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Dual Drazin inverses of dual matrices and dual Drazin-inverse solutions of systems of linear dual equations

Jin Zhong[a], Yilin Zhang[a]

[a]*Faculty of Science, Jiangxi University of Science and Technology, Ganzhou 341000, China*

Abstract. In this paper, we study a kind of dual generalized inverse, which is called the dual Drazin inverse. Unlike the real matrices case, the dual Drazin inverse of a square dual matrix may not exist. It is shown that the dual Drazin inverse is unique when it exists. Some necessary and sufficient conditions for the existence of the dual Drazin inverse are presented. A compact formula for the computation of the dual Drazin inverse is given when it exists. Moreover, we find an unexpected result that the dual Drazin inverse can be obtained by computing the Drazin inverse of a 2×2 upper triangular block matrix. We also introduce the dual Drazin-inverse solution of systems of linear dual equations. Some characterizations of the dual Drazin-inverse solution are given. In addition, some numerical examples are provided to illustrate the results.

1. Introduction

A *dual number* [9] is defined as the sum of a primal part a, and a dual part a_0, namely,

$$\widehat{a} = a + \varepsilon a_0,$$

where a and a_0 are real numbers, and ε is the *dual unit* which satisfies $\varepsilon \neq 0$, $0\varepsilon = \varepsilon 0 = 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$ and $\varepsilon^2 = 0$. Dual numbers and their algebra have been powerful and convenient tools for the analysis of mechanical systems, and have attracted a lot of attention over the last three decades because of their applicability to various areas of engineering like kinematic analysis [1, 2], robotics [4, 5], screw motion [7] and rigid body motion analysis [10]. A *dual matrix* is a matrix with dual number entries. Dual matrices can be defined likewise, i.e., if A and B are two $m \times n$ real matrices, then the $m \times n$ dual matrix is defined as $\widehat{A} = A + \varepsilon B$, where A and B are respectively called the primal part and the dual part of $\widehat{A}$.

Dual generalized inverses of dual matrices are frequently used in many problems in kinematic analysis and synthesis of machines and mechanisms. It is worth noting that unlike real matrices, the dual generalized inverses of a dual matrix may not exist. For this reason, it could be of interest for researchers to study the existence of dual generalized inverses and find efficient methods to compute them when they exist. The existence, computations and applications of dual generalized inverses have been a topic of recent interest.

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Email addresses: zhongjin1984@126.com (Jin Zhong), 2943631244@qq.com (Yilin Zhang)

de Falco et al. [3] discussed the mathematical conditions of existence for different types of dual generalized inverses. Moreover, solutions of some meaningful kinematic problems were discussed to demonstrate the usefulness and versatility of dual generalized inverses. Udwadia [14] dealt with the existence of various types of dual generalized inverses of dual matrices. Some new and foundational results on the necessary and sufficient conditions for various types of dual generalized inverses to exist are obtained. Pennestrì et al. [11] proposed novel and computationally efficient algorithms/formulas for the computation of the MPDGI. Udwadia et al. [15] investigated the question of whether all dual matrices have dual Moore-Penrose generalized inverses and showed that there are uncountably many dual matrices that do not have them. Udwadia [13] studied the properties of the DMPGI and used them to solve systems of linear dual equations. Wang [16] gave some necessary and sufficient conditions for a dual matrix to have the DMPGI, and a compact formula for the computation of the DMPGI was also given. Wang et al. [17] introduced the dual index and the dual core inverse. Zhong et al. [21] studied the existence, computation and applications of the dual group inverse.

In this paper, we extend the results of the dual group inverse to the *dual Drazin inverse*. We investigate the existence and computations of the dual Drazin inverse, and discuss the least-squares and minimal properties of the dual Drazin inverse. In Section 2, we show the uniqueness of the dual Drazin inverse when it exsits and give some necessary and sufficient conditions for a square dual matrix to have the dual Drazin inverse. If the dual Drazin inverse exists, then a compact formula for the computation of the dual Drazin inverse is given. Moreover, we find that the dual Drazin inverse of a dual matrix $\widehat{A} = A + \varepsilon B$ is closely related to the Drazin inverse of the 2×2 upper triangular block matrix $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$. In Section 3, we study the dual Drazin-inverse solution of systems of linear dual equations. Some characterizations of the dual Drazin-inverse solution are given.

Throughout this paper, we use $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$ and $\mathbb{D}^{m\times n}$ to denote the set of all $m \times n$ complex matrices, real matrices and dual matrices respectively. $\mathbb{R}^n$ and $\mathbb{D}^n$ denote the set of all n-dimensional real column vectors and n-dimensional dual column vectors respectively. For a real matrix A, $\mathcal{R}(A)$ is the range of A and $\mathcal{N}(A)$ is the null space of A. The index of a matrix $A \in \mathbb{R}^{n\times n}$ is the smallest nonnegative integer such that rank(A^k)=rank(A^{k+1}), and denoted by Ind(A). For an $n \times n$ dual matrix $\widehat{A} = A + \varepsilon B$, $\widehat{A}^T = A^T + \varepsilon B^T$, where A^T is the transpose of A. For a nonsingular real matrix Q, $Q^{-1}\widehat{A}Q = Q^{-1}AQ + \varepsilon Q^{-1}BQ$. The P-norm [21] of a dual vector $\widehat{x} = u + \varepsilon v$ is defined as

$$\| \widehat{x} \|_P = \| P^{-1}\widehat{x} \| = \sqrt{\| P^{-1}u \|_2^2 + \| P^{-1}v \|_2^2},$$

where the nonsingular real matrix P is defined in (1).

The dual Moore-Penrose generalized inverse (DMPGI) [16] of a dual matrix $\widehat{A} \in \mathbb{D}^{m\times n}$, denoted by $\widehat{A}^\dagger$, is the unique matrix $\widehat{X} \in \mathbb{D}^{n\times m}$ satisfying the following dual Penrose equations

$$\widehat{A}\widehat{X}\widehat{A} = \widehat{A}, \quad \widehat{X}\widehat{A}\widehat{X} = \widehat{X}, \quad (\widehat{A}\widehat{X})^T = \widehat{A}\widehat{X}, \quad (\widehat{X}\widehat{A})^T = \widehat{X}\widehat{A}.$$

The dual Drazin inverse can be defined on the analogy of the real case [19]. Given an $n \times n$ dual matrix $\widehat{A} = A + \varepsilon B$ with Ind$(A) = k$, if a dual matrix $\widehat{X} \in \mathbb{D}^{n\times n}$ satisfies

$$\widehat{A}^k\widehat{X}\widehat{A} = \widehat{A}^k, \quad \widehat{X}\widehat{A}\widehat{X} = \widehat{X}, \quad \widehat{A}\widehat{X} = \widehat{X}\widehat{A},$$

then $\widehat{X}$ is called the dual Drazin inverse of $\widehat{A}$, and is denoted by $\widehat{X} = \widehat{A}^D$. If Ind$(A) = 1$, then this special case is the dual group inverse [21].

Let $A \in \mathbb{C}^{n\times n}$ and Ind$(A) = k$. Then there exist nonsingular matrices P and C, and a nilpotent matrix N ($N^k = 0$) such that [19]

$$A = P\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}P^{-1} \text{ and } A^D = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}. \tag{1}$$

Lemma 1.1. [6, 8] *If* $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, *where* $A \in \mathbb{C}^{n\times n}$ *and* $C \in \mathbb{C}^{m\times m}$ *with Ind(A)=k and Ind(C)=l, then*

$$M^D = \begin{bmatrix} A^D & X \\ 0 & C^D \end{bmatrix},$$

where $X = -A^DBC^D + \sum_{i=0}^{l-1}(A^D)^{i+2}BC^iC^\pi + \sum_{i=0}^{k-1}A^\pi A^iB(C^D)^{i+2}$ *and* $A^\pi = I - AA^D$.

Lemma 1.2. [12] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times m}$, $C \in \mathbb{C}^{n\times n}$ *with Ind(B)=k and Ind(C)=l. Then*

$$rank\begin{bmatrix} A & B^k \\ C^l & 0 \end{bmatrix} = rank(B^k) + rank(C^l) + rank[(I_m - BB^D)A(I_n - C^DC)].$$

2. Dual Drazin inverses of dual matrices

In this section, we study the existence, computations and properties of the dual Drazin inverse. We first give a necessary and sufficient condition for a dual matrix to be the dual Drazin inverse of a given dual matrix.

