Filomat 36:8 (2022), 2551–2572
https://doi.org/10.2298/FIL2208551L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Pseudo-Generalized Inverse I

Asma Lahmar[a], Haïkel Skhiri[a]

[a]*Faculté des Sciences de Monastir, Département de Mathématiques, Avenue de l'environnement, 5019 Monastir, Tunisia*

Abstract. The purpose of this paper is to introduce and study a new class of operators that we call pseudo-generalized invertible operators. This class includes both the class of generalized invertible operators and the class of Drazin invertible operators. Some results connected with ascent and essential ascent are also obtained for this new class. The relationship between pseudo-generalized invertible operators, semi-Fredholm operators and generalized invertible operators is explored.

1. Introduction

In this paper, let X be an arbitrary infinite-dimensional complex Banach space and $\mathscr{B}(\mathsf{X})$ denotes the algebra of all bounded linear operators acting on X. For $T \in \mathscr{B}(\mathsf{X})$ we use $\mathsf{R}(T)$ and $\mathsf{N}(T)$, respectively, to denote the range and kernel of T.

We recall that T is called generalized invertible (g-invertible), if there is $S \in \mathscr{B}(\mathsf{X})$ for which $TST = T$. In such a case, S is called a g_1-inverse of T. Also, if T is g-invertible, then necessarily it has a generalized inverse, called also g_2-inverse, which is an operator $S \in \mathscr{B}(\mathsf{X})$ satisfying the equations $TST = T$ and $STS = S$.

Another characterization of g-invertible operators has been established by Caradus [5], in terms of range and kernel. More precisely, he proved that a bounded linear operator is g-invertible if and only if its range and kernel are both complemented. Moreover, if $S \in \mathscr{B}(\mathsf{X})$ is a g_1-inverse then TS and $I - ST$ are two projections onto $\mathsf{R}(T)$ and $\mathsf{N}(T)$, respectively. We have also $\mathsf{N}(TS) = \mathsf{N}(S)$ and $\mathsf{R}(ST) = \mathsf{R}(S)$, if S is a g_2-inverse of T. Consequently, we obtain in particular that every finite rank operator is g-invertible, and a compact operator is g-invertible if and only if it is of finite rank (see [14, Theorem 6.3.4, Theorem 6.8.5]).

The notion of generalized inverse (known in the literature also as g-inverse) appears for the first time in 1903 by I. Fredholm [8], and in 1920 by E. H. Moore, who was the first to introduce generalized invertibility for singular matrices [20]. The study of this notion has flourished since 1955, when R. Penrose redefined Moore's inverse, nowadays called the Moore-Penrose inverse, by showing that it is the only matrix satisfying four matrix equations (see [23] and its sequel [24]). The past several decades have seen an interest in this notion in various settings and the study on generalized inverse initiated a new direction of research, and many applications to linear least-squares problems have been given. As a result, this concept became a distinguished area and a great number of papers was published in which various aspects of generalized

2020 *Mathematics Subject Classification*. Primary 47A05; Secondary 47A65, 47B01, 47B02, 47A53

Keywords. Bounded operator, generalized invertible operator, g_1-inverse, g_2-inverse, nilpotent operator, ascent, descent, semi-Fredholm, Drazin invertible operator, essential ascent, essential descent.

Received: 05 July 2021; Revised: 24 December 2021; Accepted: 27 December 2021

Communicated by Dijana Mosić

This work is supported by LR18ES16 : Analyse, Géométrie et Applications, University of Monastir (Tunisia).

Email addresses: asmalahmar1996@gmail.com (Asma Lahmar), haikel.skhiri@gmail.com, haikel.skhiri@fsm.rnu.tn (Haïkel Skhiri)

inverses and their applications, see for instance, [2, 6, 25, 26, 29] for matrix theory and the case of Hilbert spaces, [5, 6, 29] for bounded and unbounded linear operators on complex Banach spaces, [4, 15] for C^*-algebras and rings. For some applications, we refer the reader to see [2, 6, 22, 26]. Further applications can be found also in numerical analysis and approximation methods, probability, statistics and econometrics, optimization, system theory, cryptography theory and operations research methods (for more details see [7], and the references therein).

Although this notion has been studied extensively and shows its usefulness considering its important applications relevant to the various linear problems, to the best of our knowledge, only a few works deal with the question of its extension. In [9], S. Guedjiba thought of generalizing the equations $TST = T$ and $STS = S$ in the case of matrices, linear operators and bounded operators by introducing the G_k-inverses, where the defined class of operators having G_k-inverse coincides with the class of g-invertible operators (see also [30]). In earlier work dating back to 1965, F. J. Beutler [3] extended the Moore-Penrose inverse, defined in the Hilbert case, to operators which need not be bounded and which may not have a closed range.

To deal with this problem for bounded operators, specifically in $\mathscr{B}(\mathsf{X})$, our work goes in the direction of generalizing the equation $TST = T$. For this reason, we suggest the following subsets :

$$\Omega_n^\ell = \Big\{T \in \mathscr{B}(\mathsf{X}) \,:\, \exists\, S \in \mathscr{B}(\mathsf{X}) :\ T^nST = T^n\Big\},$$

and

$$\Omega_n^r = \Big\{T \in \mathscr{B}(\mathsf{X}) \,:\, \exists\, S \in \mathscr{B}(\mathsf{X}) :\ TST^n = T^n\Big\},$$

with $n \in \mathbb{N}$. Furthermore, we define

$$\Omega^\ell = \bigcup_{n=0}^{\infty} \Omega_n^\ell \text{ and } \Omega^r = \bigcup_{n=0}^{\infty} \Omega_n^r.$$

Clearly, the set of g-invertible operators is Ω_1^ℓ $(= \Omega_1^r)$ and it belongs to $\Omega_n^\ell \cap \Omega_n^r$, for all $n \geq 2$. Contrary to G_k-invertible operators, in our case, we obtain larger classes. In fact, the reverse inclusion could not be true in general and so the subsets Ω_1^ℓ $(= \Omega_1^r)$ and $\Omega_n^\ell \cap \Omega_n^r$ usually do not coincide. To see that, let H be a separable Hilbert space with an orthonormal basis $(e_k)_{k>0}$, $N \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{H})$ defined as follows

$$T(e_k) = \begin{cases} \frac{1}{k}e_{k+1} & \text{if } k \text{ is even,} \\ e_k & \text{if } k \text{ is odd and } k \leq N, \\ 0 & \text{if not.} \end{cases}$$

Then,

$$T^2(e_k) = \begin{cases} \frac{1}{k}e_{k+1} & \text{if } k \text{ is even and } k+1 \leq N, \\ e_k & \text{if } k \text{ is odd and } k \leq N, \\ 0 & \text{if not.} \end{cases}$$

We have $T \notin \Omega_1^\ell$ $(= \Omega_1^r)$ being a compact operator of infinite rank and as T^2 is of finite rank then there exists $S \in \mathscr{B}(\mathsf{X})$ such that $T^2 = T^2ST^2 = T^2(ST)T = T(TS)T^2$, which implies that $T \in \Omega_2^\ell \cap \Omega_2^r$ and hence Ω_1^ℓ $(= \Omega_1^r) \subset \Omega_2^\ell \cap \Omega_2^r$.

It should be noted that not only g-invertible operators are included in this new class, but there are a few others, obviously one of which is the class of Drazin invertible operators [6, 18, 29]. Note also that every nilpotent operator of degree $k \geq 1$ belongs to $\Omega_k^\ell \cap \Omega_k^r$. Furthemore, in the Hilbert case, if the adjoint of T is a solution of one of the equations $T^nST = T^n$ and $TST^n = T^n$, then we find the case of semi-generalized partial isometries introduced by Garbouj and the second author in [10], that is to say $\mathcal{GPS}_n^\ell$ (resp. $\mathcal{GPS}_n^r$) is included in Ω_n^ℓ (resp. Ω_n^r) and similarly we have $\mathcal{GPS}^\ell \subseteq \Omega^\ell$ and $\mathcal{GPS}^r \subseteq \Omega^r$.

For $n \in \mathbb{N}$, we will call n-left (resp. n-right) pseudo-generalized invertible operator, an operator belonging to Ω_n^ℓ (resp. Ω_n^r). More generally, if $T \in \Omega^\ell$ (resp. Ω^r), then T is referred to as a left (resp. right) pseudo-generalized invertible operator. An operator in $\Omega^\ell \cup \Omega^r$ is simply called pseudo-generalized invertible.

Clearly, 0-left and 0-right pseudo-generalized invertible operators are left-invertible and right-invertible, respectively. Also, it is not difficult to see that one-to-one (resp. surjective) operators in Ω_n^ℓ (resp. Ω_n^r) belongs to Ω_0^ℓ (resp. Ω_0^r), for all $n \in \mathbb{N}$. Additionnaly, for any $\lambda \in \mathbb{C}$ and $T \in \Omega_n^\ell$ we have $\lambda T \in \Omega_n^\ell$ and similarly for Ω_n^r. While, $\lambda T \notin \Omega^\ell$ (resp. Ω^r), if $T \notin \Omega^\ell$ (resp. Ω^r) for all $\lambda \in \mathbb{C}^*$.

The paper is organized in the following way. First, we study some basic properties of pseudo-generalized invertible operators. Section 3 is devoted to the study of pseudo-generalized invertible operators having finite ascent, or finite descent. In Section 4, we focus our attention on operators both semi-Fredholm and pseudo-generalized invertible. Some decomposition results are also given. Section 5 deals with the pseudo-generalized invertible operators which possess finite essential ascent or finite essential descent. In Section 6, the topological complements of the kernel and image of pseudo-generalized invertible operators and their powers are discussed. In Section 7, we give some sufficient conditions for operators to be pseudo-generalized invertible. Finally, as applications, based on the notion of pseudo-generalized invertibility, we are able to give some results for the g-invertibility.

2. Some basic properties

In this section, we present some basic properties of pseudo-generalized invertible operators and we introduce the concept of pseudo-generalized inverses.

Our starting point is the following result, which gives easily observable remarks.

Remark 2.1. Let $T \in \mathscr{B}(\mathsf{X})$.

1) If T is a nilpotent operator of degree $k \geq 1$, then $T \in \Omega_k^\ell \cap \Omega_k^r$.

2) $\Omega_n^\ell \subseteq \Omega_{n+1}^\ell$ and $\Omega_n^r \subseteq \Omega_{n+1}^r$, for all $n \in \mathbb{N}$.

3) If $T^n \in \Omega_1^\ell$ $(= \Omega_1^r)$ for some $n \in \mathbb{N}\backslash\{0\}$, then $T \in \Omega_n^\ell \cap \Omega_n^r$.

4) If X is a Hilbert space, then $T \in \Omega_n^\ell$ if and only if $T^* \in \Omega_n^r$.

According to the previous remark, n-left (resp. n-right) pseudo-generalized invertible operators are clearly k-left (resp. k-right) pseudo-generalized invertible, for any $k \geq n$, but the converse is not true in general. In fact, there exists a nilpotent operator of degree $n \geq 2$, and thus T is n-left (resp. n-right) pseudo-generalized invertible, such that $T \notin \Omega_{n-1}^\ell \cup \Omega_{n-1}^r$. This can be seen in the next example.

