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Sequential warped product manifolds with a semi-symmetric metric connection

Semra Zeren[a], Selcen Yüksel Perktaş[b], Ahmet Yıldız[a]

[a]*Department of Mathematics, Faculty of Education, Inonu University, Malatya, Türkiye*
[b]*Department of Mathematics, Faculty of Arts and Sciences, Adıyaman University, Adıyaman, Türkiye*

Abstract. In the present paper, we study a new generalization of warped product manifolds, called sequential warped product manifolds, with respect to a semi-symmetric metric connection. We obtain relations for covariant derivatives, Riemannian curvature, Ricci curvature and scalar curvature of the sequential warped product manifolds with respect to the semi-symmetric connection, respectively, and demonstrate the relationship between them and curvatures with respect to the Levi-Civita connection. Also, we consider sequential warped product space-time models, namely sequential generalized Robertson-Walker space-times and sequential standard static space-times, with semi-symmetric metric connections and obtain conditions for such space-times to be Einstein.

1. Introduction

The concept of *singly warped product manifolds* or simply *warped product manifolds* was first defined by Bishop and O'Neill [2]. They used this concept to give examples of Riemannian manifolds admitting negative sectional curvature. Curvature of a warped product manifold in terms of the curvatures of the components included in the warped product, was released by O'Neill [9] and he also revealed the importance of warped products in physics by investigating Robertson–Walker, static, Schwarschild and Kruskal space–times as warped product manifolds, which are models using for finding exact solutions to Einstein's field equation.

As generalizations of singly warped product manifolds, *doubly warped products* and *multiwarped products*were introduced and covariant derivative formulas, geodesic equations for these spaces and applications for some generalized space-times such as generalized Robertson-Walker and generalized Kasner space-times were investigated (see [4], [5], [16], [17]).

A singly warped product was also naturally generalized to a new type product manifold, namely a twisted product, such that the warping function depends on the points of both factors [10]. Wang defined multiply twisted products by using the concept of multiply warped products and twisted products [18].

From a different perspective, a new class of warped product manifolds, called *sequential warped product manifolds*, was firstly introduced by Shenawy [12]. Sequential warped product manifolds are warped

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Email addresses: zerensemra@hotmail.com (Semra Zeren), sperktas@adiyaman.edu.tr (Selcen Yüksel Perktaş), a.yildiz@inonu.edu.tr (Ahmet Yıldız)

product manifolds where the base factor of the warped product itself is another warped product manifold. Then De, Shenawy and Ünal [3] studied the geometry of such manifolds, derived curvature tensor formulas and presented some classes of sequential warped product space-time models. In [15], Şahin initiated the study of sequential warped product submanifolds of Kaehler manifolds, gave examples and established Chen's inequality for such submanifolds.

The idea of *semi-symmetric linear connection*, a linear connection with non-zero torsion $\hat{T}(U, V) = \sigma(V)U - \sigma(U)V$, on a differentiable manifold was introduced by Friedmann and Schouten [6], where U, V are vector fields and σ is a 1-form on the manifold. Such a connection $\hat{\nabla}$ is called a semi-symmetric metric connection if there is a Riemannian metric g on the manifold satisfying $\hat{\nabla}g = 0$, otherwise it is named as non-metric. Hayden [7] defined a *semi-symmetric metric connection* on a Riemannian manifold and this concept was further developed by Yano [19]. After that, by adapting this topic to different manifolds and submanifolds, a serious resource has emerged in the literature. Of course, such a connection has physical meanings and applications. The movement of a person on the earth's surface by facing to a certain point, like Jerusalem, Mekka or North Pole, is a semi-symmetric and metric displacement (see [11]).

Sular and Özgür [13], studied warped products admitting semi-symmetric metric connection and obtained some results for such Einstein warped product manifolds. Einstein doubly warped product manifolds endowed with a semi-symmetric metric connection was investigated in [8]. Multiply warped product manifolds with respect to a semi-symmetric metric connection and applications of some results to the generalized space-times were given by [18].

Motivated by the above studies, in the present paper we consider sequential warped products with respect to a semi-symmetric metric connection.

2. Preliminaries

2.1. Semi-Symmetric Metric Connection

A linear connection $\hat{\nabla}$ on a Riemannian (as well as, semi-Riemannian) manifold N having torsion tensor $\hat{T}$ which satisfies

$$\hat{T}(U, V) = \sigma(V)U - \sigma(U)V, \tag{1}$$

is called a semi-symmetric connection. Here, σ is a 1-form associated with the vector field ξ on N defined by

$$\sigma(U) = g(U, \xi).$$

If $\hat{\nabla}g = 0$, then $\hat{\nabla}$ is said to be a semi-symmetric metric connection on the manifold. The relation between the Levi-Civita connection ∇ and the semi-symmetric metric connection $\hat{\nabla}$ of a Riemannian manifold (N, g) is given by [19]

$$\hat{\nabla}_U V = \nabla_U V + \sigma(V)U - g(U, V)\xi. \tag{2}$$

Moreover, the curvature tensors R and $\hat{R}$ of ∇ and $\hat{\nabla}$, respectively, are related by

$$\begin{aligned} \hat{R}(U, V)Y = R(U, V)Y &+ g(Y, \nabla_U \xi)V - g(Y, \nabla_V \xi)U \\ &+ g(U, Y)\nabla_V \xi - g(V, Y)\nabla_U \xi \\ &+ \sigma(\xi)[g(U, Y)V - g(V, Y)U] \\ &+ [g(V, Y)\sigma(U) - g(U, Y)\sigma(V)]\xi \\ &+ \sigma(Y)[\sigma(V)U - \sigma(U)V], \end{aligned} \tag{3}$$

for any vector fields U, V, Y on N [14].

A Riemannian manifold (N, g), $(n > 2)$, is called an *Einstein manifold* if the condition

$$S(U, V) = \varrho g(U, V) \tag{4}$$

is satisfied, where $\varrho = \frac{r}{n}$ and r denotes the scalar curvature of N. It is well-known that if $n > 2$, then ϱ is a constant [1].

2.2. Sequential Warped Product Manifolds

In this subsection, we give some basic notations for sequential warped product manifolds.

Definition 2.1. *[3] Let N_i be three semi-Riemann manifolds with metrics g_i, for $i = 1,2,3$, and $\lambda : N_1 \longrightarrow (0,\infty)$, $\mu : N_1 \times N_2 \longrightarrow (0,\infty)$ be two differentiable positive functions on N_1 and $N_1 \times N_2$, respectively. Then the sequential warped product manifold, denoted by $(N_1 \times_\lambda N_2) \times_\mu N_3$, is a triple product manifold $N = (N_1 \times N_2) \times N_3$ endowed with the metric tensor*

$$g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3. \tag{5}$$

The functions λ and μ are called warping functions. Note that if (N_i, g_i) are all Riemannian manifolds, for any $i = 1,2,3$, then the sequential warped product manifold $(N_1 \times_\lambda N_2) \times_\mu N_3$ is also a Riemannian manifold. One can easily observe that $(N_1 \times_\lambda N_2)$ is a warped product with the metric tensor $g_1 \oplus \lambda^2 g_2$ [3].

Proposition 2.2. *[3] Let $N = (N_1 \times_\lambda N_2) \times_\mu N_3$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$ and also let $U_i, V_i \in \chi(N_i)$, for any $i = 1,2,3$. Then*

1. $\nabla_{U_1} V_1 = \nabla^1_{U_1} V_1$,
2. $\nabla_{U_1} U_2 = \nabla_{U_2} U_1 = U_1(\ln\lambda) U_2$,
3. $\nabla_{U_2} V_2 = \nabla^2_{U_2} V_2 - \lambda g_2(U_2, V_2) \operatorname{grad}^1 \lambda$,
4. $\nabla_{U_3} U_1 = \nabla_{U_1} U_3 = U_1(\ln\mu) U_3$,
5. $\nabla_{U_2} U_3 = \nabla_{U_3} U_2 = U_2(\ln\mu) U_3$,
6. $\nabla_{U_3} V_3 = \nabla^3_{U_3} V_3 - \mu g_3(U_3, V_3) \operatorname{grad}\mu$,

where $\operatorname{grad}^1 \lambda$ *and* $\operatorname{grad}\mu$ *denote the gradient of λ on N_1 and the gradient of μ on $N_1 \times N_2$, respectively.*

Lemma 2.3. *[3] Let $N = (N_1 \times_\lambda N_2) \times_\mu N_3$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$ and also let $U_i, V_i, Y_i \in \chi(N_i)$, for any $i = 1,2,3$. Then*

1. $R(U_1, V_1)Y_1 = R^1(U_1, V_1)Y_1$,
2. $R(U_2, V_2)Y_2 = R^2(U_2, V_2)Y_2 - \left\|\operatorname{grad}^1 \lambda\right\|^2 \{g_2(U_2, Y_2)V_2 - g_2(V_2, Y_2)U_2\}$,
3. $R(U_1, V_2)Y_1 = -\frac{1}{\lambda} Hess^\lambda_1(U_1, Y_1)V_2$,
4. $R(U_1, V_2)Y_2 = \lambda g_2(V_2, Y_2)\nabla^1_{U_1} \operatorname{grad}^1 \lambda$,
5. $R(U_1, V_2)Y_3 = 0$,
6. $R(U_i, V_i)Y_j = 0, \quad i \neq j$,
7. $R(U_i, V_3)Y_j = -\frac{1}{\mu} Hess^\mu(U_i, Y_j)V_3, \quad i, j = 1,2$,
8. $R(U_i, V_3)Y_3 = \mu g_3(V_3, Y_3)\nabla_{U_i} \operatorname{grad}\mu, \quad i = 1,2$,
9. $R(U_3, V_3)Y_3 = R^3(U_3, V_3)Y_3 - \left\|\operatorname{grad}\mu\right\|^2 \{g_3(U_3, Y_3)V_3 - g_3(V_3, Y_3)U_3\}$,

where $\left\|\operatorname{grad}^1 \lambda\right\|^2 = g_1(\operatorname{grad}^1 \lambda, \operatorname{grad}^1 \lambda)$ *and* $\left\|\operatorname{grad}\mu\right\|^2 = g(\operatorname{grad}\mu, \operatorname{grad}\mu)$.

Lemma 2.4. *[3] Let $N = (N_1 \times_\lambda N_2) \times_\mu N_3$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$ and also let $U_i, V_i, Y_i \in \chi(N_i)$, for any $i = 1,2,3$. Then*

1. $S(U_1, V_1) = S^1(U_1, V_1) - \frac{n_2}{\lambda} Hess^\lambda_1(U_1, V_1) - \frac{n_3}{\mu} Hess^\mu(U_1, V_1)$,
2. $S(U_1, V_1) = S^2(U_2, V_2) - (\lambda\Delta^1\lambda + (n_2 - 1)\left\|\operatorname{grad}^1 \lambda\right\|^2) g_2(U_2, V_2) - \frac{n_3}{\mu} Hess^\mu(U_2, V_2)$,
3. $S(U_3, V_3) = S^3(U_3, V_3) - (\mu\Delta\mu + (n_3 - 1)\left\|\operatorname{grad}\mu\right\|^2) g_3(U_3, V_3)$,
4. $S(U_i, V_j) = 0, \quad i \neq j$.

