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Characterizations for an Invariant Submanifold of an Almost α-Cosymplectic (κ, μ, ν)-Space to be Totally Geodesic

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Abstract. The aim of the present paper is to study some geometric conditions for an invariant submanifold of an almost α-cosymplectic (κ, μ, ν)-space to be totally geodesic by means of the curvature tensors.

1. Introduction

An almost contact manifold is odd-dimensional manifold $\widetilde{M}^{2n+1}$ which carries a field ϕ of endomorphism of the tangent space, a vector field ξ, called characteristic, and a 1-form η-satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where I denote the identity mapping of tangent space of each point at M. From (1), it follows

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \;\; rank(\phi) = 2n. \tag{2}$$

An almost contact manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta)$ is said to be normal if the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denote the Nijenhuis tensor field of ϕ. It is well known that any almost contact manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta)$ has a Riemannian metric such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

for any vector fields X, Y on $\widetilde{M}$[2]. Such metric g is called compatible metric and manifold $\widetilde{M}^{2n+1}$ together with the structure (ϕ, η, ξ, g) is called an almost contact metric manifold and denoted by $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$. The 2-form Φ of $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$ is defined $\Phi(X, Y) = g(\phi X, Y)$ is called the fundamental form of $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$. If an almost contact metric manifold such that η and Φ are closed, that is, $d\eta = d\Phi = 0$, then it called cosymplectic manifold[12].

An almost α-cosymplectic manifold for any real number α which is defined as[15]

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi. \tag{4}$$

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A normal almost α-cosymplectic manifold is said to be α-cosymplectic manifold[8].

It is well known that on a contact metric manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$, the tensor h, defined by $2h = L_\xi \phi$, the following equalities satisfies;

$$\widetilde{\nabla}_X \xi = -\phi X - \phi h X, \quad h\phi + \phi h = 0, \quad trh = tr\phi h = 0, h\xi = 0, \tag{5}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}^{2n+1}$[11].

In [8], the authors studied the almost α-cosymplectic (κ, μ, ν)-spaces under different conditions and gave an example in dimension 3.

Going beyond generalized (κ, μ)-spaces, in [14], the notation of (κ, μ, ν)-contact metric manifold was introduced as follows;

$$\widetilde{R}(X, Y)\xi = \eta(Y)[\kappa I + \mu h + \nu \phi h]X - \eta(X)[\kappa I + \mu h + \nu \phi h]Y, \tag{6}$$

for some smooth functions κ, μ and ν on $\widetilde{M}^{2n+1}$, where $\widetilde{R}$ denotes the Riemannian curvature tensor of $\widetilde{M}^{2n+1}$ and X, Y are vector fields on $\widetilde{M}^{2n+1}$.

They proved that this type of manifold is intrinsically related to the harmonicity of the Reeb vector on contact metric 3-manifolds. Some authors have studied manifolds satisfying condition (6) but a non-contact metric structure. In this connection, P. Dacko and Z. Olszak defined an almost cosymplectic (κ, μ, ν)-spaces as an almost cosymplectic manifold that satisfies (6), but with κ, μ and ν functions varying exclusively in the direction of ξ in[12]. Later examples have been given for this type manifold[13].

Pseudoparallel submanifolds have been studied in different structures and working on[1, 9, 10, 18, 19]. In the present paper, we generalize the ambient space and research cases of existence or non-existence of totally geodesic submanifold in almost α-cosymplectic (κ, μ, ν)-space.

Proposition 1.1. *Given $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ an almost α-cosymplectic (κ, μ, ν)-space, then*

$$\begin{aligned}
h^2 &= (\kappa + \alpha^2)\phi^2, && (7)\\
\xi(\kappa) &= 2(\kappa + \alpha^2)(\nu - 2\alpha) && (8)\\
\widetilde{R}(\xi, X)Y &= \kappa[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX] && (9)\\
&+ \nu[g(\phi hX, Y)\xi - \eta(Y)\phi hX] && (10)\\
(\widetilde{\nabla}_X \phi)Y &= g(\alpha\phi X + hX, Y)\xi - \eta(Y)(\alpha\phi X + hX) && (11)\\
\widetilde{\nabla}_X \xi &= -\alpha\phi^2 X - \phi hX, && (12)
\end{aligned}$$

for all vector fields X, Y on $\widetilde{M}^{2n+1}$[2].