Example 2.2. Let H be a separable Hilbert space with an orthonormal basis $(e_k)_{k>0}$. For $s \in \mathbb{N}\backslash\{0, 1\}$, we define $R_s \in \mathscr{B}(\mathsf{H})$ as follows

$$R_s(e_k) = \begin{cases} \frac{1}{k} e_{k+1} & \text{if } k \in \mathbb{N}\backslash s\mathbb{N}, \\ 0 & \text{if not.} \end{cases}$$

We have, $R_s^s = 0$, then clearly $R_s \in \Omega_s^\ell \cap \Omega_s^r$. On the other hand, we see that

$$R_s^{s-1}(e_k) = \begin{cases} \dfrac{e_{k+s-1}}{\prod\limits_{m=0}^{s-2}(k+m)} & \text{if } k \in s\mathbb{N}+1, \\ 0 & \text{if not.} \end{cases}$$

Suppose that there exists $L \in \mathscr{B}(\mathsf{H})$, such that $R_s^{s-1} L R_s = R_s^{s-1}$. We set

$$L(e_k) = \sum_{n=1}^{\infty} \alpha_{n,k} e_n,$$

where $\alpha_{n,k} \in \mathbb{C}$, for any $k \in \mathbb{N}\backslash\{0\}$. Now, let $k \in s\mathbb{N}+1$, then we have

$$R_s^{s-1}LR(e_k) = \frac{1}{k}R_s^{s-1}L(e_{k+1}) = \frac{1}{k}\sum_{n\in s\mathbb{N}+1}\frac{\alpha_{n,k+1}e_{n+s-1}}{\prod_{m=0}^{s-2}(n+m)}.$$

However, since $R_s^{s-1}LR_s = R_s^{s-1}$, then

$$\frac{1}{k}\sum_{n\in s\mathbb{N}+1}\frac{\alpha_{n,k+1}e_{n+s-1}}{\prod_{m=0}^{s-2}(n+m)} = \frac{e_{k+s-1}}{\prod_{m=0}^{s-2}(k+m)}.$$

Thus, for $n = k$,

$$\frac{1}{k}\frac{\alpha_{k,k+1}}{\prod_{m=0}^{s-2}(k+m)} = \frac{1}{\prod_{m=0}^{s-2}(k+m)}.$$

It is clear that $\alpha_{k,k+1} = k$, so L is not bounded, which is a contradiction. As a consequence we see that $R_s \notin \Omega_{s-1}^{\ell}$. In the similar way, we obtain the same conclusin for Ω_{s-1}^{r}.

We also have the same result for non-nilpotent operators as can be seen in the following remark.

Remark 2.3. For $s \in \mathbb{N}\backslash\{0,1\}$, there exists a non-nilpotent operator $T \in \Omega_s^{\ell}\cap\Omega_s^{r}$ such that $T \notin \Omega_{s-1}^{\ell}\cup\Omega_{s-1}^{r}$. To see that, let H be a separable Hilbert space with an orthonormal basis $(e_k)_{k>0}$ and $N \geq s-1$. We define $T_s \in \mathscr{B}(\mathsf{H})$ as follows

$$T_s(e_k) = \begin{cases} \frac{1}{k}e_{k+1} & \text{if } k \in \mathbb{N}\backslash\{s\mathbb{N}+s-1\} \text{ and } k > N, \\ e_k & \text{if } k \in s\mathbb{N}+s-1 \text{ and } k \leq N, \\ 0 & \text{if not.} \end{cases}$$

We see that

$$T_s^{s-1}(e_k) = \begin{cases} \dfrac{e_{k+s-1}}{\prod_{l=0}^{s-2}(k+l)} & \text{if } k \in s\mathbb{N} \text{ and } k > N, \\ e_k & \text{if } k \in s\mathbb{N}+s-1 \text{ and } k \leq N, \\ 0 & \text{if not,} \end{cases}$$

and

$$T_s^{s}(e_k) = \begin{cases} e_k & \text{if } k \in s\mathbb{N}+s-1 \text{ and } k \leq N, \\ 0 & \text{if not.} \end{cases}$$

Since T_s^s is of finite rank, then $T_s^s \in \Omega_1^{\ell}$ $(=\Omega_1^{r})$ and therefore $T_s \in \Omega_s^{\ell}\cap\Omega_s^{r}$. Now, suppose that there exists $L \in \mathscr{B}(\mathsf{H})$ such that $T_s^{s-1}LT_s = T_s^{s-1}$. We consider

$$L(e_k) = \sum_{n=1}^{\infty}\alpha_{n,k}e_n,$$

where $\alpha_{n,k} \in \mathbb{C}$, for any $k \geq 1$. Now, for $k \in s\mathbb{N}$, with $k > N$, we see that

$$T_s^{s-1}LT_s(e_k) = \frac{1}{k}\Big(\sum_{n\in s\mathbb{N},\, n>N}\frac{\alpha_{n,k+1}e_{n+s-1}}{\prod_{l=0}^{s-2}(n+l)} + \sum_{n\in s\mathbb{N}+s-1,\, k\leq N}\alpha_{n,k+1}e_n\Big).$$

As $T_s^{s-1}LT_s = T_s^{s-1}$, then

$$\frac{1}{k}\Big(\sum_{n\in s\mathbb{N},\, n>N}\frac{\alpha_{n,k+1}e_{n+s-1}}{\prod_{l=0}^{s-2}(n+l)} + \sum_{n\in s\mathbb{N}+s-1,\, k\leq N}\alpha_{n,k+1}e_n\Big) = \frac{e_{k+s-1}}{\prod_{l=0}^{s-2}(k+l)}.$$

For $n = k$, we have $\alpha_{k,k+1} = k$, hence L is not bounded, and thus $T_s \notin \Omega_{s-1}^{\ell}$. Similarly, we get $T_s \notin \Omega_{s-1}^{r}$. □

This brings us directly to the following issue which remains open.

Question : Is there an infinite-dimensional Banach space X, such that $\Omega_n^\ell = \Omega_k^\ell$ (resp. $\Omega_n^r = \Omega_k^r$), for some $k, n \in \mathbb{N}\backslash\{0\}$?

Now, before proceeding with our study, we need to introduce the following subsets, for a given $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}$,

$$^T\mathsf{S}_n^\ell = \left\{S \in \mathscr{B}(\mathsf{X}) \,:\, T^nST = T^n\right\}$$

and

$$^T\mathsf{S}_n^r = \left\{S \in \mathscr{B}(\mathsf{X}) \,:\, TST^n = T^n\right\}.$$

An operator $S \in {}^T\mathsf{S}_n^\ell$ (resp. ${}^T\mathsf{S}_n^r$), will be called a n-left (resp. n-right) pseudo-generalized inverse of T or more generally, a left (resp. right) pseudo-generalized inverse. Also, for all non-negative integer n, any operator belonging to ${}^T\mathsf{S}_n^\ell \cup {}^T\mathsf{S}_n^r$ will be simply called pseudo-generalized inverse.

Clearly, ${}^T\mathsf{S}_0^\ell$ is the set of right-inverses of T and ${}^T\mathsf{S}_0^r$ is the set of left-inverses. Additionally, we have

$$^T\mathsf{S}_1^\ell = {}^T\mathsf{S}_1^r = \{S \in \mathscr{B}(\mathsf{X}) \,:\, TST = T\}$$

and for all $n \in \mathbb{N}$,

$$^T\mathsf{S}_n^\ell \subseteq {}^T\mathsf{S}_{n+1}^\ell, \quad {}^T\mathsf{S}_n^r \subseteq {}^T\mathsf{S}_{n+1}^r.$$

It is obvious to check that a n-left (resp. n-right) pseudo-generalized inverse is k-left (resp. k-right) pseudo-generalized inverse, for all $k \geq n$, but the converse need not be true in general. In fact, the following example shows it.

Example 2.4. Let H be a separable Hilbert space with an orthonormal basis $(e_k)_{k>0}$ and $N > 4$. We define $T \in \mathscr{B}(\mathsf{H})$ as follows :

$$T(e_k) = \begin{cases} e_{k+1} & \text{if } k \in 3\mathbb{N}, \\ e_{k+1} & \text{if } k \in 3\mathbb{N}+1\,,\, k < N, \\ 0 & \text{if not.} \end{cases}$$

We have $\operatorname{rank}(T^2) < +\infty$, then it is clear that $T^2 \in \Omega_1^\ell$ $(= \Omega_1^r)$ and therefore $T \in \Omega_2^\ell \cap \Omega_2^r$. Moreover, we have $T^3 = 0$. Assume that there exists $k < 3$ such that ${}^T\mathsf{S}_3^\ell = {}^T\mathsf{S}_k^\ell$ (resp. ${}^T\mathsf{S}_3^r = {}^T\mathsf{S}_2^r$), then necessarily ${}^T\mathsf{S}_3^\ell = {}^T\mathsf{S}_2^\ell$ (resp. ${}^T\mathsf{S}_3^r = {}^T\mathsf{S}_2^r$). However, clearly we have $0 \in \left({}^T\mathsf{S}_3^\ell \backslash {}^T\mathsf{S}_2^\ell\right) \cap \left({}^T\mathsf{S}_3^r \backslash {}^T\mathsf{S}_2^r\right)$, hence the result. □

Recall that in [6], S. R. Caradus proved that if T is g-invertible and S_0 is a g_2-inverse of T, then the subset $\{S_0 + U - S_0TUTS_0\,,\, U \in \mathscr{B}(\mathsf{X})\}$ coincides with ${}^T\mathsf{S}_1^\ell$ $(={}^T\mathsf{S}_1^r)$. Note that an operator of Ω_1^ℓ $(= \Omega_1^r)$ always admits a g_2-inverse (see [6, Lemma 1, P. 10]). More generally, if $S \in {}^T\mathsf{S}_n^\ell$, then clearly

$$\{S + UT^{n-1} - STUT^nS\,,\, U \in \mathscr{B}(\mathsf{X})\} \subseteq {}^T\mathsf{S}_n^\ell,$$

and if $S \in {}^T\mathsf{S}_n^r$, then we have

$$\{S + T^{n-1}U - ST^nUTS\,,\, U \in \mathscr{B}(\mathsf{X})\} \subseteq {}^T\mathsf{S}_n^r.$$

In the next remark, we further explore the nature of these inclusions.

Remark 2.5. Let $T \in \mathscr{B}(\mathsf{X})$. If $T \in \Omega_n^\ell$ (resp. Ω_n^r) and $S \in {}^T\mathsf{S}_n^\ell$ (resp. ${}^T\mathsf{S}_n^\ell$), with $n \geq 2$, then, in general,

$$\{S + UT^{n-1} - STUT^nS\,,\, U \in \mathscr{B}(\mathsf{X})\} \neq {}^T\mathsf{S}_n^\ell \tag{2.1}$$

and

$$\{S + T^{n-1}U - ST^nUTS\,,\, U \in \mathscr{B}(\mathsf{X})\} \neq {}^T\mathsf{S}_n^r. \tag{2.2}$$

Indeed, let X be a Banach space and $T \in \mathscr{B}(\mathsf{X})$ be a nilpotent operator of degree $n \geq 2$. It is clear that

$$^{T}\mathsf{S}_n^{\ell} = {}^{T}\mathsf{S}_n^{r} = \mathscr{B}(\mathsf{X}).$$

Suppose that for all $S \in \mathscr{B}(\mathsf{X})$, such that $STS = S$, we have

$$\{S + UT^{n-1}\,,\, U \in \mathscr{B}(\mathsf{X})\} = \mathscr{B}(\mathsf{X}).$$

In particular, let $S = 0$, then we have

$$\{UT^{n-1}\,,\, U \in \mathscr{B}(\mathsf{X})\} = \mathscr{B}(\mathsf{X}).$$

Hence, there exists $U \in \mathscr{B}(\mathsf{X})$, such that $I = UT^{n-1}$ and therefore it follows that $T = 0$, which is a contradiction. We then get

$$\{UT^{n-1}\,,\, U \in \mathscr{B}(\mathsf{X})\} \subsetneq {}^{T}\mathsf{S}_n^{\ell}.$$

Similarly, we can see that

$$\{T^{n-1}U\,,\, U \in \mathscr{B}(\mathsf{X})\} \subsetneq {}^{T}\mathsf{S}_n^{r}.$$

Hence the result. □

Based on the above remark, it is natural to ask the following question.

