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New Identities on Some Generalized Integral Transforms and their Applications

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Abstract. In this paper the authors gave an iteration identity for the generalized Laplace transform $\mathcal{L}_{2n}$ and the generalized Glasser transform $\mathcal{G}_{2n}$. Using this identity a Parseval-Goldstein type theorem for the $\mathcal{L}_{2n}$-transform and the $\mathcal{G}_{2n}$-transform is given. By making use of these results a number of new Parseval-Goldstein type identities are obtained for these and many other well-known integral transforms.The identities proven in this paper are shown to give rise to useful corollaries for evaluating infinite integrals of special functions. Some examples are also given.

1. Introduction

The Laplace transform, the Widder potential transform and the Glasser-transform are defined by

$$\mathcal{L}\{f(x);y\} = \int_0^\infty \exp(-xy)f(x)dx, \tag{1}$$

$$\mathcal{P}\{f(x);y\} = \int_0^\infty \frac{xf(x)}{x^2+y^2}dx, \tag{2}$$

$$\mathcal{G}\{f(x);y\} = \int_0^\infty \frac{f(x)}{\sqrt{x^2+y^2}}dx. \tag{3}$$

Yurekli [18, 19] defined the $\mathcal{L}_2$-transform

$$\mathcal{L}_2\{f(x);y\} = \int_0^\infty x\exp(-x^2y^2)f(x)dx \tag{4}$$

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as a generalization of the Laplace transform.
Dernek and Aylikci [5] introduced the $\mathcal{L}_n$ and $\mathcal{L}_{2n}$ transforms as follows:

$$\mathcal{L}_n\{f(x); y\} = \int_0^\infty x^{n-1} \exp(-x^n y^n) f(x) dx, \tag{5}$$

$$\mathcal{L}_{2n}\{f(x); y\} = \int_0^\infty x^{2n-1} \exp(-x^{2n} y^{2n}) f(x) dx. \tag{6}$$

The $\mathcal{L}_{2n}$-transform is related to the $\mathcal{L}_2$-transform and the Laplace transform by means of the following relations:

$$\mathcal{L}_{2n}\{f(x); y\} = \frac{1}{n}\mathcal{L}_2\{f(x^{1/n}); y^n\} = \frac{1}{2n}\mathcal{L}\{f(x^{1/2n}); y^{2n}\}. \tag{7}$$

Dernek and Aylikci defined in [5, 6] the $\mathcal{P}_{v,2n}$-transform, the $\mathcal{P}_{2n}$-transform and the $\mathcal{G}_{2n}$-transform as generalizations of the Widder Potential transform and the Glasser transform (see [7, 13, 16]) respectively as follows:

$$\mathcal{P}_{v,2n}\{f(x); y\} = \int_0^\infty \frac{x^{2n-1} f(x)}{(x^{2n} + y^{2n})^v} dx, \tag{8}$$

$$\mathcal{P}_{2n}\{f(x); y\} = \int_0^\infty \frac{x^{2n-1} f(x)}{x^{2n} + y^{2n}} dx \tag{9}$$

and

$$\mathcal{G}_{2n}\{f(x); y\} = \int_0^\infty \frac{f(x)}{\sqrt{x^{2n} + y^{2n}}} dx. \tag{10}$$

The $\mathcal{G}_{2n}$-transform and the $\mathcal{P}_{v,2n}$-transform are related when $v = \frac{1}{2}$ by means of the following identity:

$$\mathcal{G}_{2n}\{f(x); y\} = \mathcal{P}_{1/2,2n}\{\frac{f(x)}{x^{2n-1}}; y\}. \tag{11}$$

The $\mathcal{F}_{s,n}$-transform and the $\mathcal{F}_{c,n}$-transform are introduced in [6, 8] as generalizations Fourier sine and Fourier cosine transforms,

$$\mathcal{F}_{s,n}\{f(x); y\} = \int_0^\infty x^{n-1} \sin(x^n y^n) f(x) dx, \tag{12}$$

$$\mathcal{F}_{c,n}\{f(x); y\} = \int_0^\infty x^{n-1} \cos(x^n y^n) f(x) dx, \tag{13}$$

where Fourier sine transform and Fourier cosine transform are defined as

$$\mathcal{F}_s\{f(x); y\} = \int_0^\infty \sin(xy) f(x) dx, \tag{14}$$

$$\mathcal{F}_c\{f(x);y\} = \int_0^\infty \cos(xy)f(x)dx. \tag{15}$$

The $\mathcal{K}_{v,n}$-transform and the $H_{v,n}$-transform are defined in [6] as generalizations of the $\mathcal{K}$-transform and the Hankel transform as follows:

$$\mathcal{K}_{v,n}\{f(x);y\} = \int_0^\infty x^{n-1}(x^n y^n)^{1/2} K_v(x^n y^n) f(x)dx, \tag{16}$$

$$\mathcal{H}_{v,n}\{f(x);y\} = \int_0^\infty x^{n-1}(x^n y^n)^{1/2} J_v(x^n y^n) f(x)dx, \tag{17}$$

where K_v is the Bessel function of the second kind of order v and J_v is the Bessel function of the first kind of order v. K_v is also known as Macdonald function.
The $\mathcal{K}_{v,n}$-transform is related to the $\mathcal{K}_v$-transform

$$n\mathcal{K}_{v,n}\{f(x);y\} = \mathcal{K}_v\{f(x^{1/n});y^n\}. \tag{18}$$

The generalized Hankel transform is related to the Hankel transform with

$$n\mathcal{H}_{v,n}\{f(x);y\} = \mathcal{H}_v\{f(x^{1/n});y^n\}. \tag{19}$$

We introduce the $\mathcal{E}_{2n,1}$-transform as a generalization of the $\mathcal{E}_{2,1}$-transform [3, 4] as follows

$$\mathcal{E}_{2n,1}\{f(x);y\} = \int_0^\infty x^{2n-1}\exp(x^{2n}y^{2n})E_1(x^{2n}y^{2n})f(x)dx, \tag{20}$$

where $E_1(x)$ is the exponential integral function defined by

$$E_1(x) = -E_i(-x) = \int_x^\infty \frac{\exp(-u)}{u}du = \int_1^\infty \frac{\exp(-xt)}{t}dt. \tag{21}$$

