let* (M, d) *be a complete metric space. If* T *is a multivalued interpolative H-R-contraction operator via the simulation function* $\mathcal{Z}_{\mathcal{G}}$*, then* T *possesses a fixed point in* M.

Proof. *We start by letting* $\upsilon_0 \in M$ *and taking* $\{\upsilon_n\}$ *as* $\upsilon_n \in T^n\upsilon_0$ *for each* $n \in \mathbb{N}$. *In the case that there is a number* n_0 *so that* $\upsilon_{n_0} = \upsilon_{n_0+1}$*, the point* υ_{n_0} *is a fixed point of* T. *It terminates the proof trivially. Attendantly, we assume that*

$$\upsilon_n \neq \upsilon_{n+1} \text{ for all } n \geq 0.$$

By taking $\upsilon = \upsilon_n$ *and* $\tau = \upsilon_{n-1}$ *in* (2)*, we will get*

$$\zeta(H(T\upsilon_n, T\upsilon_{n-1}), R(\upsilon_n, \upsilon_{n-1})) \geq C_{\mathcal{G}}.$$

By using Definition 1.4*, we find*

$$C_{\mathcal{G}} \leq \zeta(H(T\upsilon_n, T\upsilon_{n-1}), R(\upsilon_n, \upsilon_{n-1})) < G(R(\upsilon_n, \upsilon_{n-1}), H(T\upsilon_n, T\upsilon_{n-1})).$$

From Definition 1.3*, we have*

$$H(T\upsilon_n, T\upsilon_{n-1}) < R(\upsilon_n, \upsilon_{n-1}). \tag{6}$$

Recall that

$$D(T\upsilon_n, \upsilon_n) \leq H(T\upsilon_n, T\upsilon_{n-1}). \tag{7}$$

Therefore,

$$\begin{aligned} D(\upsilon_n, T\upsilon_n) &< R(\upsilon_n, \upsilon_{n-1}) = \kappa\,[d(\upsilon_n, \upsilon_{n-1})]^{\alpha} \cdot [D(\upsilon_n, T\upsilon_n)]^{\beta} \cdot [D(\upsilon_{n-1}, T\upsilon_{n-1})]^{\gamma} \\ &\quad \cdot [\frac{1}{2}(D(\upsilon_n, T\upsilon_{n-1}) + D(\upsilon_{n-1}, T\upsilon_n))]^{1-\alpha-\beta-\gamma} \\ &\leq \kappa\,[d(\upsilon_n, \upsilon_{n-1})]^{\alpha} \cdot [d(\upsilon_n, \upsilon_{n+1})]^{\beta} \cdot [d(\upsilon_{n-1}, \upsilon_n)]^{\gamma} \\ &\quad \cdot [\frac{1}{2}(d(\upsilon_n, \upsilon_n) + d(\upsilon_{n-1}, \upsilon_n) + d(\upsilon_n, \upsilon_{n+1}))].^{1-\alpha-\beta-\gamma}. \end{aligned} \tag{8}$$

Suppose that $d(\upsilon_{n-1}, \upsilon_n) < d(\upsilon_n, \upsilon_{n+1})$ *for some* $n \geq 1$. *Thus,*

$$\frac{1}{2}(d(\upsilon_n, \upsilon_n) + d(\upsilon_{n-1}, \upsilon_n) + d(\upsilon_n, \upsilon_{n+1}))] < d(\upsilon_n, \upsilon_{n+1}).$$

Consequently, from (8) *we get*

$$d(\upsilon_n, \upsilon_{n+1})^{\alpha+\gamma} < \kappa d(\upsilon_{n-1}, \upsilon_n).^{\alpha+\gamma}$$

So, we get that $d(\upsilon_{n-1}, \upsilon_n) > d(\upsilon_n, \upsilon_{n+1})$*, which is a contradiction. Thus, we have* $d(\upsilon_n, \upsilon_{n+1}) \leq d(\upsilon_{n-1}, \upsilon_n)$ for all $n \geq 1$. *Hence,* $\{d(\upsilon_{n-1}, \upsilon_n)\}$ *is a non-increasing sequence with positive terms. Let* $\lim_{n\to\infty} d(\upsilon_{n-1}, \upsilon_n) = \mathcal{L}$. *Since*

$$\frac{1}{2}(d(\upsilon_n, \upsilon_n) + d(\upsilon_{n-1}, \upsilon_n) + d(\upsilon_n, \upsilon_{n+1}))] \leq d(\upsilon_{n-1}, \upsilon_n) \text{ for all } n \geq 1,$$

and by using (8)*, we find that*

$$d(\upsilon_n, \upsilon_{n+1})^{1-\beta} < \kappa d(\upsilon_{n-1}, \upsilon_n)^{1-\beta} \text{ for all } n \geq 1.$$