Now, let M be an immersed submanifold of an almost α-cosymplectic (κ, μ, ν)-space $\widetilde{M}^{2n+1}$. By $\Gamma(TM)$ and $\Gamma(T^\perp M)$, we denote the tangent and normal subspaces of M in $\widetilde{M}$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{13}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{14}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are the induced connections on M and $\Gamma(T^{\perp}M)$ and σ and A are called the second fundamental form and shape operator of M, respectively, $\Gamma(TM)$ denote the set differentiable vector fields on M. They are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V). \tag{15}$$

The first covariant derivative of the second fundamental form σ is defined by

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{16}$$

for all $X, Y, Z \in \Gamma(TM)$. If $\widetilde{\nabla}\sigma = 0$, then submanifold is said to be its second fundamental form is parallel.

By R, we denote the Riemannian curvature tensor of the submanifold M, we have the following Gauss equation

$$\widetilde{R}(X, Y)Z \;=\; R(X, Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X + (\widetilde{\nabla}_X \sigma)(Y, Z) - (\widetilde{\nabla}_Y \sigma)(X, Z), \tag{17}$$

for all $X, Y, Z \in \Gamma(TM)$.

$\widetilde{R} \cdot \sigma$ is given by

$$(\widetilde{R}(X, Y) \cdot \sigma)(U, V) \;=\; R^{\perp}(X, Y)\sigma(U, V) - \sigma(R(X, Y)U, V) - \sigma(U, R(X, Y)V), \tag{18}$$

where

$$R^{\perp}(X, Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}$$

On the hand, the concircular curvature tensor for Riemannian manifold (M^{2n+1}, g) is given by

$$C(X, Y)Z = \widetilde{R}(X, Y)Z - \frac{\tau}{2n(2n+1)}\{g(Y, Z)X - g(X, Z)Y\}, \tag{19}$$

where τ denote the scalar curvature of M.
Similarly, the tensor $C \cdot \sigma$ is defined by

$$(C(X, Y) \cdot \sigma)(U, V) \;=\; R^{\perp}(X, Y)\sigma(U, V) - \sigma(C(X, Y)U, V) - \sigma(U, C(X, Y)V), \tag{20}$$

for all $X, Y, U, V \in \Gamma(TM)$.

For a $(0, k)$-type tensor field T, $k \geq 1$ and a $(0, 2)$-type tensor field A on a Riemannian manifold (M, g), $Q(A, T)$-the Tachibana tensor field is defined by

$$\begin{aligned} Q(A, T)(X_1, X_2, ..., X_k; X, Y) \;&=\; -T((X \wedge_A Y)X_1, X_2, ..., X_k)... \\ &-\; T(X_1, X_2, ..., X_{k-1}, (X \wedge_A Y)X_k),, \end{aligned} \tag{21}$$

for all $X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$[9], where

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y. \tag{22}$$

Kowalczyk studied the semi-Riemannian manifolds satisfying $Q(S, R) = 0$ and $Q(S, g)$=0[7]. Also De and Majhi investigated the invariant submanifolds of Kenmotsu manifolds and showed that geometric conditions of invariant submanifolds of Kenmotsu manifolds are totally geodesic[16]. Recently, Hu and Wang obtained the geometric conditions of invariant submanifolds of a trans-Sasakain manifolds to be totally geodesic[5]. Furthermore, the geometry of invariant submanifolds of different manifolds was studied by many geometers[see references].

Motivated by the above studies, I make an attempt to study the invariant submanifolds of an almost α-cosymplectic (κ, μ, ν)-space satisfying some geometric conditions such that $Q(\sigma, R)$=0, $Q(S, \sigma) = 0$, $Q(S, \widetilde{\nabla}\sigma) = 0$, $(S, \widetilde{R} \cdot \sigma) = 0$, $Q(g, C \cdot \sigma) = 0$ and $Q(S, C \cdot \sigma) = 0$. In the end, we show that this conditions are equivalent to totally geodesic under the some conditions.

2. Invariant Submanifolds of an almost α-cosymplectic (κ, μ, ν)-Space

Now, let $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ be an almost α cosymplectic (κ, μ, ν)-space and M be an immersed submanifold of $\widetilde{M}^{2n+1}$. If $\phi(T_xM) \subseteq T_xM$, for each point at $x \in M$, then M is said to be an invariant submanifold of $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ with respect to ϕ. After we will easily to see that an invariant submanifold with respect to ϕ is also invariant with respect to h.