Question : If $n \geq 2$, then can we have the equalities (2.1) and (2.2) in Remark 2.5, for a certain $S \in {}^{T}\mathsf{S}_n^{\ell}\backslash\{0\}$ (resp. ${}^{T}\mathsf{S}_n^{r}\backslash\{0\}$)?

The proposition below presents some necessary and sufficient conditions to obtain pseudo-generalized invertibility.

Proposition 2.6. Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}$.

1) $T \in \Omega_n^{\ell} \Longleftrightarrow \Big(\exists S \in \mathscr{B}(\mathsf{X})\ :\ \mathsf{R}(I - ST) \subseteq \mathsf{N}(T^n)\Big)$.

2) $T \in \Omega_n^{r} \Longleftrightarrow \Big(\exists S \in \mathscr{B}(\mathsf{X})\ :\ \mathsf{R}(T^n) \subseteq \mathsf{N}(TS - I)\Big)$.

3) $T \in \Omega_n^{\ell} \Longleftrightarrow \Big(\exists S \in \mathscr{B}(\mathsf{X})\ :\ \forall\, k \in \mathbb{N}\backslash\{0\}\,, (TS - I)(\mathsf{R}(T^k)) \subseteq \mathsf{N}(T^{n-1}),\ \text{if } n \geq 1\Big)$.

4) $T \in \Omega_n^{r} \Longleftrightarrow \Big(\exists S \in \mathscr{B}(\mathsf{X})\ :\ \forall\, k \in \mathbb{N}\backslash\{0\}\,, (ST - I)(\mathsf{R}(T^{n-1})) \subseteq \mathsf{N}(T^k),\ \text{if } n \geq 1\Big)$.

Lemma 2.7. *Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}$.*

1) *If $T \in \Omega_n^{\ell}$ and let $S \in {}^{T}\mathsf{S}_n^{\ell}$, then $T^n(ST)^k = T^n$, for all $k \in \mathbb{N}$.*

2) *If $T \in \Omega_n^{r}$ and let $S \in {}^{T}\mathsf{S}_n^{r}$, then $(TS)^kT^n = T^n$, for all $k \in \mathbb{N}$.*

The next proposition aims to extend [10, Proposition 2.4] for semi-generalized partial isometries defined in the Hilbert space, to pseudo-generalized invertible operators, even in the case of Banach spaces.

Proposition 2.8.
Let $T, S \in \mathscr{B}(\mathsf{X})$ such that $\|T\|\,\|S\| < 1$ and $n \in \mathbb{N}\backslash\{0\}$. The following assertions are equivalent :

(1) $T^nST = T^n$,

(2) $TST^n = T^n$,

(3) T is a nilpotent operator of degree $k \leq n$.

Proof.
"(1) $\Longrightarrow$ (3)" If $S = 0$, the result is obvious. Now, if $S \neq 0$, then since $\|T\|\,\|S\| < 1$, we know that $I - ST$ is invertible. As $T^nST = T^n$, then

$$T^n(I - ST) = 0.$$

Therefore $T^n = 0$ and T is nilpotent of degree $k \leq n$.
"(2) $\Longrightarrow$ (3)" This implication can be proven similarly.
"(3) $\Longrightarrow$ (1)" and "(3) $\Longrightarrow$ (2)" If T is nilpotent of degree $k \leq n$, then $T^n = 0$ and $T \in \Omega_n^\ell \cap \Omega_n^r$. □

Also, we have the following topological property.

Remark 2.9. For all $n \in \mathbb{N}\backslash\{0\}$, Ω_n^ℓ (resp. Ω_n^r) is not an open subset of $\mathscr{B}(\mathsf{X})$. Indeed, let $k \in \mathbb{N}\backslash\{0\}$ and $S \notin \Omega_k^\ell$, (resp. Ω_k^r). We know that $\lambda S \notin \Omega_k^\ell$ (resp. Ω_k^r), for all $\lambda \in \mathbb{C}^*$. We set $\lambda_n = \frac{1}{n}$. It is clear that $\lim\limits_{n\to+\infty} \lambda_n S = 0$. Since $\lambda_n S \notin \Omega_k^\ell$ (resp. Ω_k^r), for all $k \in \mathbb{N}\backslash\{0\}$ and $0 \in \Omega_k^\ell \cap \Omega_k^r$, then $\mathscr{B}(\mathsf{X})\backslash\Omega_k^\ell$ (resp. $\mathscr{B}(\mathsf{X})\backslash\Omega_k^r$) is not closed and therefore Ω_k^ℓ (resp. Ω_k^r) is not open.

In the same way, we see that Ω^ℓ (resp. Ω^r) is not an open subset of $\mathscr{B}(\mathsf{X})$.

Finally, we close this section with the proposition below which generalizes [5, Section 6, P. 25].

Proposition 2.10. *Let $T, S \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$, then*

1) $I - TS \in \Omega_n^\ell \Longleftrightarrow I - ST \in \Omega_n^\ell$.

2) $I - TS \in \Omega_n^r \Longleftrightarrow I - ST \in \Omega_n^r$.

Proof.
1) " $\Longrightarrow$ " Let $W \in {}^{I-TS}\mathsf{S}_n^\ell$, then we have

$$(I - TS)^n = (I - TS)^n W(I - TS).$$

Consequently, by setting $A = (I - TS)^n - (I - TS)^n W(I - TS)$, we obtain

$$A = \sum_{k=0}^{n} C_n^k(-1)^k(TS)^k - \sum_{k=0}^{n} C_n^k(-1)^k(TS)^k W + \sum_{k=0}^{n} C_n^k(-1)^k(TS)^k WTS = 0.$$

However, if we set $B = (I - ST)^n(I + SWT)(I - ST)$, then

$$B = (I - ST)^n + \sum_{k=0}^{n} C_n^k(-1)^k(ST)^k SWT - \sum_{k=0}^{n} C_n^k(-1)^k(ST)^{k+1} - \sum_{k=0}^{n} C_n^k(-1)^k(ST)^k SWTST.$$

Consequently,

$$B = (I - ST)^n + SAT.$$

Therefore, since $A = 0$, we deduce that

$$(I - ST)^n(I + SWT)(I - ST) = (I - ST)^n.$$

Hence the result.
" $\Longleftarrow$ " If $I - ST \in \Omega_n^\ell$, then by interchanging the roles of T and S in the direct implication, we get $I - TS \in \Omega_n^\ell$.
2) Can be obtained in the same way as 1). □

3. Pseudo-generalized invertible operators with finite ascent or finite descent

In this section, we are interested in the class of pseudo-generalized invertible operators which possess finite ascent or finite descent. First, recall that for an operator $T \in \mathscr{B}(\mathsf{X})$, the ascent of T, denoted by $\boldsymbol{a}(T)$, is the smallest non-negative integer n such that $\mathsf{N}(T^n) = \mathsf{N}(T^{n+1})$. If such an integer does not exist then $\boldsymbol{a}(T) = \infty$. The descent of T, denoted by $\boldsymbol{d}(T)$, is the smallest non-negative integer n such that $\mathsf{R}(T^n) = \mathsf{R}(T^{n+1})$. If such an integer does not exist then $\boldsymbol{d}(T) = \infty$. Note that if $\boldsymbol{a}(T) < +\infty$ (resp. $\boldsymbol{d}(T) < +\infty$), we have $\mathsf{N}(T^{\boldsymbol{a}(T)}) = \mathsf{N}(T^{\boldsymbol{a}(T)+l})$ (resp. $\mathsf{R}(T^{\boldsymbol{d}(T)}) = \mathsf{R}(T^{\boldsymbol{d}(T)+l})$), for all $l \in \mathbb{N}$. We refer to [21, 27], for more results on ascent and descent of bounded operators.

Now, we start our study with the following result which extends [5, Section 6, P. 25] that will be needed in the sequel.

Lemma 3.1. *Let $T, S \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$.*

1) *$T^nST - T^n \in \Omega_1^\ell$ if and only if $T \in \Omega_n^\ell$.*

2) *$TST^n - T^n \in \Omega_1^r$ if and only if $T \in \Omega_n^r$.*

Proof.

1) If $T \in \Omega_n^\ell$, then for all $S \in {}^T\mathsf{S}_n^\ell$, we have $T^nST - T^n = 0 \in \Omega_1^\ell$. Conversely, if $T^nST - T^n \in \Omega_1^\ell$, then there exists $L \in \mathscr{B}(\mathsf{X})$, such that
$$T^nST - T^n = (T^nST - T^n)L(T^nST - T^n).$$
Consequently,
$$\begin{aligned} T^n &= T^nST - (T^nST - T^n)L(T^nST - T^n) \\ &= T^n\Big(S - (ST - I)L(T^nS - T^{n-1})\Big)T. \end{aligned}$$

2) This assertion can be proven as 1). □

Our first observation is the following proposition concerning pseudo-generalized invertible operators of finite ascent or finite descent.

Proposition 3.2. *Let $T \in \mathscr{B}(\mathsf{X})$.*

1) *If $T \in \Omega^\ell$ and $\boldsymbol{a}(T) < +\infty$, then $T \in \Omega_{\boldsymbol{a}(T)}^\ell$ and ${}^T\mathsf{S}_{\boldsymbol{a}(T)}^\ell = {}^T\mathsf{S}_n^\ell$, for all $n \geq \boldsymbol{a}(T)$.*

2) *If $T \in \Omega^r$ and $\boldsymbol{d}(T) < +\infty$, then $T \in \Omega_{\boldsymbol{d}}^r$ and ${}^T\mathsf{S}_{\boldsymbol{d}(T)}^r = {}^T\mathsf{S}_n^r$, for all $n \geq \boldsymbol{d}(T)$.*

Proof.

1) Let $n \in \mathbb{N}$, such that $T \in \Omega_n^\ell$. If $n \leq \boldsymbol{a}(T)$, the result is obvious. Now, if $n > \boldsymbol{a}(T)$, then for all $S \in {}^T\mathsf{S}_n^\ell$, we have
$$\mathsf{R}(I - ST) \subseteq \mathsf{N}(T^n) = \mathsf{N}(T^{\boldsymbol{a}(T)}).$$
Consequently, we see that $T \in \Omega_{\boldsymbol{a}(T)}^\ell$ and ${}^T\mathsf{S}_{\boldsymbol{a}(T)}^\ell = {}^T\mathsf{S}_m^\ell$, for all $m \geq \boldsymbol{a}(T)$.

2) Can be proven in the same way as the first assertion. □

As a direct consequence of Proposition 3.2, we obtain the following corollary concerning one-to-one pseudo-generalized invertible operators.