Lemma 2.1. *Let* $\widehat{A} = A + \varepsilon B$ *be a dual matrix with* $A, B \in \mathbb{R}^{n\times n}$ *and* $Ind(A) = k$. *Denote* $\widehat{A^k} = A^k + \varepsilon(\sum_{i=1}^{k} A^{k-i}BA^{i-1})$ *as* $\widehat{A^k} = A^k + \varepsilon M$. *Then an* $n \times n$ *dual matrix* $\widehat{G} = G + \varepsilon R$ *is a dual Drazin inverse of* $\widehat{A}$ *if and only if* $G = A^D$ *and*

$$M = A^kA^DB + A^kRA + MA^DA, \tag{2}$$

$$R = A^DAR + A^DBA^D + RAA^D, \tag{3}$$

$$AR + BA^D = RA + A^DB. \tag{4}$$

Proof. According to the definition of the dual Drazin inverse, $\widehat{G} = G + \varepsilon R$ is a dual Drazin inverse of $\widehat{A} = A + \varepsilon B$ if and only if

$$(A + \varepsilon B)^k(G + \varepsilon R)(A + \varepsilon B) = (A + \varepsilon B)^k,$$

$$(G + \varepsilon R)(A + \varepsilon B)(G + \varepsilon R) = (G + \varepsilon R),$$

$$(A + \varepsilon B)(G + \varepsilon R) = (G + \varepsilon R)(A + \varepsilon B).$$

Since $\widehat{A^k} = A^k + \varepsilon(\sum_{i=1}^{k} A^{k-i}BA^{i-1}) = A^k + \varepsilon M$, then the above three equalities can be simplified to

$$A^kGA + \varepsilon(A^kGB + A^kRA + MGA) = A^k + \varepsilon M,$$

$$GAG + \varepsilon(GAR + GBG + RAG) = G + \varepsilon R,$$

$$AG + \varepsilon(AR + BG) = GA + \varepsilon(GB + RA).$$

Hence, we can observe from the primal parts of the above equalities that $A^kGA = A^k$, $GAG = G$, $AG = GA$, i.e., $G = A^D$, and it can be seen from the dual parts of the above equalities that the equations (2)-(4) are satisfied. □

It is known that the Drazin inverse of a real square matrix exists and is unique. However, the dual Drazin inverse of a square dual matrix may not exist. For example, let $\widehat{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. If $\widehat{A}^D$ exists, then by Lemma 2.1, it is of the form $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}$. In this case, a direct calculation shows that $A^kA^DB + A^kRA + MA^DA = \begin{bmatrix} r_1 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = M$. Hence, we can see from (2) that the dual Drazin inverse of $\widehat{A}$ does not exist.

We will show in the following that the dual Drazin inverse of a square dual matrix is unique when it exists.

Theorem 2.1. *Let $\widehat{A} = A + \varepsilon B$ be a dual matrix with $A, B \in \mathbb{R}^{n\times n}$ and $Ind(A) = k$. If the dual Drazin inverse of $\widehat{A}$ exists, then it is unique.*

Proof. According to Lemma 2.1, if the dual Drazin inverse of $\widehat{A} = A + \varepsilon B$ exists, then it has the form $A^D + \varepsilon R$. Let $\widehat{G_1} = A^D + \varepsilon R_1$ and $\widehat{G_2} = A^D + \varepsilon R_2$ be two dual Drazin inverses of $\widehat{A}$. To show the uniqueness of $\widehat{A}^D$, we need only to show that $R_1 = R_2$.

It follows from (2) that

$$M = A^kA^DB + A^kR_1A + MA^DA \quad \text{and} \quad M = A^kA^DB + A^kR_2A + MA^DA.$$

Then

$$A^k(R_1 - R_2)A = 0. \tag{5}$$

Similarly, we can see from (3) that

$$R_1 = A^DAR_1 + A^DBA^D + R_1AA^D \quad \text{and} \quad R_2 = A^DAR_2 + A^DBA^D + R_2AA^D,$$

which gives

$$R_1 - R_2 = A^DA(R_1 - R_2) + (R_1 - R_2)AA^D. \tag{6}$$

Furthermore, it can be seen from (4) that

$$AR_1 + BA^D = R_1A + A^DB \quad \text{and} \quad AR_2 + BA^D = R_2A + A^DB,$$

which implies that

$$A(R_1 - R_2) = (R_1 - R_2)A. \tag{7}$$

The equalities (5) and (7) tell us that $0 = A^k(R_1 - R_2)A = A^{k+1}(R_1 - R_2) = 0$, thus $\mathcal{R}(R_1 - R_2) \subset \mathcal{N}(A^{k+1}) = \mathcal{N}(A^k) = \mathcal{N}(A^D)$, i.e., $A^k(R_1-R_2) = 0$ and $A^D(R_1-R_2) = 0$. On the other hand, we can see from $A^k(R_1-R_2) = 0$ and (7) that $(R_1 - R_2)A^k = 0$. Postmultiplying $(R_1 - R_2)A^k = 0$ by $(A^D)^k$ yields $(R_1 - R_2)AA^D = 0$. Now, substituting $A^D(R_1-R_2) = 0$ and $(R_1-R_2)AA^D = 0$ into (6) we get $R_1-R_2 = 0$, which is the desired result. □

We now present some necessary and sufficient conditions for the existence of the dual Drazin inverse in the following theorem. A compact formula for the computation of the dual Drazin inverse is also given.

Theorem 2.2. *Let $\widehat{A} = A+\varepsilon B$ be a dual matrix with $A, B \in \mathbb{R}^{n\times n}$ and $Ind(A) = k$. Denote $\widehat{A}^k = A^k+\varepsilon(\sum_{i=1}^{k} A^{k-i}BA^{i-1})$ as $\widehat{A}^k = A^k + \varepsilon M$. Then the following conditions are equivalent:*

(i) The dual Drazin inverse of $\widehat{A}$ exists;

(ii) $\widehat{A} = P\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}P^{-1}$, where C and P are nonsingular matrices, N is a nilpotent matrix with $N^k = 0$, and $\sum_{i=1}^{k} N^{k-i}B_4N^{i-1} = 0$;

(iii) $(I - AA^D)M(I - AA^D) = 0$;

(iv) $rank\begin{bmatrix} M & A^k \\ A^k & 0 \end{bmatrix} = 2rank(A^k)$;

(v) $(\widehat{A}^k)^\dagger$ *exists.*

Furthermore, if the dual Drazin inverse of $\widehat{A}$ *exists, then*

$$\widehat{A}^D = A^D + \varepsilon R, \tag{8}$$

where

$$R = -A^D B A^D + \sum_{i=0}^{k-1} (A^D)^{i+2} B A^i A^\pi + \sum_{i=0}^{k-1} A^\pi A^i B (A^D)^{i+2}, A^\pi = I - AA^D. \tag{9}$$

Proof. $(i)\Longrightarrow(ii)$: If $\mathrm{Ind}(A) = k$, then A and A^D have the block representations in (1). Let $B = P\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}P^{-1}$ and $R = P\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}P^{-1}$. Then

$$M = \sum_{i=1}^{k} A^{k-i} B A^{i-1} = P\begin{bmatrix} \sum\limits_{i=1}^{k} C^{k-i} B_1 C^{i-1} & \sum\limits_{i=1}^{k} C^{k-i} B_2 N^{i-1} \\ \sum\limits_{i=1}^{k} N^{k-i} B_3 C^{i-1} & \sum\limits_{i=1}^{k} N^{k-i} B_4 N^{i-1} \end{bmatrix}P^{-1}.$$

If the dual Drazin inverse of $\widehat{A}$ exists, then it follows from (2) that

$$\begin{aligned} \begin{bmatrix} \sum\limits_{i=1}^{k} C^{k-i} B_1 C^{i-1} & \sum\limits_{i=1}^{k} C^{k-i} B_2 N^{i-1} \\ \sum\limits_{i=1}^{k} N^{k-i} B_3 C^{i-1} & \sum\limits_{i=1}^{k} N^{k-i} B_4 N^{i-1} \end{bmatrix} &= \begin{bmatrix} C^{k-1}B_1 & C^{k-1}B_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} C^k R_1 C & C^k R_2 N \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sum\limits_{i=1}^{k} C^{k-i} B_1 C^{i-1} & 0 \\ \sum\limits_{i=1}^{k} N^{k-i} B_3 C^{i-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} C^{k-1}B_1 + C^k R_1 C + \sum\limits_{i=1}^{k} C^{k-i} B_1 C^{i-1} & C^{k-1}B_2 + C^k R_2 N \\ \sum\limits_{i=1}^{k} N^{k-i} B_3 C^{i-1} & 0 \end{bmatrix}. \end{aligned}$$

It can be seen from the above equality that $\sum_{i=1}^{k} N^{k-i} B_4 N^{i-1} = 0$.