Setting $v = \frac{1}{2}$, $v = -\frac{1}{2}$ in the definitions (16) and (17), we have

$$\mathcal{H}_{1/2,n}\{f(x);y\} = \sqrt{\frac{2}{\pi}}\mathcal{F}_{s,n}\{f(x);y\}, \tag{22}$$

$$\mathcal{H}_{-1/2,n}\{f(x);y\} = \sqrt{\frac{2}{\pi}}\mathcal{F}_{c,n}\{f(x);y\}, \tag{23}$$

$$\mathcal{K}_{1/2,n}\{f(x);y\} = \sqrt{\frac{\pi}{2}}\mathcal{L}_n\{f(x);y\}, \tag{24}$$

where

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x), \tag{25}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos(x) \tag{26}$$

and

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\exp(-x). \tag{27}$$

The beta function is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, x>0, y>0 \tag{28}$$

and it is related to the gamma function through

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y,x). \tag{29}$$

2. The Main Theorem

Lemma 2.1. *The following identity holds true,*

$$2n\mathcal{L}_{2n}\{u^{-n}\mathcal{L}_{2n}\{f(x);u\};y\} = \sqrt{\pi}\mathcal{G}_{2n}\{x^{2n-1}f(x);y\}, \tag{30}$$

provided that the integrals involved converge absolutely.
Proof. Using definition (6) of the $\mathcal{L}_{2n}$-transform, we get

$$\mathcal{L}_{2n}\{u^{-n}\mathcal{L}_{2n}\{f(x);u\},y\}$$
$$= \int_0^\infty u^{n-1}\exp(-u^{2n}y^{2n})\Big[\int_0^\infty x^{2n-1}\exp(-x^{2n}u^{2n})f(x)dx\Big]du. \tag{31}$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, then changing the variable of the integration from u to t where $u^n(x^{2n}+y^{2n})^{1/2} = t$ and using definition (10) of the $\mathcal{G}_{2n}$-transform, we obtain

$$\mathcal{L}_{2n}\{u^{-n}\mathcal{L}_{2n}\{f(x);u\};y\} = \frac{1}{n}\int_0^\infty \frac{x^{2n-1}f(x)}{\sqrt{x^{2n}+y^{2n}}}\Big[\int_0^\infty \exp(-t^2)dt\Big]dx$$
$$= \frac{\sqrt{\pi}}{2n}\mathcal{G}_{2n}\{x^{2n-1}f(x);y\}. \tag{32}$$

Corollary 2.2. *We have for* $0 < Re(\mu) < 1$,

$$n\mathcal{G}_{2n}\{x^{n\mu-1};y\} = 2^{-\mu}y^{n(\mu-1)}B\Big(\mu, \frac{1}{2}-\frac{\mu}{2}\Big), \tag{33}$$

where $B(x,y)$ *is the beta function (28).*
Proof. Putting

$$f(x) = x^{n(\mu-2)}, \ \ 0 < Re(\mu) < 1 \tag{34}$$

in Lemma 2.1 and using relation (7) and the known formula [9, p.137, Entry (2)] twice, we find that

$$\mathcal{G}_{2n}\{x^{n\mu-1};y\} = \frac{2n}{\sqrt{\pi}}\mathcal{L}_{2n}\{u^{-n}\mathcal{L}_{2n}\{x^{n(\mu-2)};u\};y\}, \tag{35}$$

$$\mathcal{L}_{2n}\{x^{n(\mu-2)};u\} = \frac{\Gamma(\mu/2)}{2nu^{n\mu}} \tag{36}$$

and

$$\mathcal{L}_{2n}\{u^{-n(\mu+1)};y\} = \frac{1}{2n}\frac{\Gamma(\frac{1}{2}-\frac{\mu}{2})}{y^{-n(\mu-1)}}. \tag{37}$$

Substituting (36) and (37) into (35) and utilizing the following well known formulas for the gamma function [2, p.414, Eq(43)],

$$\Gamma\Big(\frac{\mu}{2}\Big)\Gamma\Big(\frac{\mu}{2}+\frac{1}{2}\Big) = \sqrt{\pi}2^{1-\mu}\Gamma(\mu) \tag{38}$$

and for the Beta function (29),

$$B\Big(\mu,\frac{1}{2}-\frac{\mu}{2}\Big) = \frac{\Gamma(\mu)\Gamma(\frac{1}{2}-\frac{\mu}{2})}{\Gamma(\frac{1}{2}+\frac{\mu}{2})}, \tag{39}$$

we obtain the assertion (33).

Corollary 2.3. *We have for* $-1 < Re(v) < \frac{1}{2}$,

$$n\sqrt{\frac{\pi}{2}}\mathcal{G}_{2n}\{x^{n(v+2)-1}J_v(z^nx^n);y\} = z^{-n/2}y^{n(v+\frac{1}{2})}K_{v+\frac{1}{2}}(z^ny^n). \tag{40}$$

Proof. Setting

$$f(x) = x^{nv}J_v(z^nx^n) \tag{41}$$

in Lemma 2.1, then we get

$$\mathcal{G}_{2n}\{x^{n(v+2)-1}J_v(z^nx^n);y\} = \frac{2n}{\sqrt{\pi}}\mathcal{L}_{2n}\{u^{-n}\mathcal{L}_{2n}\{x^{nv}J_v(z^nx^n);u\};y\}. \tag{42}$$

Using relation (7) and the known formulas [9, p.185, Entry(30); p.146, Entry(29)], we have

$$\mathcal{L}_{2n}\{x^{nv}J_v(z^nx^n);u\} = \frac{1}{2n}\Big(\frac{z^n}{2}\Big)^v u^{-2n(v+1)}\exp\Big(-\frac{z^{2n}}{4u^{2n}}\Big) \tag{43}$$

and

$$\mathcal{L}_{2n}\{u^{-n(2v+3)}\exp\Big(-\frac{z^{2n}}{4u^{2n}}\Big);y\} = \frac{1}{2n}\frac{2^{v+\frac{3}{2}}y^{nv+\frac{n}{2}}}{z^{nv+\frac{n}{2}}}K_{v+\frac{1}{2}}(z^ny^n). \tag{44}$$

Now, substituting the results (43) and (44) into equation (42), we get assertion (40).