Corollary 3.3. *Let $T \in \mathscr{B}(\mathsf{X})$ be a one-to-one operator. If $T \in \Omega^\ell$, then $T \in \Omega_0^\ell$ and for all $k \in \mathbb{N}$,*
$$S \in {}^T\mathsf{S}_k^\ell \Longleftrightarrow S \in {}^T\mathsf{S}_0^\ell.$$

Similarly, for surjective pseudo-generalized invertible operators we obtain :

Corollary 3.4. *Let $T \in \mathscr{B}(\mathsf{X})$, be a surjective operator. If $T \in \Omega^r$, then $T \in \Omega^r_0$ and for all $k \in \mathbb{N}$*

$$S \in {}^T\mathsf{S}^r_k \Longleftrightarrow S \in {}^T\mathsf{S}^r_0.$$

From [21, 29] recall that an operator $T \in \mathscr{B}(\mathsf{X})$ is said to be Drazin invertible if there exists $S \in \mathscr{B}(\mathsf{X})$ such that

$$T^nST = T^n,\ STS = S \text{ and } TS = ST. \tag{3.1}$$

Also, the Drazin invertibility of T is equivalent to the fact $\boldsymbol{a}(T) < +\infty$ and $\boldsymbol{d}(T) < +\infty$. Recall also that the index of T is defined by

$$\boldsymbol{i}(T) = \inf\{n \in \mathbb{N} \ : T^nT^DT = T^n\}.$$

From [18, Theorem 4], we know that

$$\boldsymbol{i}(T) = \boldsymbol{a}(T) = \boldsymbol{d}(T).$$

Particularly, if $\boldsymbol{i}(T) \leq 1$, then T is called group invertible.

A direct application of Proposition 3.2 to Drazin invertible operators gives the following result.

Corollary 3.5. *Let $T \in \mathscr{B}(\mathsf{X})$ be a Drazin invertible operator, then $T \in \Omega^\ell_{\boldsymbol{i}(T)} \cap \Omega^r_{\boldsymbol{i}(T)}$ and we have*

$${}^T\mathsf{S}^\ell_{\boldsymbol{i}(T)} = {}^T\mathsf{S}^\ell_n \text{ and } {}^T\mathsf{S}^r_{\boldsymbol{i}(T)} = {}^T\mathsf{S}^r_n, \text{ for any } n \geq \boldsymbol{i}(T).$$

Particularly, if T is group invertible then every pseudo-generalized inverse of T is a g_1-inverse of T.

In the sequel we need the following result due to Grabiner and Zemánek.

Lemma 3.6 ([13, Lemma 1.1]). *Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}$.*

1) *The following assertions are equivalent :*
   - (i) $\boldsymbol{a}(T) \leq n$,
   - (ii) $\exists\, m \geq 1 \ : \ \mathsf{R}(T^n) \cap \mathsf{N}(T^m) = \{0\}$,
   - (iii) $\mathsf{R}(T^n) \cap \mathsf{N}(T^m) = \{0\}$, $\forall m \geq 1$.
2) *The following assertions are equivalent :*
   - (i) $\boldsymbol{d}(T) \leq n$,
   - (ii) $\exists\, m \geq 1 \ : \ \mathsf{R}(T^n) + \mathsf{N}(T^m) = \mathsf{X}$,
   - (iii) $\mathsf{R}(T^n) + \mathsf{N}(T^m) = \mathsf{X}$, $\forall m \geq 1$.

We can now state the following proposition concerning the relationship between g-invertibility and pseudo-generalized invertibility.

Proposition 3.7. *Let $T \in \mathscr{B}(\mathsf{X})$.*

1) *If $T \in \Omega^\ell$ and $\dim\Big(\mathsf{R}(T) \cap \mathsf{N}(T)\Big) < +\infty$, then $T \in \Omega^\ell_1$. In particular, if $\mathsf{R}(T) \cap \mathsf{N}(T) = \{0\}$, then $T \in \Omega^\ell_{\boldsymbol{a}(T)}$, $\boldsymbol{a}(T) \leq 1$ and*
$${}^T\mathsf{S}^\ell_{\boldsymbol{a}(T)} = {}^T\mathsf{S}^\ell_k,\ \forall k \geq \boldsymbol{a}(T).$$
2) *If $T \in \Omega^r$ and $\operatorname{codim}\Big(\mathsf{R}(T) + \mathsf{N}(T)\Big) < +\infty$, then $T \in \Omega^r_1$. In particular, if $\mathsf{R}(T) + \mathsf{N}(T) = \mathsf{X}$, then $\boldsymbol{d}(T) \leq 1$ and we have*
$${}^T\mathsf{S}^r_{\boldsymbol{d}(T)} = {}^T\mathsf{S}^r_k,\ \forall k \geq \boldsymbol{d}(T).$$

Proof.
1) First, if $T \in \Omega_1^\ell$, then the result is obvious. Now, if $T \in \Omega_n^\ell$, where $n \in \mathbb{N}\backslash\{0, 1\}$, then there exists an operator $S \in \mathscr{B}(\mathsf{X})$ such that $T^n = T^n ST$. We see that

$$\mathsf{R}(T^{n-1} - T^{n-1}ST) \subseteq \mathsf{R}(T) \cap \mathsf{N}(T).$$

Using the fact that $\dim\Big(\mathsf{R}(T) \cap \mathsf{N}(T)\Big) < +\infty$, we deduce that $T^{n-1} - T^{n-1}ST \in \Omega_1^\ell$. So by Lemma 3.1, we see that $T \in \Omega_{n-1}^\ell$.
If $n - 1 > 1$, we can similarly see that $T \in \Omega_{n-2}^\ell$. Thus we finally obtain $T \in \Omega_1^\ell$.
In particular, if $\mathsf{R}(T) \cap \mathsf{N}(T) = \{0\}$, then by Lemma 3.6, we have $\boldsymbol{a}(T) \leq 1$ and hence the Proposition 3.2 allows us to conclude.
2) Using arguments similar to 1) we can show this assertion. □

From the previous results we then have the following.

Remark 3.8. Let $T \in \mathscr{B}(\mathsf{X})$ such that $T \in \Omega^\ell$. It is clear that T is one-to-one if and only if $\boldsymbol{a}(T) = 0$. Hence, from Corollary 3.3, we get

$$\boldsymbol{a}(T) = 0 \Longleftrightarrow T \in \Omega_0^\ell.$$

Therefore, Lemma 3.6 and Proposition 3.7, show that if $\boldsymbol{a}(T) \leq 1$, then

$$\boldsymbol{a}(T) = 1 \Longleftrightarrow T \in \Omega_1^\ell\backslash\Omega_0^\ell \Longleftrightarrow T \in \Omega^\ell\backslash\Omega_0^\ell.$$

Similarly, for right pseudo-generalized invertible operators, we have :

Remark 3.9. Let $T \in \mathscr{B}(\mathsf{X})$ such that $T \in \Omega^r$. It is clear that T is onto if and only if $\boldsymbol{d}(T) = 0$. By Corollary 3.4, we can see

$$\boldsymbol{d}(T) = 0 \Longleftrightarrow T \in \Omega_0^r.$$

Therefore, Lemma 3.6 and and Proposition 3.7 show that if $\boldsymbol{d}(T) \leq 1$, then

$$\boldsymbol{d}(T) = 1 \Longleftrightarrow T \in \Omega_1^r\backslash\Omega_0^r \Longleftrightarrow T \in \Omega^r\backslash\Omega_0^r.$$

We conclude this section with the following corollary which is simple to verify.

Corollary 3.10. *Let $T \in \mathscr{B}(\mathsf{X})$.*

1) *If $T \in \Omega^\ell$ such $\mathsf{N}(T)$ is complemented and X_1 a topological complement of $\mathsf{N}(T)$ such that $\mathsf{R}(T) \subseteq \mathsf{X}_1$, then $\boldsymbol{a}(T) \leq 1$, $T \in \Omega_{\boldsymbol{a}(T)}$ and*
$$^T\mathsf{S}_{\boldsymbol{a}(T)}^\ell = {}^T\mathsf{S}_k^\ell,\ \forall k \geq \boldsymbol{a}(T).$$

2) *If $T \in \Omega^r$ such that $\mathsf{R}(T)$ is complemented and X_2 a topological complement of $\mathsf{R}(T)$ such that $\mathsf{X}_2 \subseteq \mathsf{N}(T)$, then $\boldsymbol{d}(T) \leq 1$, $T \in \Omega_{\boldsymbol{d}(T)}^r$ and*
$$^T\mathsf{S}_{\boldsymbol{d}(T)}^r = {}^T\mathsf{S}_k^r,\ \forall k \geq \boldsymbol{d}(T).$$

4. Semi-Fredholm and pseudo-generalized invertibility

In this section, we are interested in studying operators both semi-Fredholm and pseudo-generalized invertible. Recall that an operator $T \in \mathscr{B}(\mathsf{X})$ is said to be left semi-Fredholm if T is of closed range and $\alpha(T) < +\infty$, where $\alpha(T)$ denotes the dimension of $\mathsf{N}(T)$, and we call T right semi-Fredholm if $\beta(T) < +\infty$, where $\beta(T)$ denotes the codimension of $\mathsf{R}(T)$. An operator is called semi-fredholm if it is left or right semi-Fredholm. Notice that if $\beta(T) < +\infty$, then $\mathsf{R}(T)$ must be closed [1, Corollary 1.15]. For more strong results about semi-Fredholm operators one can see [1, 11, 21].

Clearly, from Proposition 3.7 all left (resp. right) semi-Fredholm operators belonging to Ω^ℓ (resp. Ω^r) are g-invertible. More precisely we have

Proposition 4.1. *Let $T \in \mathscr{B}(\mathsf{X})$.*

1) *If $T \in \Omega^\ell$ and $\alpha(T) < +\infty$, then $T \in \Omega_1^\ell$ and so T is left semi-Fredholm.*

2) *If $T \in \Omega^r$ and T is right semi-Fredholm, then $T \in \Omega_1^r$.*

Recall the following result which will be used to prove Theorem 4.3.

Lemma 4.2 ([28, Lemma 3.4]). *Let $T \in \mathscr{B}(\mathsf{X})$.*

1) *If $\alpha(T) < +\infty$, then $\alpha(T^n) < n\alpha(T)$, for all $n \geq 1$.*

2) *If $\beta(T) < +\infty$, then $\beta(T^n) < n\beta(T)$, for all $n \geq 1$.*

We can now state some of the main results concerning operators both semi-Fredholm and pseudo-generalized invertible.