$(ii)\Longrightarrow(iii)$: If $\widehat{A} = P\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}P^{-1}$ and $\sum_{i=1}^{k} N^{k-i} B_4 N^{i-1} = 0$, then

$$\begin{aligned} (I - AA^D)M(I - AA^D) &= P\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \sum\limits_{i=1}^{k} C^{k-i} B_1 C^{i-1} & \sum\limits_{i=1}^{k} C^{k-i} B_2 N^{i-1} \\ \sum\limits_{i=1}^{k} N^{k-i} B_3 C^{i-1} & \sum\limits_{i=1}^{k} N^{k-i} B_4 N^{i-1} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}P^{-1} \\ &= P\begin{bmatrix} 0 & 0 \\ 0 & \sum\limits_{i=1}^{k} N^{k-i} B_4 N^{i-1} \end{bmatrix}P^{-1} = 0. \end{aligned}$$

$(iii)\Longrightarrow(i)$: If $(I - AA^D)M(I - AA^D) = 0$, then it is easy to see from the block representations of A, A^D and M that $\sum_{i=1}^{k} N^{k-i} B_4 N^{i-1} = 0$.

Denote

$$\widehat{G} = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1} B_1 C^{-1} & \sum\limits_{i=0}^{k-1} C^{-i-2} B_2 N^i \\ \sum\limits_{i=0}^{k-1} N^i B_3 C^{-i-2} & 0 \end{bmatrix}P^{-1}.$$

Then we will show that the dual matrix $\widehat{G}$ satisfies the three conditions in the definition of the dual Drazin inverse.

$$\begin{aligned}
\widehat{A}\widehat{G} &= \left(P\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}P^{-1}\right)\times\left(P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & \sum_{i=0}^{k-1} C^{-i-2}B_2N^i \\ \sum_{i=0}^{k-1} N^iB_3C^{-i-2} & 0 \end{bmatrix}P^{-1}\right) \\
&= P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} 0 & \sum_{i=0}^{k-1} C^{-i-1}B_2N^i \\ \sum_{i=0}^{k-1} N^{i+1}B_3C^{-i-2} + B_3C^{-1} & 0 \end{bmatrix}P^{-1} \\
&= P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} 0 & \sum_{i=0}^{k-1} C^{-i-1}B_2N^i \\ \sum_{i=0}^{k-1} N^iB_3C^{-i-1} & 0 \end{bmatrix}P^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{G}\widehat{A} &= \left(P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & \sum_{i=0}^{k-1} C^{-i-2}B_2N^i \\ \sum_{i=0}^{k-1} N^iB_3C^{-i-2} & 0 \end{bmatrix}P^{-1}\right)\times\left(P\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}P^{-1}\right) \\
&= P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} 0 & \sum_{i=0}^{k-1} C^{-i-2}B_2N^{i+1} + C^{-1}B_2 \\ \sum_{i=0}^{k-1} N^iB_3C^{-i-1} & 0 \end{bmatrix}P^{-1} \\
&= P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} 0 & \sum_{i=0}^{k-1} C^{-i-1}B_2N^i \\ \sum_{i=0}^{k-1} N^iB_3C^{-i-1} & 0 \end{bmatrix}P^{-1}.
\end{aligned}$$

Hence, $\widehat{A}\widehat{G} = \widehat{G}\widehat{A}$.

Moreover,

$$\begin{aligned}
\widehat{A}^k\widehat{G}\widehat{A} &= \left(P\begin{bmatrix} C^k & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} \sum_{i=1}^{k} C^{k-i}B_1C^{i-1} & \sum_{i=1}^{k} C^{k-i}B_2N^{i-1} \\ \sum_{i=1}^{k} N^{k-i}B_3C^{i-1} & 0 \end{bmatrix}P^{-1}\right) \\
&\quad\times\left(P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} 0 & \sum_{i=0}^{k-1} C^{-i-1}B_2N^i \\ \sum_{i=0}^{k-1} N^iB_3C^{-i-1} & 0 \end{bmatrix}P^{-1}\right) \\
&= P\begin{bmatrix} C^k & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} \sum_{i=1}^{k} C^{k-i}B_1C^{i-1} & \sum_{i=0}^{k-1} C^{k-i-1}B_2N^i \\ \sum_{i=1}^{k} N^{k-i}B_3C^{i-1} & 0 \end{bmatrix}P^{-1}
\end{aligned}$$

$$= P\begin{bmatrix} C^k & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} \sum_{i=1}^{k} C^{k-i}B_1C^{i-1} & \sum_{i=1}^{k} C^{k-i}B_2N^{i-1} \\ \sum_{i=1}^{k} N^{k-i}B_3C^{i-1} & 0 \end{bmatrix}P^{-1}$$
$$= \widehat{A}^k$$

and

$$\begin{aligned} \widehat{G}\widehat{A}\widehat{G} &= \left(P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} 0 & \sum_{i=0}^{k-1} C^{-i-1}B_2N^i \\ \sum_{i=0}^{k-1} N^iB_3C^{-i-1} & 0 \end{bmatrix}P^{-1}\right) \\ &\times \left(P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & \sum_{i=0}^{k-1} C^{-i-2}B_2N^i \\ \sum_{i=0}^{k-1} N^iB_3C^{-i-2} & 0 \end{bmatrix}P^{-1}\right) \\ &= P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & \sum_{i=0}^{k-1} C^{-i-2}B_2N^i \\ \sum_{i=0}^{k-1} N^iB_3C^{-i-2} & 0 \end{bmatrix}P^{-1} \\ &= \widehat{G}. \end{aligned}$$

Therefore, if $(I - AA^D)M(I - AA^D) = 0$, then $\widehat{A}^D$ exists and $\widehat{A}^D = \widehat{G}$.

Since

$$A^DBA^D = P\begin{bmatrix} C^{-1}B_1C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}, \quad \sum_{i=0}^{k-1} (A^D)^{i+2}BA^iA^\pi = P\begin{bmatrix} 0 & \sum_{i=0}^{k-1} C^{-i-2}B_2N^i \\ 0 & 0 \end{bmatrix}P^{-1}$$

and

$$\sum_{i=0}^{k-1} A^\pi A^iB(A^D)^{i+2} = P\begin{bmatrix} 0 & 0 \\ \sum_{i=0}^{k-1} N^iB_3C^{-i-2} & 0 \end{bmatrix}P^{-1},$$

then $\widehat{G} = \widehat{A}^D = A^D + \varepsilon[-A^DBA^D + \sum_{i=0}^{k-1} (A^D)^{i+2}BA^iA^\pi + \sum_{i=0}^{k-1} A^\pi A^iB(A^D)^{i+2}]$, which completes the proofs of (8) and (9).

Since Ind(A)=k, it can be deduced from Lemma 1.2 that

$$rank\begin{bmatrix} M & A^k \\ A^k & 0 \end{bmatrix} = 2rank(A^k) + rank[(I - AA^D)M(I - AA^D)]. \tag{10}$$

Hence, we can conclude from (10) that $(I - AA^D)M(I - AA^D) = 0$ if and only if $rank\begin{bmatrix} M & A^k \\ A^k & 0 \end{bmatrix} = 2rank(A^k)$. Thus $(iii)\Leftrightarrow(iv)$ follows.

