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Some relations between Hewitt-Stromberg premeasure and Hewitt-Stromberg measure

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Abstract. Let K be a compact set of $\mathbb{R}^n$ and $t \geq 0$. In this paper, we discuss the relation between the t-dimensional Hewitt-Stromberg premeasure and measure denoted by $\overline{\mathsf{H}}^t$ and H^t respectively. We prove : if $\overline{\mathsf{H}}^t(K) < +\infty$ then $\overline{\mathsf{H}}^t(K) = \mathsf{H}^t(K)$ and if $\overline{\mathsf{H}}^t(K) = +\infty$, there exists a compact subset F of K such that $\overline{\mathsf{H}}^t(F) = \mathsf{H}^t(F)$ and $\mathsf{H}^t(F)$ is close as we like to $\mathsf{H}^t(K)$.

1. Introduction

Hewitt-Stromberg measures were introduced in [13, Exercise (10.51)]. Since then, they have been investigated by several authors, highlighting their importance in the study of local properties of fractals and products of fractals. One can cite, for example [2, 3, 9–12]. In particular, Edgar's textbook [6, pp. 32-36] provides an excellent and systematic introduction to these measures. Such measures also appears explicitly, for example, in Pesin's monograph [18, 5.3] and implicitly in Mattila's text [16]. The reader can be referred to [15]for a class of generalization of these measures).

For $t \geq 0$, let $\overline{\mathsf{H}}^t$, H^t denote the t-dimensional Hewitt-Stromberg premeasure and measure, respectively (see Section 2 for the definitions). In this paper, we discuss the relation between $\overline{\mathsf{H}}^t$ and H^t. We prove, for $n \geq 1$ and any compact subset K of $\mathbb{R}^n$, that

$$\overline{\mathsf{H}}^t(K) = \mathsf{H}^t(K)$$

provided that $\overline{\mathsf{H}}^t(K) < +\infty$ (Theorem 3.3). As a consequence, we prove, for $E \subseteq \mathbb{R}^n$, that if $\overline{\mathsf{H}}^t(E) \in (0, \infty)$ then

$$\mathsf{H}^t(\overline{E}) \in (0, \infty).$$

Moreover, if E is compact then, for $t > \dim_{MB}(E)$, we have either $\overline{\mathsf{H}}^t(E) = 0$ or $\overline{\mathsf{H}}^t(E) = +\infty$ (Corollary 3.4), where $\dim_{MB}$ denote the Hewitt-Stomberg dimension (see definition in Section 2). We prove also, as an

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application, some semifiniteness property of $\overline{\mathsf{H}}^t$. A measure μ is said to be semifinite if every set of infinite measure has a subset of finite positive measure. This property was be studied in [4, 5] for Hausdorff measure and in [14] for packing measure, but this does not hold for the Hewitt-Stromberg premeasure (Corollary 3.5). More precisely, there exists a compact set K and $t > 0$ with $\overline{\mathsf{H}}^t(K) = +\infty$ such that K contains no subset with positive finite Hewitt-Stromberg premeasure. In addition, we study in Theorem 4.1 the compact sets of infnite Hewitt-Stromberg premeasure. We prove that if $\overline{\mathsf{H}}^t(K) = +\infty$, there exists a compact subset F of K such that

$$\overline{\mathsf{H}}^t(F) = \mathsf{H}^t(F)$$

and $\mathsf{H}^t(F)$ is close as we like to $\mathsf{H}^t(K)$.

2. Preliminary

First we recall briefly the definitions of Hausdorff dimension, packing dimension and Hewitt-Stromberg dimension and the relationship linking these three notions. Let $\mathcal{F}$ be the class of dimension functions, i.e., the functions $h : \mathbb{R}_+^* \to \mathbb{R}_+^*$ which are right continuous, monotone increasing with $\lim_{r\to 0} h(0) = 0$.