In the modern differential geometry, the geometry of submanifold has turned into a subject of growing interest for its significant applications in applied mathematics and theoretical physics. For instance, the notation of invariant submanifold is used to study the properties of non-linear autonomous systems. Also the notion of geodesic plays an important role in the theory of relativity. For totally geodesic submanifolds, the geodesic of the ambient manifolds remain geodesic in the submanifolds. Therefore totally geodesic submanifolds are very much important in mathematic as well as physical sciences. The study of the geometry of invariant submanifolds was introduced by Bejancu and Papaghuic[1]. In general, the geometry of an invariant submanifold inherits almost all properties of the ambient manifold.

Proposition 2.1. *Let M be an invariant submanifold of an almost α-cosymplectic (κ, μ, ν)-space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ such that ξ tangent to M. Then the following equalities hold on M;*

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] + \nu[\eta(Y)\phi hX - \eta(X)\phi hY] \quad (23)$$

$$(\nabla_X \phi)Y = g(\alpha\phi X + hX, Y)\xi - \eta(Y)(\alpha\phi X + hX) \quad (24)$$

$$\nabla_X \xi = -\alpha\phi^2 X - \phi hX \quad (25)$$

$$\phi\sigma(X,Y) = \sigma(\phi X, Y) = \sigma(X, \phi Y), \quad \sigma(X,\xi) = 0, \quad (26)$$

where ∇, σ and R denote the induced Levi-Civita connection on M, the shape operator and Riemannian curvature tensor of M, respectively.

Proof. We will not give the proof as it is a result of direct calculations. □

In the rest of this paper, we will assume that M is an invariant submanifold of an α-cosymplectic (κ, μ, ν)-space $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$. In this case, from (5), we have

$$\varphi hX = -h\varphi X, \quad (27)$$

for all $X \in \Gamma(TM)$, that is, M is also invariant with respect to the tensor field h.

We need the following lemma to quarante for the second fundamental form σ is not always identically zero.

Lemma 2.2. *Let M be an invariant submanifold of an almost α-cosymplectic (κ, μ, ν)-space $M^{2n+1}(\phi, \xi, \eta, g)$. Then the second fundamental form σ of M is parallel M is totally geodesic provided $\kappa \neq 0$.*

Proof. Let us suppose that σ is parallel. From (16), we have

$$(\widetilde{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = 0, \quad (28)$$

for all vector fields X, Y and Z on M^{2n+1}. Setting $Z = \xi$ in (28) and taking into account (25) and (26), we have

$$\sigma(\nabla_X \xi, Y) = \sigma(\alpha\phi^2 X + \phi hX, Y) = 0,$$

that is,

$$-\alpha\sigma(X,Y) + \phi\sigma(hX,Y) = 0. \quad (29)$$

Writing hX of X in (29) and by using (7) and (26), we obtain

$$-\alpha\sigma(hX,Y) + \phi\sigma(h^2X,Y) = \alpha\sigma(hX,Y) - (\alpha^2 + \kappa)\phi\sigma(X,Y) = 0. \quad (30)$$

From (29) and (30), we conclude that $\kappa\sigma(X,Y) = 0$, which proves our assertion. □

Theorem 2.3. *Let M be an invariant submanifold of an almost α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(S,\sigma)=0$ if and only if M is totally geodesic provided $\kappa\neq 0$.*

Proof.

$$Q(S,\sigma)(U,V;X,Y) \quad = \quad -\sigma((X\wedge_S Y)U,V)-\sigma(U,(X\wedge_S Y)V)=0, \tag{31}$$

for all $X,Y,U,V\in\Gamma(TM)$. Expanding the (31) and inserting $X=U=\xi$, by using (26), (31) implies that $\kappa\sigma(Y,V)=0$. This proves our assertion. The converse is obvious. □

Theorem 2.4. *Let M be an invariant submanifold of an almost α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(S,\widetilde{\nabla}\cdot\sigma)=0$ if and only if M is totally geodesic provided $\kappa\neq 0$.*

Proof.