This implies that

$$d(\upsilon_n, \upsilon_{n+1}) < \kappa d(\upsilon_{n-1}, \upsilon_n) < \kappa^2 d(\upsilon_{n-2}, \upsilon_{n-1}) < ... < \kappa^n d(\upsilon_0, \upsilon_1)$$

for all $n \geq 1$. *Recall that* $\kappa < 1$, *so as* $n \to \infty, d(\upsilon_n, \upsilon_{n+1}) \to 0$. *Thus,* $\mathcal{L} = 0$. *Now, we assert that* $\{\upsilon_n\}$ *is a Cauchy sequence. For this purpose, let* $n, m \in \mathbb{N}$ *with* $m > n$. *Employing the inequality*

$$\begin{aligned} d(\upsilon_n, \upsilon_m) &\leq d(\upsilon_n, \upsilon_{n+1}) + ... + d(\upsilon_{m-1}, \upsilon_m) \\ &< \kappa^n d(\upsilon_0, \upsilon_1) + ... + \kappa^{m-1} d(\upsilon_0, \upsilon_1) \\ &\leq \frac{\kappa^n}{1-\kappa} d(\upsilon_0, \upsilon_1), \end{aligned} \tag{9}$$

thus, on account of (9)*, the sequence* $\{\upsilon_n\}$ *is Cauchy. Regarding the completeness, we guarantee the existing of* $\upsilon \in M^*$ *so that* $\lim_{n\to\infty} d(\upsilon_n, \upsilon) = 0$. *Suppose* $\upsilon \notin T\upsilon$. *It means that* $\upsilon_n \notin T\upsilon_n$ *for each* $n \geq 0$. *By taking* $\upsilon = \upsilon_n$ *and* $\tau = \upsilon$ *in inequality* (2) *and Definition* 1.4*, one writes*

$$C_{\mathcal{G}} \leq \lim_{n\to\infty} sup \zeta(H(T\upsilon_n, T\upsilon), R(\upsilon_n, \upsilon)) < C_{\mathcal{G}}, \tag{10}$$

which is a contradiction. It terminates the proof, that is, $\upsilon \in T\upsilon$. □

Example 2.4. *Let* $M = [0, 1]$ *and* $d(\upsilon, \tau) = |\upsilon - \tau|$. *Define*

$$T\upsilon = [\frac{6 + 2\upsilon}{12}, \frac{7 + 2\upsilon}{12}] \tag{11}$$

for all $\upsilon, \tau \in M$. *Let* $\zeta(t, r) = \frac{12}{13} r - t$, $\mathcal{G}(r, t) = r - t$ *for all* $r, t \in [0, \infty)$ *and* $C_(\mathcal{G}) = 0$. *Then, for* $\upsilon, \tau \in M \setminus Fix(T)$ *and* $\upsilon \neq \tau$ *and* $\kappa \in [0, 1)$*, we get*

$$\begin{aligned} R(\upsilon, \tau) = &\ \kappa\, [d(\upsilon, \tau)]^{\alpha} \cdot [D(\upsilon, T\upsilon)]^{\beta} \cdot [D(\tau, T\tau)]^{\gamma} \\ &\cdot [\frac{1}{2}(D(\upsilon, T\tau) + D(\tau, T\upsilon))]^{1-\alpha-\beta-\gamma}. \end{aligned} \tag{12}$$

Therefore,

$$\zeta(H(T\upsilon, T\tau)), R(\upsilon, \tau)) = \frac{12}{13} R(\upsilon, \tau) - H(T\upsilon, T\tau),$$

and

$$\mathcal{G}(R(\upsilon, \tau), H(T\upsilon, T\tau)) = R(\upsilon, \tau) - H(T\upsilon, T\tau).$$

The nonnegative values of α, β, γ *are such that* $\alpha + \beta + \gamma < 1$, *so we get that*

$$0 \leq \zeta(H(T\upsilon, T\tau), R(\upsilon, \tau)) < \mathcal{G}(R(\upsilon, \tau), H(T\upsilon, T\tau)). \tag{13}$$

Thus, from (13)*, it is clear that* T *is a multivalued interpolative H-R-contraction operator via the simulation function* $\mathcal{Z}_{\mathcal{G}}$ *and all the axioms of Theorem 2.3 are satisfied. Here,* $Fix(T) = [0.6, 0.7]$.