3. Sequential Warped Product Manifolds With Respect to A Semi-Symmetric Metric Connection

In this section, we consider a sequential warped product manifold endowed with a semi-symmetric metric connection and give the fundamental relations between the curvature, Ricci curvature and scalar curvature of the manifold with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively.

Lemma 3.1. *Let $N = (N_1 \times_\lambda N_2) \times_\mu N_3$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$ and the associated vector field $\xi \in \chi(N_1)$. Then we have*

1. $\hat{\nabla}_{U_1} V_1 = \hat{\nabla}^1_{U_1} V_1$,
2. $\hat{\nabla}_{U_1} U_2 = U_1 (\ln \lambda)\, U_2, \quad \hat{\nabla}_{U_2} U_1 = (U_1 (\ln \lambda) + \sigma(U_1))\, U_2$,
3. $\hat{\nabla}_{U_1} U_3 = U_1 (\ln \mu)\, U_3, \quad \hat{\nabla}_{U_3} U_1 = (U_1 (\ln \mu) + \sigma(U_1))\, U_3$,
4. $\hat{\nabla}_{U_2} V_2 = \nabla^2_{U_2} V_2 - \lambda g_2(U_2, V_2)\, \mathrm{grad}^1 \lambda - g(U_2, V_2)\xi$,
5. $\hat{\nabla}_{U_2} U_3 = \hat{\nabla}_{U_3} U_2 = U_2(\ln \mu) U_3$
6. $\hat{\nabla}_{U_3} V_3 = \nabla^3_{U_3} V_3 - \mu g_3(U_3, V_3)\, \mathrm{grad}\, \mu - g(U_3, V_3)\xi$,

where $U_i, V_i \in \chi(N_i)$, for any $i = 1, 2, 3$, $\hat{\nabla}^1$ is the lift of semi-symmetric metric connection $\hat{\nabla}$ on N_1, ∇^2 and ∇^3 are the lifts of ∇ on N_2 and N_3, respectively.

Proof. By using Kozsul formula, we write

$$2g(\nabla_{U_1} V_1, U_2) = U_1 g(V_1, U_2) + V_1 g(U_1, U_2) - U_2 g(U_1, V_1) - g(U_1, [V_1, U_2]) - g(V_1, [U_1, U_2]) + g(U_2, [U_1, V_1]), \tag{6}$$

for all vector fields $U_1, V_1 \in \chi(N_1)$, and $U_2 \in \chi(N_2)$, where ∇ is the Levi-Civita connection on N. From (2) the last equation reduces to

$$\begin{aligned} 2g(\hat{\nabla}_{U_1} V_1, U_2) = {} & U_1 g(V_1, U_2) + V_1 g(U_1, U_2) - U_2 g(U_1, V_1) \\ & - g(U_1, [V_1, U_2]) - g(V_1, [U_1, U_2]) + g(U_2, [U_1, V_1]) \\ & - 2\sigma(U_2) g(U_1, V_1) + 2\sigma(V_1) g(U_1, U_2). \end{aligned} \tag{7}$$

It is well known that $[V_1, U_2] = 0 = [U_1, U_2]$, $[U_1, V_1] \in \chi(N_1)$ and $g(V_1, U_2) = 0 = g(U_1, U_2)$. So we get

$$2g(\hat{\nabla}_{U_1} V_1, U_2) = -U_2 g(U_1, V_1) - 2\sigma(U_2) g(U_1, V_1),$$

which implies $g(\hat{\nabla}_{U_1} V_1, U_2) = 0$, by using the fact that $g(U_1, V_1)$ is constant on N_2 and $\xi \in \chi(N_1)$. Similarly, since we have $g(\hat{\nabla}_{U_1} V_1, U_3) = 0$, for any $U_3 \in \chi(N_3)$, then we obtain (1).

We know that g is a metric connection with respect to $\hat{\nabla}$ and $g(U_2, V_1) = 0$. So we write

$$g(\hat{\nabla}_{U_1} U_2, V_1) = -g(\hat{\nabla}_{U_1} V_1, U_2) = 0.$$

Also from Kozsul formula, we have

$$\begin{aligned} 2g(\hat{\nabla}_{U_1} U_2, U_3) = {} & U_1 g(U_2, U_3) + U_2 g(U_1, U_3) - U_3 g(U_1, U_2) \\ & - g(U_1, [U_2, U_3]) - g(U_2, [U_1, U_3]) + g(U_3, [U_1, U_2]) \\ & - 2\sigma(U_3) g(U_1, U_2) + 2\sigma(U_2) g(U_1, U_3), \end{aligned} \tag{8}$$

which implies $g(\hat{\nabla}_{U_1} U_2, U_3) = 0$, via $g(U_i, U_j) = 0$, $\left[U_i, U_j\right] = 0$, $i \neq j$, for $U_i \in \chi(N_i)$, $i = 1, 2, 3$, and $\xi \in \chi(N_1)$.

By taking V_2 instead of U_3 in (8), we get

$$2g(\hat{\nabla}_{U_1} U_2, V_2) = U_1 g(U_2, V_2), \quad \text{for any } V_2 \in \chi(N_2).$$

Since $g_2(U_2, V_2)$ is constant on N_1, from (5), we obtain

$$U_1 g(U_2, V_2) = 2\lambda U_1(\lambda)\, g_2(U_2, V_2) = 2U_1(\ln\lambda)\, g(U_2, V_2),$$

which implies the first part of (2). Changing the roles of U_1 and U_2 in (8) and taking $U_3 = V_1$ gives $g(\hat{\nabla}_{U_2} U_1, V_1) = 0 = g(\hat{\nabla}_{U_2} U_1, U_3)$. If we substitute U_3 for V_2 with U_1 and U_2 interchanged in (8) and use (5), we get

$$\begin{aligned} 2g(\hat{\nabla}_{U_2} U_1, V_2) &= U_1 g(U_2, V_2) + 2\sigma(U_1) g(U_2, V_2) \\ &= 2\left(U_1(\ln\lambda) + \sigma(U_1)\right) g(U_2, V_2), \end{aligned}$$

which gives the second part of (2).

From Kozsul formula, we again have

$$\begin{aligned} 2g(\hat{\nabla}_{U_1} U_3, V_1) &= U_1 g(U_3, V_1) + U_3 g(U_1, V_1) - V_1 g(U_1, U_3) \\ &\quad - g(U_1, [U_3, V_1]) - g(U_3, [U_1, V_1]) + g(V_1, [U_1, U_3]) \\ &\quad - 2\sigma(V_1) g(U_1, U_3) + 2\sigma(U_3) g(U_1, V_1), \end{aligned} \tag{9}$$

which implies $2g(\hat{\nabla}_{U_1} U_3, V_1) = U_3 g(U_1, V_1) = 0$, for any $U_1, V_1 \in \chi(N_1)$ and $U_3 \in \chi(N_3)$. Taking $V_1 = U_2$ in (9) we get $g(\hat{\nabla}_{U_1} U_3, U_2) = 0$. Also, by writing V_3 instead of V_1 in (9) we obtain

$$2g(\hat{\nabla}_{U_1} U_3, V_3) = U_1 g(U_3, V_3),$$

which gives $2g(\hat{\nabla}_{U_1} U_3, V_3) = U_1(\ln\mu)\, g(U_3, V_3)$. So we have the first part of (3). For the second part of (3), we change the roles of U_1 and U_3 in (9) and we get $2g(\hat{\nabla}_{U_3} U_1, V_1) = U_3 g(U_1, V_1) = 0$. Taking $V_1 = U_2$ and $V_1 = V_3$ in (9), respectively, with U_1 and U_3 interchanged gives $g(\hat{\nabla}_{U_3} U_1, U_2) = 0$ and

$$\begin{aligned} 2g(\hat{\nabla}_{U_3} U_1, V_3) &= U_1 g(U_3, V_3) + 2\sigma(U_1) g(U_3, V_3) \\ &= 2\left(U_1(\ln\mu) + \sigma(U_1)\right) g(U_3, V_3), \end{aligned}$$

respectively.

For any $U_1 \in \chi(N_1)$ and $U_2, V_2 \in \chi(N_2)$, we write

$$\begin{aligned} 2g(\hat{\nabla}_{U_2} V_2, U_1) &= U_2 g(V_2, U_1) + V_2 g(U_2, U_1) - U_1 g(U_2, V_2) \\ &\quad - g(U_2, [V_2, U_1]) - g(V_2, [U_2, U_1]) + g(U_1, [U_2, V_2]) \\ &\quad - 2\sigma(U_1) g(U_2, V_2) + 2\sigma(V_2) g(U_2, U_1), \end{aligned} \tag{10}$$

which implies

$$\begin{aligned} 2g(\hat{\nabla}_{U_2} V_2, U_1) &= -U_1 g(U_2, V_2) - 2\sigma(U_1) g(U_2, V_2) \\ &= -2\lambda g(\operatorname{grad}\lambda, U_1) g_2(U_2, V_2) - 2g(\xi, U_1) g(U_2, V_2) \\ &= -2g(\lambda g_2(U_2, V_2)\operatorname{grad}\lambda + g(U_2, V_2)\xi, U_1). \end{aligned}$$

We also have $g(\hat{\nabla}_{U_2} V_2, U_3) = 0$, by taking $U_1 = U_3$ in (10). By taking $U_1 = Y_2$ in (10), we conclude the proof of (4).

Since $g(U_1, U_3) = 0 = g(V_2, U_3)$, then we get $g(\hat{\nabla}_{U_2} U_3, U_1) = -g(U_3, \hat{\nabla}_{U_2} U_1) = 0$ and $g(\hat{\nabla}_{U_2} U_3, V_2) = -g(U_3, \hat{\nabla}_{U_2} V_2) = 0$, by use of (2) and (4), respectively. Also, from Kozsul formula we have

$$\begin{aligned} 2g(\hat{\nabla}_{U_2} U_3, V_3) &= U_2 g(U_3, V_3) + U_3 g(U_2, V_3) - V_3 g(U_2, U_3) \\ &\quad - g(U_2, [U_3, V_3]) - g(U_3, [U_2, V_3]) + g(V_3, [U_2, U_3]) \\ &\quad - 2\sigma(V_3) g(U_2, U_3) + 2\sigma(U_3) g(U_2, V_3), \end{aligned} \tag{11}$$

which implies

$$2g(\hat{\nabla}_{U_2}U_3, V_3) = U_2 g(U_3, V_3) = g(U_2(\ln\mu)U_3, V_3).$$

Hence we obtain (5).