Corollary 2.4. *We have for* $Re(v) > -1$,

$$\mathcal{G}_{2n}\{x^{n-1}J_v(z^nx^n);y\} = \frac{1}{n}I_{v/2}\Big(\frac{1}{2}z^ny^n\Big)K_{v/2}\Big(\frac{1}{2}z^ny^n\Big). \tag{45}$$

Proof. Setting

$$f(x) = x^{-n}J_v(z^nx^n), \quad Re(v) > -1 \tag{46}$$

in Lemma 2.1, we get

$$\mathcal{G}_{2n}\{x^{n-1}J_v(z^nx^n);y\} = \frac{2n}{\sqrt{\pi}}\mathcal{L}_{2n}\{u^{-n}\mathcal{L}_{2n}\{x^{-n}J_v(z^nx^n);u\};y\}. \tag{47}$$

Using relation (7) and the known formulas [9, p.185, Entry(29)] and [14, p.325, Entry(10)], we have

$$2n\mathcal{L}_{2n}\{x^{-n}J_v(z^n x^n);u\} = \sqrt{\pi}u^{-n}\exp\Big(-\frac{z^{2n}}{8u^{2n}}\Big)I_{v/2}\Big(\frac{z^{2n}}{8u^{2n}}\Big) \tag{48}$$

and

$$\mathcal{L}_{2n}\Big\{u^{-2n}\exp\Big(-\frac{z^{2n}}{8u^{2n}}\Big)I_{v/2}\Big(\frac{z^{2n}}{8u^{2n}}\Big);y\Big\} = \frac{1}{n}I_{v/2}\Big(\frac{1}{2}z^n y^n\Big)K_{v/2}\Big(\frac{1}{2}z^n y^n\Big). \tag{49}$$

Substituting (48) and (49) into (47), we obtain the assertion (45).

Theorem 2.5. *If the conditions stated in Lemma 2.1 are satisfied, then the Parseval-Goldstein type relations hold true:*

$$\int_0^\infty y^{n-1}\mathcal{L}_{2n}\{f(x);y\}\mathcal{L}_{2n}\{g(u);y\}dy = \frac{\sqrt{\pi}}{2n}\int_0^\infty x^{2n-1}f(x)\mathcal{G}_{2n}\{u^{2n-1}g(u);x\}dx, \tag{50}$$

$$\int_0^\infty y^{n-1}\mathcal{L}_{2n}\{f(x);y\}\mathcal{L}_{2n}\{g(u);y\}dy = \frac{\sqrt{\pi}}{2n}\int_0^\infty u^{2n-1}g(u)\mathcal{G}_{2n}\{x^{2n-1}f(x);u\}du \tag{51}$$

and

$$\int_0^\infty x^{2n-1}f(x)\mathcal{G}_{2n}\{u^{2n-1}g(u);x\}dx = \int_0^\infty u^{2n-1}g(u)\mathcal{G}_{2n}\{x^{2n-1}f(x);u\}du. \tag{52}$$

Proof. We only give the proof of (50), as the proof of (51) is similar. Identity (52) follows from the identities (50) and (51). Using definition (6) of the $\mathcal{L}_{2n}$-transform and changing the order of integration which is permissible by absolute convergence of the integrals involved, we have

$$\int_0^\infty y^{n-1}\mathcal{L}_{2n}\{f(x);y\}\mathcal{L}_{2n}\{g(u);y\}dy$$

$$= \int_0^\infty x^{2n-1}f(x)\Big[\int_0^\infty y^{n-1}\exp(-x^{2n}y^{2n})\mathcal{L}_{2n}\{g(u);y\}dy\Big]dx$$

$$= \int_0^\infty x^{2n-1}f(x)\mathcal{L}_{2n}\{y^{-n}\mathcal{L}_{2n}\{g(u);y\};x\}dx. \tag{53}$$

Now the assertion (50) easily follows from (53) and (30) of Lemma 2.1.

Corollary 2.6. *If the integrals involved converge absolutely then we have for* $0 < Re(\mu) < 1$,

$$2n\int_0^\infty y^{-n(\mu-1)-1}\mathcal{L}_{2n}\{f(x);y\}dy = \Gamma\Big(\frac{1}{2}-\frac{\mu}{2}\Big)\int_0^\infty x^{n(\mu+1)-1}f(x)dx, \tag{54}$$

$$\Gamma\Big(\frac{\mu}{2}\Big)\int_0^\infty y^{-n(\mu+1)-1}\mathcal{L}_{2n}\{f(x);y\}dy = \sqrt{\pi}\int_0^\infty u^{n\mu-1}\mathcal{G}_{2n}\{x^{2n-1}f(x);u\}du \tag{55}$$

and

$$2n\int_0^\infty u^{n\mu-1}\mathcal{G}_{2n}\{x^{2n-1}f(x);u\}du = B\Big(\frac{\mu}{2},-\frac{\mu}{2}+\frac{1}{2}\Big)\int_0^\infty x^{n(\mu+1)-1}f(x)dx. \tag{56}$$