Since $\widehat{A}^k = A^k + \varepsilon M$, then by the equivalence of (a) and (c) in [16, Theorem 2.2], $(\widehat{A}^k)^\dagger$ exists if and only if $rank\begin{bmatrix} M & A^k \\ A^k & 0 \end{bmatrix} = 2rank(A^k)$. Hence, (iv) is equivalent to (v). □

It is an interesting phenomenon that if the dual Drazin inverse of $\widehat{A} = A + \varepsilon B$ exists, then by Lemma 1.1 and Theorem 2.2, the primal part and the dual part of $\widehat{A}^D$ are respectively the (1,1)-entry and the (1,2)-entry of the Drazin inverse of the 2×2 block matrix $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$. Hence, in order to obtain $\widehat{A}^D$, we need only to compute $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^D$, which is an efficient way to compute $\widehat{A}^D$.

Corollary 2.1. *Let $\widehat{A} = A + \varepsilon B$ be a dual matrix. If $\widehat{A}^D$ exists, then*

$$\widehat{A}^D = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^D \begin{bmatrix} I \\ \varepsilon I \end{bmatrix}.$$

Corollary 2.2. *Let $\widehat{A} = A + \varepsilon B$ be a dual matrix with $A, B \in \mathbb{R}^{n\times n}$ and $Ind(A) = k$. Denote $\widehat{A}^k = A^k + \varepsilon(\sum_{i=1}^{k} A^{k-i}BA^{i-1})$ as $\widehat{A}^k = A^k + \varepsilon M$. Then the following conditions are equivalent:*
(i) The dual Drazin inverse of $\widehat{A}$ exists and $\widehat{A}^D = A^D - \varepsilon A^D BA^D$;
(ii) $AA^D M = MAA^D = M$;
(iii) $\mathcal{R}(M) \subset \mathcal{R}(A^k)$ and $\mathcal{N}(A^k) \subset \mathcal{N}(M)$.

Proof. $(i) \Longrightarrow (ii)$: If $\widehat{A}^D$ exists, then by (ii) of Theorem 2.2,

$$\widehat{A} = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1} + \varepsilon P \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} P^{-1},$$

where C and P are nonsingular matrices, N is a nilpotent matrix with $N^k = 0$, and $\sum_{i=1}^{k} N^{k-i}B_4N^{i-1} = 0$;

Moreover, if $\widehat{A}^D = A^D - \varepsilon A^D BA^D$, then by (8) and (9),

$$\sum_{i=0}^{k-1} (A^D)^{i+2}BA^iA^\pi + \sum_{i=0}^{k-1} A^\pi A^i B(A^D)^{i+2} = P \begin{bmatrix} 0 & \sum\limits_{i=0}^{k-1} C^{-i-2}B_2N^i \\ \sum\limits_{i=0}^{k-1} N^iB_3C^{-i-2} & 0 \end{bmatrix} P^{-1} = 0,$$

i.e., $\sum\limits_{i=0}^{k-1} C^{-i-2}B_2N^i = 0$ and $\sum\limits_{i=0}^{k-1} N^iB_3C^{-i-2} = 0$. In this case,

$$\sum_{i=1}^{k} C^{k-i}B_2N^{i-1} = C^{k+1}\sum_{i=0}^{k-1} C^{-i-2}B_2N^i = 0, \sum_{i=1}^{k} N^{k-i}B_3C^{i-1} = (\sum_{i=0}^{k-1} N^iB_3C^{-i-2})C^{k+1} = 0$$

and then

$$M = \sum_{i=1}^{k} A^{k-i}BA^{i-1} = P \begin{bmatrix} \sum\limits_{i=1}^{k} C^{k-i}B_1C^{i-1} & \sum\limits_{i=1}^{k} C^{k-i}B_2N^{i-1} \\ \sum\limits_{i=1}^{k} N^{k-i}B_3C^{i-1} & \sum\limits_{i=1}^{k} N^{k-i}B_4N^{i-1} \end{bmatrix} P^{-1} = P \begin{bmatrix} \sum\limits_{i=1}^{k} C^{k-i}B_1C^{i-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Now, it is not difficult to see that $AA^DM = MAA^D = M$.

$(ii) \Longrightarrow (i)$: If $AA^DM = MAA^D = M$, then $(I - AA^D)M(I - AA^D) = 0$. Thus, by Theorem 2.2, the dual Drazin inverse of $\widehat{A}$ exists. Moreover, the condition $AA^DM = MAA^D = M$ implies that $\sum\limits_{i=1}^{k} C^{k-i}B_2N^{i-1} = 0$ and $\sum\limits_{i=1}^{k} N^{k-i}B_3C^{i-1} = 0$. Then we can see from the proof of $(i) \Longrightarrow (ii)$ that $\sum\limits_{i=0}^{k-1} C^{-i-2}B_2N^i = 0$ and $\sum\limits_{i=0}^{k-1} N^iB_3C^{-i-2} = 0$. Hence,

$$\sum_{i=0}^{k-1} (A^D)^{i+2}BA^iA^\pi + \sum_{i=0}^{k-1} A^\pi A^i B(A^D)^{i+2} = P \begin{bmatrix} 0 & \sum\limits_{i=0}^{k-1} C^{-i-2}B_2N^i \\ \sum\limits_{i=0}^{k-1} N^iB_3C^{-i-2} & 0 \end{bmatrix} P^{-1} = 0.$$

Now, by (8) and (9), $\widehat{A}^D = A^D - \varepsilon A^D B A^D$.

(ii)⇔ (iii): Since $AA^D = P_{\mathcal{R}(A^k),\mathcal{N}(A^k)}$ is a projector, then $AA^D M = M$ if and only if $\mathcal{R}(M) \subset \mathcal{R}(A^k)$, and $MAA^D = M$ if and only if $\mathcal{N}(A^k) \subset \mathcal{N}(M)$. □

Example 2.1. Let

$$\widehat{A} = \begin{bmatrix} 2 & 4 & 6 & 5 \\ 1 & 4 & 5 & 4 \\ 0 & -1 & -1 & 0 \\ -1 & -2 & -3 & -3 \end{bmatrix} + \varepsilon \begin{bmatrix} -4 & 3 & -3 & 2 \\ 5 & 4 & 0 & 2 \\ -7 & 7 & 1 & 0 \\ 2 & -3 & 2 & 1 \end{bmatrix} := A + \varepsilon B.$$

It is clear that $rank(A) = 3$ and $rank(A^2) = rank(A^3) = 2$, thus Ind($A$)=2. In this case,

$$M = AB + BA = \begin{bmatrix} -27 & 44 & -2 & 3 \\ 1 & 74 & 54 & 49 \\ -5 & -12 & -9 & -9 \\ 9 & -31 & -14 & -14 \end{bmatrix}.$$

Since

$$rank \begin{bmatrix} M & A^2 \\ A^2 & 0 \end{bmatrix} = 4 = 2rank(A^2),$$

then by (iv) of Theorem 2.2, $\widehat{A}^D$ exists.