$$-Q(S,\widetilde{\nabla}\cdot\sigma)(U,V,Z;X,Y) \quad = \quad (\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,Z)+(\widetilde{\nabla}_U\sigma)((X\wedge_S Y)V,Z)+(\widetilde{\nabla}_U\sigma)(V,(X\wedge_S Y)Z),$$

for all $X,Y,U,V,Z\in\Gamma(TM)$. In last equality, taking $Y=Z=\xi$, we have

$$(\widetilde{\nabla}_{(X\wedge_S\xi)U}\sigma)(V,\xi)+(\widetilde{\nabla}_U\sigma)((X\wedge_S\xi)V,\xi)+(\widetilde{\nabla}_U\sigma)(V,(X\wedge_S\xi)\xi)=0. \tag{32}$$

Thus by virtue of Proposition 2.1, we obtain

$$\begin{aligned}(\widetilde{\nabla}_{(X\wedge_S\xi)U}\sigma)(V,\xi) &= -\sigma(\nabla_{(X\wedge_S\xi)U}\xi,V)\\ &= \sigma(\alpha\phi^2(X\wedge_S\xi)U+\phi hX\wedge_S\xi)U,V)\\ &= -\alpha\sigma(S(U,\xi)X-S(U,X)\xi,V)+\sigma(\phi h[S(U,\xi)X-S(X,U)\xi],V)\\ &= 2n\kappa\eta(U)\Big[\sigma(\phi hX,V)-\alpha\sigma(X,V)\Big],\end{aligned} \tag{33}$$

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)((X\wedge_S\xi)V,\xi) &= -\sigma(\nabla_U\xi,(X\wedge_S\xi)V)\\ &= \sigma(\alpha\phi^2U+\phi hU,S(\xi,V)X-S(X,V)\xi)\\ &= -2n\kappa\alpha\eta(V)\Big\{\sigma(\phi hU,X)-\alpha\sigma(U,X)\Big\},\end{aligned} \tag{34}$$

and

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(V,(X\wedge_S\xi)\xi) &= (\widetilde{\nabla}_U\sigma)(V,S(\xi,\xi)X-S(X,\xi)\xi)\\ &= (\widetilde{\nabla}_U\sigma)(V,2n\kappa X-2n\kappa\eta(X)\xi)\\ &= 2n\Big\{(\widetilde{\nabla}_U\sigma)(V,\kappa X)-(\widetilde{\nabla}_U\sigma)(\kappa\eta(X)\xi,V)\Big\}\\ &= 2n\Big\{(\widetilde{\nabla}_U\sigma)(V,\kappa X)+\sigma(\nabla_U\kappa\eta(X)\xi,V)\Big\}\\ &= 2n\{(\widetilde{\nabla}_U\sigma)(V,\kappa X)+\sigma(U[\kappa\eta(X)]\xi+\kappa\eta(X)\nabla_U\xi,V)\}\\ &= 2n\Big\{(\widetilde{\nabla}_U\sigma)(V,\kappa X)-\kappa\eta(X)\sigma(\alpha\phi^2U+\phi hU,V)\Big\}\\ &= 2n\{(\widetilde{\nabla}_U\sigma)(V,\kappa X)+\alpha\kappa\eta(X)\sigma(U,V)-\kappa\eta(X)\sigma(\phi hU,V)\}.\end{aligned} \tag{35}$$

Substituting (33), (34) and (35) in (32), we arrive at

$$\begin{aligned}2n\kappa\eta(U)[\sigma(\phi hU,V) &- \alpha\sigma(X,V)]+2n\kappa\eta(V)[\sigma(\phi hU,X)-\alpha\sigma(X,U)]+\alpha\kappa\eta(X)\sigma(U,V)\\ &- \kappa\eta(X)\sigma(\phi hU,V)+2n\kappa(\widetilde{\nabla}_U\sigma)(X,V)=0.\end{aligned} \tag{36}$$

In (36), putting $V=\xi$ and taking into account of Proposition 2.1, we conclude provided $\kappa\neq 0$

$$(\widetilde{\nabla}_U\sigma)(X,\xi)+\sigma(\phi hU,X)-\alpha\sigma(U,X)=0. \tag{37}$$

Here,

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(X,\xi) &= -\sigma(\nabla_U\xi, X) = \sigma(\alpha\phi^2U + \phi hU, X)\\ &= -\alpha\sigma(U,X) + \sigma(\phi hU, X).\end{aligned} \tag{38}$$