Proof. Setting

$$g(u) = u^{n(\mu-2)} \tag{57}$$

into (50) of Theorem 2.5 and utilizing the relations (33) and (36) of Corollary 2.2, we have

$$\int_0^\infty y^{n-1}\mathcal{L}_{2n}\{f(x);y\}\mathcal{L}_{2n}\{u^{n(\mu-2)};y\}dy$$

$$= \Gamma\Big(\frac{\mu}{2}\Big)\int_0^\infty y^{-n(\mu-1)-1}\mathcal{L}_{2n}\{f(x);y\}dy = \sqrt{\pi}\int_0^\infty x^{2n-1}f(x)\mathcal{G}_{2n}\{u^{n\mu-1};x\}dx$$

$$= \frac{\sqrt{\pi}}{n}2^{-\mu}B\Big(\mu,\frac{1}{2}-\frac{\mu}{2}\Big)\int_0^\infty x^{n(\mu+1)-1}f(x)dx. \tag{58}$$

Using the duplication formula (38) for the gamma function on the right-hand side of (58), we arrive at the assertion (54).
Similarly, the proof of assertion (55) could be obtained with setting $g(u) = u^{n(\mu-2)}$ into (51) of Theorem 2.5 and using the formula (36) of Corollary 2.2.
The assertion (56) follows from the identities (54), (55) and the relation (39).

Corollary 2.7. *If the integrals involved converge absolutely then we have the following identities for $-1 < Re(v) < \frac{1}{2}$:*

$$n\mathcal{L}_{2n}\{y^{n(2v-1)}\mathcal{L}_{2n}\Big\{f(x);\frac{1}{2^{1/n}y}\Big\};z\} = 2^{-v-\frac{1}{2}}z^{-n(v+1)}\mathcal{K}_{v+\frac{1}{2},n}\{x^{n(v+1)}f(x);z\}, \tag{59}$$

$$2^{v+1}\mathcal{L}_{2n}\{y^{n(2v-1)}\mathcal{L}_{2n}\Big\{f(x);\frac{1}{2^{1/n}y}\Big\};z\}$$

$$= \sqrt{\pi}z^{-n(v+\frac{1}{2})}\mathcal{H}_{v,n}\{u^{n(v+\frac{1}{2})}\mathcal{G}_{2n}\{x^{2n-1}f(x);u\};z\} \tag{60}$$

and

$$\sqrt{2}\mathcal{K}_{v+\frac{1}{2},n}\{x^{n(v+1)}f(x);z\} = n\sqrt{\pi}z^{\frac{n}{2}}\mathcal{H}_{v,n}\{u^{n(v+\frac{1}{2})}\mathcal{G}_{2n}\{x^{2n-1}f(x);u\};z\}. \tag{61}$$

Proof. We start with the proof of assertion (59) by setting

$$g(u) = u^{nv}J_v(z^n u^n), \tag{62}$$

into (50) of Theorem 2.5. Then, utilizing identity (40) of Corollary 2.3 and the known formula [9, p.185, Entry (30)] and setting $y = \frac{1}{2^{1/n}y}$, we get

$$z^{nv}2^{v+1}\int_0^\infty y^{n(2v+1)-1}\exp(-z^{2n}y^{2n})\mathcal{L}_{2n}\{f(x);\frac{1}{2^{1/n}y}\}dy$$

$$= \frac{\sqrt{2}}{nz^n}\int_0^\infty x^{n-1}(x^nz^n)^{\frac{1}{2}}K_{v+\frac{1}{2}}(x^nz^n)x^{n(v+1)}f(x)dx. \tag{63}$$

Using the definitions (6) of the $\mathcal{L}_{2n}$-transform and (16) of the $\mathcal{K}_{v,n}$-transform respectively, we obtain the assertion (59).
Assertion (60) is obtained similarly with setting $g(u) = u^{nv}J_v(z^n u^n)$ into (51) of Theorem 2.5 and using definitions (6) of the $\mathcal{L}_{2n}$-transform and (17) of the $\mathcal{H}_{v,n}$-transform, respectively.
Assertion (61) immediately follows from the relations (59) and (60).

Remark 2.8. Setting $v = 0$ in Corollary 2.7 and then using the formula (24) and definitions of the $\mathcal{L}_n$-transform (5), $\mathcal{K}_{v,n}$-transform (16) and the $\mathcal{H}_{v,n}$-transform (17), we have

$$2nz^n \mathcal{L}_{2n}\{y^{-n}\mathcal{L}_{2n}\Big\{f(x); \frac{1}{2^{1/n}y}\Big\}; z\} = \sqrt{\pi}\mathcal{L}_n\{x^n f(x); z\}, \tag{64}$$

$$2z^{n/2}\mathcal{L}_{2n}\{y^{-n}\mathcal{L}_{2n}\Big\{f(x); \frac{1}{2^{1/n}y}\Big\}; z\} = \sqrt{\pi}\mathcal{H}_{0,n}\{u^{\frac{n}{2}}\mathcal{G}_{2n}\{x^{2n-1}f(x); u\}; z\} \tag{65}$$

and

$$\mathcal{L}_n\{x^n f(x); z\} = nz^{n/2}\mathcal{H}_{0,n}\{u^{\frac{n}{2}}\mathcal{G}_{2n}\{x^{2n-1}f(x); u\}; z\}. \tag{66}$$