Theorem 4.3. *Let $k \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{X})$.*
1) *If $T \in \Omega^\ell$ and $\alpha(T) < +\infty$.*

i) *For all $S \in {}^T\mathsf{S}_k^\ell$, there exists an operator $F \in \mathscr{B}(\mathsf{X})$, such that* $\mathrm{rank}(F) < k\alpha(T)$ *and $T = TST + F$.*

ii) *For all $S \in {}^T\mathsf{S}_k^\ell$, there exists $L \in {}^T\mathsf{S}_k^\ell$, such that $T = TLT + TP$, where $P \in \mathscr{B}(\mathsf{X})$ is a projection of range $\mathsf{R}(I - ST)$.*

2) *If $T \in \Omega^r$ and $\beta(T) < +\infty$.*

i) *For all $S \in {}^T\mathsf{S}_k^r$, there exists an operator $F \in \mathscr{B}(\mathsf{X})$, such that* $\mathrm{rank}(F) < k\beta(T)$ *and $T = TST + F$.*

ii) *For all $S \in {}^T\mathsf{S}_k^r$, there exists $L \in {}^T\mathsf{S}_k^r$, such that $T = TLT + PT$, where $P \in \mathscr{B}(\mathsf{X})$ is a projection of kernel $\mathsf{N}(P) = \mathsf{N}(I - TS)$.*

Proof.
1) If $T \in \Omega^\ell$ and $\alpha(T) < +\infty$, then $T \in \Omega_1^\ell$ (see Proposition 4.1). Moreover, by Lemma 4.2, we have $\alpha(T^k) < k\alpha(T) < +\infty$. Let $S \in {}^T\mathsf{S}_k^\ell$.

i) If we set $F = T - TST$ and since $\mathsf{R}(I - ST) \subseteq \mathsf{N}(T^k)$, then

$$\mathrm{rank}(F) \leq \mathrm{rank}(I - ST) \leq \alpha(T^k) < k\alpha(T).$$

ii) Since $\mathrm{rank}(ST - I) < +\infty$, then $ST - I \in \Omega_1^\ell$ (see [5, Theorem 3.1] or [14, Theorem 6.2.6, Theorem 6.8.5]). Let L_0 be a g_1-inverse of $ST - I$, then

$$(ST - I)L_0(ST - I) = ST - I.$$

Hence,

$$\begin{aligned} I &= ST - (ST - I)L_0(ST - I) \\ &= ST - STL_0ST + STL_0 + L_0ST - L_0 \\ &= (S - STL_0S + L_0S)T + (ST - I)L_0. \end{aligned}$$

Now, we set $L = S - STL_0S + L_0S$ and $P = (ST - I)L_0$. So

$$T = TLT + TP.$$

Moreover, we know that P is a projection of range $\mathsf{R}(ST - I)$. This implies that $T^kP = 0$ and therefore

$$T^k = T^kLT + T^kP = T^kLT.$$

Consequently, $L \in {}^T\mathsf{S}_k^\ell$. Hence the result follows.

2) Can be proven in the same way. □

Theorem 4.4. *Let $n \in \mathbb{N}\backslash\{0\}$, $k \in \{0, \cdots, n-1\}$, $T \in \Omega^\ell$ such that $\alpha(T) < +\infty$ and $S \in {}^T\mathsf{S}^\ell_n$.*

1) *There exists $F \in \mathscr{B}(\mathsf{X})$, such that* $\operatorname{rank}(F) < k\alpha(T)$ *and*

$$T^{n-k} = T^{n-k}ST + F.$$

2) *There exists $L \in {}^T\mathsf{S}^\ell_n$, such that $T^{n-k} = T^{n-k}LT + PT^{n-k}$, where $P \in \mathscr{B}(\mathsf{X})$, is a projection of range* $\mathsf{R}(P) = \mathsf{R}(T^{n-k}ST - T^{n-k})$.

Proof.
If $T \in \Omega^\ell$ and $\alpha(T) < +\infty$, then $T \in \Omega^\ell_1$ (see Proposition 4.1). Moreover, by Lemma 4.2, we have $\alpha(T^k) < k\alpha(T) < +\infty$.
1) Since $\mathsf{R}(T^{n-k} - T^{n-k}ST) \subseteq \mathsf{N}(T^k)$, then $\operatorname{rank}(T^{n-k} - T^{n-k}ST) \leq \alpha(T^k) < k\alpha(T)$ and by setting $F = T^{n-k} - T^{n-k}ST$, we obtain

$$T^{n-k} = T^{n-k}ST + F.$$

2) As $\operatorname{rank}(T^{n-k} - T^{n-k}ST) < +\infty$, then by [5, Theorem 3.1] (see also [14, Theorem 6.2.6, Theorem 6.8.5]), we have $T^{n-k} - T^{n-k}ST \in \Omega^\ell_1$. Let $L_0 \in \mathscr{B}(\mathsf{X})$ be a g_1-inverse of $T^{n-k}ST - T^{n-k}$. So,

$$(T^{n-k}ST - T^{n-k})L_0(T^{n-k}ST - T^{n-k}) = T^{n-k}ST - T^{n-k}.$$

Hence,

$$T^{n-k} = T^{n-k}ST - (T^{n-k}ST - T^{n-k})L_0(T^{n-k}ST - T^{n-k}).$$

If we set $P = (T^{n-k}ST - T^{n-k})L_0$ and $L = S + L_0T^{n-k}S - STL_0T^{n-k}S$, then

$$\begin{aligned} T^{n-k} &= T^{n-k}(S + L_0T^{n-k}S - STL_0T^{n-k}S)T + (T^{n-k}ST - T^{n-k})L_0T^{n-k} \\ &= T^{n-k}LT + PT^{n-k}. \end{aligned}$$

Clearly, $T^kP = 0$, and so $L \in {}^T\mathsf{S}^\ell_n$. Since L_0 is a g_1-inverse of $T^{n-k}ST - T^{n-k}$, then P is a projection of range $\mathsf{R}(P) = \mathsf{R}(T^{n-k}ST - T^{n-k})$. □

Finally, by a similar argument to the one in the proof of Theorem 4.4, we obtain :

Theorem 4.5. *Let $n \in \mathbb{N}\backslash\{0\}$, $k \in \{0, \cdots, n-1\}$, $T \in \Omega^r$ such that $\beta(T) < +\infty$ and $S \in {}^T\mathsf{S}^r_n$.*

1) *There exists $F \in \mathscr{B}(\mathsf{X})$, such that* $\operatorname{rank}(F) < k\beta(T)$ *and*

$$T^{n-k} = TST^{n-k} + F.$$

2) *There exists $L \in {}^T\mathsf{S}^r_n$, such that $T^{n-k} = TLT^{n-k} + T^{n-k}P$, where $P \in \mathscr{B}(\mathsf{X})$ is a projection of kernel* $\mathsf{N}(P) = \mathsf{N}(TST^{n-k} - T^{n-k})$.

5. Pseudo-generalized invertible operators with finite essential ascent or finite essential descent

This section aims to present some results for pseudo-generalized invertible operators having finite essential ascent or finite essential descent. Note that operators with finite essential ascent or descent were first studied in 1974 by S. Grabiner [12]. Recall that $T \in \mathscr{B}(\mathsf{X})$ is said to be of finite essential ascent if there exists $n \in \mathbb{N}$, such that $\dim\left(\mathsf{N}(T^{n+1})/\mathsf{N}(T^n)\right) < +\infty$. In this case the essential ascent of T denoted by $\boldsymbol{a}_e(T)$ is defined by

$$\boldsymbol{a}_e(T) = \inf\{n \in \mathbb{N} : \dim\left(\mathsf{N}(T^{n+1})/\mathsf{N}(T^n)\right) < +\infty\}.$$

Similarly, if there exists $n \in \mathbb{N}$ such that $\dim\Big(\mathsf{R}(T^n)/\mathsf{R}(T^{n+1})\Big) < +\infty$, we say that T is of finite essential descent. In this case the essential descent of T denoted by $\boldsymbol{d}_e(T)$ is defined by

$$\boldsymbol{d}_e(T) = \inf\{n \in \mathbb{N} \,:\, \dim\Big(\mathsf{R}(T^n)/\mathsf{R}(T^{n+1})\Big) < +\infty\}.$$

Notice that $\boldsymbol{a}_e(T) = 0$ if and only if $\alpha(T) < +\infty$ and $\boldsymbol{d}_e(T) = 0$ if and only if $\beta(T) < +\infty$. We refer to [12, 13, 19, 21], for more results on essential ascent and essential descent of bounded operators. Let us also mention the lemma below, which is needed in the following.

Lemma 5.1. [13, Lemma 5.1] Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}$.

1) The following assertions are equivalent :

   (i) $\dim\Big(\mathsf{N}(T^{n+1})/\mathsf{N}(T^n)\Big) < +\infty$,

   (ii) $\exists\, m \geq 1 \,:\, \dim\Big(\mathsf{R}(T^n) \cap \mathsf{N}(T^m)\Big) < +\infty$,

   (iii) $\dim\Big(\mathsf{R}(T^n) \cap \mathsf{N}(T^m)\Big) < +\infty$, $\forall m \geq 1$.

2) The following assertions are equivalent :

   (i) $\dim\Big(\mathsf{R}(T^n)/\mathsf{R}(T^{n+1}\Big) < +\infty$,

   (ii) $\exists\, m \geq 1 \,:\, \operatorname{codim}\Big(\mathsf{N}(T^n) + \mathsf{R}(T^m)\Big) < +\infty$,

   (iii) $\operatorname{codim}\Big(\mathsf{N}(T^n) + \mathsf{R}(T^m)\Big) < +\infty$, $\forall m \geq 1$.

The next lemma will be used to prove Theorem 5.3.

Lemma 5.2. *Let $n \geq 2$, $k \in \{1, \cdots, n-1\}$ and $T \in \mathscr{B}(\mathsf{X})$.*

1) *If $T \in \Omega_n^\ell$ and* $\dim\Big(\mathsf{N}(T^k) \cap \mathsf{R}(T^{n-k})\Big) < +\infty$, *then $T \in \Omega_{n-k}^\ell$.*

2) *If $T \in \Omega_n^r$ and* $\operatorname{codim}\Big(\mathsf{R}(T^k) + \mathsf{N}(T^{n-k})\Big) < +\infty$, *then $T \in \Omega_{n-k}^r$.*

Proof.
1) Let $S \in {}^T\mathsf{S}_n^\ell$. Since

$$\mathsf{R}(T^{n-k}ST - T^{n-k}) \subseteq \mathsf{N}(T^k)$$

and

$$\mathsf{R}(T^{n-k}ST - T^{n-k}) \subseteq \mathsf{R}(T^{n-k}),$$

it follows that

$$\mathsf{R}(T^{n-k}ST - T^{n-k}) \subseteq \mathsf{R}(T^{n-k}) \cap \mathsf{N}(T^k).$$

Consequently, by hypothesis we see that $\operatorname{rank}(T^{n-k}ST - T^{n-k}) < +\infty$ and from [14, Theorem 6.3.4], we get

$$T^{n-k}ST - T^{n-k} \in \Omega_1^\ell.$$

Finally, by Lemma 3.1, we conclude that $T \in \Omega_{n-k}^\ell$.
2) Can be seen in the same way. □

The main result of this section is the following theorem.

Theorem 5.3. *Let $T \in \mathscr{B}(\mathsf{X})$.*

1) *If $T \in \Omega^\ell$ and $1 \leq \boldsymbol{a}_e(T) < +\infty$, then $T \in \Omega_{\boldsymbol{a}_e(T)}^\ell$.*

2) *If* $T \in \Omega^r$ *and* $1 \leq \boldsymbol{d}_e(T) < +\infty$, *then* $T \in \Omega^r_{\boldsymbol{d}_e(T)}$.

Proof.

1) Let $n \in \mathbb{N}$, such that $T \in \Omega^\ell_n$. If $n \leq \boldsymbol{a}_e(T)$, the result is obvious. Suppose that $n > \boldsymbol{a}_e(T)$, then $n \geq 2$ and there exists $k \in \{1, \cdots, n-1\}$, such that $\boldsymbol{a}_e(T) = n - k$. Consequently, by Lemma 5.1, we have $\dim\Big(\mathsf{R}(T^{n-k}) \cap \mathsf{N}(T^k)\Big) < +\infty$ and finally Lemma 5.2 can be applied to conclude the result.