Finally, by putting $U_2 = U_3$ and $V_2 = V_3$ in (10) and following the similar steps in the proof of (4) we conclude the proof of (6). □

In the following two lemmas, we present the results for a semi-symmetric metric connection in case of $\xi \in \chi(N_2)$ and $\xi \in \chi(N_3)$, respectively, by omitting the proofs to avoid repetition.

Lemma 3.2. *Let $N = (N_1 \times_\lambda N_2) \times_\mu N_3$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$ and the associated vector field $\xi \in \chi(N_2)$. Then we have*

1. $\hat{\nabla}_{U_1}V_1 = \nabla^1_{U_1}V_1 - g(U_1, V_1)\xi$,
2. $\hat{\nabla}_{U_1}U_2 = U_1(\ln\lambda)U_2 + \sigma(U_2)U_1, \quad \hat{\nabla}_{U_2}U_1 = U_1(\ln\lambda)U_2$,
3. $\hat{\nabla}_{U_1}U_3 = \hat{\nabla}_{U_3}U_1 = U_1(\ln\mu)U_3$,
4. $\hat{\nabla}_{U_2}V_2 = \hat{\nabla}^2_{U_2}V_2 - \lambda g_2(U_2, V_2)\,\mathrm{grad}^1\lambda$,
5. $\hat{\nabla}_{U_2}U_3 = U_2(\ln\mu)U_3, \quad \hat{\nabla}_{U_3}U_2 = (U_2(\ln\mu) + \sigma(U_2))U_3$
6. $\hat{\nabla}_{U_3}V_3 = \nabla^3_{U_3}V_3 - \mu g_3(U_3, V_3)\,\mathrm{grad}\,\mu - g(U_3, V_3)\xi$,

where $U_i, V_i \in \chi(N_i)$, for any $i = 1, 2, 3$, $\hat{\nabla}^2$ is the lift of semi-symmetric metric connection $\hat{\nabla}$ on N_2, ∇^1 and ∇^3 are the lifts of ∇ on N_1 and N_3, respectively.

Lemma 3.3. *Let $N = (N_1 \times_\lambda N_2) \times_\mu N_3$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$ and the associated vector field $\xi \in \chi(N_3)$. Then we have*

1. $\hat{\nabla}_{U_1}V_1 = \nabla^1_{U_1}V_1 - g(U_1, V_1)\xi$,
2. $\hat{\nabla}_{U_1}U_2 = \hat{\nabla}_{U_2}U_1 = U_1(\ln\lambda)U_2$,
3. $\hat{\nabla}_{U_1}U_3 = U_1(\ln\mu)U_3 + \sigma(U_3)U_1, \quad \hat{\nabla}_{U_3}U_1 = U_1(\ln\mu)U_3$,
4. $\hat{\nabla}_{U_2}V_2 = \nabla^2_{U_2}V_2 - \lambda g_2(U_2, V_2)\,\mathrm{grad}^1\lambda - g(U_2, V_2)\xi$,
5. $\hat{\nabla}_{U_2}U_3 = U_2(\ln\mu)U_3 + \sigma(U_3)U_1, \quad \hat{\nabla}_{U_3}U_2 = U_2(\ln\mu)U_3$,
6. $\hat{\nabla}_{U_3}V_3 = \hat{\nabla}^3_{U_3}V_3 - \mu g_3(U_3, V_3)\,\mathrm{grad}\,\mu$,

where $U_i, V_i \in \chi(N_i)$, for any $i = 1, 2, 3$, $\hat{\nabla}^3$ is the lift of semi-symmetric metric connection $\hat{\nabla}$ on N_3, ∇^1 and ∇^2 are the lifts of ∇ on N_1 and N_2, respectively.

Lemma 3.4. *Let $N = (N_1 \times_\lambda N_2) \times_\mu N_3$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$, $\xi \in \chi(N_1)$, R and $\hat{R}$ be the Riemannian curvature tensors of N with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively. Then we have*

1. $\hat{R}(U_1, V_1)Y_1 \in \chi(N_1)$ *is the lift of* $\hat{R}^1(U_1, V_1)Y_1$ *on* N_1,
2. $\hat{R}(U_1, U_2)V_1 = -\begin{pmatrix} \frac{1}{\lambda}Hess_1^\lambda(U_1, V_1) - (\xi(\ln\lambda) + \eta(\xi))g(U_1, V_1) \\ -g(V_1, \nabla^1_{U_1}\xi) + \sigma(U_1)\sigma(V_1) \end{pmatrix} U_2$,
3. $\hat{R}(U_1, U_2)V_2 = g(U_2, V_2)\begin{pmatrix} \frac{1}{\lambda}(\nabla^1_{U_1}\mathrm{grad}^1\lambda) - \xi(\ln\lambda)U_1 - \nabla^1_{U_1}\xi \\ -\sigma(\xi)U_1 + \sigma(U_1)\xi \end{pmatrix}$,
4. $\hat{R}(U_2, V_2)Y_2 = R^2(U_2, V_2)Y_2 - \begin{pmatrix} \frac{1}{\lambda^2}\left\|\mathrm{grad}^1\lambda\right\|^2 \\ -2\xi(\ln\lambda) - \sigma(\xi) \end{pmatrix}(g(Y_2, U_2)V_2 - g(Y_2, V_2)U_2)$,
5. $\hat{R}(U_1, U_2)U_3 = 0$,

6. $\hat{R}(U_i, V_i)Y_j = 0, \quad i \neq j,$

7. $\hat{R}(U_1, U_3)V_1 = -\begin{pmatrix} \frac{1}{\mu}Hess^{\mu}(U_1, V_1) - g(V_1, \nabla_{U_1}\xi) \\ -(\xi(\ln\mu) + \sigma(\xi))\, g(U_1, V_1) + \sigma(U_1)\sigma(V_1) \end{pmatrix} U_3,$

8. $\hat{R}(U_1, U_3)U_2 = -\frac{1}{\mu}Hess^{\mu}(U_1, U_2)U_3,$

9. $\hat{R}(U_2, U_3)U_1 = -\frac{1}{\mu}Hess^{\mu}(U_2, U_1)U_3,$

10. $\hat{R}(U_2, U_3)V_2 = -\begin{pmatrix} \frac{1}{\mu}Hess^{\mu}(U_2, V_2) \\ -(\xi(\ln\lambda) + \xi(\ln\mu) + \eta(\xi))\, g(U_2, V_2) \end{pmatrix} U_3,$

11. $\hat{R}(U_1, U_3)V_3 = \begin{matrix} \mu g_3(U_3, V_3)\nabla_{U_1}\operatorname{grad}\mu \\ -g(U_3, V_3)\left((\xi(\ln\mu) + \sigma(\xi))\, U_1 + \nabla^1_{U_1}\xi - \sigma(U_1)\xi\right) \end{matrix},$

12. $\hat{R}(U_2, U_3)V_3 = \begin{matrix} \mu g_3(U_3, V_3)\nabla_{U_2}\operatorname{grad}\mu \\ -g(U_3, V_3)\,(\xi(\ln\mu) + \xi(\ln\lambda) + \sigma(\xi))\, U_2 \end{matrix},$

13. $\hat{R}(U_3, V_3)Y_3 = R^3(U_3, V_3)Y_3 - \begin{pmatrix} \left\|\operatorname{grad}\mu\right\|^2 \\ -2\mu^2\xi(\ln\mu) - \mu^2\eta(\xi) \end{pmatrix} (g_3(U_3, Y_3)V_3 - g_3(V_3, Y_3)U_3),$

for all $U_i, V_i, Y_i \in \chi(N_i)$.

Proof. 1. Since $\hat{\nabla}_{U_1}V_1 = \hat{\nabla}^1_{U_1}V_1$, for any $U_1, V_1 \in \chi(N_1)$, then from (3), one can easily see that $\hat{R}(U_1, V_1)Y_1 \in \chi(N_1)$ is the lift of $\hat{R}^1(U_1, V_1)Y_1$ on N_1.

2. By using (3) and $\xi \in \chi(N_1)$, we write

$$\begin{aligned}\hat{R}(U_1, U_2)V_1 = {} & R(U_1, U_2)V_1 + g(V_1, \nabla_{U_1}\xi)U_2 - g(V_1, \nabla_{U_2}\xi)U_1 \\ & + g(U_1, V_1)\nabla_{U_2}\xi - g(U_2, V_1)\nabla_{U_1}\xi \\ & + \sigma(\xi)[g(U_1, V_1)U_2 - g(U_2, V_1)U_1] \\ & + [g(U_2, V_1)\sigma(U_1) - g(U_1, V_1)\sigma(U_2)]\xi \\ & + \sigma(V_1)[\sigma(U_2)U_1 - \sigma(U_1)U_2],\end{aligned}$$

which gives

$$\begin{aligned}\hat{R}(U_1, U_2)V_1 &= R(U_1, U_2)V_1 + g(V_1, \nabla^1_{U_1}\xi)U_2 - g(V_1, \xi(\ln\lambda)U_2)U_1 \\ &\quad + g(U_1, V_1)\xi(\ln\lambda)U_2 - g(U_2, V_1)\nabla^1_{U_1}\xi \\ &\quad + \sigma(\xi)[g(U_1, V_1)U_2 - g(U_2, V_1)U_1] \\ &\quad + [g(U_2, V_1)\sigma(U_1) - g(U_1, V_1)\sigma(U_2)]\xi \\ &\quad + \sigma(V_1)[\sigma(U_2)U_1 - \sigma(U_1)U_2] \\ &= -\frac{1}{\lambda}Hess^{\lambda}_1(U_1, V_1)U_2 + g(V_1, \nabla^1_{U_1}\xi)U_2 + g(U_1, V_1)\xi(\ln\lambda)U_2 \\ &\quad + \sigma(\xi)g(U_1, V_1)U_2 - \sigma(U_1)\sigma(V_1)U_2\end{aligned}$$

via Proposition 2.2 (1). From Lemma 2.3, we obtain

$$\hat{R}(U_1, U_2)V_1 = -\left\{ \begin{matrix} \frac{1}{\lambda}Hess^{\lambda}_1(U_1, V_1) - (\xi(\ln\lambda) + \sigma(\xi))g(U_1, V_1) \\ -g(V_1, \nabla^1_{U_1}\xi) + \sigma(U_1)\sigma(V_1) \end{matrix} \right\} U_2.$$

3. From (3), Lemma 2.3 and Proposition 2.2, we get

$$\begin{aligned}
\hat{R}(U_1,U_2)V_2 &= R(U_1,U_2)V_2 + g(V_2,\nabla_{U_1}\xi)U_2 - g(V_2,\nabla_{U_2}\xi)U_1 \\
&\quad + g(U_1,V_2)\nabla_{U_2}\xi - g(U_2,V_2)\nabla_{U_1}\xi \\
&\quad + \sigma(\xi)[g(U_1,V_2)U_2 - g(U_2,V_2)U_1] \\
&\quad + [g(U_2,V_2)\sigma(U_1) - g(U_1,V_2)\sigma(U_2)]\xi \\
&\quad + \sigma(V_2)[\sigma(U_2)U_1 - \sigma(U_1)U_2], \\
&= \lambda g_2(U_2,V_2)\nabla^1_{U_1}\operatorname{grad}^1\lambda - g(V_2,\xi(\ln\lambda)U_2)U_1 - g(U_2,V_2)\nabla^1_{U_1}\xi \\
&\quad - \sigma(\xi)g(U_2,V_2)U_1 + g(U_2,V_2)\sigma(U_1)\xi,
\end{aligned}$$

which implies (3).