Remark 2.9. Setting $v = -\frac{1}{2}$ in Corollary 2.7 and then using the formula (23) and definition (13) of the generalized Fourier cosine transform, we obtain

$$nz^{n/2}\mathcal{L}_{2n}\{y^{-2n}\mathcal{L}_{2n}\Big\{f(x); \frac{1}{2^{1/n}y}\Big\}; z\} = \mathcal{K}_{0,n}\{x^{\frac{n}{2}}f(x); z\}, \tag{67}$$

$$\mathcal{L}_{2n}\{y^{-2n}\mathcal{L}_{2n}\Big\{f(x); \frac{1}{2^{1/n}y}\Big\}; z\} = \frac{\sqrt{\pi}}{\sqrt{2}}\mathcal{H}_{-1/2,n}\{\mathcal{G}_{2n}\{x^{2n-1}f(x); u\}; z\}$$

$$= \mathcal{F}_{c,n}\{\mathcal{G}_{2n}\{x^{2n-1}f(x); u\}; z\} \tag{68}$$

and

$$\mathcal{K}_{0,n}\{x^{\frac{n}{2}}f(x); z\} = nz^{\frac{n}{2}}\mathcal{F}_{c,n}\{\mathcal{G}_{2n}\{x^{2n-1}f(x); u\}; z\}. \tag{69}$$

Remark 2.10. Setting $v = \frac{1}{2}$ in Corollary 2.7 and then using the formula (22) and the definition (12) of the generalized Fourier sine transform, we have

$$\mathcal{L}_{2n}\{\mathcal{L}_{2n}\Big\{f(x); \frac{1}{2^{1/n}y}\Big\}; z\} = \frac{z^{-3n/2}}{2n}\mathcal{K}_{1,n}\{x^{\frac{3n}{2}}f(x); z\}, \tag{70}$$

$$\mathcal{L}_{2n}\{\mathcal{L}_{2n}\Big\{f(x); \frac{1}{2^{1/n}y}\Big\}; z\} = \frac{\sqrt{\pi}}{2^{3/2}z^n}\mathcal{H}_{1/2,n}\{u^n\mathcal{G}_{2n}\{x^{2n-1}f(x); u\}; z\}$$

$$= \frac{1}{2z^n}\mathcal{F}_{s,n}\{u^n\mathcal{G}_{2n}\{x^{2n-1}f(x); u\}; z\} \tag{71}$$

and

$$\mathcal{K}_{1,n}\{x^{\frac{3n}{2}}f(x); z\} = nz^{\frac{n}{2}}\mathcal{F}_{s,n}\{u^n\mathcal{G}_{2n}\{x^{2n-1}f(x); u\}; z\}. \tag{72}$$

Corollary 2.11. *If the integrals involved converge absolutely, then we have*

$$\int_0^\infty y^{-1}\exp\Big(-\frac{z^{2n}}{8y^{2n}}\Big)I_{v/2}\Big(\frac{z^{2n}}{8y^{2n}}\Big)\mathcal{L}_n\{f(x); y\}dy$$

$$= \frac{1}{n}\int_0^\infty x^{2n-1} f(x) I_{v/2}\Big(\frac{z^{2n}}{8y^{2n}}\Big) K_{v/2}(z^n x^n)dx, \tag{73}$$

$$\int_0^\infty y^{-1}\exp\Big(-\frac{z^{2n}}{8y^{2n}}\Big) I_{v/2}\Big(\frac{z^{2n}}{8y^{2n}}\Big)\mathcal{L}_n\{f(x);y\}dy$$
$$= z^{-n/2}\mathcal{H}_{v,n}\{u^{-n/2}\mathcal{G}_{2n}\{x^{2n-1}f(x);u\};z\} \tag{74}$$

and

$$\int_0^\infty x^{2n-1} f(x) I_{v/2}\Big(\frac{z^{2n}}{8y^{2n}}\Big) K_{v/2}(z^n x^n)dx$$
$$= nz^{-n/2}\mathcal{H}_{v,n}\{u^{-n/2}\mathcal{G}_{2n}\{x^{2n-1}f(x);u\};z\}. \tag{75}$$

Proof. Setting

$$g(u) = u^{-n} J_v(z^n u^n) \tag{76}$$

into (50) and (51) respectively and using the relations (45), (48) of Corollary 2.4 and the definition (17) of $\mathcal{H}_{v,n}$-transform, we obtain the assertions (73) and (74). The assertion (75) immediately follows from (73) and (74).

Corollary 2.12. *If the integrals involved converge absolutely, then we have*

$$\mathcal{E}_{2n,1}\{y^{-n}\mathcal{L}_{2n}\{f(x);y\};z\} = \sqrt{\pi}\mathcal{P}_{2n}\{\mathcal{G}_{2n}\{x^{2n-1}f(x);u\};z\}. \tag{77}$$

Proof. Setting

$$g(u) = (u^{2n} + z^{2n})^{-1} \tag{78}$$

into (51) of Theorem 2.5 and using the known formula for [9, p.137; Entry(4)] for $v = -1$, we have

$$2n\int_0^\infty y^{n-1}\mathcal{L}_{2n}\{f(x);y\}\mathcal{L}_{2n}\{(u^{2n}+z^{2n})^{-1};y\}dy$$

$$= \sqrt{\pi}\int_0^\infty u^{2n-1}\frac{\mathcal{G}_{2n}\{x^{2n-1}f(x);u\}}{u^{2n}+z^{2n}}du. \tag{79}$$

Using definitions (20) of the $\mathcal{E}_{2n,1}$-transform and (9) of the $\mathcal{P}_{2n}$-transform, we obtain the assertion (77).