Suppose that, for $n \geq 1$, $\mathbb{R}^n$ is endowed with the Euclidean distance. For $E \subset \mathbb{R}^n$, $h \in \mathcal{F}$ and $\varepsilon > 0$, we write

$$\mathcal{H}_\varepsilon^h(E) = \inf\left\{\sum_i h\Big(|E_i|\Big) \;\; E \subseteq \bigcup_i E_i, \;\; |E_i| < \varepsilon\right\},$$

where $|A|$ is the diameter of the set A defined as $|A| = \sup\Big\{|x-y|, \;\; x, y \in A\Big\}$. This allows to define the Hausdorff measure, with respect to h, of E by

$$\mathcal{H}^h(E) = \sup_{\varepsilon>0} \mathcal{H}_\varepsilon^h(E).$$

The reader can be referred to Rogers' classical text [20] for a systematic discussion of $\mathcal{H}^h$.

We define, for $\varepsilon > 0$,

$$\overline{\mathcal{P}}_\varepsilon^h(E) = \sup\left\{\sum_i h\Big(2r_i\Big)\right\},$$

where the supremum is taken over all disjoint closed balls $\Big(B(x_i, r_i)\Big)_i$ such that $r_i \leq \varepsilon$ and $x_i \in E$. The h-dimensional packing premeasure, with respect to h, of E is now defined by

$$\overline{\mathcal{P}}^h(E) = \sup_{\varepsilon>0} \overline{\mathcal{P}}_\varepsilon^h(E).$$

This makes us able to define the packing measure, with respect to h, of E as

$$\mathcal{P}^h(E) = \inf\left\{\sum_i \overline{\mathcal{P}}^h(E_i) \;\Big|\; E \subseteq \bigcup_i E_i\right\}.$$

While Hausdorff and packing measures are defined using coverings and packings by families of sets with diameters less than a given positive number ε, the Hewitt-Stromberg measures are defined using covering of balls with the same diameter ε. Now, we define

$$\overline{\mathsf{H}}_0^h(E) = \limsup_{r\to 0} \overline{\mathsf{H}}_r^h \quad \text{where} \quad \overline{\mathsf{H}}_r^h(E) = N_r(E)\, h(2r)$$

and the covering number $N_r(E)$ of E is defined by

$$N_r(E) = \inf\Big\{\sharp\{I\} \;\Big|\; \big(B(x_i, r)\big)_{i\in I} \text{ is a family of closed balls with } x_i \in E \text{ and } E \subseteq \bigcup_i B(x_i, r)\Big\}.$$

Since $\overline{\mathsf{H}}_0^h$ is not increasing and not countably subadditive, one needs a standard modification to get an outer measure. Hence, we modify the definition as follows, first we define the Hewitt-Stromberg premeasure

$$\overline{\mathsf{H}}^h(E) = \sup_{F\subseteq E} \overline{\mathsf{H}}_0^h(F)$$

and, by applying now the standard construction ([17, 20, 21]), we obtain the Hewitt-Stromberg measure, with respect to h, defined by

$$\mathsf{H}^h(E) = \inf\left\{\sum_i \overline{\mathsf{H}}^h(E_i) \;\Big|\; E \subseteq \bigcup_i E_i \text{ and } E_i \text{ is closed}\right\}.$$

In the following, we illustrate the basic inequalities satisfied by the Hewitt-Stromberg, the Hausdorff and the packing measures (the proof is straightforward and mimics that in [15, Proposition 2.1]

$$\begin{array}{ccccc} & & \overline{\mathsf{H}}^h(E) & \leq & \overline{\mathcal{P}}^h(E) \\ & & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} \\ \mathcal{H}^h(E) & \leq & \mathsf{H}^h(E) & \leq & \mathcal{P}^h(E). \end{array}$$