4. For any $U_2, V_2, Y_2 \in \chi(N_2)$, we have

$$\begin{aligned}
\hat{R}(U_2,V_2)Y_2 &= R(U_2,V_2)Y_2 + g(Y_2,\nabla_{U_2}\xi)V_2 - g(Y_2,\nabla_{V_2}\xi)U_2 \\
&\quad + g(U_2,Y_2)\nabla_{V_2}\xi - g(V_2,Y_2)\nabla_{U_2}\xi \\
&\quad + \sigma(\xi)[g(U_2,Y_2)V_2 - g(V_2,Y_2)U_2] \\
&\quad + [g(V_2,Y_2)\sigma(U_2) - g(U_2,Y_2)\sigma(V_2)]\xi \\
&\quad + \sigma(Y_2)[\sigma(V_2)U_2 - \sigma(U_2)V_2] \\
&= R^2(U_2,V_2)Y_2 - \left\|\operatorname{grad}^1\lambda\right\|^2\{g_2(U_2,Y_2)V_2 - g_2(V_2,Y_2)U_2\} \\
&\quad + g(Y_2,\xi(\ln\lambda)U_2)V_2 - g(Y_2,\xi(\ln\lambda)V_2)U_2 \\
&\quad + g(U_2,Y_2)\xi(\ln\lambda)V_2 - g(Y_2,V_2)\xi(\ln\lambda)U_2 \\
&\quad + \sigma(\xi)[g(U_2,Y_2)V_2 - g(V_2,Y_2)U_2] \\
&= R^2(U_2,V_2)Y_2 - \frac{1}{\lambda^2}\left\|\operatorname{grad}^1\lambda\right\|^2\{g(U_2,Y_2)V_2 - g(V_2,Y_2)U_2\} \\
&\quad + (2\xi(\ln\lambda) + \sigma(\xi))\{g(Y_2,U_2)V_2 - g(Y_2,V_2)U_2\}.
\end{aligned}$$

5. By using (3) and the orthogonality of vector fields, we obtain

$$\begin{aligned}
\tilde{R}(U_1,U_2)U_3 &= R(U_1,U_2)U_3 + g(U_3,\nabla_{U_1}\xi)U_2 - g(U_3,\nabla_{U_2}\xi)U_1 \\
&\quad + g(U_1,U_3)\nabla_{U_2}\xi - g(U_2,U_3)\nabla_{U_1}\xi \\
&\quad + \sigma(\xi)[g(U_1,U_3)U_2 - g(U_2,U_3)U_1] \\
&\quad + [g(U_2,U_3)\sigma(U_1) - g(U_1,U_3)\sigma(U_2)]\xi \\
&\quad + \sigma(U_3)[\sigma(U_2)U_1 - \sigma(U_1)U_2], \\
&= 0.
\end{aligned}$$

6. For any $U_1, V_1 \in \chi(N_1)$ and $Y_2 \in \chi(N_2)$, since $R(U_1,V_1)Y_2 = 0$, then we have

$$\begin{aligned}
\hat{R}(U_1,V_1)Y_2 &= R(U_1,V_1)Y_2 + g(Y_2,\nabla_{U_1}\xi)V_1 - g(Y_2,\nabla_{V_1}\xi)U_1 \\
&\quad + g(U_1,Y_2)\nabla_{V_1}\xi - g(V_1,Y_2)\nabla_{U_1}\xi \\
&\quad + \sigma(\xi)[g(U_1,Y_2)V_1 - g(V_1,Y_2)U_1] \\
&\quad + [g(V_1,Y_2)\sigma(U_1) - g(U_1,Y_2)\sigma(V_1)]\xi \\
&\quad + \sigma(Y_2)[\sigma(V_1)U_1 - \sigma(U_1)V_1], \\
&= 0.
\end{aligned}$$

7. Using (3) and Lemma 2.3, we have

$$\begin{aligned}\hat{R}(U_1,U_3)V_1 &= R(U_1,U_3)V_1 + g(V_1,\nabla_{U_1}\xi)U_3 - g(V_1,\nabla_{U_3}\xi)U_1\\ &+ g(U_1,V_1)\nabla_{U_3}\xi - g(U_3,V_1)\nabla_{U_1}\xi\\ &+ \sigma(\xi)[g(U_1,V_1)U_3 - g(U_3,V_1)U_1]\\ &+ [g(U_3,V_1)\sigma(U_1) - g(U_1,V_1)\sigma(U_3)]\xi\\ &+ \sigma(V_1)[\sigma(U_3)U_1 - \sigma(U_1)U_3]\\ &= -\frac{1}{\mu}Hess^{\mu}(U_1,V_1)U_3 + g(V_1,\nabla_{U_1}\xi)U_3\\ &+ g(U_1,V_1)\xi(\ln\mu)U_3 + \sigma(\xi)g(U_1,V_1)U_3 - \sigma(U_1)\sigma(V_1)U_3\end{aligned}$$

8. From (3) and Lemma 2.3, we write

$$\begin{aligned}\hat{R}(U_1,U_3)U_2 &= R(U_1,U_3)U_2 + g(U_2,\nabla_{U_1}\xi)U_3 - g(U_2,\nabla_{U_3}\xi)U_1\\ &+ g(U_1,U_2)\nabla_{U_3}\xi - g(U_3,U_2)\nabla_{U_1}\xi\\ &+ \sigma(\xi)[g(U_1,U_2)U_3 - g(U_3,U_2)U_1]\\ &+ [g(U_3,U_2)\sigma(U_1) - g(U_1,U_2)\sigma(U_3)]\xi\\ &+ \sigma(U_2)[\sigma(U_3)U_1 - \sigma(U_1)U_3],\end{aligned}$$

which implies (8).

9. Changing the roles of U_1 and U_2 in the last equation above we get (9).

10. For any $U_2, V_2 \in \chi(N_2)$, $U_3 \in \chi(N_3)$, we have

$$\begin{aligned}\hat{R}(U_2,U_3)V_2 &= R(U_2,U_3)V_2 + g(V_2,\nabla_{U_2}\xi)U_3 - g(V_2,\nabla_{U_3}\xi)U_2\\ &+ g(U_2,V_2)\nabla_{U_3}\xi - g(U_3,V_2)\nabla_{U_2}\xi\\ &+ \sigma(\xi)[g(U_2,V_2)U_3 - g(U_3,V_2)U_2]\\ &+ [g(U_3,V_2)\sigma(U_2) - g(U_2,V_2)\sigma(U_3)]\xi\\ &+ \sigma(V_2)[\sigma(U_3)U_2 - \sigma(U_2)U_3]\\ &= -\frac{1}{\mu}Hess^{\mu}(U_2,V_2)U_3 + \xi(\ln\lambda)g(U_2,V_2)U_3\\ &+ g(U_2,V_2)\xi(\ln\mu)U_3 + \sigma(\xi)g(U_2,V_2)U_3,\end{aligned}$$

and we complete proof of (10).

11. By use of (3) we get

$$\begin{aligned}\tilde{R}(U_1,U_3)V_3 &= R(U_1,U_3)V_3 + g(V_3,\nabla_{U_1}\xi)U_3 - g(V_3,\nabla_{U_3}\xi)U_1\\ &+ g(U_1,V_3)\nabla_{U_3}\xi - g(U_3,V_3)\nabla_{U_1}\xi\\ &+ \sigma(\xi)[g(U_1,V_3)U_3 - g(U_3,V_3)U_1]\\ &+ [g(U_3,V_3)\sigma(U_1) - g(U_1,V_3)\sigma(U_3)]\xi\\ &+ \sigma(V_3)[\sigma(U_3)U_1 - \sigma(U_1)U_3]\\ &= \mu g_3(U_3,V_3)\nabla_{U_1}\,\mathrm{grad}\,\mu - \xi(\ln\mu)g(U_3,V_3)U_1 - g(U_3,V_3)\nabla^1_{U_1}\xi\\ &- \sigma(\xi)g(U_3,V_3)U_1 + g(U_3,V_3)\sigma(U_1)\xi,\end{aligned}$$

which gives (11).

12. If we replace U_1 with U_2 in the last equation, we obtain (12).

13. Putting $U_1 = U_3$ in the equation (11) then we can easily get (13). □

The relations between R and $\hat{R}$ when $\xi \in \chi(N_2)$ and $\xi \in \chi(N_3)$ are given by the following lemmas.

Lemma 3.5. *Let $N = (N_1 \times_\lambda N_2) \times_\mu N_3$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$, $\xi \in \chi(N_2)$, R and $\hat{R}$ be the Riemannian curvature tensors of N with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively. Then we have*

$$\begin{aligned}\hat{R}(U_1, V_1)Y_1 &= R^1(U_1, V_1)Y_1 + \sigma(\xi)\left(g(U_1, Y_1)V_1 - g(V_1, Y_1)U_1\right)\\ &\quad + \left(g(U_1, Y_1)V_1(\ln\lambda) - g(V_1, Y_1)U_1(\ln\lambda)\right)\xi,\end{aligned}$$

$$\begin{aligned}\hat{R}(U_1, U_2)V_1 &= \lambda g_2(U_2, \xi)g(\mathrm{grad}^1\lambda, V_1)U_1 - \lambda g_2(U_2, \xi)g(U_1, V_1)\,\mathrm{grad}^1\lambda\\ &\quad - \frac{1}{\lambda}Hess_1^\lambda(U_1, V_1)U_2 + g(U_1, V_1)\left(\nabla^2_{U_2}\xi + \sigma(\xi)U_2 - \sigma(U_2)\xi\right),\end{aligned}$$

$$\begin{aligned}\hat{R}(U_1, U_2)V_2 &= g(U_2, V_2)\left(\frac{1}{\lambda}\nabla_{U_1}\mathrm{grad}\,\lambda - \sigma(\xi)U_1\right) - \left(g(V_2, \nabla^2_{U_2}\xi) - \sigma(V_2)\sigma(U_2)\right)U_1\\ &\quad + U_1(\ln\lambda)\left(\sigma(V_2)U_2 - g(U_2, V_2)\xi\right),\end{aligned}$$

$$\begin{aligned}\tilde{R}(U_2, V_2)Y_2 &= R^2(U_2, V_2)Y_2 - \left(\frac{1}{\lambda^2}\left\|\mathrm{grad}^1\lambda\right\|^2 - \sigma(\xi)\right)\left(g(U_2, Y_2)V_2 - g(V_2, Y_2)U_2\right)\\ &\quad + g(Y_2, \nabla^2_{U_2}\xi)V_2 - g(Y_2, \nabla^2_{V_2}\xi)U_2 + g(U_2, Y_2)\nabla_{V_2}\xi - g(V_2, Y_2)\nabla_{U_2}\xi\\ &\quad + \left(g(V_2, Y_2)\sigma(U_2) - g(U_2, Y_2)\sigma(V_2)\right)\xi + \sigma(Y_2)\left(\sigma(V_2)U_2 - \sigma(U_2)V_2\right),\end{aligned}$$