3. Illustrative Examples

Example 3.1. *We show for* $2Re(\mu) > |Re(v)| - 2$ *and* $Re(y) > 0$,

$$\int_0^\infty \frac{x^{2n-1}Ei(-\frac{a^{2n}}{x^{2n}})}{\sqrt{x^{2n}+y^{2n}}}dx = -\frac{2\sqrt{\pi}}{n}\frac{y^{2n}}{a^n}\exp\Big(\frac{a^{2n}}{2y^{2n}}\Big)W_{1,0}\Big(\frac{a^{2n}}{y^{2n}}\Big), \tag{80}$$

where the the exponential integral function defined as

$$E_1(x) = -Ei(-x) = \int_x^\infty \frac{\exp(-u)}{u}du = \int_1^\infty \frac{\exp(-xt)}{t}dt, \tag{81}$$

and $W_{\alpha,\beta}(x) = W_{\alpha,-\beta}(x)$ denotes the Whittaker's second function and it is represented as

$$W_{\alpha-\frac{1}{2},\alpha}(x) = \exp(\frac{x}{2})x^{\frac{1-\alpha}{2}}\Gamma(\alpha, x). \tag{82}$$

Demonstration. Putting

$$f(x) = Ei(-\frac{a^{2n}}{x^{2n}}) \tag{83}$$

in the relation (30) of Lemma 2.1, then using the relation (7), the known formulas [14, p.136, Entry(13)] and [14, p.353, Entry(3)] respectively, we have

$$\mathcal{L}_{2n}\Big\{Ei\Big(-\frac{a^{2n}}{x^{2n}}\Big); u\Big\} = -\frac{1}{n}u^{-2n}K_0(2a^n u^n), \tag{84}$$

$$\mathcal{L}_{2n}\{u^{-3n}K_0(2a^n u^n); y\} = \pi\frac{y^{2n}}{na^n}\exp\Big(\frac{a^{2n}}{2y^{2n}}\Big)W_{1,0}\Big(\frac{a^{2n}}{y^{2n}}\Big). \tag{85}$$

Substituting the formulas (84) and (85) into identity (30) of Lemma 2.1 and using the definition (10), we obtain the result (80).

Example 3.2. *We show for $Re(z) > 0$, $Re(v) > |Re(\mu)| - 2$, $Re(\mu) < 1$, $Re(\mu + v) > 0$,*

$$\int_0^\infty y^{-n(\mu+1)}\exp\Big(-\frac{z^{2n}}{8y^{2n}}\Big)I_{v/2}\Big(\frac{z^{2n}}{8y^{2n}}\Big)dy = \frac{2^{\mu-1}}{z^{n\mu}n\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-\frac{\mu}{2})\Gamma(\frac{\mu}{2}+\frac{v}{2})}{\Gamma(\frac{v}{2}-\frac{\mu}{2}+1)} \tag{86}$$

and

$$\int_0^\infty y^{n\mu-1}I_{v/2}\Big(\frac{1}{2}z^n y^n\Big)K_{v/2}\Big(\frac{1}{2}z^n y^n\Big)dy = \frac{1}{n}B\Big(\frac{\mu}{2}, -\frac{\mu}{2}+\frac{1}{2}\Big)\frac{2^{\mu-2}}{z^{n\mu}}\frac{\Gamma(\frac{\mu}{2}+\frac{v}{2})}{\Gamma(\frac{v}{2}-\frac{\mu}{2}+1)}, \tag{87}$$

where $K_v(x)$ and $I_v(x)$ denote the modified Bessel function of order v and the modified Bessel function of the first kind, respectively and $B(x, y)$ is the beta function (28).
Demonstration. Setting

$$f(x) = x^{-n}J_v(z^n x^n) \tag{88}$$

into the identity (54) of Corollary 2.6, we get

$$\int_0^\infty y^{-n\mu+n-1}\mathcal{L}_{2n}\{x^{-n}J_v(z^n x^n); y\}dy = \frac{1}{2n}\Gamma\Big(\frac{1}{2}-\frac{\mu}{2}\Big)\int_0^\infty x^{n\mu-1}J_v(z^n x^n)dx. \tag{89}$$

Utilizing the formulas (48) of Corollary 2.4 and [9, p.326, Entry(1)] together with (89) we obtain the assertion (86).
Similarly, using the relation (45) of Corollary 2.4 and the formula [9, p.326, Entry(1)] together with (56) of Corollary 2.6, we arrive at the assertion (87) of Example 3.2.

Example 3.3. *We show for $Re(a^{2n}) > 0$, $Re(y^n) < \frac{Re(a^n)}{2}$,*

$$\int_0^\infty \frac{x^{n-1}\exp(\frac{a^{2n}}{4x^{2n}})}{\sqrt{x^{2n}+y^{2n}}}dx = -\frac{y^n}{na^n}\exp\Big(\frac{a^{2n}}{4y^{2n}}\Big)\sum_{m=0}^\infty\frac{2^m y^{mn}}{a^{mn}}\Gamma\Big(\frac{m+1}{2}, \frac{a^{2n}}{4y^{2n}}\Big). \tag{90}$$

Demonstration. Putting

$$f(x) = x^{-n}\exp(-\frac{a^{2n}}{4x^{2n}}) \tag{91}$$

in (30), we have

$$\int_0^\infty \frac{x^{n-1}\exp(-\frac{a^{2n}}{4x^{2n}})}{\sqrt{x^{2n}+y^{2n}}}dx = \frac{2n}{\sqrt{\pi}}\mathcal{L}_{2n}\Big\{u^{-n}\mathcal{L}_{2n}\Big\{x^{-n}\exp\Big(-\frac{a^{2n}}{4x^{2n}}\Big);u\Big\};y\Big\}. \tag{92}$$

Using the relation (7) and the formula [9, p.146, Entry(27)], we find

$$\mathcal{L}_{2n}\Big\{x^{-n}\exp\Big(-\frac{a^{2n}}{4x^{2n}}\Big);u\Big\} = \frac{1}{2n}\sqrt{\pi}u^{-n}\exp(-a^n u^n), \tag{93}$$

where $Re(u^{2n}) > 0$. Making use of the relation (7) and the definition (1) of the classic Laplace transform, we have

$$\mathcal{L}_{2n}\{u^{-2n}\exp(-a^n u^n);y\} = \frac{1}{2n}\int_0^\infty \frac{1}{u}\exp(-uy^{2n})\exp(-a^n u^{1/2})du. \tag{94}$$