Let $t \geq 0$ and h_t is the dimension function defined by

$$h_t(r) = r^t.$$

In this case we will denote simply $\mathcal{H}^{h_t}$ by $\mathcal{H}^t$, also $\mathcal{P}^{h_t}$ will be denoted by $\mathcal{P}^t$, $\overline{\mathsf{H}}^{h_t}$ will be denoted by $\overline{\mathsf{H}}^t$ and H^{h_t} will be denoted by H^t. Now we define the Hausdorff dimension, packing dimension and Hewitt-Stromberg dimension of a set $E \subseteq \mathbb{R}^n$ respectively by

$$\dim_H E = \sup\Big\{t \geq 0,\;\; \mathcal{H}^t(E) = +\infty\Big\} = \inf\Big\{t \geq 0,\;\; \mathcal{H}^t(E) = 0\Big\},$$

$$\dim_P E = \sup\Big\{t \geq 0,\;\; \mathcal{P}^t(E) = +\infty,\Big\} = \inf\Big\{t \geq 0,\;\; \mathcal{P}^t(E) = 0\Big\}$$

and

$$\dim_{MB} E = \sup\Big\{t \geq 0,\;\; \mathsf{H}^t(E) = +\infty\Big\} = \inf\Big\{t \geq 0,\;\; \mathsf{H}^t(E) = 0\Big\}.$$

It follows that

$$\dim_H(E) \leq \dim_{MB}(E) \leq \dim_P(E).$$

Lemma 2.1. *Let $E \subset \mathbb{R}^n$ and $t \geq 0$. Then*

$$\overline{\mathsf{H}}^t(\overline{E}) \leq 2^t \overline{\mathsf{H}}^t(E),$$

where $\overline{E}$ is the closure of E.

Proof. Let $r > 0$ and $\Big\{B_i := B(x_i, r)\Big\}_i$ be a covering of E and let $A \subset \overline{E}$. Now, we consider

$$I = \Big\{i \;:\; B_i \cap A \neq \emptyset\Big\}.$$

For each $i \in I$, let $y_i \in B_i \cap A$. Therefore, $B_i \subseteq B(y_i, 2r)$ and then $\Big\{B(y_i, 2r)\Big\}_i$ is a covering of A. It follows that

$$N_{2r}(A)(4r)^t \leq 2^t N_r(E)(2r)^t.$$

Thus, $\overline{\mathsf{H}}_0^t(A) \leq 2^t \overline{\mathsf{H}}_0^t(E) \leq 2^t \overline{\mathsf{H}}^t(E)$. Since A is arbitrarily, we get the desired result. □

We finish this section by a lemma which will be useful in the following.

Lemma 2.2. *Let $\{E_n\}$ be a decreasing sequence of compact subsets of $\mathbb{R}^n$ and $F = \bigcap_n E_n$. Then, for $t \geq 0$ and $\gamma > 1$, there exist n_0 such that*

$$\overline{\mathsf{H}}^t(E_n) \leq \gamma^t \overline{\mathsf{H}}^t(F), \qquad \forall\ n \geq n_0.$$

Proof. Let $\delta > 0$ and $\left\{B_i := B(x_i, \delta)\right\}_i$ be any covering of F. We claim that there exists n_0 such that $E_n \subset U = \bigcup_i B(x_i, \gamma\delta)$, for all $n \geq n_0$. Indeed, otherwise, $\left\{E_n \backslash U\right\}$ is a decreasing sequence of non-empty compact sets, which, by an elementary consequence of compactness, has a non-empty limit set $(\lim E_n)\backslash U$. Then, for $t \geq 0$ and $n \geq n_0$,

$$\overline{\mathsf{H}}^t_{\gamma\delta}(E_n) = N_{\gamma\delta}(E_n)(2\gamma\delta)^t \quad \leq \quad \gamma^t N_\delta(F)(2\delta)^t = \gamma^t \overline{\mathsf{H}}^t_\delta(F).$$

It follows, for all $n \geq n_0$, that

$$\overline{\mathsf{H}}^t_0(E_n) \leq \gamma^t \overline{\mathsf{H}}^t_0(F) \leq \gamma^t \overline{\mathsf{H}}^t(F). \tag{2.1}$$

Now, let $A \subseteq E_n$, we only have to prove that $\overline{\mathsf{H}}^t_0(A) \leq \gamma^t \overline{\mathsf{H}}^t(F)$. We may suppose that $F \subseteq A \subseteq E_n$. Indeed, otherwise,

$$\overline{\mathsf{H}}^t_0(A) \leq \overline{\mathsf{H}}^t(F) \leq \gamma^t \overline{\mathsf{H}}^t(F).$$