3. Strict fixed points and well-posedness

The set $Fix(T) := \{\upsilon \in M | \upsilon \in T(\upsilon)\}$ is called the fixed point set of T. $SFix(T) := \{\upsilon \in M : \{\upsilon\} = T(\upsilon)\}$ is called the strict fixed point set of T. Notice that $SFix(T) \subseteq Fix(T)$. Before starting, let us state the definition of the well-posedness of a fixed point problem (in short, FPP).

Definition 3.1. *[18, 22] Let (M, d) be a metric space, $Y \in P(M)$ and $T : M \to P_c(M)$ be a multivalued operator. Then FPP is well-posed for T with respect to d when:*
(a_2) $SFix(T) = \{\upsilon\}$;
(b_2) If $\{\upsilon_n\} \in Y$, $n \in N$ and $H(\upsilon_n, T\upsilon_n) \to 0$ as $n \to \infty$, then $d(\upsilon_n, \upsilon) \to 0$ as $n \to \infty$.

Moţ and Petruşel [15] proved the following theorem.

Theorem 3.2. *[15] Let (M, d) be a complete metric space and $T : M \to P_c(M)$ be a multivalued operator. Assume that*
(i) T is an a-KS multivalued operator, that is, if $a \in [0, 1)$ and

$$\upsilon, \tau \in M \quad with \ \frac{1}{1+a} D(\upsilon, T\upsilon) \le d(\upsilon, \tau) \quad implies \quad H(T\upsilon, T\tau) \le a d(\upsilon, \tau),$$

(ii) $SFix(T) \neq \emptyset$.
Then
(a) $Fix(T) = SFix(T) = \{\upsilon\}$;
(b) FPP is well posed with respect to H.

Inspired by Theorem 3.2, we propose the following result.

Theorem 3.3. *Let (M, d) be a complete metric space. Assume that*
(i) $T : M \to P_c(M)$ is a multivalued interpolative H-R-contraction via the simulation function $\mathcal{Z}_{\mathcal{G}}$, with $\gamma = 0$;
(ii) $SFix(T) \neq \emptyset$,
then
(a) $Fix(T) = SFix(T) = \{\upsilon\}$;
(b) FPP is well posed with respect to H.

Proof. *(a) We will prove that $Fix(T) = \upsilon$. We suppose that $u, \upsilon \in Fix(T)$ with $u \neq \upsilon$. Since*

$$\zeta(H(Tu, T\upsilon), R(u, \upsilon)) \ge C_{\mathcal{G}},$$

we find using Definition 1.4,

$$C_G \le \zeta(H(Tu, T\upsilon), R(u, \upsilon)) < G(R(u, \upsilon), H(Tu, T\upsilon)).$$

From Definition 1.3, we have

$$H(Tu, T\upsilon) < R(u, \upsilon). \tag{14}$$

Therefore,

$$\begin{aligned} D(T\upsilon, u) = d(\upsilon, u) \quad \le \quad & H(Tu, T\upsilon) < R(u, \upsilon) \\ & \kappa\, [d(u, \upsilon)]^{\alpha} \cdot [D(u, Tu)]^{\beta} \cdot [D(\upsilon, T\upsilon)]^{\gamma} \\ & \cdot [\frac{1}{2}(D(u, T\upsilon) + D(\upsilon, Tu))]^{1-\alpha-\beta-\gamma} = 0. \end{aligned}$$

This implies that $0 < d(\upsilon, u) < 0$, which is a contradiction. Hence, $u = \upsilon$ and so $Fix(T) = \{\upsilon\}$.
(b) Let $\upsilon \in SFix(T)$ and $\{\upsilon_n\}_{n \in N}$ be such that $D(\upsilon_n, T\upsilon_n) \to 0$, as $n \to \infty$. Since T is a multivalued interpolative H-R-contraction via the simulation function $\mathcal{Z}_{\mathcal{G}}$ with $\gamma = 0$, by taking $\upsilon = \upsilon_n$ and $\tau = \upsilon$ in (2), one writes

$$\zeta(H(T\upsilon_n, T\upsilon), R(\upsilon_n, \upsilon)) \ge C_{\mathcal{G}}.$$