On the other hand, a direct computation shows that

$$\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^D = \left[\begin{array}{cccc|cccc} 3 & -1 & 2 & 2 & 192 & -55 & 163 & 213 \\ 2 & 1 & 3 & 3 & 97 & -114 & -8 & 17 \\ -1 & 0 & -1 & -1 & -68 & 76 & 1 & -14 \\ -1 & 0 & -1 & -1 & -54 & 15 & -46 & -61 \\ \hline 0 & 0 & 0 & 0 & 3 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \end{array}\right].$$

Therefore, it follows from Corollary 2.1 that

$$\widehat{A}^D = \begin{bmatrix} 3 & -1 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix} + \varepsilon \begin{bmatrix} 192 & -55 & 163 & 213 \\ 97 & -114 & -8 & 17 \\ -68 & 76 & 1 & -14 \\ -54 & 15 & -46 & -61 \end{bmatrix}.$$

We next give some properties of the dual Drazin inverse, which are similar to those of real square matrices. Define the range and the null space of a dual matrix $\widehat{A} = A + \varepsilon B \in \mathbb{D}^{n\times n}$ as follows.

$$\mathcal{R}(\widehat{A}) = \{\widehat{w} \in \mathbb{D}^n : \widehat{w} = \widehat{A}\widehat{z}, \widehat{z} \in \mathbb{D}^n\} = \{Ax + \varepsilon(Ay + Bx) : x, y \in \mathbb{R}^n\}, \tag{11}$$

$$\mathcal{N}(\widehat{A}) = \{\widehat{z} \in \mathbb{D}^n : \widehat{A}\widehat{z} = 0\} = \{x + \varepsilon y : Ax = 0, Ay + Bx = 0, x, y \in \mathbb{R}^n\}. \tag{12}$$

Theorem 2.3. *Let $\widehat{A} = A + \varepsilon B$ be a dual matrix with $A, B \in \mathbb{R}^{n\times n}$ and $Ind(A) = k$. If the dual Drazin inverse of $\widehat{A}$ exists, then*
(i) $\mathcal{R}(\widehat{A}^D) = \mathcal{R}(\widehat{A}^k)$;
(ii) $\mathcal{N}(\widehat{A}^D) = \mathcal{N}(\widehat{A}^k) = \mathcal{N}(\widehat{A}^l)$, *$l \geq k$, l is a positive integer;*
(iii) $\mathcal{R}(\widehat{A}^k) \bigcap \mathcal{N}(\widehat{A}^k) = \{0\}$.

Proof. (*i*) If the dual Drazin inverse of $\widehat{A}$ exists, then it can be seen from the definition of the dual Drazin inverse that $\widehat{A}^k = \widehat{A}^k\widehat{A}^D\widehat{A} = \widehat{A}^D\widehat{A}^{k+1}$. Thus, $\mathcal{R}(\widehat{A}^k) \subset \mathcal{R}(\widehat{A}^D)$. On the other hand, since $\widehat{A}^D = \widehat{A}^D\widehat{A}\widehat{A}^D = \widehat{A}^D(\widehat{A}\widehat{A}^D)^k = \widehat{A}^D\widehat{A}^k(\widehat{A}^D)^k = \widehat{A}^k(\widehat{A}^D)^{k+1}$, then $\mathcal{R}(\widehat{A}^D) \subset \mathcal{R}(\widehat{A}^k)$. Therefore, $\mathcal{R}(\widehat{A}^D) = \mathcal{R}(\widehat{A}^k)$.

(*ii*) It can be easily seen from $\widehat{A}^k = \widehat{A}^{k+1}\widehat{A}^D$ that $\mathcal{N}(\widehat{A}^D) \subset \mathcal{N}(\widehat{A}^k)$. On the other hand, it follows from $\widehat{A}^D = \widehat{A}^D\widehat{A}\widehat{A}^D = \widehat{A}^D(\widehat{A}\widehat{A}^D)^k = \widehat{A}^D\widehat{A}^k(\widehat{A}^D)^k = \widehat{A}^D(\widehat{A}^D)^k\widehat{A}^k = (\widehat{A}^D)^{k+1}\widehat{A}^k$ that $\mathcal{N}(\widehat{A}^k) \subset \mathcal{N}(\widehat{A}^D)$. Hence, $\mathcal{N}(\widehat{A}^D) = \mathcal{N}(\widehat{A}^k)$.

It is clear that $\mathcal{N}(\widehat{A}^k) \subset \mathcal{N}(\widehat{A}^l)$ for any positive integer $l \geq k$. Conversely, denote $\widehat{A}^l = A^l + \varepsilon \sum_{i=1}^{l} A^{l-i}BA^{i-1}$ as $\widehat{A}^l = A^l + \varepsilon N$. For any $\widehat{z} = x + \varepsilon y \in \mathcal{N}(\widehat{A}^l)$, it can be seen from (12) that $A^l x = 0$ and $A^l y + Nx = 0$. Since $\mathcal{N}(A^k) = \mathcal{N}(A^l)$, then $A^l x = 0$ implies $A^k x = 0$. Moreover,

$$0 = A^l y + Nx = A^l y + [A^{l-k}M + (\sum_{i=k+1}^{l} A^{l-i}BA^{i-k-1})A^k]x = A^{l-k}(A^k y + Mx),$$

i.e., $A^k y + Mx \in \mathcal{N}(A^{l-k}) \subset \mathcal{N}(A^l) = \mathcal{N}(A^k)$. On the other hand, since $\widehat{A}^D$ exists, then by (*iii*) of Theorem 2.2, $(I - AA^D)M(I - AA^D) = 0$, i.e.,

$$M = AA^DM + MAA^D - AA^DMAA^D. \tag{13}$$

Substituting (13) into $A^k y + Mx$ we get

$$\begin{aligned} A^k y + Mx &= A^k y + (AA^DM + MAA^D - AA^DMAA^D)x = A^k y + AA^DMx \\ &= A^k y + (AA^D)^k Mx = A^k y + A^k(A^D)^k Mx = A^k[y + (A^D)^k Mx] \in \mathcal{R}(A^k). \end{aligned}$$

Thus $A^k y + Mx \in \mathcal{R}(A^k) \bigcap \mathcal{N}(A^k) = \{0\}$, i.e., $A^k y + Mx = 0$, which implies that $\mathcal{N}(\widehat{A}^l) \subset \mathcal{N}(\widehat{A}^k)$. Therefore, $\mathcal{N}(\widehat{A}^k) = \mathcal{N}(\widehat{A}^l)$ for $l \geq k$.

(*iii*) For any $\widehat{z} \in \mathcal{R}(\widehat{A}^k) \bigcap \mathcal{N}(\widehat{A}^k)$, by (11) and (12), there exist $x, y \in \mathbb{R}^n$ such that $\widehat{z} = \widehat{A}^k(x + \varepsilon y) = A^k x + \varepsilon(A^k y + Mx)$, and

$$(A^k + \varepsilon M)[A^k x + \varepsilon(A^k y + Mx)] = A^{2k}x + \varepsilon[A^{2k}y + (A^kM + MA^k)x] = 0.$$

Thus $A^{2k}x = 0$ and $A^{2k}y + (A^kM + MA^k)x = 0$.

It can be seen from $A^{2k}x = A^k(A^k x) = 0$ that $A^k x \in \mathcal{R}(A^k) \bigcap \mathcal{N}(A^k) = \{0\}$. Hence, $A^k x = 0$. In this case, $0 = A^{2k}y + (A^kM + MA^k)x = A^{2k}y + A^kMx = A^k(A^k y + Mx)$, i.e, $A^k y + Mx \in \mathcal{N}(A^k)$. Moreover, it follows from the proof of (*ii*) that $A^k y + Mx \in \mathcal{R}(A^k)$. Thus $A^k y + Mx \in \mathcal{R}(A^k) \bigcap \mathcal{N}(A^k) = \{0\}$, i.e., $A^k y + Mx = 0$. Therefore, $\widehat{z} = A^k x + \varepsilon(A^k y + Mx) = 0$, which implies that $\mathcal{R}(\widehat{A}^k) \bigcap \mathcal{N}(\widehat{A}^k) = \{0\}$. □

3. Dual Drazin-inverse solution of systems of dual linear equations

In this section, we consider the linear dual equation

$$\widehat{A}\widehat{x} = \widehat{b}, \tag{14}$$

where $\widehat{A} \in \mathbb{D}^{n\times n}$ and $\widehat{x}, \widehat{b} \in \mathbb{D}^n$.

Dual generalized inverses are important tools in the solutions and least-squares solutions of systems of linear dual equations [13]. In this section, we will show some applications of the dual Drazin inverse in solving systems of linear dual equations. We first give the general solution to the equation (14) under some assumptions, which is analogous to the real case [20]. We omit the proof.

Theorem 3.1. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}^{n\times n}$ be a dual matrix with $A, B \in \mathbb{R}^{n\times n}$ and $Ind(A) = k$. If $\widehat{A}^D$ exists and $\widehat{b} \in \mathcal{R}(\widehat{A}^k)$, then the general solution to (14) is*

$$\widehat{x} = \widehat{A}^D\widehat{b} + \widehat{A}^{k-1}(I - \widehat{A}\widehat{A}^D)\widehat{z},$$

where $\widehat{z} \in \mathbb{D}^n$ *is an arbitrary dual vector.*

We call $\widehat{A}^D\widehat{b}$ the *dual Drazin-inverse solution* of the linear dual equation (14), although it may not be a solution to the linear dual equation (14). Next, we present some characterizations of the dual Drazin-inverse solution $\widehat{A}^D\widehat{b}$.