$$\hat{R}(U_1, U_2)U_3 = \hat{R}(U_1, V_1)U_3 = \hat{R}(U_2, V_2)U_3 = \hat{R}(U_3, V_3)U_1 = \hat{R}(U_3, V_3)U_2 = 0,$$

$$\hat{R}(U_1, V_1)U_2 = \sigma(U_2)[U_1(\ln\lambda)V_1 - V_1(\ln\lambda)U_1],$$

$$\hat{R}(U_2, V_2)U_1 = U_1(\ln\lambda)[\sigma(V_2)U_2 - \sigma(U_2)V_2],$$

$$\hat{R}(U_1, U_3)V_1 = -\frac{1}{\mu}Hess^\mu(U_1, V_1)U_3 + g(U_1, V_1)\left[\xi(\ln\mu) + \sigma(\xi)\right]U_3,$$

$$\begin{aligned}\hat{R}(U_2, U_3)V_2 &= -\frac{1}{\mu}Hess^\mu(U_2, V_2) + g(V_2, \nabla^2_{U_2}\xi)\\ &\quad + g(U_2, V_2)\xi(\ln\mu) + \sigma(\xi)g(U_2, V_2) - \sigma(V_2)\sigma(U_2)U_3,\end{aligned}$$

$$\hat{R}(U_1, U_3)U_2 = -\frac{1}{\mu}Hess^\mu(U_1, U_2)U_3 + \sigma(U_2)U_1(\ln\lambda)U_3,$$

$$\hat{R}(U_2, U_3)U_1 = \left(-\frac{1}{\mu}Hess^\mu(U_2, U_1) - U_1(\ln\lambda)\sigma(U_2)\right)U_3,$$

$$\hat{R}(U_1, U_3)V_3 = \mu g_3(U_3, V_3)\nabla_{U_1} \operatorname{grad} \mu - (\xi (\ln \mu) U_1 + \sigma(\xi)U_1 + U_1(\ln \lambda)\xi) g(U_3, V_3),$$

$$\hat{R}(U_2, U_3)V_3 = \mu g_3(U_3, V_3)\nabla_{U_2} \operatorname{grad} \mu - (\xi (\ln \mu) + \sigma(\xi)) g(U_3, V_3)U_2 - \left(\nabla^2_{U_2}\xi - \lambda g_2(U_2, \xi) \operatorname{grad}^1 \lambda - \sigma(U_2)\xi\right) g(U_3, V_3),$$

$$\hat{R}(U_3, V_3)Y_3 = R^3(U_3, V_3)Y_3 - \left(\left\|\operatorname{grad} \mu\right\|^2 - 2\mu^2\xi(\ln \mu) - \mu^2\sigma(\xi)\right)\{g_3(U_3, Y_3)V_3 - g_3(V_3, Y_3)U_3\},$$

for all $U_i, V_i, Y_i \in \chi(N_i)$.

Lemma 3.6. *Let* $N = (N_1 \times_\lambda N_2) \times_\mu N_3$ *be a sequential warped product manifold with metric* $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$, $\xi \in \chi(N_3)$, *R and* $\hat{R}$ *be the Riemannian curvature tensors of N with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively. Then we have*

$$\hat{R}(U_1, V_1)Y_1 = R^1(U_1, V_1)Y_1 + (g(U_1, Y_1)V_1(\ln \mu) - g(V_1, Y_1)U_1(\ln \mu))\, \xi + \sigma(\xi) (g(U_1, Y_1)V_1 - g(V_1, Y_1)U_1),$$

$$\hat{R}(U_1, U_2)V_1 = \left(-\frac{1}{\lambda} Hess_1^\lambda(U_1, V_1) + \sigma(\xi)g(U_1, V_1)\right)U_2 + g(U_1, V_1)U_2(\ln \mu)\xi,$$

$$\hat{R}(U_1, U_2)V_2 = g(U_2, V_2)\left(\frac{1}{\lambda}\nabla^1_{U_1} \operatorname{grad}^1 \lambda - U_1(\ln \mu)\xi - \sigma(\xi)U_1\right),$$

$$\hat{R}(U_2, V_2)Y_2 = R^2(U_2, V_2)Y_2 - \left(\frac{1}{\lambda^2}\left\|\operatorname{grad}^1 \lambda\right\|^2 - \sigma(\xi)\right)[g(U_2, Y_2)V_2 - g(V_2, Y_2)U_2] + (g(U_2, Y_2)V_2(\ln \mu) - g(V_2, Y_2)U_2(\ln \mu))\, \xi,$$

$$\tilde{R}(U_1, U_2)U_3 = \sigma(U_3)[U_1(\ln \mu)U_2 - U_2(\ln \mu)U_1],$$

$$\hat{R}(U_i, V_i)Y_j = 0, i \neq j, \quad i = 1, 2,$$

$$\hat{R}(U_i, V_i)U_3 = \sigma(U_3)[U_i(\ln \mu)V_i - V_i(\ln \mu)U_i], \quad i = 1, 2,$$

$$\hat{R}(U_3, V_3)Y_i = Y_i(\ln \mu)[\sigma(V_3)U_3 - \sigma(U_3)V_3], \quad i = 1, 2,$$

$$\hat{R}(U_i, U_3)V_i = \left(-\frac{1}{\mu}Hess^{\mu}(U_i, V_i) + \sigma(\xi)g(U_i, V_i)\right)U_3 + V_i(\ln\mu)\sigma(U_3)U_i + \left(\nabla^3_{U_3}\xi - \frac{1}{\mu}\sigma(U_3)\operatorname{grad}\mu - \sigma(U_3)\xi\right)g(U_i, V_i),$$

$$\hat{R}(U_1, U_3)U_2 = -\frac{1}{\mu}Hess^{\mu}(U_1, U_2)U_3 + U_2(\ln\mu)\sigma(U_3)U_1,$$

$$\hat{R}(U_2, U_3)U_1 = -\frac{1}{\mu}Hess^{\mu}(U_2, U_1)U_3 + U_1(\ln\mu)\sigma(U_3)U_2,$$

$$\hat{R}(U_i, U_3)V_3 = \mu g_3(U_3, V_3)\nabla_{U_i}\operatorname{grad}\mu - g(U_3, V_3)U_i(\ln\mu)\xi + \sigma(V_3)\left(U_i(\ln\mu)U_3 + \sigma(U_3)U_i\right) - \left(g(V_3, \nabla^3_{U_3}\xi) + \sigma(\xi)g(U_3, V_3)\right)U_i,$$

$$\begin{aligned}\hat{R}(U_3, V_3)Y_3 = {} & R^3(U_3, V_3)Y_3 - \left\|\operatorname{grad}\mu\right\|^2\{g_3(U_3, Y_3)V_3 - g_3(V_3, Y_3)U_3\} \\ & + g(Y_3, \nabla^3_{U_3}\xi)V_3 - g(Y_3, \nabla^3_{V_3}\xi)U_3 + g(U_3, Y_3)[\nabla^3_{V_3}\xi - \mu g_3(V_3, \xi)\operatorname{grad}\mu] \\ & - g(V_3, Y_3)[\nabla^3_{U_3}\xi - \mu g_3(U_3, \xi)\operatorname{grad}\mu] + \sigma(\xi)[g(U_3, Y_3)V_3 - g(V_3, Y_3)U_3] \\ & + [g(V_3, Y_3)\sigma(U_3) - g(U_3, Y_3)\sigma(V_3)]\xi + \sigma(Y_3)[\sigma(V_3)U_3 - \sigma(U_3)V_3],\end{aligned}$$

for all $U_i, V_i, Y_i \in \chi(N_i)$, $i = 1, 2, 3$.

Let $\{e_i\}_{i=1}^{n_1+n_2+n_3}$ be an orthonormal basis of $\chi(N)$, where $\{e_i\}_{i=1}^{n_1}$, $\{e_i\}_{i=n_1+1}^{n_1+n_2}$, and $\{e_i\}_{i=n_1+n_2+1}^{n_1+n_2+n_3}$ are the orthonormal bases of $\chi(N_1)$, $\chi(N_2)$, and $\chi(N_3)$, respectively. Then, by using Lemma 3.4, Lemma 3.5, Lemma 3.6, respectively and a contraction of the curvature tensors we obtain the Ricci tensors as follows:

Corollary 3.7. *Let* $N = (N_1^{n_1} \times_\lambda N_2^{n_2}) \times_\mu N_3^{n_3}$ *be a sequential warped product manifold with metric* $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$, $\xi \in \chi(N_1)$. *Then the Ricci tensor* $\hat{S}$ *of the sequential warped product with respect to the semi-symmetric metric connection is given by*

$$\begin{aligned}\hat{S}(U_1, V_1) = {} & \hat{S}^1(U_1, V_1) + \frac{n_2}{\lambda}Hess_1^{\lambda}(U_1, V_1) + \frac{n_3}{\mu}Hess^{\mu}(U_1, V_1) \\ & - (n_2 + n_3)\left(\sigma(\xi)g(U_1, V_1) + g(V_1, \nabla_{U_1}\xi) - \sigma(U_1)\sigma(V_1)\right) \\ & - \left(n_2\xi(\ln\lambda) + n_3\xi(\ln\mu)\right)g(U_1, V_1),\end{aligned}$$

$$\hat{S}(U_1, U_2) = \hat{S}(U_2, U_1) = \frac{n_3}{\mu}Hess^{\mu}(U_1, U_2),$$

$$\hat{S}(U_i, U_3) = \hat{S}(U_3, U_i) = 0, \quad i = 1, 2,$$

$$\hat{S}(U_2, V_2) = \hat{S}^2(U_2, V_2) + \frac{n_3}{\mu} Hess^{\mu}(U_2, V_2) - \{\frac{\Delta\lambda}{\lambda} + (n_1 + 2n_2 + n_3 - 2)\xi(\ln\lambda)$$
$$+ (n_1 + n_2 + n_3 - 2)\sigma(\xi) + \sum_{i=1}^{n_1} g(\nabla^1_{e_i}\xi, e_i)$$
$$- \frac{n_2 - 1}{\lambda^2} \left\|\text{grad}^1 \lambda\right\|^2 + n_3\xi(\ln\mu)\} g(U_2, V_2),$$

$$\hat{S}(U_3, V_3) = \hat{S}^3(U_3, V_3) - \{\frac{\Delta\mu}{\mu} + (n_1 + n_2 + 2n_3 - 2)\xi(\ln\mu) + n_2\xi(\ln\lambda)$$
$$+ (n_1 + n_2 + n_3 - 2)\sigma(\xi)$$
$$- \frac{n_3 - 1}{\mu^2} \left\|\text{grad}\,\mu\right\|^2 + \sum_{i=1}^{n_1} g(\nabla^1_{e_i}\xi, e_i)\} g(U_3, V_3).$$