Thus, without loss of generality we may suppose that, $A = E_m$, for some $m \geq n$. Therefore, using (2.1), we have $\overline{\mathsf{H}}^t_0(A) \leq \gamma^t \overline{\mathsf{H}}^t(F)$. □

3. Main results

We can see, from the definition, that estimating $\overline{\mathsf{H}}^t$ is much easier than estimating the Hewitt-Sttromberg measure H^t. It is therefore natural to look for relationships between these two quantities. The reader can also see [1, 8, 14, 22] for a similar result for Hausdorff and packing measures.

Lemma 3.1. *Let K be compact set in $\mathbb{R}^n$ and $t \geq 0$. Suppose that for every $\epsilon > 0$ and closed subset E of K one can find an open set U such that $E \subset U$ and $\overline{\mathsf{H}}^t(U \cap K) \leq \overline{\mathsf{H}}^t(E) + \epsilon$, then*

$$\mathsf{H}^t(K) = \overline{\mathsf{H}}^t(K).$$

Proof. Let $\epsilon > 0$ and let $\{E_i\}$ be a sequence of closed sets such that $K \subseteq \bigcup_i E_i$. Take, for each i, an open set U_i such that $E_i \subset U_i$ and

$$\overline{\mathsf{H}}^t(U_i \cap K) \leq \overline{\mathsf{H}}^t(E_i) + 2^{-i-1}\epsilon.$$

Since K is compact, the cover $\{U_i\}$ of K has a finite subcover. So we may use the fact that, for all $F_1, F_2 \subset \mathbb{R}^n$,

$$\overline{\mathsf{H}}^t(F_1 \cup F_2) \leq \overline{\mathsf{H}}^t(F_1) + \overline{\mathsf{H}}^t(F_2)$$

to infer that

$$\overline{\mathsf{H}}^t(K) \leq \sum_i \overline{\mathsf{H}}^t(U_i \cap K) \leq \sum_i (\overline{\mathsf{H}}^t(E_i) + 2^{-i-1}\epsilon) \leq \sum_i \overline{\mathsf{H}}^t(E_i) + \epsilon.$$

This is true for all $\epsilon > 0$ and $\{E_i\}$ such that $K \subseteq \bigcup_i E_i$. Thus

$$\mathsf{H}^t(K) \geq \overline{\mathsf{H}}^t(K).$$

The opposite inequality is obvious. □

Theorem 3.2. *Let $K \subset \mathbb{R}^n$ be a compact set and $t \geq 0$ such that $\overline{\mathsf{H}}^t(K) < +\infty$. Then, for any closed subset E of K and any $\epsilon > 0$, there exists an open set U such that $E \subset U$ and*

$$\overline{\mathsf{H}}^t(U \cap K) < \overline{\mathsf{H}}^t(E) + \epsilon.$$

Proof. For $n \geq 1$, define the n-parallel body E_n of E by

$$E_n = \Big\{x \in \mathbb{R}^n, \quad |x - y| < 1/n, \ \text{ for some } y \in E\Big\}.$$

It is clear that E_n is an open set and $E \subseteq E_n$, for all n. Denote by $\overline{E}_n$ the closure of E_n and let $\gamma > 1$. Using Lemma 2.2 and Lemma 2.1, there exists n such that

$$\overline{\mathsf{H}}^t(\overline{E}_n \cap K) \leq \gamma^t \overline{\mathsf{H}}^t(E)$$

For $\epsilon > 0$, we can choose γ such that $\gamma^t \overline{\mathsf{H}}^t(E) \leq \overline{\mathsf{H}}^t(E) + \epsilon$. Finally, we get

$$\overline{\mathsf{H}}^t(E_n \cap K) \leq \overline{\mathsf{H}}^t(\overline{E}_n \cap K) \leq \overline{\mathsf{H}}^t(E) + \epsilon.$$

□

As a direct consequence, we get the following result.