Theorem 3.2. *If the dual Drazin inverse of a dual matrix* $\widehat{A} = A + \varepsilon B \in \mathbb{D}^{n\times n}$ *exists, then* $\widehat{A}^D\widehat{b}$ *is the unique solution in* $\mathcal{R}(\widehat{A}^k)$ *of*

$$\widehat{A}^{k+1}\widehat{x} = \widehat{A}^k\widehat{b}. \tag{15}$$

Proof. If $\widehat{A}^D$ exists, then it is obvious that $\widehat{A}^D\widehat{b}$ is a solution to the equation (15).

By (*i*) of Theorem 2.3, $\widehat{A}^D\widehat{b} \in \mathcal{R}(\widehat{A}^D) = \mathcal{R}(\widehat{A}^k)$. Suppose that $\widehat{u}$ is another solution in $\mathcal{R}(\widehat{A}^k)$ of (15). Then $\widehat{u}-\widehat{A}^D\widehat{b} \in \mathcal{R}(\widehat{A}^k)$. On the other hand, since $\widehat{u}$ and $\widehat{A}^D\widehat{b}$ are solutions of (15), then $\widehat{A}^{k+1}(\widehat{u}-\widehat{A}^D\widehat{b}) = 0$. Thus, by (*ii*) of Theorem 2.3, $\widehat{u}-\widehat{A}^D\widehat{b} \in \mathcal{N}(\widehat{A}^{k+1}) = \mathcal{N}(\widehat{A}^k)$. Therefore, by (*iii*) of Theorem 2.3, $\widehat{u}-\widehat{A}^D\widehat{b} \in \mathcal{R}(\widehat{A}^k) \bigcap \mathcal{N}(\widehat{A}^k) = \{0\}$, i.e., $\widehat{u} = \widehat{A}^D\widehat{b}$. □

We call the linear dual equation (15) the *dual Drazin normal equation*.

Theorem 3.3. *Let* $\widehat{A} = A + \varepsilon B \in \mathbb{D}^{n\times n}$, $\widehat{b} \in \mathbb{D}^n$ *be such that* $\widehat{A}^D$ *exists. Then*
(i) The solutions of the restricted equation $\widehat{A}^{k+1}\widehat{x} = \widehat{A}^k\widehat{b}$, $\widehat{x} \in \mathcal{N}(\widehat{A}) + \mathcal{R}(\widehat{A}^{k-1})$ *give small P-norm of the error of the inconsistent equation* $\widehat{A}\widehat{x} = \widehat{b}$*. The P-norm of the error*

$$\| \widehat{e} \|_P = \| \widehat{A}\widehat{x} - \widehat{b} \|_P = \| \widehat{A}\widehat{A}^D\widehat{b} - \widehat{b} \|_P.$$

(ii) The dual Drazin-inverse solution $\widehat{A}^D\widehat{b}$ *has a small P-norm among the solutions of the dual Drazin normal equation.*

Proof. (*i*) Denote $\widehat{w}_1 = \widehat{A}\widehat{A}^D\widehat{b} - \widehat{b} = u_1 + \varepsilon v_1$ and $\widehat{w}_2 = \widehat{A}\widehat{x} - \widehat{A}\widehat{A}^D\widehat{b} = u_2 + \varepsilon v_2$. Then

$$\| \widehat{e} \|_P = \| \widehat{A}\widehat{x} - \widehat{b} \|_P = \| \widehat{w}_1 + \widehat{w}_2 \|_P = \sqrt{\| u_1 + u_2 \|_P^2 + \| v_1 + v_2 \|_P^2}. \tag{16}$$

If $\widehat{x} \in \mathcal{N}(\widehat{A}) + \mathcal{R}(\widehat{A}^{k-1})$, then $\widehat{x}$ can be represented as $\widehat{x} = \widehat{y} + \widehat{A}^{k-1}\widehat{z}$, where $\widehat{y} \in \mathcal{N}(\widehat{A})$ and $\widehat{z} \in \mathbb{D}^n$. Since $\widehat{A}\widehat{A}^D\widehat{b} \in \mathcal{R}(\widehat{A}\widehat{A}^D) = \mathcal{R}(\widehat{A}^D) = \mathcal{R}(\widehat{A}^k)$, then $\widehat{A}\widehat{A}^D\widehat{b} = \widehat{A}^k\widehat{w}$ for some $\widehat{w} \in \mathbb{D}^n$. Hence,

$$\begin{aligned}
(P^{-1}\widehat{w}_1)^T(P^{-1}\widehat{w}_2) &= [P^{-1}(\widehat{A}\widehat{A}^D\widehat{b} - \widehat{b})]^T[P^{-1}(\widehat{A}\widehat{x} - \widehat{A}\widehat{A}^D\widehat{b})] \\
&= [P^{-1}(\widehat{A}\widehat{A}^D\widehat{b} - \widehat{b})]^T[P^{-1}\widehat{A}^k(\widehat{z} - \widehat{w})] \\
&= [P^{-1}(\widehat{A}\widehat{A}^D - I)PP^{-1}\widehat{b}]^T[P^{-1}\widehat{A}^kPP^{-1}(\widehat{z} - \widehat{w})] \\
&= (P^{-1}\widehat{b})^T[P^{-1}(\widehat{A}\widehat{A}^D - I)P]^T(P^{-1}\widehat{A}^kP)[P^{-1}(\widehat{z} - \widehat{w})].
\end{aligned}$$

It can be seen from the block representations of $\widehat{A}$ and $\widehat{A}^D$ in Theorem 2.2 that

$$\begin{aligned}
[P^{-1}(\widehat{A}\widehat{A}^D - I)P]^T(P^{-1}\widehat{A}^kP) &= \left(\begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & (\sum\limits_{i=0}^{k-1} N^iB_3C^{-i-1})^T \\ (\sum\limits_{i=0}^{k-1} C^{-i-1}B_2N^i)^T & 0 \end{bmatrix}\right) \\
&\times \left(\begin{bmatrix} C^k & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} \sum\limits_{i=1}^{k} C^{k-i}B_1C^{i-1} & \sum\limits_{i=1}^{k} C^{k-i}B_2N^{i-1} \\ \sum\limits_{i=1}^{k} N^{k-i}B_3C^{i-1} & 0 \end{bmatrix}\right) \\
&= \varepsilon \begin{bmatrix} 0 & 0 \\ (\sum\limits_{i=0}^{k-1} C^{-i-1}B_2N^i)^TC^k - \sum\limits_{i=1}^{k} N^{k-i}B_3C^{i-1} & 0 \end{bmatrix}.
\end{aligned}$$

Then the primal part of $(P^{-1}\widehat{w}_1)^T(P^{-1}\widehat{w}_2)$ is zero, i.e., $(P^{-1}u_1)^T(P^{-1}u_2) = 0$, which implies that

$$\| u_1 + u_2 \|_P^2 = \| u_1 \|_P^2 + \| u_2 \|_P^2. \tag{17}$$

Substituting the equality (17) into (16) we get an upper bound for $\| \widehat{e} \|_P$, i.e,

$$\| \widehat{e} \|_P = \| \widehat{A}\widehat{x} - \widehat{b} \|_P \leq \sqrt{\| u_1 \|_P^2 + \| u_2 \|_P^2 + (\| v_1 \|_P + \| v_2 \|_P)^2} := \delta_1. \tag{18}$$

Notice that the dual vector $\widehat{w}_1$ depends only on $\widehat{A}$ and $\widehat{b}$, we can choose $\widehat{w}_2 = 0$ so that the upper bound δ_1 given in (18) will attain its minimum.