Corollary 3.8. *Let $N = (N_1^{n_1} \times_\lambda N_2^{n_2}) \times_\mu N_3^{n_3}$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$, $\xi \in \chi(N_2)$. Then the Ricci tensor $\hat{S}$ of the sequential warped product with respect to the semi-symmetric metric connection is given by*

$$\hat{S}(U_1, V_1) = \hat{S}^1(U_1, V_1) + \frac{n_2}{\lambda} Hess_1^{\lambda}(U_1, V_1) + \frac{n_3}{\mu} Hess^{\mu}(U_1, V_1)$$
$$- \{\sum_{i=n_1+1}^{n_1+n_2} g(\nabla_{e_i}\xi, e_i) + n_3\xi(\ln\mu) + (n_1 + n_2 + n_3 - 2)\sigma(\xi)\} g(U_1, V_1),$$

$$\hat{S}(U_1, U_2) = \frac{n_3}{\mu} Hess^{\mu}(U_1, U_2) - (n_1 + n_2 + n_3 - 2)\, U_1(\ln\lambda)\sigma(U_2),$$

$$\hat{S}(U_2, U_1) = \frac{n_3}{\mu} Hess^{\mu}(U_1, U_2) + (n_1 + n_2 + n_3 - 2)\, U_1(\ln\lambda)\sigma(U_2),$$

$$\hat{S}(U_i, U_3) = \hat{S}(U_3, U_i) = 0, \quad i = 1, 2,$$

$$\tilde{S}(U_2, V_2) = \tilde{S}^2(U_2, V_2) + \frac{n_3}{\mu} Hess^{\mu}(U_2, V_2)$$
$$+ \sum_{i=n_1+1}^{n_1+n_2} (g(V_2, \nabla_{e_i}\xi) g(U_2, e_i) - g(\nabla_{e_i}\xi, e_i) g(U_2, V_2))$$
$$- \{\frac{\Delta\lambda}{\lambda} - \frac{n_2 - 1}{\lambda^2} \left\|\text{grad}\,\lambda\right\|^2 + (n_1 + n_2 + n_3 - 2)\sigma(\xi)$$
$$+ n_3\xi(\ln\mu)\} g(U_2, V_2)$$
$$- (n_1 + n_2 + n_3 - 1) g(V_2, \nabla^2_{U_2}\xi) + (n_1 + n_2 + n_3 - 2)\sigma(U_2)\sigma(V_2),$$

$$\tilde{S}(U_3, Y_3) = \tilde{S}^3(U_3, V_3) - \{\frac{\Delta\mu}{\mu} + (n_1 + n_2 + 2n_3 - 2)\xi(\ln\mu) - \sum_{i=n_1+1}^{n_1+n_2} g(\nabla_{e_i}\xi, e_i)$$
$$- \frac{n_3 - 1}{\mu^2} \left\|\text{grad}\,\mu\right\|^2 + (n_1 + n_2 + n_3 - 2)\sigma(\xi)\} g(U_3, V_3).$$

Corollary 3.9. *Let $N = (N_1^{n_1} \times_\lambda N_2^{n_2}) \times_\mu N_3^{n_3}$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$, $\xi \in \chi(N_3)$. Then the Ricci tensor $\hat{S}$ of the sequential warped product with respect to the semi-symmetric metric connection is given by*

$$\begin{aligned}\hat{S}(U_1, V_1) = \hat{S}^1(U_1, V_1) + \frac{n_2}{\lambda} Hess_1^\lambda(U_1, V_1) + \frac{n_3}{\mu} Hess^\mu(U_1, V_1)\\ - \{\sum_{i=n_1+n_2+1}^{n_1+n_2+n_3} g(\nabla_{e_i}\xi, e_i) + (n_1 + n_2 + n_3 - 2)\sigma(\xi)\} g(U_1, V_1),\end{aligned}$$

$$\begin{aligned}\hat{S}(U_2, V_2) = \hat{S}^2(U_2, V_2) + \frac{n_3}{\mu} Hess^\mu(U_2, V_2) - \{\frac{\Delta\lambda}{\lambda} + (n_1 + n_2 + n_3 - 2)\sigma(\xi)\\ - \frac{n_2 - 1}{\lambda^2} \left\|\text{grad}\,\lambda\right\|^2 + \sum_{i=n_1+n_2+1}^{n_1+n_2+n_3} g(\nabla_{e_i}\xi, e_i)\} g(U_2, V_2),\end{aligned}$$

$$\hat{S}(U_1, U_2) = \hat{S}(U_2, U_1) = \frac{n_3}{\mu} Hess^\mu(U_1, U_2),$$

$$\hat{S}(U_i, U_3) = -(n_1 + n_2 + n_3 - 2)\,\sigma(U_3) U_i(\ln\mu) = -\hat{S}(U_3, U_i), \quad i = 1, 2,$$

$$\begin{aligned}\hat{S}(U_3, V_3) = \hat{S}^3(U_3, V_3) - (n_1 + n_2 + n_3 - 1)\left(g(V_3, \nabla^3_{U_3}\xi) + (n_1 + n_2 + n_3 - 2)\sigma(U_3)\sigma(V_3)\right)\\ - \{\frac{\Delta\mu}{\mu} - \frac{n_3 - 1}{\mu^2} \left\|\text{grad}\,\mu\right\|^2 + (n_1 + n_2 + n_3 - 2)\sigma(\xi)\} g(U_3, V_3)\\ + \sum_{i=n_1+n_2+1}^{n_1+n_2+n_3} \left(g(V_3, \nabla_{e_i}\xi) g(U_3, e_i) - g(e_i, \nabla_{e_i}\xi) g(U_3, V_3)\right).\end{aligned}$$

In this section, finally we give the scalar curvature $\hat{r}$ of the sequential warped product with respect to the semi-symmetric metric connection.

Corollary 3.10. *Let $N = (N_1^{n_1} \times_\lambda N_2^{n_2}) \times_\mu N_3^{n_3}$ be a sequential warped product manifold with metric $g = (g_1 \oplus \lambda^2 g_2) \oplus \mu^2 g_3$. Then the scalar curvature $\hat{r}$ of the sequential warped product with respect to the semi-symmetric metric connection is given by followings:*

$$\begin{aligned}\hat{r} = r_1 + \frac{r_2}{\lambda^2} + \frac{r_3}{\mu^2} + 2n_2\frac{\Delta\lambda}{\lambda} + 4n_3\frac{\Delta\mu}{\mu} + \frac{n_2(n_2 - 1)}{\lambda^2} \left\|\text{grad}\,\lambda\right\|^2\\ + \frac{n_3(n_3 - 1)}{\mu^2} \left\|\text{grad}\,\mu\right\|^2 - 2n_2(n-1)\xi(\ln\lambda)\\ - 2n_3(n-1)\xi(\ln\mu) + (n-1)(n-2)\sigma(\xi) - 2(n-1)\sum_{i=1}^{n_1} g(\nabla_{e_i}\xi, e_i),\end{aligned} \tag{12}$$

where $\xi \in \chi(N_1)$,

$$\begin{aligned}\hat{r} = r_1 &+ \frac{r_2}{\lambda^2} + \frac{r_3}{\mu^2} + 2n_2\frac{\Delta\lambda}{\lambda} + 4n_3\frac{\Delta\mu}{\mu} + \frac{n_2(n_2-1)}{\lambda^2}\left\|\operatorname{grad}\lambda\right\|^2 \\ &+ \frac{n_3(n_3-1)}{\mu^2}\left\|\operatorname{grad}\mu\right\|^2 - 2n_3(n-1)\xi(\ln\lambda) \\ &- 2(n-1)\sum_{i=n_1+1}^{n_1+n_2} g(\nabla_{e_i}\xi, e_i) - (n-1)(n-2)\sigma(\xi),\end{aligned} \tag{13}$$

where $\xi \in \chi(N_2)$,

$$\begin{aligned}\hat{r} = r_1 &+ \frac{r_2}{\lambda^2} + \frac{r_3}{\mu^2} + 2n_2\frac{\Delta\lambda}{\lambda} + 4n_3\frac{\Delta\mu}{\mu} + \frac{n_2(n_2-1)}{\lambda^2}\left\|\operatorname{grad}\lambda\right\|^2 \\ &+ \frac{n_3(n_3-1)}{\mu^2}\left\|\operatorname{grad}\mu\right\|^2 \\ &- 2(n-1)\sum_{i=n_1+n_2+1}^{n_1+n_2+n_3} g(\nabla_{e_i}\xi, e_i) - (n-1)(n-2)\sigma(\xi),\end{aligned} \tag{14}$$

where $\xi \in \chi(N_3)$. *Here* r_i *denotes the scalar curvature of* N_i *with respect to the Levi-Civita connection* ∇ *and* $n = n_1 + n_2 + n_3$.

4. Geometry of Generalized Robertson-Walker Space-times with respect to the semi-symmetric metric connection

In this section, we consider two types of space-times, generalized Robertson-Walker space-time and standard static space-times, as sequential warped products with semi-symmetric metric connections and investigate geometric conditions for such space-times to be Einstein.

4.1. Sequential Generalized Robertson-Walker Space-times with respect to the semi-symmetric metric connection

Proposition 4.1. *Let* $N = (I \times_\lambda N_2) \times_\mu N_3$ *be a sequential generalized Robertson-Walker space-time with metric* $g = (-dt^2 + \lambda^2 g_2) + \mu^2 g_3$ *and* $\xi = \partial_t$. *Then we have*

1. $\hat{\nabla}_{\partial_t}\partial_t = 0$,
2. $\hat{\nabla}_{\partial_t}U = \frac{\dot{\lambda}}{\lambda}U$, $\quad \hat{\nabla}_U\partial_t = \left(\frac{\dot{\lambda}}{\lambda} - 1\right)U$,
3. $\hat{\nabla}_{\partial_t}W = \frac{\dot{\mu}}{\mu}W$, $\quad \hat{\nabla}_W\partial_t = \left(\frac{\dot{\mu}}{\mu} - 1\right)W$,
4. $\hat{\nabla}_U V = \hat{\nabla}^2_U V - \lambda\left(\dot{\lambda} + \lambda\right)g_2(U,V)\partial_t$
5. $\hat{\nabla}_U W = \hat{\nabla}_W U = U(\ln\mu)W$,
6. $\hat{\nabla}_W Z = \hat{\nabla}^3_W Z - \mu g_3(W,Z)\operatorname{grad}\mu - g(W,Z)\partial_t$,

where $U, V \in \chi(N_2)$ *and* $W, Z \in \chi(N_3)$.