Theorem 3.3. *Let $K \subset \mathbb{R}^n$ be a compact set and $t \geq 0$. Assume that $\overline{\mathsf{H}}^t(K) < +\infty$ then*

$$\overline{\mathsf{H}}^t(K) = \mathsf{H}^t(K).$$

From Theorem 3.3, we immediately obtain the following corollary.

Corollary 3.4. *Let $E \subset \mathbb{R}^n$ and $t \geq 0$*

1. *Assume that $0 < \overline{\mathsf{H}}^t(E) < +\infty$. Then $0 < \mathsf{H}^t(\overline{E}) < \infty$. In particular,*

$$\dim_{\overline{MB}} E = \dim_{MB} \overline{E} = t,$$

where $\dim_{\overline{MB}} E = \sup\Big\{t \geq 0, \ \overline{\mathsf{H}}^t(E) = +\infty\Big\} = \inf\Big\{t \geq 0, \ \overline{\mathsf{H}}^t(E) = 0\Big\}$.

2. *Assume that E is compact and $t > \dim_{MB} E$. Then either $\overline{\mathsf{H}}^t(E) = 0$ or $\overline{\mathsf{H}}^t(E) = +\infty$.*

The following corollary shows that the theorems of Besicovitch [4] and Davies [5] for Hausdorff measures and the theorem of Joyce and Preiss [14] for packing measures does not hold for the Hewitt-Stromberg premeasure.

Corollary 3.5. *There exists a compact set K and $t > 0$ with $\overline{\mathsf{H}}^t(K) = +\infty$ such that K contains no subset with positive finite Hewitt-Stromberg premeasure.*

Proof. Consider for $n \geq 1$, the set $A_n = \{0\} \bigcup \{1/k, \ \ k \leq n\}$ and

$$K = \bigcup_n A_n = \Big\{0\Big\} \bigcup \Big\{1/n, \ \ n \in \mathbb{N}\Big\}.$$

Now, we will prove that $\dim_{\overline{MB}} K = 1/2$. For $n \geq 1$ and $\delta_n = \frac{1}{n+n^2}$, remark that

$$N_{\delta_n}(A_n) = n + 1.$$

It follows that

$$\overline{\mathsf{H}}^{1/2}_{\delta_n}(K) \geq \overline{\mathsf{H}}^{1/2}_{\delta_n}(A_n) = \sqrt{2}\frac{n+1}{\sqrt{n+n^2}}.$$

Thereby, $\overline{\mathsf{H}}^{1/2}(K) > 0$ which implies that $\dim_{\overline{MB}} K \geq 1/2$. In the other hand, if $\overline{\dim}_p(K)$ denote the box-counting dimension of K, i.e.,

$$\overline{\dim}_p(K) = \sup\{t;\ \overline{\mathcal{P}}^t(K) = +\infty\} = \inf\{t;\ \overline{\mathcal{P}}^t(K) = 0\}$$

then $\overline{\dim}_p(K) = \frac{1}{2}$ (see Corollary 2.5 in [8]) and thus

$$\dim_{\overline{MB}} K \leq \overline{\dim}_p(K) = 1/2.$$

As a consequence, we have $\dim_{\overline{MB}} K = 1/2$. Take $t = 1/3$, it is cleat that $\mathsf{H}^t(K) = 0$. Moreover, $\overline{\mathsf{H}}^t(K) = +\infty$. It follows, for any subset F of K, that $\overline{\mathsf{H}}^t(F) = 0$ or $+\infty$. Otherwise, assume that $0 < \overline{\mathsf{H}}^t(F) < +\infty$. Then $0 < \overline{\mathsf{H}}^t(\overline{F}) < +\infty$ and thus, by using Theorem 3.3, $0 < \mathsf{H}^t(\overline{F}) < +\infty$, which is impossible since F is a subset of K. □

4. Compact sets of infnite Hewitt-Stromberg premeasure

Now, we discuss the compact sets of infnite Hewitt-Stromberg premeasure.