By using Definition 1.4, we find

$$C_{\mathcal{G}} \le \zeta(H(T\upsilon_n, T\upsilon), R(\upsilon_n, \upsilon)) < G(R(\upsilon_n, \upsilon), H(T\upsilon_n, T\upsilon)).$$

Using Definition 1.3, *we have*

$$H(Tv_n, Tv) < R(v_n, v). \tag{15}$$

Also,

$$\begin{aligned} d(v_n, v) = D(v_n, Tv) &< R(v_n, v) = \kappa\, [d(v_n, v)]^{\alpha} \cdot [D(v_n, Tv_n)]^{\beta} \\ &\quad \cdot [\frac{1}{2}(D(v_n, Tv) + D(v, Tv_n))]^{1-\alpha-\beta} \\ &\leq \kappa\, [d(v_n, v)]^{\alpha} \cdot [D(v_n, Tv_n)]^{\beta} \\ &\quad \cdot [D(v_n, Tv_n)]^{1-\alpha-\beta} \\ d(v_n, v) &< \kappa[D(v_n, Tv_n)]. \end{aligned} \tag{16}$$

Letting $n \to \infty$ in (16), *we find that $d(v_n, v) \to 0$.* □

4. Data dependence of the fixed point set

This section is devoted to the work on the data dependence of fixed point sets for the multivalued interpolative H-R-contraction operator via the simulation function $\mathcal{Z}_{\mathcal{G}}$.

Theorem 4.1. *(cf [15]) Let (M, d) be a metric space and T_1, T_2 be two operators. Assume that*
(a_1) T_i is an a_i-KS multivalued operator for $i \in \{1, 2\}$;
(b_1) There exists a real number $\kappa' > 0$ such that $H(T_1v, T_2v) \leq \kappa'$ for all $v \in M$.
Then
(a_2) $Fix(T_i) \in P_c(M)$, for $i \in \{1, 2\}$;
(b_2) T_1 and T_2 are weakly multivalued operators and

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\kappa'}{1 - \max\{a_1, a_2\}}.$$

Inspired by the above result, we propose the following theorem.

Theorem 4.2. *Let (M, d) be a metric space and T_1, T_2 be two operators so that*

1. *T_i is a multivalued interpolative H-R-contraction via the simulation function $\mathcal{Z}_{\mathcal{G}}$ for $i \in \{1, 2\}$;*
2. *There exists a real number $\kappa' > 0$ such that $H(T_1v, T_2v) \leq \kappa'$ for all $v \in M$.*

Then

1. *$Fix(T_i) \in P_c(M)$, for $i \in \{1, 2\}$;*
2. *T_1 and T_2 are multivalued interpolative H-R-contractions via the simulation function $\mathcal{Z}_{\mathcal{G}}$ and*

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\kappa'}{1 - \max\{r_1, r_2\}}.$$

Proof. *From Theorem 2.3, $Fix(T_i)$ is nonempty for $i \in \{1, 2\}$. First of all, we will show that set of fixed points of the multivalued interpolative H-R-contraction operator T via the simulation function $\mathcal{Z}_{\mathcal{G}}$ is closed. Let $\{v_n\}$ be a sequence in $Fix(T)$ such that $v_n \to u$ as $n \to \infty$, and*

$$\zeta(H(Tv_n, Tv_{n-1}), R(v_n, v_{n-1})) \geq C_{\mathcal{G}}. \tag{17}$$

By using Definition 1.4, *we find that*

$$C_{\mathcal{G}} \leq \zeta(H(Tv_n, Tv_{n-1}), R(v_n, v_{n-1})) < G(R(v_n, v_{n-1}), H(Tv_n, Tv_{n-1})).$$