Proposition 4.2. *Let* $N = (I \times_\lambda N_2) \times_\mu N_3$ *be a sequential generalized Robertson-Walker space-time with metric* $g = (-dt^2 + \lambda^2 g_2) + \mu^2 g_3$ *and* $\xi \in \chi(N_2)$. *Then we have*

1. $\hat{\nabla}_{\partial_t}\partial_t = \xi$,
2. $\hat{\nabla}_{\partial_t}U = \frac{\dot{\lambda}}{\lambda}U + \sigma(U)\partial_t$, $\quad \hat{\nabla}_U\partial_t = \frac{\dot{\lambda}}{\lambda}U$,
3. $\hat{\nabla}_{\partial_t}W = \hat{\nabla}_W\partial_t = \frac{\dot{\mu}}{\mu}W$,
4. $\hat{\nabla}_U V = \hat{\nabla}^2_U V - \lambda\dot{\lambda}g_2(U,V)\partial_t$,

5. $\hat{\nabla}_U W = U(\ln\mu)W, \qquad \hat{\nabla}_W U = (U(\ln\mu) + \sigma(U))\,W,$
6. $\hat{\nabla}_W Z = \hat{\nabla}^3_W Z - \mu g_3(W,Z)\,\mathrm{grad}\,\mu - g(W,Z)\xi,$

where $U, V \in \chi(N_2)$ *and* $W, Z \in \chi(N_3)$.

Proposition 4.3. *Let* $N = (I \times_\lambda N_2) \times_\mu N_3$ *be a sequential generalized Robertson-Walker space-time with metric* $g = (-dt^2 + \lambda^2 g_2) + \mu^2 g_3$ *and* $\xi \in \chi(N_3)$. *Then we have*

1. $\hat{\nabla}_{\partial_t}\partial_t = \xi,$
2. $\hat{\nabla}_{\partial_t} U = \hat{\nabla}_U \partial_t = \frac{\dot{\lambda}}{\lambda} U,$
3. $\hat{\nabla}_{\partial_t} W = \frac{\dot{\mu}}{\mu} W + \sigma(W)\partial_t, \qquad \hat{\nabla}_W \partial_t = \frac{\dot{\mu}}{\mu} W,$
4. $\hat{\nabla}_U V = \hat{\nabla}^2_U V - \lambda\dot{\lambda} g_2(U,V)\partial_t - g(U,V)\xi,$
5. $\hat{\nabla}_U W = U(\ln\mu)W + \sigma(W)U, \qquad \hat{\nabla}_W U = U(\ln\mu)W,$
6. $\hat{\nabla}_W Z = \hat{\nabla}^3_W Z - \mu g_3(W,Z)\,\mathrm{grad}\,\mu,$

where $U, V \in \chi(N_2)$ *and* $W, Z \in \chi(N_3)$.

Proposition 4.4. *Let* $N = (I \times_\lambda N_2) \times_\mu N_3$ *be a sequential generalized Robertson-Walker space-time with metric* $g = (-dt^2 + \lambda^2 g_2) + \mu^2 g_3$ *and* $\xi = \partial_t$. *Then we have*

1. $\hat{R}(\partial_t, \partial_t)\partial_t = 0,$
2. $\hat{R}(\partial_t, V_2)\partial_t = \frac{1}{\lambda}\left(\ddot{\lambda} + \dot{\lambda}\right) V_2,$
3. $\hat{R}(\partial_t, \partial_t)Y_i = 0,$
4. $\hat{R}(\partial_t, V_2)Y_2 = \lambda\left(\ddot{\lambda} - \dot{\lambda}\right) g_2(V_2, Y_2)\partial_t$
5. $\hat{R}(U_2, V_2)Y_2 = R^2(U_2, V_2)Y_2 - \left(\dot{\lambda}^2 - 2\dot{\lambda}\lambda + \lambda^2\right)\{g_2(Y_2, U_2)V_2 - g_2(Y_2, V_2)U_2\},$
6. $\hat{R}(\partial_t, V_2)Y_3 = 0,$
7. $\hat{R}(\partial_t, V_3)\partial_t = \frac{1}{\mu}\left(\frac{\partial^2\mu}{\partial t^2} + \frac{\partial\mu}{\partial t}\right) V_3,$
8. $\hat{R}(\partial_t, V_3)Y_2 = -\frac{1}{\mu} Hess^\mu(\partial_t, Y_2)V_3,$
9. $\hat{R}(U_2, V_3)\partial_t = -\frac{1}{\mu} Hess^\mu(\partial_t, U_2)V_3,$
10. $\hat{R}(U_2, V_3)Y_2 = \left(-\frac{1}{\mu} Hess^\mu(U_2, Y_2) + \left(\frac{\dot{\lambda}}{\lambda} + \frac{1}{\mu}\frac{\partial\mu}{\partial t} - 1\right) g(U_2, Y_2)\right) V_3,$
11. $\hat{R}(\partial_t, V_3)Y_3 = \mu\left(\hat{\nabla}_{\partial_t}\,\mathrm{grad}\,\mu - \frac{\partial\mu}{\partial t}\right) g_3(V_3, Y_3)\partial_t,$
12. $\hat{R}(U_2, V_3)Y_3 = \mu\left(\hat{\nabla}_{U_2}\,\mathrm{grad}\,\mu - \frac{\partial\mu}{\partial t} - \mu\frac{\dot{\lambda}}{\lambda} + \mu\right) g_3(V_3, Y_3)U_2,$
13. $\hat{R}(U_3, V_3)Y_3 = R^3(U_3, V_3)Y_3 + \left(2\mu\frac{\partial\mu}{\partial t} - \mu^2 - \left\|\mathrm{grad}\,\mu\right\|^2\right)(g_3(U_3, Y_3)V_3 - g_3(V_3, Y_3)U_3),$

where $U_i, V_i, Y_i \in \chi(N_i)$, $i = 2, 3$.

Proposition 4.5. *Let* $N = (I \times_\lambda N_2) \times_\mu N_3$ *be a sequential generalized Robertson-Walker space-time with metric* $g = (-dt^2 + \lambda^2 g_2) + \mu^2 g_3$ *and* $\xi = \partial_t$. *Then we have*

$$\hat{S}(\partial_t, \partial_t) = -\frac{n_2}{\lambda}\left(\ddot{\lambda} - \dot{\lambda}\right) - \frac{n_3}{\mu}\left(\frac{\partial^2\mu}{\partial t^2} - \frac{\partial\mu}{\partial t}\right),$$

$$\begin{aligned}\hat{S}(U_2, V_2) = \hat{S}^2(U_2, V_2) &+ \frac{n_3}{\mu} Hess^\mu(U_2, V_2)\\ &+ \{\lambda(\ddot{\lambda}) - (2n_2 + n_3 - 1)\lambda\dot{\lambda} - (n_2 + n_3 - 1)\lambda^2\\ &\quad + (n_2 - 1)(\dot{\lambda})^2 - \frac{n_3}{\mu}\lambda^2\frac{\partial\mu}{\partial t}\}g_2(U_2, V_2),\end{aligned}$$

$$\hat{S}(U_3,V_3) = \hat{S}^3(U_3,V_3) - \{\mu(\Delta\mu) + (n_2+2n_3-1)\mu\frac{\partial\mu}{\partial t} + n_2\frac{\dot{\lambda}}{\lambda}\mu^2 - (n_2+n_3-1)\mu^2 - (n_3-1)\left\|\mathrm{grad}\,\mu\right\|^2\}g_3(U_3,V_3),$$

$$\hat{S}(\partial_t,U_2) = \frac{n_3}{\mu}Hess^{\mu}(\partial_t,U_2), \qquad \hat{S}(\partial_t,U_3) = 0,$$

where $U_i, V_i \in \chi(N_i)$, $i = 2,3$.

Now, consider that the sequential generalized Robertson-Walker space-time $N = (I \times_\lambda N_2) \times_\mu N_3$ with metric $g = (-dt^2 + \lambda^2 g_2) + \mu^2 g_3$ and $\xi = \partial_t$ is Einstein with respect to the semi-symmetric metric connection. Then we write

$$\hat{S}(U,V) = \varrho g(U,V).$$

By using the previous theorem, we get

$$\frac{n_2}{\lambda}\left(\ddot{\lambda} - \dot{\lambda}\right) + \frac{n_3}{\mu}\left(\frac{\partial^2\mu}{\partial t^2} - \frac{\partial\mu}{\partial t}\right) = \varrho,$$

$$\left\{\begin{array}{c} \hat{S}^2(U_2,V_2) + \frac{n_3}{\mu}Hess^{\mu}(U_2,V_2) \\ + \left(\begin{array}{c} \mu\left((\Delta\mu) + (n_2+2n_3-1)\frac{\partial\mu}{\partial t} + n_2\frac{\dot{\lambda}}{\lambda}\mu\right) \\ +(n_2-1)(\dot{\lambda})^2 - \frac{n_3}{\mu}\lambda^2\frac{\partial\mu}{\partial t} \end{array}\right)g_2(U_2,V_2) \end{array}\right. = \varrho\lambda^2 g_2(U_2,V_2),$$

$$\hat{S}^3(U_3,V_3) - \left(\begin{array}{c} \mu(\Delta\mu) + (n_2+2n_3-1)\mu\frac{\partial\mu}{\partial t} + n_2\frac{\dot{\lambda}}{\lambda}\mu^2 \\ -(n_2+n_3-1)\mu^2 - (n_3-1)\left\|\mathrm{grad}\,\mu\right\|^2 \end{array}\right)g_3(U_3,V_3) = \varrho\mu^2 g_3(U_3,V_3),$$

and

$$Hess^{\mu}(\partial_t,U_2) = 0,$$

for any $U_i, V_i \in \chi(N_i)$, $i = 2,3$.

Hence we give

Theorem 4.6. *Let* $N = (I \times_\lambda N_2) \times_\mu N_3$ *be a sequential generalized Robertson-Walker space-time with* $g = (-dt^2 + \lambda^2 g_2) + \mu^2 g_3$ *and* $\xi = \partial_t$. *If* N *is Einstein with respect to the semi-symmetric metric connection with the Einstein constant* ϱ, *then we have the followings:*

1. $\frac{n_2}{\lambda}\left(\ddot{\lambda} - \dot{\lambda}\right) + \frac{n_3}{\mu}\left(\frac{\partial^2\mu}{\partial t^2} - \frac{\partial\mu}{\partial t}\right) = \varrho$,
2. (N_2, g_2) *is Einstein with respect to the semi-symmetric metric connection if* $Hess^{\mu}(U_2,V_2) = 0$, *for any* $U_2, V_2 \in \chi(N_2)$,
3. (N_3, g_3) *is Einstein with respect to the semi-symmetric metric connection,*
4. $Hess^{\mu}(\partial_t,U_2) = 0$.