Theorem 4.1. *Let K be a compact subset of* $\mathbb{R}^n$; $t \geq 0$ *and* $\overline{\mathsf{H}}^t(K) = +\infty$. *Then, for any* $\epsilon > 0$; *there exists a compact subset F of K such that* $\overline{\mathsf{H}}^t(F) = \mathsf{H}^t(F)$ *and*

$$\mathsf{H}^t(F) \geq \mathsf{H}^t(K) - \epsilon.$$

Proof. The case $\mathsf{H}^t(K) = +\infty$ is trivial, then we assume that $\mathsf{H}^t(K) < +\infty$. Take a closed sets $\{F_i\}$ such that $K = \bigcup_i F_i$ and

$$\sum_i \overline{\mathsf{H}}^t(F_i) \leq \mathsf{H}^t(K) + \frac{\epsilon}{2}. \tag{4.1}$$

Since we have $\sum_i \overline{\mathsf{H}}^t(F_i) \geq \mathsf{H}^t(K)$, there exists $m \in \mathbb{N}$ such that

$$\sum_{i=1}^m \overline{\mathsf{H}}^t(F_i) \geq \mathsf{H}^t(K) - \frac{\epsilon}{2}. \tag{4.2}$$

Therefore, from (4.1) and (4.2), we obtain

$$\sum_{i=m+1}^{+\infty} \overline{\mathsf{H}}^t(F_i) \leq \epsilon. \tag{4.3}$$

We condiser the set $F = \bigcup_{i=1}^m F_i$. Then, by the finite subadditivity of $\overline{\mathsf{H}}^t$ and (4.1), we have

$$\overline{\mathsf{H}}^t(F) \leq \sum_{i=1}^m \overline{\mathsf{H}}^t(F_i) < +\infty.$$

Finally, using Theorem 3.3 we have $\overline{\mathsf{H}}^t(F) = \mathsf{H}^t(F)$ and, by (4.3), we get

$$\mathsf{H}^t(K) - \mathsf{H}^t(F) \leq \mathsf{H}^t\Big(\bigcup_{i=m+1}^{+\infty} F_i\Big) \leq \sum_{i=m+1}^{+\infty} \overline{\mathsf{H}}^t(F_i) \leq \epsilon.$$

□

Remark 4.2. *One can check that the proof of Theorem 3.3 and Theorem 4.1 works for every dimension function h and the corresponding Hewitt-Stromberg measure and premeasure* H^h *and* $\overline{\mathsf{H}}^h$ *repectively, provided that for every* $\epsilon > 0$ *there are* $\delta > 0$ *and* $r_0 > 0$ *such that*

$$\frac{h\Big((1+\delta)r\Big)}{h(r)} < 1+\epsilon \qquad \forall r < r_0.$$

Especially, if $h(r) = x^t L(r)$ *where L is slowly varying in the sense of Karamata, that is,*

$$\lim_{r\to 0} \frac{L(ar)}{L(r)} = 1$$

for every $a > 0$ *([19]). Then, for every compact set K,*

$$\overline{\mathsf{H}}^h(K) < +\infty \Longrightarrow \mathsf{H}^h(K) = \overline{\mathsf{H}}^h(K) \tag{4.4}$$

and if $\overline{\mathsf{H}}^h(K) = +\infty$ *then there exists a compact set* $F \subseteq K$ *such that*

$$\overline{\mathsf{H}}^h(F) = \mathsf{H}^h(F) \ \ and \ \ \mathsf{H}^h(F) \geq \mathsf{H}^h(K) - \epsilon. \tag{4.5}$$

Open problems :

1. We ask if (4.4) and (4.5) remain true for any dimension function h or even for h satisfies the doubling condition, that is, for all $r > 0$

   $$h(2r) \leq kh(r),$$

   for some positive constant k.
2. We ask if Theorem 3.3 remains true if the Hewitt-Stromberg measure of a set E is defined with

   $$M_r(E) = \sup\Big\{\sharp\{I\} \ \mid \ \Big(B(x_i, r)\Big)_{i\in I} \text{ is a family of disjoint closed balls with } x_i \in E\Big\}.$$

   instead of $N_r(E)$.

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