From (1)) *of Definition* 1.3, *we have*

$$H(Tv_n, Tv_{n-1}) < R(v_n, v_{n-1}). \tag{18}$$

Since

$$D(Tv_n, v_n) \leq H(Tv_n, Tv_{n-1}), \tag{19}$$

we have

$$\begin{aligned} D(v_n, Tv_n) &< R(v_n, v_{n-1}) = \kappa\, [d(v_n, v_{n-1})]^{\alpha} \cdot [D(v_n, Tv_n)]^{\beta} \cdot [D(v_{n-1}, Tv_{n-1})]^{\gamma} \\ &\quad \cdot [\frac{1}{2}(D(v_n, Tv_{n-1}) + D(v_{n-1}, Tv_n))]^{1-\alpha-\beta-\gamma} \\ &\leq \kappa\, [d(v_n, v_{n-1})]^{\alpha} \cdot [d(v_n, v_{n+1})]^{\beta} \cdot [d(v_{n-1}, v_n)]^{\gamma} \\ &\quad \cdot [\frac{1}{2}(d(v_n, v_n) + d(v_{n-1}, v_n) + d(v_n, v_{n+1}))]^{1-\alpha-\beta-\gamma}. \end{aligned} \tag{20}$$

As $n \to \infty$, we get that $0 < D(u, Tu) < 0$. Thus, $D(u, Tu) = 0$. Since $Tu \in P_{cb}(M)$, we get that $u \in Tu$. Hence $u \in Fix(T)$. Secondly, from Theorem 2.3, the multivalued interpolative H-R-contraction operator via the simulation function $\mathcal{Z}_{\mathcal{G}}$ possesses a fixed point. Let $q > 1$ be a real number and $v_0 \in Fix(T_1)$ be arbitrary. Then, there exists $v_1 \in T_2v_0$ such that $d(v_0, v_1) \leq qH(T_1v_0, T_2v_0)$. Next, for $v_1 \in T_2v_0$ there exists $v_2 \in T_2v_1$ such that $d(v_1, v_2) \leq qH(T_2v_0, T_2v_1)$. Since $v_1 \in T_2v_0, D(v_1, T_2v_0) = 0 \leq d(v_0, v_1)$, we have

$$d(v_1, v_2) \leq qH(T_2v_0, T_2v_1) \leq qr_2 d(v_0, v_1).$$

Similarly, we find a sequence of successive approximations for T_2 starting from v_0 and satisfying the following assertions:

$$v_{n+1} \in Tv_n \ \textit{ and } \ d(v_n, v_{n+1}) \leq (qr_2)^2 d(v_0, v_1) \ \text{ for all } \ n \geq 1.$$

Hence, for all $n \geq N$ and $p \geq 1$, we have

$$\begin{aligned} d(v_{n+p}, v_n) &\leq d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+2}) + ... + d(v_{n+p-1}, v_{n+p}) \\ &\leq (qr_2)^n d(v_0, v_1) + (qr_2)^{n+1} d(v_0, v_1) + ... + (qr_2)^{n+p-1} d(v_0, v_1) \\ &\leq \frac{(qr_2)^n}{1 - qr_2} d(v_0, v_1). \end{aligned} \tag{21}$$

Choosing $1 < q < \min\{\frac{1}{r_1}, \frac{1}{r_2}\}$ and letting $n \to \infty$, we find that the sequence $\{v_n\}$ is Cauchy in (M^*, d). Then there exists $u \in M$ such that $v_n \to u$ as $n \longrightarrow \infty$. We will prove that u is a fixed point for T_2. Suppose, on the contrary, that $u \notin T_2u$ and $v_{n(k)} \notin T_2v_{n(k)}$. By taking $v = v_{n(k)}$, $\tau = u$ in inequality (2) *and using Definition* 1.4,

$$C_{\mathcal{G}} \leq \lim_{n\to\infty} Sup[\zeta(H(T_2u, T_2v_{n(k)}), R(u, v_{n(k)}))] < C_{\mathcal{G}}. \tag{22}$$

From this contradiction, we easily find that $u \in T_2u$. Hence, $u \in Fix(T_2)$.

By Taking $p \to \infty$ in (21)*, we have $d(u, v_n) \leq \frac{(qr_2)^n}{1-qr_2} d(v_0, v_1)$ for each $n \in N$. Then $d(v_0, u) \leq \frac{1}{1-qr_2} d(v_0, v_1) \leq \frac{q\kappa'}{1-qr_2}$. In a similar way, we get that for each $u_0 \in Fix(T_2)$, there exists $v \in Fix(T_1)$ such that $d(u_0, v) \leq \frac{1}{1-qr_2} d(u_0, u_1) \leq \frac{q\kappa'}{1-qr_2}$. Hence,*

$$H(Fix(T_1), Fix(T_2)) \leq \frac{q\kappa'}{1 - \max\{qr_1, qr_2\}}.$$

Letting $q \to 1$, we find the conclusion. Moreover, we get that T_i is $\frac{1}{1-r_i}$ operator (for $i \in \{1, 2\}$). □

5. Homotopy result

The main result of this section is the following.