In view of (37) and (38), we have

$$\phi\sigma(hU,X) - \alpha\sigma(U,X) = 0. \tag{39}$$

Writing hU instead of U in (39) and taking account of (7) and (26), we have

$$\phi\sigma(h^2U,X) - \alpha\sigma(hU,X) = -(\kappa+\alpha^2)\phi\sigma(U,X) - \alpha\sigma(hU,X) = 0. \tag{40}$$

From (39) and (40), we conclude that $\kappa\sigma(U,X) = 0$. The converse is obvious. The proof is completed. □

Theorem 2.5. *Let M be an invariant submanifold of an almost α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(S,\widetilde{R}\cdot\sigma) = 0$ if and only if M is either totally geodesic or $(\kappa+\alpha^2)(\mu^2-\nu^2)+\kappa^2 = \mu\nu = 0$.*

Proof.

$$\begin{aligned}-Q(S,\widetilde{R}(X,Y)\cdot\sigma)(U,V,W,Z) &= (\widetilde{R}(X,Y)\cdot\sigma)((W\wedge_S Z)U, V)\\ &+ (\widetilde{R}(X,Y)\cdot\sigma)(U,(W\wedge_S Z)V) = 0,\end{aligned}$$

for all $X,Y,U,V,Z,W \in \Gamma(TM)$. For $Y = U = V = Z = \xi$, it follows that

$$(\widetilde{R}(X,\xi)\cdot\sigma)((W\wedge_S\xi)\xi,\xi) = 0,$$

that is,

$$\begin{aligned}(\widetilde{R}(X,\xi)\cdot\sigma)(S(\xi,\xi)W - S(W,\xi)\xi) &= 2n\kappa(\widetilde{R}(X,\xi)\cdot\sigma)(W-\eta(W)\xi,\xi)\\ &= 2n\kappa\left\{(\widetilde{R}(X,\xi)\cdot\sigma)(W,\xi) - (\widetilde{R}(X,\xi)\cdot\sigma)(\eta(W)\xi,\xi)\right\}\\ &= 2n\kappa\{R^\perp(X,\xi)\sigma(W,\xi) - \sigma(R(X,\xi)W,\xi) - \sigma(W,R(X,\xi)\xi)\\ &- R^\perp(X,\xi)\sigma(\eta(W)\xi,\xi) + \sigma(\eta(W)R(X,\xi)\xi,\xi) + \sigma(\eta(W)\xi,R(X,\xi)\xi)\}\\ &= -2n\kappa\sigma(W,R(X,\xi)\xi) = 0.\end{aligned}$$

Thus we have

$$2n\kappa\sigma(W,\kappa[X-\eta(X)\xi] + \mu hX + \phi hX) = 2n\kappa\{\kappa\sigma(W,X) + \mu\sigma(W,hX) + \nu\phi\sigma(W,hX)\} = 0. \tag{41}$$

Here, replacing hX instead of X in (41) and by virtue of (7) and (26), we reach at

$$\kappa\sigma(W,hX) - \mu(\kappa+\alpha^2)\sigma(W,X) - \nu(\kappa+\alpha^2)\phi\sigma(W,X) = 0. \tag{42}$$

From (41) and (42), we conclude that

$$[(\kappa+\alpha^2)(\mu^2-\nu^2)+\kappa^2]\sigma(X,W) + 2\mu\nu\phi\sigma(X,W) = 0. \tag{43}$$

This proves our assertion because σ and $\phi\sigma$ are orthogonal vector fields. The converse is trivial. □

Theorem 2.6. *Let M be an invariant submanifold of an almost α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(g,C\cdot\sigma) = 0$ if and only if M is either totally geodesic or the scalar curvature τ of M^{2n+1} satisfies $\tau = 2n(2n+1)[\kappa \mp \sqrt{(\kappa+\alpha^2)(\nu^2-\mu^2)}]$, $\mu\nu = 0$.*

Proof. Let us assume $Q(g, C\cdot\sigma) = 0$. This means that

$$(C(X,Y)\cdot\sigma)((Z\wedge_g W)U,V) + (C(X,Y)\cdot\sigma)(U,(Z\wedge_g W)V) = 0,$$

for all $X, Y, U, V, Z, W \in \Gamma(TM)$. This implies that

$$(C(X,Y)\cdot\sigma)(g(U,W)Z - g(Z,U)W, V) + (C(X,Y)\cdot\sigma)(U, g(V,W)Z - g(Z,V)W) = 0. \tag{44}$$