2) Let $n \in \mathbb{N}$, such that $T \in \Omega^r_n$. If $n \leq \boldsymbol{d}_e(T)$, the assertion is obvious. Suppose that $n > \boldsymbol{d}_e(T)$, then $n \geq 2$ and there exists $k \in \{1, \cdots, n-1\}$, such that $\boldsymbol{d}_e(T) = n - k$. Therefore, by Lemma 5.1, we have $\operatorname{codim}\Big(\mathsf{N}(T^{n-k}) + \mathsf{R}(T^k)\Big) < +\infty$ and finally we can conclude the result by using Lemma 5.2. □

Clearly, if $T \in \mathscr{B}(\mathsf{X})$ is of finite ascent, then $\boldsymbol{a}_e(T) \leq \boldsymbol{a}(T)$. Similarly, if T is of finite descent, then $\boldsymbol{d}_e(T) \leq \boldsymbol{d}(T)$. Hence, we deduce the following result.

Corollary 5.4. *Let* $T \in \mathscr{B}(\mathsf{X})$.

1) *If* $T \in \Omega^\ell$ *of finite ascent and* $\boldsymbol{a}_e(T) \geq 1$, *then* $T \in \Omega^\ell_{\boldsymbol{a}_e(T)}$.

2) *If* $T \in \Omega^r$ *of finite descent and* $\boldsymbol{d}_e(T) \geq 1$, *then* $T \in \Omega^r_{\boldsymbol{d}_e(T)}$.

Also, as a consequence of Theorem 5.3, we have the following result.

Corollary 5.5. *Let* $T \in \mathscr{B}(\mathsf{X})$. *If* $T \in \Omega^\ell$ *and* $\alpha(T)$ *is infinite and there exists* $p, m \geq 1$ *such that* $\dim\Big(\mathsf{R}(T^p) \cap \mathsf{N}(T^m)\Big) < +\infty$, *then* $1 \leq \boldsymbol{a}_e(T) < +\infty$ *and* $T \in \Omega^\ell_{\boldsymbol{a}_e(T)}$.
In particular, if $\mathsf{R}(T^p) \cap \mathsf{N}(T^m) = \{0\}$, *we have* $\boldsymbol{a}(T) \leq p$, $T \in \Omega^\ell_{\boldsymbol{a}(T)}$ *and*

$$^T\mathsf{S}^\ell_{\boldsymbol{a}(T)} = {}^T\mathsf{S}^\ell_p = {}^T\mathsf{S}^\ell_k, \ \forall k \geq \boldsymbol{a}(T).$$

Proof.
By Lemma 5.1, it is clear that $\boldsymbol{a}_e(T) < +\infty$. Also, since $\alpha(T)$ is infinite, then we deduce that $1 \leq \boldsymbol{a}_e(T) < +\infty$. Consequently, the result follows from Theorem 5.3.
Now, if $\mathsf{R}(T^p) \cap \mathsf{N}(T^m) = \{0\}$, then by Lemma 3.6, we have $\boldsymbol{a}(T) \leq p$. Hence, we deduce the result from Proposition 3.2. This completes the proof. □

In the same way, we obtain :

Corollary 5.6. *Let* $T \in \mathscr{B}(\mathsf{X})$. *If* $T \in \Omega^r$ *and* $\beta(T)$ *is infinite and there exists* $p, m \geq 1$ *such that* $\operatorname{codim}\Big(\mathsf{N}(T^p) + \mathsf{R}(T^m)\Big) < +\infty$, *then* $1 \leq \boldsymbol{d}_e(T) < +\infty$ *and* $T \in \Omega^r_{\boldsymbol{d}_e(T)}$.
In particular, if $\mathsf{N}(T^p) + \mathsf{R}(T^m) = \mathsf{X}$, *we have* $\boldsymbol{d}(T) \leq p$, $T \in \Omega^r_{\boldsymbol{d}(T)}$ *and*

$$^T\mathsf{S}^r_{\boldsymbol{d}(T)} = {}^T\mathsf{S}^r_p = {}^T\mathsf{S}^r_k, \ \forall k \geq \boldsymbol{a}(T).$$

Corollary 5.7. *Let* $T \in \mathscr{B}(\mathsf{X})$.

1) *If* $T \in \Omega^\ell$ *and there exist* p, $m \geq 1$ *such that* $\mathsf{N}(T^m)$ *is complemented and there exists* $\mathsf{X}_m \subseteq \mathsf{X}$, *a topological complement of* $\mathsf{N}(T^m)$ *such that* $\mathsf{R}(T^p) \subseteq \mathsf{X}_m$, *then* $\boldsymbol{a}(T) \leq p$, $T \in \Omega^\ell_{\boldsymbol{a}(T)}$ *and*

$$^T\mathsf{S}^\ell_{\boldsymbol{a}(T)} = {}^T\mathsf{S}^\ell_p = {}^T\mathsf{S}^\ell_k, \ \forall k \geq \boldsymbol{a}(T).$$

2) *If* $T \in \Omega^r$ *and there exist* p, $m \geq 1$ *such that* $\mathsf{R}(T^m)$ *is complemented and there exists* $\mathsf{X}_m \subseteq \mathsf{X}$, *a topological complement* $\mathsf{R}(T^m)$ *such that* $\mathsf{X}_m \subseteq \mathsf{N}(T^p)$, *then* $\boldsymbol{d}(T) \leq p$, $T \in \Omega^r_{\boldsymbol{d}(T)}$ *and*

$$^T\mathsf{S}^r_{\boldsymbol{d}(T)} = {}^T\mathsf{S}^r_p = {}^T\mathsf{S}^r_k, \ \forall k \geq \boldsymbol{d}(T).$$

Proof.

1) This result is a simple consequence of Corollary 5.5, it suffices to see that the fact $\mathsf{R}(T^p) \subseteq \mathsf{X}_m$ implies that $\mathsf{N}(T^m) \cap \mathsf{R}(T^p) = \{0\}$.

2) Since $\mathsf{X}_m \subseteq \mathsf{N}(T^p)$, then $\mathsf{R}(T^m) + \mathsf{N}(T^p) = \mathsf{X}$, so the result follows directly from Corollary 5.6. □

However, the following question remains open.

Question :

1) If $T \in \Omega^\ell$ and $\boldsymbol{a}_e(T) < +\infty$, as in the case of finite ascent, we don't know if

$$^T\mathsf{S}^\ell_{\boldsymbol{a}_e(T)} = {}^T\mathsf{S}^\ell_k,\ \forall k \geq \boldsymbol{a}_e(T)\,?$$

2) If $T \in \Omega^r$ and $\boldsymbol{d}_e(T) < +\infty$, as in the case of finite descent, we don't know if

$$^T\mathsf{S}^\ell_{\boldsymbol{d}_e(T)} = {}^T\mathsf{S}^\ell_k,\ \forall k \geq \boldsymbol{d}_e(T)\,?$$

6. Pseudo-generalized invertible operator : topological complements of the kernel and image

First, we recall that an operator in $\mathscr{B}(\mathsf{X})$ is g-invertible if and only if its range and kernel are complemented. In fact, if S is a g_1-inverse of T, then we have

$$\mathsf{X} = \mathsf{R}(T) \dotplus \mathsf{N}(TS) \text{ and } \mathsf{X} = \mathsf{N}(T) \dotplus \mathsf{R}(ST),$$

where the symbol $\dotplus$ denotes the topological direct sum. Particularly, we obtain $\mathsf{R}(S) = \mathsf{R}(ST)$ and $\mathsf{N}(S) = \mathsf{N}(TS)$, if S is a g_2-inverse [5]. On the contrary, the closure of $\mathsf{R}(T)$ is not always guaranteed for pseudo-generalized invertible operators. For example, from Remark 2.3 and Example 2.2, we have $R_s, T_s \in \Omega^\ell_s \cap \Omega^r_s$, for $s \geq 2$, while their ranges $\mathsf{R}(R_s)$ and $\mathsf{R}(T_s)$ are not closed because, as we know, the range of compact operators of infinite rank is not closed. In this section, we discuss various special cases when this fact remains valid for pseudo-generalized operators and their powers.

First we present in the following result some sufficient conditions under which the range of left pseudo-generalized invertible operators remains closed.

Proposition 6.1. *Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$. If $T \in \Omega^\ell_n$ and $\mathsf{N}(T^k) \subseteq \mathsf{R}(T)$, for some $k \geq n-1$, then $\mathsf{R}(T)$ is closed.*

Proof.
Let $S \in {}^T\mathsf{S}^\ell_n$ and $(x_i)_{i\in\mathbb{N}}$ be a sequence of X such that

$$\lim_{i\to+\infty} Tx_i = y,$$

then we have

$$\lim_{i\to+\infty} T^n x_i = \lim_{i\to+\infty} T^n STx_i = T^{n-1}y = T^n Sy.$$

Consequently, we obtain

$$y - TSy \in \mathsf{N}(T^{n-1}).$$

Now, as $\mathsf{N}(T^{n-1}) \subseteq \mathsf{N}(T^k) \subseteq \mathsf{R}(T)$, then we deduce that $y \in \mathsf{R}(T)$ and therefore $\mathsf{R}(T)$ is closed. □

For the powers of right pseudo-generalized invertible operators, we state the following results.

Proposition 6.2. *Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$. If $T \in \Omega^r_n$ and there exists $S \in {}^T\mathsf{S}^r_n\backslash\{0\}$, such that $\mathsf{R}(ST^n)$ is closed, then $\mathsf{R}(T^n)$ is closed.*

Proof.
Let $S \in {}^{T}\mathsf{S}_n^r$ and $(x_i)_{i\in\mathbb{N}}$ be a sequence of X such that

$$\lim_{i\to+\infty} T^n x_i = y,$$

then we have

$$\lim_{i\to+\infty} ST^n x_i = Sy.$$

Now, since $\mathsf{R}(ST^n)$ is closed, then there exists $z \in \mathsf{X}$ such that $Sy = ST^n z$. Therefore, we obtain

$$\lim_{i\to+\infty} T^n x_i = \lim_{i\to+\infty} TST^n x_i = TST^n z = T^n z,$$

hence the result. □

Corollary 6.3. *Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$. If $T \in \Omega_n^r$ and there exists $S \in {}^{T}\mathsf{S}_n^r\backslash\{0\}$, $k \in \mathbb{N}\backslash\{0\}$, such that $\mathsf{R}((ST)^k) \subseteq \mathsf{R}(ST^n)$, then $\mathsf{R}(T^n)$ is closed.*

Proof.
It is clear that $ST^n = STST^n$ and hence $\mathsf{R}(ST^n) \subseteq \mathsf{N}(I - ST)$. Conversely, let $x \in \mathsf{N}(I - ST)$, then

$$x = STx = (ST)^k x \in \mathsf{R}(ST^n)$$

and therefore $\mathsf{R}(ST^n) = \mathsf{N}(I - ST)$, which implies that $\mathsf{R}(ST^n)$ is closed. Finally, by Proposition 6.2, we deduce that $\mathsf{R}(T^n)$ is closed. □

Proposition 6.4. *Let $T \in \Omega^r$. If there exists a non-zero operator $S \in {}^{T}\mathsf{S}_n^r$, for some $n \in \mathbb{N}\backslash\{0\}$, such that $\mathsf{X} = \mathsf{R}(T^n) + \mathsf{N}(S)$, then $\mathsf{R}(T^n)$ is complemented, $S = STS$ and*

$$\mathsf{X} = \mathsf{R}(T^n) \dotplus \mathsf{N}(S).$$

Proof.
Let $n \in \mathbb{N}\backslash\{0\}$, such that $T \in \Omega_n^r$. Clearly, $ST^n = STST^n$ and consequently

$$\mathsf{X} = \mathsf{R}(T^n) + \mathsf{N}(S) \subseteq \mathsf{N}(S - STS).$$

Hence, $STS = S$ and so TS is a projection of kernel $\mathsf{N}(S)$. This implies that

$$\mathsf{R}(T^n) \subseteq \mathsf{N}(I - TS) = \mathsf{R}(TS).$$

Conversely, let $x \in \mathsf{R}(TS)$, then by hypothesis, there exists $x_1 \in \mathsf{R}(T^n)$, such that $x - x_1 \in \mathsf{N}(S)$. Therefore

$$x - x_1 \in \mathsf{R}(TS) \cap \mathsf{N}(S) = \{0\}.$$

Consequently,

$$x = x_1 \in \mathsf{R}(T^n).$$

The proof is therefore complete. □

Recall that the reduced minimum modulus of the operator $T \in \mathscr{B}(\mathsf{X})$, denoted by $\gamma(T)$, is defined to be

$$\gamma(T) = \inf\left\{\frac{\|T(x)\|}{\operatorname{dist}(x, \mathsf{N}(T))} : x \notin \mathsf{N}(T)\right\}. \tag{6.1}$$

We see also

$$\gamma(T) = \sup\left\{a \geq 0 : \|T(x)\| \geq a \operatorname{dist}(x, \mathsf{N}(T))\right\} \tag{6.2}$$

and

$$\gamma(T) = \inf\Big\{\|T(x)\|,\ x \in \mathsf{X}, \mathrm{dist}(x, \mathsf{N}(T)) = 1\Big\}. \tag{6.3}$$

For more details about the reduced minimum modulus see [11, 16, 17, 21].
It is well-know that for all $T \in \mathscr{B}(\mathsf{X})$,

$$\gamma(T) > 0 \Longleftrightarrow \mathsf{R}(T) \text{ is closed.}$$

In the remainder of this section, we discuss the relationship between the notion of n-left pseudo-generalized invertibility of an operator T and the complementarity of the subspace $\mathsf{N}(T^n)$.