Theorem 5.1. *Let $v_0 \in (M^*, d)$ and $a > 0$. If $T : B(v_0, a) \to P_{cb}(M)$ is a multivalued interpolative H-R-contraction via the simulation function $\mathcal{Z}_{\mathcal{G}}$ so that $d(v_0, Tv_0) < a(1-\kappa)$, then T possesses a fixed point in $B(v_0, a)$.*

Proof. Set $0 < a_1 < a$ in a way that $\widetilde{B}(v_0, a_1) \subset B(v_0, a)$ and $d(v_0, Tv_0) < (1-\kappa)a_1 < (1-\kappa)a$. Let $v_1 \in Tv_0$ be such that $d(v_0, v_1) < (1-\kappa)a_1$. Then, for $h = \frac{1}{\sqrt{\kappa}} > 1$ and $v_1 \in Tv_0$ there exists $v_2 \in Tv_1$ such that

$$d(v_1, v_2) \le h\, H(Tv_0, Tv_1).$$

Since for $v = v_1$ and $\tau = v_2$ in (2), we will get

$$\zeta(H(Tv_1, Tv_2), R(v_1, v_2)) \ge C_{\mathcal{G}}.$$

By using Definition 1.4, we find

$$C_{\mathcal{G}} \le \zeta(H(Tv_1, Tv_2), R(v_1, v_2)) < G(R(v_1, v_2), H(Tv_1, Tv_2)).$$

From Definition 1.3, we have

$$H(Tv_1, Tv_2) < R(v_1, v_2). \tag{23}$$

Therefore,

$$\begin{aligned} d(v_1, v_2) &\le \frac{1}{\sqrt{\kappa}}\, H(Tv_0, Tv_1) \\ &< \frac{1}{\sqrt{\kappa}}\, R(v_0, v_1) = \frac{1}{\sqrt{\kappa}}\, \kappa\, [d(v_0, v_1)]^{\alpha} \cdot [D(v_0, Tv_0)]^{\beta} \cdot [D(v_1, Tv_1)]^{\gamma} \\ &\quad \cdot [\frac{1}{2}(D(v_0, Tv_1) + D(v_1, Tv_0))]^{1-\alpha-\beta-\gamma} \\ &\le \sqrt{\kappa}\, [d(v_0, v_1)]^{\alpha} \cdot [d(v_0, v_1)]^{\beta} \cdot [d(v_1, v_2)]^{\gamma} \\ &\quad \cdot [\frac{1}{2}(d(v_0, v_0) + d(v_0, v_1) + d(v_1, v_2))].^{1-\alpha-\beta-\gamma} \end{aligned} \tag{24}$$

Suppose that $d(v_0, v_1) < d(v_1, v_2)$. Thus,

$$\frac{1}{2}(d(v_0, v_1) + d(v_1, v_2))] < d(v_1, v_2).$$

Consequently, from (24) we get

$$d(v_1, v_2)^{\alpha+\beta} < \sqrt{\kappa}\, d(v_0, v_1).^{\alpha+\beta}$$

So, we conclude that $d(v_0, v_1) > d(v_1, v_2)$. It is a contradiction. As a result, we find $d(v_1, v_2) \le d(v_0, v_1)$. So,

$$\frac{1}{2}(d(v_0, v_1) + d(v_1, v_2))] \le d(v_0, v_1).$$

Hence, from inequality (24),

$$d(v_1, v_2) < \sqrt{\kappa}\, d(v_0, v_1) < \sqrt{\kappa}\, (1-\kappa)\, a_1.$$

Also, we have $v_2 \in B(v_0, a)$ because

$$d(v_0, v_2) \le d(v_0, v_1) + d(v_1, v_2) < (1-\kappa)a_1 + \sqrt{\kappa}(1-\kappa)\, a_1 = (1-\kappa)(1+\sqrt{\kappa})\, a_1.$$

In this way, we find inductively a sequence $\{v_n\}_{n\in\mathbb{N}}$ satisfying:
(*i*) $v_n \in B(v_0, a)$ for each $n \in \mathbb{N}$;
(*ii*) $v_{n+1} \in Tv_n$ for each $n \in \mathbb{N}$;
(*iii*) $d(v_n, v_{n+1}) < (\sqrt{\kappa})^n(1-\kappa)a_1$.
From (*iii*), the sequence $\{v_n\}_{n\in\mathbb{N}}$ is Cauchy, and so it is convergent to some $v \in B(v_0, a)$. By following the same steps as in Theorem 2.3, we find $v \in Tv$. □

References

[1] H. Aydi, E. Karapınar, A. Francisco Roldan Lopez de Hierro, w-interpolative Ciric-Reich-Rus type contractions, Mathematics, 2019, 7(1), 57.