Changing the variable of the integration on the right-hand side from u to z where $u = z^2$, equation (94) becomes

$$\mathcal{L}_{2n}\{u^{-2n}\exp(-a^n u^n);y\} = \frac{\exp(\frac{a^{2n}}{4y^{2n}})}{n}\int_0^\infty \frac{1}{z}\exp\Big(-y^{2n}\Big(z+\frac{a^n}{2y^{2n}}\Big)^2\Big)dz. \tag{95}$$

Changing the variable of the integration on the right-hand side of (95) from y to t where $y^n(z+\frac{a^n}{2y^{2n}}) = t$, we get

$$\mathcal{L}_{2n}\{u^{-2n}\exp(-a^n u^n);y\} = \frac{\exp(\frac{a^{2n}}{4y^{2n}})}{n}\int_{a^n/2y^n}^\infty \frac{\exp(-t^2)}{t-\frac{a^n}{2y^n}}dt. \tag{96}$$

Substituting

$$\frac{1}{t-\frac{a^n}{2y^n}} = -\frac{2y^n}{a^n}\sum_{m=0}^\infty\Big(\frac{2y^n}{a^n}t\Big)^m, \tag{97}$$

into (96), we obtain

$$\begin{aligned}&\mathcal{L}_{2n}\{u^{-2n}\exp(-a^n u^n);y\}\\ &= -\frac{1}{n}\exp\Big(\frac{a^{2n}}{4y^{2n}}\Big)\frac{2y^n}{a^n}\sum_{m=0}^\infty\frac{2^m y^{mn}}{a^{mn}}\int_{a^n/2y^n}^\infty t^m\exp(-t^2)dt\end{aligned} \tag{98}$$

where

$$\begin{aligned}\int_{a^n/2y^n}^\infty t^m\exp(-t^2)dt &= \frac{1}{2}\int_{a^{2n}/4y^{2n}}^\infty z^{\frac{m+1}{2}-1}\exp(-z)dz\\ &= \frac{1}{2}\Gamma\Big(\frac{m+1}{2},\frac{a^{2n}}{4y^{2n}}\Big).\end{aligned} \tag{99}$$

Now, inserting equations (96), (98), (99) and (95) into equation (92), we arrive at the the assertion (90).

Example 3.4. *We show for* $-1 < Re(v) < \frac{1}{2}$ *and* $|arg(-a^{2n})| < \pi$,

$$\int_0^\infty \frac{x^{2n-1}\exp(a^{2n}x^{2n})\Gamma(v,a^{2n}x^{2n})}{\sqrt{x^{2n}+y^{2n}}}dx = \frac{ia^{-n}}{2n}\exp(-a^{2n}y^{2n})\Big[-\sqrt{\pi}\Gamma(v)erfc(ia^n y^n)$$

$$+(-1)^{-v}B\Big(v,\frac{1}{2}-v\Big)\Gamma\Big(v+\frac{1}{2},-a^{2n}y^{2n}\Big)\Big]. \tag{100}$$

Demonstration. In the relation (30), we take

$$f(x) = \exp(a^{2n}x^{2n})\Gamma(v,a^{2n}x^{2n}). \tag{101}$$

Making use the identity (7) and the formula [9, p.178, Entry(31)], we get

$$\mathcal{L}_{2n}\{\exp(a^{2n}x^{2n})\Gamma(v,a^{2n}x^{2n});u\}$$

$$= \frac{1}{2n}\Gamma(v)(u^{2n}-a^{2n})^{-1}\Big(1-\frac{a^{2nv}}{u^{2nv}}\Big), \quad Re(v) > -1. \tag{102}$$

Substituting relation (102) into (30) and using (7) once again, we find

$$\mathcal{G}_{2n}\{x^{2n-1}\exp(a^{2n}x^{2n})\Gamma(v,a^{2n}x^{2n});y\}$$

$$= \frac{\Gamma(v)}{\sqrt{\pi}}\mathcal{L}_{2n}\Big\{\frac{u^{-n}}{u^{2n}-a^{2n}}\Big(1-\frac{a^{2nv}}{u^{2nv}}\Big);y\Big\}$$

$$= \frac{\Gamma(v)}{2n\sqrt{\pi}}\Big[\mathcal{L}\Big\{\frac{u^{-1/2}}{u-a^{2n}};y^{2n}\Big\} - a^{2nv}\mathcal{L}\Big\{\frac{u^{-v-\frac{1}{2}}}{u-a^{2n}};y^{2n}\Big\}\Big]. \tag{103}$$

Using the known formula [12, p.348, (10)], we have

$$\mathcal{G}_{2n}\{x^{2n-1}\exp(a^{2n}x^{2n})\Gamma(v,a^{2n}x^{2n});y\}$$

$$= \frac{\Gamma(v)}{2n\sqrt{\pi}}\Big[\frac{\Gamma(v)}{i}a^{-n}\exp(-a^{2n}y^{2n})\Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{1}{2},-a^{2n}y^{2n}\Big)$$

$$-\frac{1}{i}(-1)^{-v}a^{-n}\exp(-a^{2n}y^{2n})\Gamma\Big(\frac{1}{2}-v\Big)\Gamma\Big(v+\frac{1}{2},-a^{2n}y^{2n}\Big)\Big]. \tag{104}$$

Making use the following known formulas into (104),

$$\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}, \quad \Gamma(v)\Gamma\Big(\frac{1}{2}-v\Big) = \sqrt{\pi}B\Big(v,\frac{1}{2}-v\Big), \quad \Gamma\Big(\frac{1}{2},x\Big) = \sqrt{\pi}erfc(\sqrt{x}) \tag{105}$$

we arrive at the identity (100).