Proposition 6.5. *Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$. If $T \in \Omega_n^{\ell}$ and there exists a non-zero operator $S \in {}^{T}\mathsf{S}_n^{\ell}$, such that $\mathsf{N}(T^n) = \mathsf{N}(ST)$, then $\mathsf{N}(T^n)$ is complemented and*

$$\mathsf{X} = \mathsf{N}(T^n) \dotplus \mathsf{R}(ST).$$

Proof.
First, since $\mathsf{N}(ST) = \mathsf{N}(T^n)$, then

$$\begin{array}{rcl}\gamma(ST) & = & \inf\Big\{\|STx\|,\ x \in \mathsf{X},\ \mathrm{dist}\Big(x, \mathsf{N}(ST)\Big) = 1\Big\} \\ & = & \inf\Big\{\|STx\|,\ x \in \mathsf{X},\ \mathrm{dist}\Big(x, \mathsf{N}(T^n)\Big) = 1\Big\}.\end{array}$$

Now, since $\mathsf{R}(I - ST) \subseteq \mathsf{N}(T^n)$, we deduce that

$$\mathrm{dist}(x, \mathsf{N}(T^n)) \le \|x - (x - STx)\| \le \|STx\|,\ \forall x \in \mathsf{X}.$$

Hence, for all $x \in \mathsf{X}$ such that $\mathrm{dist}\Big(x, \mathsf{N}(ST)\Big) = 1$, we have $\|STx\| \ge 1$ and therefore,

$$\gamma(ST) = \inf\Big\{\|STx\|,\ x \in \mathsf{X},\ \mathrm{dist}\Big(x, \mathsf{N}(T^n)\Big) = 1\Big\} \ge \|STx\| \ge 1.$$

From [21, Theorem 2, P. 93], we conclude that $\mathsf{R}(ST)$ is closed. Now, let $x \in \mathsf{N}(ST) \cap \mathsf{R}(ST)$, then there exists $y \in \mathsf{X}$, such that $x = STy$. Since $\mathsf{N}(T^n) = \mathsf{N}(ST)$, it follows that

$$0 = T^n x = T^n y.$$

Now, as $y \in \mathsf{N}(T^n)$, then $y \in \mathsf{N}(ST)$ and $x = STy = 0$. Moreover, for all $x \in \mathsf{X}$, we have $x = (I - ST)x + STx$. Finally, since $(I - ST)x \in \mathsf{N}(T^n) = \mathsf{N}(ST)$, we deduce that

$$\mathsf{X} = \mathsf{N}(ST) \dotplus \mathsf{R}(ST).$$

This completes the proof. □

As a consequence we obtain the following corollary.

Corollary 6.6. *Let $T \in \Omega^{\ell}$. If there exists a non-zero operator $S \in {}^{T}\mathsf{S}_n^{\ell}$, for some $n \in \mathbb{N}\backslash\{0\}$, such that $\mathsf{N}(T^n) \cap \mathsf{R}(S) = \{0\}$, then $\mathsf{N}(T^n)$ is complemented, $S = STS$ and*

$$\mathsf{X} = \mathsf{N}(T^n) \dotplus \mathsf{R}(S).$$

Proof.
Since $T^n ST = T^n$, then clearly $\mathsf{N}(ST) \subseteq \mathsf{N}(T^n)$. Now, let $x \in \mathsf{N}(T^n)$, then we have $STx \in \mathsf{N}(T^n) \cap \mathsf{R}(ST) = \{0\}$ and therefore $x \in \mathsf{N}(ST)$ and $\mathsf{N}(T^n) = \mathsf{N}(ST)$. Consequently, by Proposition 6.5, we obtain

$$\mathsf{X} = \mathsf{N}(T^n) \dotplus \mathsf{R}(ST).$$

Now, since $T^nS = T^nSTS$, it follows that

$$\mathsf{R}(S - STS) \subseteq \mathsf{N}(T^n) \cap \mathsf{R}(S) = \{0\}.$$

Hence, $STS = S$, and this implies that ST is a projection of range $\mathsf{R}(S)$. Consequently,

$$\mathsf{X} = \mathsf{N}(T^n) \dotplus \mathsf{R}(S),$$

which completes the proof. □

By combining [5, Theorem 3.1], Proposition 6.4 and Corollary 6.6, we have the following result :

Corollary 6.7. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{X})$. If $T \in \Omega_n^\ell \cap \Omega_n^r$ and there exists a non-zero operator $S \in {}^T\mathsf{S}_n^\ell \cap {}^T\mathsf{S}_n^r$, such that*

(1) $\mathsf{R}(T^n) + \mathsf{N}(S) = \mathsf{X}$,

(2) $\mathsf{N}(T^n) \cap \mathsf{R}(S) = \{0\}$,

then $T^n \in \Omega_1^\ell$ $(= \Omega_1^r)$.

Motivated by the last results, it is natural to ask the following question.

Question :

1) If $T \in \Omega_n^\ell \cup \Omega_n^r$, with $n \geq 2$, can we find $m \in \mathbb{N}\backslash\{0\}$ such that $\mathsf{R}(T^m)$ is closed ?

2) If $T \in \Omega_n^\ell \cup \Omega_n^r$, with $n \geq 2$, can we find $m \in \mathbb{N}\backslash\{0\}$ such that $\mathsf{N}(T^m)$ is complemented ?

7. Pseudo-generalized invertibility

In this section, some special cases giving pseudo-generalized invertibility are discussed.
We start with the following result.

Proposition 7.1. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{X})$. If there exists $S \in \mathscr{B}(\mathsf{X})$, such that $I - ST \in \Omega_1^\ell$ and $U \in \mathscr{B}(\mathsf{X})$, with $UT^n \in {}^{I-ST}\mathsf{S}_1^\ell$, then $T \in \Omega_n^\ell$.*

Proof.
As $UT^n \in {}^{I-ST}\mathsf{S}_1^\ell$, then we have

$$I - ST = (I - ST)UT^n(I - ST). \tag{$*$}$$

Hence, by multiplying the equation $(*)$ from the left-hand side by T^n, we obtain

$$T^n - T^nST = T^n(I - ST)UT^n(I - ST).$$

Consequently,

$$T^n - T^nST = (T^n - T^nST)U(T^n - T^nST).$$

This proves that $T^n - T^nST \in \Omega_1^\ell$ and by Lemma 3.1, we see that $T \in \Omega_n^\ell$. □

A similar argument can be used to show the next proposition.

Proposition 7.2. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{X})$. If there exists $S \in \mathscr{B}(\mathsf{X})$, such that $I - TS \in \Omega_1^r$ and $U \in \mathscr{B}(\mathsf{X})$, with $T^nU \in {}^{I-TS}\mathsf{S}_1^r$, then $T \in \Omega_n^r$.*

Lemma 7.3. *Let $n, k \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{X})$.*

1) *If there exists $S \in \mathscr{B}(\mathsf{X})$, such that $T^k - T^kST \in \Omega_n^\ell$ and $T^kS = ST^k$, then $T \in \Omega_{nk}^\ell$.*

2) *If there exists $S \in \mathscr{B}(\mathsf{X})$, such that $T^k - TST^k \in \Omega_n^r$ and $T^kS = ST^k$, then $T \in \Omega_{nk}^r$.*

Proof.
1) Let us first prove by induction that, for all $n \in \mathbb{N}\backslash\{0\}$, there exists $U_n \in \mathscr{B}(\mathsf{X})$ such that

$$(T^k - T^kST)^n = T^{nk} + T^{nk}U_nT.$$

• For $n = 1$. We can take $U_1 = -S$.
• Pour $n = 2$. We see that

$$\begin{array}{rcl}(T^k - T^kST)^2 &=& T^{2k} - T^{2k}ST - T^kST^{k+1} + T^kST^{k+1}ST \\ &=& T^{2k} - T^{2k}ST - T^{2k}ST + T^{2k}(ST)^2 \\ &=& T^{2k} - T^{2k}(2S - STS)T.\end{array}$$

Hence the first point is established.
• Now, assume that for $n \geq 2$, there exists $U_n \in \mathscr{B}(\mathsf{X})$, such that

$$(T^k - T^kST)^n = T^{nk} + T^{nk}U_nT.$$

Our goal is to prove that there exists $U_{n+1} \in \mathscr{B}(\mathsf{X})$, such that

$$(T^k - T^kST)^{n+1} = T^{(n+1)k} + T^{(n+1)k}U_{n+1}T.$$

First, we see that

$$\begin{array}{rcl}(T^k - T^kST)^{n+1} &=& (T^k - T^kST)(T^k - T^kST)^n \\ &=& (T^k - T^kST)(T^{nk} + T^{nk}U_nT) \\ &=& T^{(n+1)k} + T^{(n+1)k}U_nT - T^kST^{nk+1} - T^kST^{nk+1}U_nT.\end{array}$$

Now, as $T^{nk}S = ST^{nk}$, then

$$\begin{array}{rcl}(T^k - T^kST)^{n+1} &=& T^{(n+1)k} + T^{(n+1)k}U_nT - T^{(n+1)k}ST - T^{(n+1)k}STU_nT \\ &=& T^{(n+1)k} + T^{(n+1)k}(U_n - S - STU_n)T,\end{array}$$

and by setting $U_{n+1} = U_n - S - STU_n$ we obtain the desired result.
Now, let $U \in \mathscr{B}(\mathsf{X})$, such that

$$(T^k - T^kST)^n = (T^k - T^kST)^nU(T^k - T^kST).$$

From the above, it can be seen that there exists $U_n \in \mathscr{B}(\mathsf{X})$, which satisfies

$$(T^k - T^kST)^n = T^{nk} + T^{nk}U_nT.$$

As a result, we get

$$\begin{array}{rcl}T^{nk} &=& -T^{nk}U_nT + (T^{nk} + T^{nk}U_nT)U(T^k - T^kST) \\ &=& T^{nk}\Big(- U_n + (I + U_nT)U(T^{k-1} - T^kS)\Big)T.\end{array}$$

Consequently, $T \in \Omega_{nk}^\ell$.
2) This assertion can be proved in the same way as 1). □

Proposition 7.4. *Let $n, k \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{X})$. If there exist S, $U \in \mathscr{B}(\mathsf{X})$, such that $I - ST \in \Omega_n^\ell$, $T^kS = ST^k$ and $UT^k \in {}^{I-ST}\mathsf{S}_n^\ell$, then $T \in \Omega_{nk}^\ell$.*

Proof.
Since $T^kS = ST^k$, then it is clear that

$$\begin{array}{rcl}(T^k - T^kST)^n & = & T^{nk}(I - ST)^n \\ & = & T^{nk}(I - ST)^n UT^k(I - ST) \\ & = & (T^k - T^kST)^n U(T^k - T^kST).\end{array}$$

Therefore, by Lemma 7.3, we deduce the result. □

As in Proposition 7.4, we can obtain :

Proposition 7.5. *Let $n, k \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{X})$. If there exists S, $U \in \mathscr{B}(\mathsf{X})$, such that $I - TS \in \Omega_n^r$, $T^kS = ST^k$ and $T^kU \in {}^{I-ST}\mathsf{S}_n^r$, then $T \in \Omega_{nk}^r$.*

Motivated by the last results, it is natural to ask the following question.