By means of (20), (19), (6), expanding (44) and inserting $Y = U = Z = V = \xi$ in (44), after the necessary revisions, we arrive at

$$\begin{aligned}(C(X,\xi)\cdot\sigma)(\eta(W)\xi - W,\xi) &= (C(X,\xi)\cdot\sigma)(\eta(W)\xi,\xi) - (C(X,\xi)\cdot\sigma)(W,\xi)\\ &= R^{\perp}(X,\xi)\sigma(\eta(W)\xi,\xi) - \sigma(\eta(W)C(X,\xi)\xi,\xi)\\ &- \sigma(\eta(W)\xi, C(X,\xi)\xi) - R^{\perp}(X,\xi)\sigma(\xi,W) + \sigma(C(X,\xi)W,\xi)\\ &+ \sigma(W, C(X,\xi)\xi) = 0.\end{aligned} \tag{45}$$

In view of (6) and (17), non-zero components of (45) vectors give us

$$\sigma(C(X,\xi)\xi, W) = \left(\kappa - \frac{\tau}{2n(2n+1)}\right)\sigma(X,W) + \mu\sigma(hX,W) + \nu\phi\sigma(hX,W) = 0. \tag{46}$$

Taking hX instead of X in (46) and by virtue of Proposition 1.1 and Proposition 2.1, we have

$$\left(\kappa - \frac{\tau}{2n(2n+1)}\right)\sigma(X,W) - \mu(\kappa+\alpha^2)\sigma(hX,W) - \nu(\kappa+\alpha^2)\phi\sigma(hX,W) = 0. \tag{47}$$

From (46) and (47), we conclude that

$$\left[\left(\kappa - \frac{\tau}{2n(2n+1)}\right)^2 - (\kappa+\alpha^2)(\nu^2-\mu^2)\right]\sigma(X,W) + 2\mu\nu\phi\sigma(X,W) = 0,$$

which completes of the proof. □

Theorem 2.7. *Let M be an invariant submanifold of an almost α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(S, C\cdot\sigma) = 0$ if and only if M is either totally geodesic or the scalar curvature τ of M^{2n+1} satisfies* $\tau = 2n(2n+1)[\kappa \mp \sqrt{(\kappa+\alpha^2)(\nu^2-\mu^2)}]$, $\mu\nu = 0$ *provided* $\kappa \neq 0$.

Proof. Assuming that $Q(S, C\cdot\sigma) = 0$. It follows that

$$Q(S, C(X,Y)\cdot\sigma)(U,V;Z,W) = 0,$$

for all $X, Y, U, V, Z, W \in \Gamma(TM)$. By virtue of (20) and (21), we have

$$\begin{aligned}S(Z,U)(C(X,Y)\cdot\sigma)(W,V) &- S(W,U)(C(X,Y)\cdot\sigma)(Z,V)\\ S(Z,V)(C(X,Y)\cdot\sigma)(U,W) &- S(W,V)(C(X,Y)\cdot\sigma)(U,Z).\end{aligned} \tag{48}$$

Expanding (48) and putting $Y = U = V = Z = \xi$ in (48), non-zero components is

$$2n\kappa\sigma(W, C(X,\xi)\xi).$$

By means of (46), we obtain provided $\kappa \neq 0$

$$\left(\kappa - \frac{\tau}{2n(2n+1)}\right)\sigma(X,W) + \mu\sigma(hX,W) + \nu\phi\sigma(hX,W) = 0.$$

Similar to (46) and (47), we get desired result. □

Next, we will give a non-trivial example to verify the obtained results of my paper.