[2] H. Aydi, C.M. Chen, E. Karapınar, Interpolative Ciric-Reich-Rus type contractions via the Branciari distance, Mathematics, 2019, 7(1), 84.

[3] A.H. Ansari, Note on φ-ψ-contractive type mappings and related fixed point, The 2nd Regional Conference on Math. Appl. PNU, (2014), 377-380.

[4] A.H. Ansari, H. Isik, S. Radenović, Coupled fixed point theorems for contractive mappings involving new function classes and applications, Filomat, 31:7 (2017), 1893-1907.

[5] H. Covitz, S.B. Nadler, Multi-valued contraction mappings in generalized metric spaces, Israel J. Math. 8(1970) 5-11.

[6] J. Dugundji, A. Granas, Fixed Point Theory, Springer-Verlag, Berlin, 2003.

[7] Z.M. Fadail, A.G.B. Ahmad, A.H. Ansari, S. Radenović, M. Rajović, Some common fixed point results of mappings in 0-σ-complete metric-like spaces, Appl. Math. Sci. 9(2015), 5009-5027.

[8] E. Karapınar, R.P. Agarwal, H. Aydi, Interpolative Reich-Rus-Ciric type contractions on partial metric spaces, Mathematics 2018,6,256.

[9] E. Karapınar, Revisting the Kannan type contractions via interpolation, Adv. Theory Nonlinear Anal. Appl. 2018,2, 85-87.

[10] E. Karapınar, O. Alqahtani, H. Aydi, On interpolative Hardy-Rogers type contractions, Symmetry, 2018, 11, 8.

[11] F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theorems via simulation functions, Filomat, 29 (6)(2015), 1189-1194.

[12] X.L. Liu, A.H. Ansari, S. Chandok, S. Radenović, On some results in metric spaces using auxiliary simulation functions via new functions, J. Comput. Anal. Appl. 24(6) (2018), 1103-1114.

[13] B. Mohammadi, V. Parvaneh, H. Aydi, On extended interpolative Ciric-Reich-Rus type F-contractions and an application, Journal of Inequalities and Applications, Volume 2019, 2019:290.

[14] G. Mastroianni, G. Milovanović, Interpolation Process: Basic Theory and Applications, Springer Monographs in Mathemamtics, 1st edition, (2018), 446 pages.

[15] G. Moţ, A. Petruşel, Fixed point theory for a new type of contractive multivalued operators, Nonlinear Anal. 70 (2009), 3371-3377.

[16] A. Nastasi, P. Vetro, Fixed point results on metric and partial metric spaces via simulation functions, J. Nonlinear Sci. Appl. 8(2015), 1059-1069 .

[17] A. Petruşel, Operational Inclusions, House of the Book of the Sience, Cluj-Napoca, 2002.

[18] A. Petruşel, I.A. Rus, Well-posedness of the fixed point problem for multivalued operators, Applied Analysis and Differential Equations, World Sci. Publ., Hackensack, NJ, (2007), 295-306.

[19] A. Petruşel, G. Petruşel, Some variants of the contraction principle for multi-valued operators, generalizations and applications, J. Nonlinear Convex Anal. 20 (10) (2019), 2187-2203.

[20] A. Roldan, E. Karapınar, C. Roldan, J. Martinez, Coincidence point theorems on metric spaces via simulation function, J. Comput. Appl. Math. 275 (2015), 345-355.

[21] S. Radenović, S. Chandok, Simulation type functions and coincidence points, Filomat, 32(1) (2018), 141-147.

[22] I.A. Rus, Generalized Contractions and Applications, Cluj Univ. Press,Cluj-Napoca, 2001.

[23] S. Wang, A.H. Ansari, S. Chandok, Some fixed point results for non-decreasing and mixed monotone mappings with auxiliary functions, Fixed Point Theory Appl. (2015) 2015:209.