Theorem 4.7. *Let $N=(I\times_\lambda N_2)\times_\mu N_3$ be a sequential generalized Robertson-Walker space-time with $g=(-dt^2+\lambda^2 g_2)+\mu^2 g_3$ and $\xi=\partial_t$. Then N is an Einstein manifold with Einstein constant ϱ with respect to the semi-symmetric metric connection if*

1. $Hess^\mu(\partial_t,U_2)=0,\quad Hess^\mu(U_2,V_2)=0,$ *for any* $U_2,V_2\in\chi(N_2)$,
2. (N_i,g_i) *is Einstein manifold with Einstein factor* ϱ_i, $i=2,3$,
3. $\frac{n_2}{\lambda}\left(\ddot\lambda-\dot\lambda\right)+\frac{n_3}{\mu}\left(\frac{\partial^2\mu}{\partial t^2}-\frac{\partial\mu}{\partial t}\right)=\varrho$,
4. $\varrho_2+\begin{pmatrix}\mu\left((\Delta\mu)+(n_2+2n_3-1)\frac{\partial\mu}{\partial t}+n_2\frac{\dot\lambda}{\lambda}\mu\right)\\+(n_2-1)(\dot\lambda)^2-\frac{n_3}{\mu}\lambda^2\frac{\partial\mu}{\partial t}\end{pmatrix}=\varrho\lambda^2$,
5. $\varrho_3-\begin{pmatrix}\mu\,(\Delta\mu)+(n_2+2n_3-1)\mu\frac{\partial\mu}{\partial t}+n_2\frac{\dot\lambda}{\lambda}\mu^2\\-(n_2+n_3-1)\mu^2-(n_3-1)\left\|\operatorname{grad}\mu\right\|^2\end{pmatrix}=\varrho\mu^2$.

4.2. Sequential Standard Static Space-times with respect to the semi-symmetric metric connection

Proposition 4.8. *Let $N=(N_1\times_\lambda N_2)\times_\mu I$ be a sequential standard static space-time with metric $g=(g_1+\lambda^2 g_2)+\mu^2\left(-dt^2\right)$ and $\xi=\partial_t$. Then we have*

1. $\hat\nabla_X Y=\nabla^1_X Y-g(X,Y)\partial_t$,
2. $\hat\nabla_X U=\hat\nabla_U X=X(\ln\lambda)U$,
3. $\hat\nabla_X\partial_t=X(\ln\mu)\partial_t-\mu^2X,\qquad \hat\nabla_{\partial_t}X=X(\ln\mu)\partial_t$,
4. $\hat\nabla_U V=\nabla^2_U V-\lambda g_2(U,V)\operatorname{grad}^1\lambda-g(U,V)\partial_t$,
5. $\hat\nabla_U\partial_t=U(\ln\mu)\partial_t-\mu^2U,\qquad \hat\nabla_{\partial_t}U=U(\ln\mu)\partial_t$,
6. $\hat\nabla_{\partial_t}\partial_t=\mu\operatorname{grad}\mu$,

where $X,Y\in\chi(N_1)$ and $U,V\in\chi(N_2)$.

Proposition 4.9. *Let $N=(N_1\times_\lambda N_2)\times_\mu I$ be a sequential standard static space-time with metric $g=(g_1+\lambda^2 g_2)+\mu^2\left(-dt^2\right)$ and $\xi=\partial_t$. Then we have*

1. $\hat R(U_1,V_1)Y_1=R^1(U_1,V_1)Y_1+\begin{bmatrix}[g(U_1,Y_1)V_1(\ln\mu)-g(V_1,Y_1)U_1(\ln\mu)]\,\partial_t\\-\mu^2[g(U_1,Y_1)V_1-g(V_1,Y_1)U_1]\end{bmatrix}$,
2. $\hat R(U_1,U_2)V_1=\left(\frac{-1}{\lambda}Hess^\lambda_1(U_1,V_1)-\mu^2g(U_1,V_1)\right)U_2+g(U_1,V_1)U_2(\ln\mu)\partial_t$,
3. $\hat R(U_1,V_1)U_2=0=\hat R(U_2,V_2)V_1$,
4. $\hat R(U_1,U_2)V_2=g(U_2,V_2)\left(\frac{1}{\lambda}\hat\nabla^1_{U_1}\operatorname{grad}^1\lambda-U_1(\ln\mu)\partial_t+\mu^2U_1\right)$,
5. $\hat R(U_2,V_2)Y_2=R^2(U_2,V_2)Y_2-\begin{pmatrix}\left(\frac{1}{\lambda^2}\left\|\operatorname{grad}^1\lambda\right\|^2+\mu^2\right)(g(U_2,Y_2)V_2-g(V_2,Y_2)U_2)\\-\left(g(U_2,Y_2)V_2(\ln\mu)+g(V_2,Y_2)U_2(\ln\mu)\right)\partial_t\end{pmatrix}$,
6. $\hat R(U_1,U_2)\partial_t=-\mu^2[U_1(\ln\mu)U_2-U_2(\ln\mu)U_1]$,
7. $\hat R(U_1,\partial_t)V_1=-\frac{1}{\mu}Hess^\mu(U_1,V_1)\partial_t-\mu^2V_1(\ln\mu)U_1+\mu g(U_1,V_1)\operatorname{grad}\mu$,
8. $\hat R(U_1,\partial_t)U_2=-\frac{1}{\mu}Hess^\mu(U_1,U_2)\partial_t-\mu^2U_2(\ln\mu)U_1$,
9. $\hat R(U_2,\partial_t)U_1=-\frac{1}{\mu}Hess^\mu(U_2,U_1)\partial_t-\mu^2U_1(\ln\mu)U_2$,
10. $\hat R(U_2,\partial_t)V_2=-\frac{1}{\mu}Hess^\mu(U_2,V_2)\partial_t-\mu^2V_2(\ln\mu)U_2+\mu g(U_2,V_2)\operatorname{grad}\mu$,
11. $\hat R(U_i,\partial_t)\partial_t=-\mu\nabla_{U_i}\operatorname{grad}\mu,\quad i=1,2$,
12. $\hat R(U_i,V_i)\partial_t=-\mu^2\left(U_i(\ln\mu)V_i-V_i(\ln\mu)U_i\right),\quad i=1,2$,
13. $\hat R(\partial_t,\partial_t)\partial_t=0$,

where $U_i,V_i,Y_i\in\chi(N_i)$, $i=1,2$.

Proposition 4.10. *Let $N = (N_1 \times_\lambda N_2) \times_\mu I$ be a sequential standard static space-time with metric $g = (g_1 + \lambda^2 g_2) + \mu^2\left(-dt^2\right)$ and $\xi = \partial_t$. Then we have*

$$\hat{S}(U_1, V_1) = \hat{S}^1(U_1, V_1) + \frac{n_2}{\lambda} Hess_1^\lambda(U_1, V_1) + \frac{1}{\mu} Hess^\mu(U_1, V_1) + (n_1 + n_2 - 1)\mu^2 g(U_1, V_1),$$

$$\hat{S}(U_2, V_2) = \hat{S}^2(U_2, V_2) + \frac{1}{\mu} Hess^\mu(U_2, V_2) - \left(\frac{\Delta\lambda}{\lambda} - (n_1 + n_2 - 1)\mu^2 - \frac{n_2 - 1}{\lambda^2}\left\|\text{grad}^1 \lambda\right\|^2\right) g(U_2, V_2),$$

$$\hat{S}(\partial_t, \partial_t) = \mu\left(\Delta\mu - \mu^3\right),$$

$$\hat{S}(U_1, U_2) = \frac{1}{\mu} Hess^\mu(U_1, U_2),$$

$$\hat{S}(U_i, \partial_t) = \mu^2 (n_1 + n_2 - 1)\, U_i(\ln \mu) = -\hat{S}(\partial_t, U_i), \quad i = 1, 2,$$

where $U_i, V_i \in \chi(N_i)$, $i = 1, 2$.

Now, consider that the sequential standard static space-time $N = (N_1 \times_\lambda N_2) \times_\mu I$ with metric $g = (g_1 + \lambda^2 g_2) + \mu^2\left(-dt^2\right)$ and $\xi = \partial_t$ is Einstein with respect to the semi-symmetric metric connection. Then we write

$$\hat{S}(U, V) = \rho g(U, V).$$

By using the previous theorem, we get

$$\left\{ \begin{array}{c} \hat{S}^1(U_1, V_1) + \frac{n_2}{\lambda} Hess_1^\lambda(U_1, V_1) \\ + \frac{1}{\mu} Hess^\mu(U_1, V_1) + (n_1 + n_2 - 1)\mu^2 g_1(U_1, V_1) \end{array} \right. = \rho g_1(U_1, V_1),$$

$$\left\{ \begin{array}{c} \hat{S}^2(U_2, V_2) + \frac{1}{\mu} Hess^\mu(U_2, V_2) \\ -\left(\lambda\,(\Delta\lambda) - (n_1 + n_2 - 1)\lambda^2\mu^2 - (n_2 - 1)\left\|\text{grad}^1 \lambda\right\|^2\right) g_2(U_2, V_2) \end{array} \right. = \rho\lambda^2 g_2(U_2, V_2),$$

$$\Delta\mu - \mu^3 = -\rho\mu,$$

$$Hess^\mu(U_1, U_2) = 0,$$

and

$$\mu^2 (n_1 + n_2 - 1)\, U_i(\ln \mu) = 0, \quad i = 1, 2,$$

for any $U_i, V_i \in \chi(N_i)$, $i = 1, 2$.

Hence we give

Theorem 4.11. *Let $N=(N_1\times_\lambda N_2)\times_\mu I$ be a sequential standard static space-time with metric $g=(g_1+\lambda^2g_2)+\mu^2\left(-dt^2\right)$ and $\xi=\partial_t$. If N is Einstein with respect to the semi-symmetric metric connection with the Einstein constant ρ, then the scalar curvature of N with respect to the semi-symmetric metric connection is given by*

$$\hat{r}=-(n_1+n_2+1)\left(\frac{\Delta\mu}{\mu}-\mu^2\right).$$

Theorem 4.12. *Let $N=(N_1\times_\lambda N_2)\times_\mu I$ be an Einstein sequential standard static space-time with metric $g=(g_1+\lambda^2g_2)+\mu^2\left(-dt^2\right)$ and $\xi=\partial_t$. Then we have*

1. *(N_1,g_1) is Einstein with the Einstein constant $\rho-(n_1+n_2-1)\mu^2$ with respect to the semi-symmetric metric connection provided $\frac{n_2}{\lambda}Hess_1^\lambda(U_1,V_1)+\frac{1}{\mu}Hess^\mu(U_1,V_1)=0$, for any $U_1,V_1\in\chi(N_1)$,*
2. *(N_2,g_2) is Einstein with the Einstein constant*

$$\rho\lambda^2+\left(\lambda\Delta\lambda-(n_1+n_2-1)\lambda^2\mu^2-(n_2-1)\left\|\mathrm{grad}^1\lambda\right\|^2\right)$$

   with respect to the semi-symmetric metric connection provided $Hess^\mu(U_2,V_2)=0$, for any $U_2,V_2\in\chi(N_2)$,
3. *$Hess^\mu(U_1,U_2)=0$,*
4. *$U_i(\ln\mu)=0,\quad i=1,2.$*

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