We conclude that $\widehat{A}^{k+1}\widehat{x} = \widehat{A}^k\widehat{b}$, $\widehat{x} \in \mathcal{N}(\widehat{A}) + \mathcal{R}(\widehat{A}^{k-1})$ is equivalent to $\widehat{A}\widehat{x} = \widehat{A}\widehat{A}^D\widehat{b}$, $\widehat{x} \in \mathcal{N}(\widehat{A}) + \mathcal{R}(\widehat{A}^{k-1})$. If $\widehat{A}\widehat{x} = \widehat{A}\widehat{A}^D\widehat{b}$, then premultiplying both sides of $\widehat{A}\widehat{x} = \widehat{A}\widehat{A}^D\widehat{b}$ by $\widehat{A}^k$ yields $\widehat{A}^{k+1}\widehat{x} = \widehat{A}^k\widehat{b}$. On the other hand, premultiplying both sides of $\widehat{A}^{k+1}\widehat{x} = \widehat{A}^k\widehat{b}$ by $(\widehat{A}^D)^{k+1}$ leads to $\widehat{A}\widehat{A}^D\widehat{x} = \widehat{A}^D\widehat{b}$. Since $\widehat{x} \in \mathcal{N}(\widehat{A}) + \mathcal{R}(\widehat{A}^{k-1})$, then there exist $\widehat{y} \in \mathcal{N}(\widehat{A})$ and $\widehat{z} \in \mathbb{D}^n$ such that $\widehat{x} = \widehat{y} + \widehat{A}^{k-1}\widehat{z}$. Hence, $\widehat{A}(\widehat{x} - \widehat{A}^D\widehat{b}) = \widehat{A}(\widehat{x} - \widehat{A}\widehat{A}^D\widehat{x}) = \widehat{A}^k\widehat{z} - \widehat{A}^D\widehat{A}^{k+1}\widehat{z} = \widehat{A}^k\widehat{z} - \widehat{A}^k\widehat{z} = 0$. Therefore, $\widehat{A}\widehat{x} = \widehat{A}\widehat{A}^D\widehat{b}$.

Hence, the choices of $\widehat{x}$ that satisfy $\widehat{A}^{k+1}\widehat{x} = \widehat{A}^k\widehat{b}$, $\widehat{x} \in \mathcal{N}(\widehat{A}) + \mathcal{R}(\widehat{A}^{k-1})$ will cause $\widehat{w}_2$ to vanish. The P-norm of the error is given by

$$\| \widehat{e} \|_P = \| \widehat{w}_1 \|_P = \sqrt{\| u_1 \|_P^2 + \| v_1 \|_P^2} = \| \widehat{A}\widehat{A}^D\widehat{b} - \widehat{b} \|_P.$$

(*ii*) The general solution to the linear dual equation (15) is $\widehat{x} = \widehat{A}^D\widehat{b} + \mathcal{N}(\widehat{A}^k) = \widehat{A}^D\widehat{b} + (I - \widehat{A}\widehat{A}^D)\widehat{z}$, where $\widehat{z} \in \mathbb{D}^n$ is an arbitrary dual vector. Denote $\widehat{A}^D\widehat{b} = \widehat{\mu}_1 = \alpha_1 + \varepsilon\beta_1$ and $(I - \widehat{A}\widehat{A}^D)\widehat{z} = \widehat{\mu}_2 = \alpha_2 + \varepsilon\beta_2$. Then

$$\begin{aligned}(P^{-1}\widehat{\mu}_1)^T(P^{-1}\widehat{\mu}_2) &= (P^{-1}\widehat{A}^D\widehat{b})^T[P^{-1}(I - \widehat{A}\widehat{A}^D)\widehat{z}]\\ &= (P^{-1}\widehat{A}^DPP^{-1}\widehat{b})^T[P^{-1}(I - \widehat{A}\widehat{A}^D)PP^{-1}\widehat{z}]\\ &= (P^{-1}\widehat{b})^T(P^{-1}\widehat{A}^DP)^T[P^{-1}(I - \widehat{A}\widehat{A}^D)P](P^{-1}\widehat{z})\end{aligned}$$

and it can be seen from the block representations of $\widehat{A}$ and $\widehat{A}^D$ that the primal part of $(P^{-1}\widehat{A}^DP)^T[P^{-1}(I - \widehat{A}\widehat{A}^D)P]$ is zero. Thus the primal part of $(P^{-1}\widehat{\mu}_1)^T(P^{-1}\widehat{\mu}_2)$ is also zero, i.e., $(P^{-1}\alpha_1)^T(P^{-1}\alpha_2) = 0$. Therefore, $\| \alpha_1 + \alpha_2 \|_P^2 = \| \alpha_1 \|_P^2 + \| \alpha_2 \|_P^2$,

Thus, as before, we obtain an upper bound for the P-norm of the dual vector $\widehat{x}$ given by

$$\| \widehat{x} \|_P \leq \sqrt{\| \alpha_1 \|_P^2 + \| \alpha_2 \|_P^2 + (\| \beta_1 \|_p + \| \beta_2 \|_p)^2} := \delta_2. \tag{19}$$

To make δ_2 in (19) as small as possible, we can choose $\widehat{z} \in \mathcal{R}(\widehat{A}^k)$ such that $\widehat{\mu}_2 = (I - \widehat{A}\widehat{A}^D)\widehat{z} = 0$. In this case,

$$\| \widehat{x} \|_P = \| \widehat{\mu}_1 \|_P = \sqrt{\| \alpha_1 \|_P^2 + \| \beta_1 \|_P^2} = \| \widehat{A}^D\widehat{b} \|_P.$$

□

Example 3.1. Consider the linear dual equation $\widehat{A}\widehat{x} = \widehat{b}$, where

$$\widehat{A} = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} + \varepsilon \begin{bmatrix} 3 & 7 & 1 \\ -2 & -4 & 2 \\ 1 & 2 & 0 \end{bmatrix} := A + \varepsilon B, \ \widehat{b} = \begin{bmatrix} 23.1 \\ -12.2 \\ -2.8 \end{bmatrix} + \varepsilon \begin{bmatrix} 13.2 \\ -14.8 \\ -0.5 \end{bmatrix}.$$

Then Ind(A)=2, and the Jordan canonical form of A is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{5}{9} & \frac{2}{3} & \frac{5}{9} \\ -\frac{1}{9} & -\frac{2}{3} & \frac{1}{9} \end{bmatrix} \left[\begin{array}{c|cc} 3 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} 1 & 0 & 0 \\ -\frac{5}{9} & \frac{2}{3} & \frac{5}{9} \\ -\frac{1}{9} & -\frac{2}{3} & \frac{1}{9} \end{bmatrix}^{-1} := P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}.$$

Since $rank\begin{bmatrix} AB+BA & A^2 \\ A^2 & 0 \end{bmatrix} = 2 = 2rank(A^2)$, then by Theorem 2.2 and Corollary 2.1, $\widehat{A}^D$ exists and

$$\widehat{A}^D = \begin{bmatrix} I & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^D \begin{bmatrix} I \\ \varepsilon I \end{bmatrix} = \begin{bmatrix} 0.3333 & 0 & 0 \\ -0.1852 & 0 & 0 \\ -0.0370 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0.7037 & 1.0000 & 0.3333 \\ -0.4815 & -0.5556 & -0.1852 \\ -0.0741 & -0.1111 & -0.0370 \end{bmatrix}.$$

The P-norm of the error is

$$\| \widehat{e} \|_P = \| \widehat{A}\widehat{A}^D\widehat{b} - \widehat{b} \|_P = \sqrt{\| P^{-1}u \|_2^2 + \| P^{-1}v \|_2^2} = 2.2890,$$

where $u = [-0.0023, -0.6321, 0.2336]^T$ and $v = [1.6615, 0.2713, -0.8673]^T$.

Moreover,

$$\widehat{A}^D\widehat{b} = \begin{bmatrix} 7.6992 \\ -4.2781 \\ -0.8547 \end{bmatrix} + \varepsilon \begin{bmatrix} 7.5218 \\ -6.2704 \\ -0.7411 \end{bmatrix} := x + \varepsilon y.$$

Then

$$\| \widehat{A}^D\widehat{b} \|_P = \sqrt{\| P^{-1}x \|_2^2 + \| P^{-1}y \|_2^2} = 11.1910.$$

We will show in the following that if $\widehat{A}^D$ exists and $\widehat{A}^D = A^D - \varepsilon A^D B A^D$, then we can obtain some results which is analogous to those in [18] and [20].