Example 2.8. *Let* $M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5, x_5 \neq \mp 1, 0\}$ *and we take*

$$\begin{aligned} e_1 &= (x_5+1)\frac{\partial}{\partial x_1}, \; e_2 = \frac{1}{x_5-1}\frac{\partial}{\partial x_2}, \; e_3 = \frac{1}{2}(x_5+1)^2\frac{\partial}{\partial x_3}, \\ e_4 &= \frac{5}{x_5-1}\frac{\partial}{\partial x_4}, \; e_5 = \xi = (x_5^2-1)\frac{\partial}{\partial x_5} \end{aligned}$$

are linearly independent vector fields on M. We also definite $(1,1)$*-type tensor field* ϕ *by* $\phi e_1 = e_2$, $\phi e_2 = -e_1$, $\phi e_3 = e_4$, $\phi e_4 = -e_3$ *and* $\phi e_5 = 0$. *Furthermore, the Riemannian metric tensor g is given by*

$$g(e_i, e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

By direct computations, we can easily to see that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X, \xi)$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus $M^5(\phi, \xi, \eta, g)$ *is a 5-dimensional almost contact metric manifold. From the Lie-operatory, we have the non-zero components*

$$[e_1, e_5] = -(x_5-1)e_1, \quad [e_2, e_5] = (x_5+1)e_2, \quad [e_3, e_5] = -(x_5-1)e_3$$

and

$$[e_4, e_5] = (x_5+1)e_4.$$

Furthermore, By ∇, *we denote the Levi-Civita connection on M, by using Kozsul formulae, we can reach at the non-zero components*

$$\nabla_{e_1} e_5 = -(x_5-1)e_1, \quad \nabla_{e_2} e_5 = (x_5+1)e_2, \quad \nabla_{e_3} e_5 = -(x_5-1)e_3, \quad \nabla_{e_4} e_5 = (x_5+1)e_4.$$

Comparing the above relations with

$$\nabla_X e_5 = X - \eta(X)e_5 - \phi h X, \tag{49}$$

we can observe

$$he_1 = -x_5 e_2, he_2 = -x_5 e_1, he_3 = -x_5 e_4, \quad he_4 = -x_5 e_3, \quad \text{and} \quad he_5 = 0.$$

By direct calculations, we get

$$\begin{aligned} R(e_1, e_5)e_5 &= \kappa e_1 + \mu h e_1 + \nu \phi h e_1 = 2(x_5-1)e_1, \\ R(e_2, e_5)e_5 &= \kappa e_2 + \mu h e_2 + \nu \phi h e_2 = -2x_5(x_5+1)e_2 \\ R(e_3, e_5)e_5 &= \kappa e_3 + \mu h e_3 + \nu \phi h e_3 = 2(x_5-1)e_3 \end{aligned}$$

and

$$R(e_4, e_5)e_5 = \kappa e_4 + \mu h e_4 + \nu \phi h e_4 = -2x_5(x_5+1)e_4,$$

which imply that

$$\kappa = -(x_5^2+1), \quad \mu = 0 \;\; \text{and} \;\; \nu = 2 - \frac{1}{x_5} + x_5.$$

Thus contact metric manifold $M^5(\phi, \xi, \eta, g)$ *is an almost* 1*-cosymplectic* $(-(x_5^2+1), 0, 2 - \frac{1}{x_5} + x_5)$*-space.*

Now, we define the submanifold $\widetilde{M}^3$ whose the tangent space $\Gamma(T\bar{M}^3)$ spanned vector fields by

$$E_1 = (x_5+1)\frac{\partial}{\partial x_1} + \frac{1}{2}(x_5+1)^2\frac{\partial}{\partial x_3}, \quad E_2 = \frac{1}{x_5-1}\frac{\partial}{\partial x_2} + \frac{5}{x_5-1}\frac{\partial}{\partial x_5},$$
$$E_3 = \xi = (x_5^2-1)\frac{\partial}{\partial x_5}.$$

One can easily to see that $\phi E_1 = E_2$, $\phi E_2 = -E_1$ and $\phi E_3 = 0$. This tells us $\widetilde{M}^3$ is a 3-dimensional invariant submanifold of an almost 1-cosymplectic $(-(x_5^2+1), 0, 2-\frac{1}{x_5}+x_5)$-space.

By direct calculations, we may calculate

$$\nabla_{E_1}E_2 = \nabla_{E_2}E_1 = 0, \quad \nabla_{E_1}E_3 = -(x_5-1)E_1, \quad \nabla_{E_2}E_3 = (x_5+1)E_2.$$

Consequently, we can say that $\widetilde{M}^3$ is totally geodesic submanifold and $\kappa = -(x_5^2+1) \neq 0$, $(\kappa+1)(2-\frac{1}{x_5}+x_5)^2+\kappa^2 \neq 0$. This shows that the space we are working on is not empty.

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