Question :

1) Let $n \in \mathbb{N}\backslash\{0\}$, $T \in \mathscr{B}(\mathsf{X})$ and $k \in \{1, \cdots, n-1\}$. Suppose there exists S, $U \in \mathscr{B}(\mathsf{X})$, such that $I - ST \in \Omega_k^\ell$ and $UT^n \in {}^{I-ST}\mathsf{S}_k^\ell$. Can we prove that $T \in \Omega_n^\ell$?

2) Let $n \in \mathbb{N}\backslash\{0\}$, $T \in \mathscr{B}(\mathsf{X})$ and $k \in \{1, \cdots, n-1\}$. Suppose there exist S, $U \in \mathscr{B}(\mathsf{X})$, such that $I - TS \in \Omega_k^r$ and $T^nU \in {}^{I-TS}\mathsf{S}_k^r$. Can we prove that $T \in \Omega_n^r$?

8. Some applications to the g-invertibility

In this last section, we prove some cases in which the pseudo-generalized invertibility allows us to obtain the g-invertibility. In the following, for $T \in \mathscr{B}(\mathsf{X})$ and $\mathsf{M} \subseteq \mathsf{X}$, we denote by $T_{|\mathsf{M}}$ the restriction of T from M onto M.

Let us start with the following result.

Proposition 8.1. *Let $T \in \mathscr{B}(\mathsf{X})$, such that $\mathsf{N}(T)$ be a complemented subspace of X and let X_1 be a topological complement of $\mathsf{N}(T)$. If $T(\mathsf{X}_1) \subseteq \mathsf{X}_1$ and there exists $n \in \mathbb{N}$, such that $T_1 = T_{|\mathsf{X}_1}$ is n-left pseudo-generalized invertible, then $T \in \Omega_1^\ell$.*

Proof.
It is clear that T_1 is one-to-one. Now, since T_1 is n-left pseudo-generalized invertible operator, then T_1 is left-invertible and so $\mathsf{R}(T_1) = \mathsf{R}(T)$ is complemented in X_1. Therefore $\mathsf{R}(T)$ is complemented in X and $T \in \Omega_1^\ell$. □

Proposition 8.2. *Let $T \in \mathscr{B}(\mathsf{X})$, such that $\mathsf{R}(T)$ is complemented in X and let X_2 be a topological complement of $\mathsf{R}(T)$. If $\mathsf{X}_2 \subseteq \mathsf{N}(T)$ and there exists $n \in \mathbb{N}$, such that $T_2 = T_{|\mathsf{R}(T)}$ is n-right pseudo-generalized invertible, then $T \in \Omega_1^r$.*

Proof.
First, as $\mathsf{X}_2 \subseteq \mathsf{N}(T)$, we have

$$\mathsf{R}(T) = T(\mathsf{X}) = T(\mathsf{R}(T) \dotplus \mathsf{X}_2) = T(\mathsf{R}(T)) = T_2(\mathsf{R}(T)),$$

and so, T_2 is onto. Now, since T_2 is n-right pseudo-generalized invertible operator then T_2 is g-invertible. Hence, there exists a subspace $\mathsf{X}_1 \subseteq \mathsf{X}$, such that

$$\mathsf{R}(T) = \mathsf{N}(T_2) \dotplus \mathsf{X}_1.$$

Now, it is clear that

$$\mathsf{N}(T_2) = \mathsf{N}(T) \cap \mathsf{R}(T)$$

and

$$\mathsf{N}(T) = (\mathsf{N}(T) \cap \mathsf{R}(T)) \dotplus (\mathsf{X}_2 \cap \mathsf{N}(T)).$$

As $\mathsf{X}_2 \subseteq \mathsf{N}(T)$, then

$$\mathsf{N}(T) = \mathsf{N}(T) \cap \mathsf{R}(T) \dotplus \mathsf{X}_2$$

and consequently

$$\mathsf{X} = \mathsf{R}(T) \dotplus \mathsf{X}_2 = \mathsf{N}(T_2) \dotplus \mathsf{X}_1 \dotplus \mathsf{X}_2 = (\mathsf{N}(T) \cap \mathsf{R}(T)) \dotplus \mathsf{X}_1 \dotplus \mathsf{X}_2 = \mathsf{N}(T) \dotplus \mathsf{X}_1.$$

Therefore $\mathsf{N}(T)$ is complemented in X and $T \in \Omega_1^r$. □

Proposition 8.3. *Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}$. If $T \in \Omega_n^\ell$ and there exists $S \in {}^T\mathsf{S}_n^\ell$ such that $\mathsf{R}(I - ST) \subseteq \mathsf{N}(I - T)$, then $T \in \Omega_0^\ell$ and $S \in {}^T\mathsf{S}_0^\ell$.*

Proof.
If $n = 0$, the result follows from Corollary 3.3.
Now, if $n \in \mathbb{N}\backslash\{0\}$. Suppose that there exists $S \in {}^T\mathsf{S}_n^\ell$, such that $\mathsf{R}(I - ST) \subseteq \mathsf{N}(I - T)$, then

$$(I - T)(I - ST) = 0.$$

Therefore,

$$T = I - ST + TST.$$

So,

$$\begin{aligned} T^n &= T^{n-1} - T^{n-1}ST + T^nST \\ &= T^{n-1} - T^{n-1}ST + T^n. \end{aligned}$$

This implies that,

$$T^{n-1} = T^{n-1}ST.$$

Consequently, $S \in {}^T\mathsf{S}_{n-1}^\ell$ and $T \in \Omega_{n-1}^\ell$. By repeating the same process $n - 1$ times, we obtain $T \in \Omega_0^\ell$ and $S \in {}^T\mathsf{S}_0^\ell$. □

Similarly, we can show the final proposition.

Proposition 8.4. *Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$. If $T \in \Omega_n^r$ and there exists $S \in {}^T\mathsf{S}_n^r$ such that $\mathsf{R}(I - T) \subseteq \mathsf{N}(I - TS)$, then $T \in \Omega_0^r$ and $S \in {}^T\mathsf{S}_0^r$.*

Acknowledgements
We want to thank the referee for reading this paper carefully, whose generous and valuable remarks brought improvements to the paper and enhance clarity.

References

[1] P. Aiena, Fredholm and Local Spectral Theory, With Applications to Multipliers, Springer, 2004.
[2] A. Ben-Israel, T. N. E. Greville, Generalized Inverses : Theory and Applications, Second edition, Springer-Verlag, New York, 103, 2003.
[3] F. J. Beutler, The operator Theory of the Pseudo-inverse. II Unbounded operators with arbitrary range, J. Math. Anal. Appl. 10 (1965) , 471–493.
[4] K. P. S. Bhaskara Rao, The Theory of Generalized Inverses Over Commutative Rings, Taylor & Francis, 2002.
[5] S. R. Caradus, Operator theory of the pseudo-inverse, Queen's Paper in Pure and Applied Mathematics, 38. Kingston, Ontario, Canada: Queen's University. II, 67p. 1974.
[6] S. R. Caradus, Generalized Inverse and Operator Theory, Queen's Paper in Pure and Applied Mathematics, no. 50, 1978.
[7] S. D. Cvetković Ilić, Y. Wei, Algebraic Properties of Generalized Inverses, Springer, Singapore, 2017.
[8] I. Fredholm, Sur une classe déquations fonctionnelles, Acta Math. 27 (1903) 365–390.
[9] S. Guedjiba, Sur les G_k-inverses d'opérateurs linéaires, Thesis, 2005.
[10] Z. Garbouj, H. Skhiri, Semi-generalized Partial Isometries, Results Math. 75, no. 1 (2020) Art. 15, 25 pp.
[11] S. Goldberg, Unbounded Linear operators, McGraw-Hill, New York, 1966.

[12] S. Grabiner, Ranges of products of operators, Canad. J. Math. 26 (1974) 1430–1441.
[13] S. Grabiner, J. Zemanek, Ascent, Descent and Ergodic Properties of Linear Operators, J. Operator Theory 48 (2002) 69–81.
[14] R. Harte, Invertibility and Singularity for Bounded Linear Operators 109, 1988.
[15] R. Harte, M. Mbekhta, In Generelized Inverses in $\mathbb{C}^*$-algebras, Studia Math. 103 (1992) 71–77.
[16] T. Kato, Perturbation theory for nullity, deficiency, and other quantities of linear operators, J. Anal. Math. 6 (1958) 261–322.
[17] T. Kato, Perturbation theory for linear operators, Springer-Verlag, Berlin, 1966.
[18] C. F. King, A note on Drazin inverses, Pacific J. Math. 70 (1977) 383–390.
[19] M. Mbekhta, V. Müller, On the axiomatic theory of spectrum II, Studia Math. 119 (1996) 129–147.
[20] E. H. Moore, On reciprocal of the general algebraic matrix, Bull. Amer. Math. Soc. 26 (1920) 394–395.
[21] V. Müller, Spectral Theory of Linear Operators and Spectral Systems on Banach Algebras, Birkhäuser Verlag, Berlin, 2003.
[22] M. Z. Nashed, X. Chen, Generalized Inverses and Applications, Academic Press, 1976.
[23] R. Penrose, A generalized inverse for matrices 51 (1955) 406–413.
[24] R. Penrose, The best approximation to the solution of matrix equations, Proc. Cambridge Phil. Soc., 52 (1956) 17–19.
[25] R. Piziak, P. L. Odell, Matrix Theory, From Generalized Inverses to Jordan Form, Chapman & HalI/CRC, 2007.
[26] I. Stanimirović, Computation of Generalized Matrix Inverses and Applications, Apple Academic Press, 2017.
[27] A. E. Taylor, Introduction to functional analysis, John Wiley & Sons Inc., New York, 1958.
[28] A. E. Taylor, Theorems on Ascent, Descent, Nullity and Defect of Linear Operators, Math. Ann. 163 (1966), 18–49.
[29] G. Wang, Y. Wei, S. Qiao, Generalized Inverses: Theory and Computations, Science Press, New York, 2018.
[30] H. Zerkraoui, Sur les propriétés algébriques des G_k-inverses des matrices, Thesis, 2011.