Theorem 3.4. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}^{n\times n}$, $\widehat{b} \in \mathbb{D}^n$ be such that $\widehat{A}^D$ exists and $\widehat{A}^D = A^D - \varepsilon A^D B A^D$. Then $\widehat{x}^*$ satisfies*

$$\| \widehat{b} - \widehat{A}\widehat{x}^* \|_P = \min_{\widehat{x} \in \mathcal{N}(\widehat{A}) + \mathcal{R}(\widehat{A}^{k-1})} \| \widehat{b} - \widehat{A}\widehat{x} \|_P$$

if and only if $\widehat{x}^*$ is the solution of $\widehat{A}^{k+1}\widehat{x} = \widehat{A}^k\widehat{b}$, $\widehat{x} \in \mathcal{N}(\widehat{A}) + \mathcal{R}(\widehat{A}^{k-1})$. Moreover, the dual Drazin-inverse solution $\widehat{A}^D\widehat{b}$ is the unique minimal P-norm solution of (15).

Proof. If $\widehat{A}^D$ exists and $\widehat{A}^D = A^D - \varepsilon A^D B A^D$, then it follows from (8) and (9) that $\sum_{i=0}^{k-1} (A^D)^{i+2} B A^i A^\pi + \sum_{i=0}^{k-1} A^\pi A^i B (A^D)^{i+2} = 0$, i.e.,

$$\begin{bmatrix} 0 & \sum\limits_{i=0}^{k-1} C^{-i-2}B_2N^i \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \sum\limits_{i=0}^{k-1} N^iB_3C^{-i-2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sum\limits_{i=0}^{k-1} C^{-i-2}B_2N^i \\ \sum\limits_{i=0}^{k-1} N^iB_3C^{-i-2} & 0 \end{bmatrix} = 0.$$

Thus

$$\sum_{i=0}^{k-1} C^{-i-2}B_2N^i = C^{-k-1}(\sum_{i=1}^{k} C^{k-i}B_2N^{i-1}) = 0$$

and

$$\sum_{i=0}^{k-1} N^iB_3C^{-i-2} = (\sum_{i=1}^{k} N^{k-i}B_3C^{i-1})C^{-k-1} = 0,$$

which implies that $\sum_{i=1}^{k} C^{k-i}B_2N^{i-1} = 0$ and $\sum_{i=1}^{k} N^{k-i}B_3C^{i-1} = 0$. In this case,

$$\widehat{A}^k = P\begin{bmatrix} C^k & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} \sum\limits_{i=1}^{k} C^{k-i}B_1C^{i-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}. \tag{20}$$

For any $\widehat{x} \in \mathcal{N}(\widehat{A}) + \mathcal{R}(\widehat{A}^{k-1})$, $\widehat{A}\widehat{x}$ can be represented as $\widehat{A}\widehat{x} = \widehat{A}^k\widehat{y}$ for some $\widehat{y} \in \mathbb{D}^n$. Moreover, since $\widehat{A}\widehat{A}^D\widehat{b} \in \mathcal{R}(\widehat{A}\widehat{A}^D) = \mathcal{R}(\widehat{A}^k)$, then $\widehat{A}\widehat{A}^D\widehat{b} = \widehat{A}^k\widehat{z}$ for some $\widehat{z} \in \mathbb{D}^n$. Write $\widehat{b} = \widehat{A}\widehat{A}^D\widehat{b} + (I - \widehat{A}\widehat{A}^D)\widehat{b}$. Then

$$\begin{aligned} \| \widehat{b} - \widehat{A}\widehat{x} \|_P^2 &= \| \widehat{A}\widehat{A}^D\widehat{b} - \widehat{A}\widehat{x} \|_P^2 + \| (I - \widehat{A}\widehat{A}^D)\widehat{b} \|_P^2 \\ &+ 2[P^{-1}(\widehat{A}\widehat{A}^D\widehat{b} - \widehat{A}\widehat{x})]^T[P^{-1}(I - \widehat{A}\widehat{A}^D)\widehat{b}] \\ &= \| \widehat{A}\widehat{A}^D\widehat{b} - \widehat{A}\widehat{x} \|_P^2 + \| (I - \widehat{A}\widehat{A}^D)\widehat{b} \|_P^2 \\ &+ 2[P^{-1}(\widehat{z} - \widehat{y})]^T(P^{-1}\widehat{A}^kP)^T[P^{-1}(I - \widehat{A}\widehat{A}^D)P](P^{-1}\widehat{b}). \end{aligned} \tag{21}$$

If $\widehat{A}^D = A^D - \varepsilon A^DBA^D$, then $\widehat{A}^D$ is of the form

$$\widehat{A}^D = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}. \tag{22}$$

It can be deduced from (20) and (22) that the third term of the right hand side of (21) vanishes. Hence,

$$\| \widehat{b} - \widehat{A}\widehat{x} \|_P^2 = \| \widehat{A}\widehat{A}^D\widehat{b} - \widehat{A}\widehat{x} \|_P^2 + \| (I - \widehat{A}\widehat{A}^D)\widehat{b} \|_P^2 \geq \| (I - \widehat{A}\widehat{A}^D)\widehat{b} \|_P^2,$$

the equality holds if and only if $\widehat{A}\widehat{x} = \widehat{A}\widehat{A}^D\widehat{b}$. If $\widehat{x} \in \mathcal{N}(\widehat{A}) + \mathcal{R}(\widehat{A}^{k-1})$, then it was shown in the proof of Theorem 3.3 that $\widehat{A}\widehat{x} = \widehat{A}\widehat{A}^D\widehat{b}$ is equivalent to the dual Drazin normal equation (15).

On the other hand, the general solution to (15) is

$$\widehat{x} = \widehat{A}^D\widehat{b} + \mathcal{N}(\widehat{A}^k) = \widehat{A}^D\widehat{b} + (I - \widehat{A}\widehat{A}^D)\widehat{z}.$$

Therefore,

$$\| \widehat{A}^D\widehat{b} + (I - \widehat{A}\widehat{A}^D)\widehat{z} \|_P^2 = \| \widehat{A}^D\widehat{b} \|_P^2 + \| (I - \widehat{A}\widehat{A}^D)\widehat{z} \|_P^2 \geq \| \widehat{A}^D\widehat{b} \|_P^2.$$

Equality in the above relation holds if and only if $(I - \widehat{A}\widehat{A}^D)\widehat{z} = 0$, i.e., $\widehat{A}^D\widehat{b}$ is the unique minimal P-norm solution of (15). □

4. Conclusions

In this paper, we obtained some results of the dual Drazin inverse, especially in the existence and computations. As we have stated, the dual Drazin inverse of a square dual matrix may not exist. We gave some necessary and sufficient conditions for the existence of the dual Drazin inverse. If the dual Drazin inverse exists, then a compact formula was given. In particular, we found an unexpected result that the dual Drazin inverse can be easily obtained by computing the Drazin inverse of a 2×2 upper triangular block matrix.

The least-squares and minimal properties of the dual Drazin inverse were also discussed. It was shown that if the dual Drazin inverse of the coefficient dual matrix of the linear dual equation $\widehat{A}\widehat{x} = \widehat{b}$ exists and $\widehat{A}^D = A^D + \varepsilon[-A^DBA^D + \sum_{i=0}^{k-1}(A^D)^{i+2}BA^iA^\pi + \sum_{i=0}^{k-1}A^\pi A^iB(A^D)^{i+2}]$, where $A^\pi = I - AA^D$, then the least-squares and minimal properties of the linear dual equation $\widehat{A}\widehat{x} = \widehat{b}$ are somewhat different from those of the real case. On the other hand, if the dual Drazin inverse of the coefficient dual matrix of the linear dual equation $\widehat{A}\widehat{x} = \widehat{b}$ exists and $\widehat{A}^D = A^D - \varepsilon A^DBA^D$, then the least-squares and minimal properties of the linear dual equation $\widehat{A}\widehat{x} = \widehat{b}$ are almost the same as those of the real case.

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