Example 3.5. *We show for* $Re(a^{2n}) > 0$, $Re(y^{2n}) > 0$, $Re(u^{2n}) > Im(a^{2n})$,

$$\int_0^\infty \frac{x^{2n-1}\sin(a^{2n}x^{2n})}{\sqrt{x^{2n}+y^{2n}}}dx = -\frac{a^{-n}}{2n}\Big[\exp\Big(ia^{2n}y^{2n}-i\frac{3\pi}{4}\Big)\Gamma\Big(\frac{3}{2},ia^{2n}y^{2n}\Big)+\exp\Big(-ia^{2n}y^{2n}+i\frac{3\pi}{4}\Big)\Gamma\Big(\frac{3}{2},-ia^{2n}y^{2n}\Big)$$

$$= \frac{iy^n}{n} - \frac{\sqrt{\pi}}{4na^n}\Big[\exp\Big(ia^{2n}y^{2n}-i\frac{3\pi}{4}\Big)erfc(\sqrt{i}a^n y^n)+\exp\Big(-ia^{2n}y^{2n}+i\frac{3\pi}{4}\Big)erfc(\sqrt{-i}a^n y^n)\Big]. \tag{106}$$

Demonstration. Setting

$$f(x) = \sin(a^{2n}x^{2n}) \tag{107}$$

in (30) of Lemma 2.1, we have

$$\mathcal{G}_{2n}\{x^{2n-1}\sin(a^{2n}x^{2n});y\}=\frac{2n}{\sqrt{\pi}}\mathcal{L}_{2n}\{u^{-n}\mathcal{L}_{2n}\{\sin(a^{2n}x^{2n});u\};y\}. \tag{108}$$

Using relation (7) and the known formulas [9, p.150, Entry(1)],[12, p.352, 6.3.(389)], we get the following relations

$$\mathcal{L}_{2n}\{\sin(a^{2n}x^{2n});u\}=\frac{1}{2n}\frac{a^{2n}}{u^{4n}+a^{4n}},\quad Re(u^{2n})>Im(a^{2n}) \tag{109}$$

and

$$\mathcal{L}_{2n}\Big\{\frac{u^{-n}}{u^{4n}+a^{4n}};y\Big\}=-\frac{a^{-3n}\sqrt{\pi}}{2n}\Big[\exp\Big(ia^{2n}y^{2n}-i\frac{3\pi}{4}\Big)\Gamma\Big(\frac{3}{2},ia^{2n}y^{2n}\Big)$$
$$+\exp\Big(-ia^{2n}y^{2n}+i\frac{3\pi}{4}\Big)\Gamma\Big(\frac{3}{2},-ia^{2n}y^{2n}\Big)\Big]. \tag{110}$$

Using the definition of the incomplete gamma function, we have

$$\Gamma\Big(\frac{3}{2},x\Big)=\int_x^\infty t^{1/2}\exp(-t)dt=\sqrt{x}\exp(-x)+\frac{\sqrt{\pi}}{2}erfc(\sqrt{x}). \tag{111}$$

Now, inserting relations (109), (110) and (111) into (108), we arrive at the the assertion (106).

Example 3.6. *We show for $Re(a^{-2n})>0$, $Re(z^{2n})>0$, $Re(v)>-\frac{1}{2}$,*

$$\int_0^\infty x^{n(v+\frac{5}{2})-1}\cos(a^{2n}x^{2n})K_{v+1/2}(x^nz^n)dx$$
$$=\frac{z^{n(v+\frac{1}{2})}\Gamma(v+\frac{1}{2})}{2^{v+\frac{7}{2}}na^{n(2v+3)}}\Big[\exp\Big(i\frac{z^{2n}}{4a^{2n}}+i\frac{(v-\frac{1}{2})\pi}{2}\Big)\Gamma\Big(\frac{1}{2}-v,i\frac{z^{2n}}{4a^{2n}}\Big)$$
$$+\exp\Big(-i\frac{z^{2n}}{4a^{2n}}-i\frac{(v-\frac{1}{2})\pi}{2}\Big)\Gamma\Big(\frac{1}{2}-v,-i\frac{z^{2n}}{4a^{2n}}\Big)\Big]. \tag{112}$$

Demonstration. Setting

$$f(x)=\cos(a^{2n}x^{2n}) \tag{113}$$

in relation (59) of Corollary 2.7 and using the identity (7) and known formulas [9, p.154, Entry(43)] and [12, p.352, 6.3.389] respectively, we find

$$\mathcal{L}_{2n}\Big\{\cos(a^{2n}x^{2n});\frac{1}{2^{1/n}y}\Big\}=\frac{1}{8na^{4n}}\frac{y^{2n}}{y^{4n}+\frac{1}{16a^{4n}}} \tag{114}$$

$$\mathcal{L}_{2n}\Big\{\frac{y^{n(2v+1)}}{y^{4n}+\frac{1}{16a^{4n}}};z\Big\}=\frac{1}{4n}\Gamma\Big(v+\frac{1}{2}\Big)\Big(\frac{1}{4a^{2n}}\Big)^{v-\frac{1}{2}}\Big[\exp\Big(i\frac{z^{2n}}{4a^{2n}}+i\frac{(v-\frac{1}{2})\pi}{2}\Big)\Gamma\Big(\frac{1}{2}-v,i\frac{z^{2n}}{4a^{2n}}\Big)$$
$$+\exp\Big(-i\frac{z^{2n}}{4a^{2n}}-i\frac{(v-\frac{1}{2})\pi}{2}\Big)\Gamma\Big(\frac{1}{2}-v,-i\frac{z^{2n}}{4a^{2n}}\Big)\Big]. \tag{115}$$

Substituting (114) and (115) into (59) we obtain the assertion (112).

4. Conclusion

The new Parseval-Goldstein type relations in this paper could be used in many other areas such as applied mathematics, physics, statistics, medicine, etc. In literature, there are a lot of examples. The equations for new generalized integral transforms in this manuscript could be used to evaluate many infinite integrals which have not been calculated before. Thus, using the above theorems and lemmas, the integral tables could be extented.

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