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Hankel and Toeplitz operators, block matrices and derivations

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Abstract. Hankel and Toeplitz operators are the compressions of Laurent and bilateral Hankel operators, which in turn can be presented as two-by-two operator matrices with Toeplitz and Hankel entries.

0. Introduction Toeplitz operators

0.1 $$\mathrm{T}_\varphi : f \mapsto \varphi \cdot f : H_2(\mathbb{S}) \to H_2(\mathbb{S}) \subseteq L_2(\mathbb{S})$$

are compressions of a simpler "bilateral" version, known as *Laurent operators*,

0.2 $$L_\varphi : f \mapsto \varphi \cdot f : L_2(\mathbb{S}) \to L_2(\mathbb{S}),$$

multiplication by $\varphi \in L_\infty(\mathbb{S})$, essentially bounded on the unit circle $\mathbb{S} \subseteq \mathbb{C}$. Hilbert space isomorphism

0.3 $$H_2(\mathbb{S}) \cong L_2(\mathbb{S}) \cong \ell_2(\mathbb{Z}) \cong \ell_2(\mathbb{N})$$

enables us to work with infinite matrices throughout.

Here, (0.1) just reminds the reader that we are looking at the same Toeplitz operators as all those complex analysts. The relation (0.3) just recalls the fact that every two infinite dimensional separable Hilbert spaces are mutually isomorphic, which allows us to switch from $L_2(\mathbb{S})$ to $\ell_2(\mathbb{N})$ and $\ell_2(\mathbb{Z})$.

1. Hankel and Toeplitz operators

On the linear space $X = \mathbb{C}^{\mathbb{N}}$ all complex sequences $x = (x_1, x_2, \ldots)$ the *forward shift*

1.1 $$u : (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots)$$

and the *backward shift*

1.2 $$v : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$$

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satisfy

1.3 $$vu = 1 \neq uv\ ;$$

the difference is the rank one operator

1.4 $$1 - uv = \delta_1 \otimes \delta_1 := \delta_1^* \odot \delta_1 : (x_1, x_2, x_3, \ldots) \mapsto (x_1, 0, 0, \ldots)$$

where $\delta_n := (\delta_{nj})$ is the Kronecker delta:

1.5 $$\delta_{nj} = 1,\ \ (j = n),\ \ \delta_{nj} = 0\ (j \neq n).$$

A linear operator $a : X \to X$ can be represented by an infinite matrix (a_{ij}) where, for each $\{i, j\} \subseteq \mathbb{N} \times \mathbb{N}$,

1.6 $$a_{ij} = (a\delta_i)_j,$$

and has an *adjoint*, $a^* : X \to X$, with

1.7 $$(a^*)_{ij} = (\overline{a}_{ji});$$

there is a partially defined inner product:

1.8 $$\sum_{n=1}^{\infty} |x_n|\,|y_n| < \infty \Longrightarrow \langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y}_n.$$

When (1.8) holds then we write

1.9 $$x \otimes y = y^* \odot x : w \mapsto y^*(w)x = \langle w, y\rangle x$$

for the induced rank one operator. We can extend this discussion to subspaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ for which

1.10 $$u(X) + v(X) \subseteq X\ ;\ \{\delta_n : n \in \mathbb{N}\} \subseteq X,$$

in particular the familiar spaces ℓ_p and c_0, which of course carry norms. (1.8) holds in particular when $(x, y) \in \ell_p \times \ell_q$ with $1/p + 1/q = 1$. When $a : \ell_2 \to \ell_2$ is bounded then of course for arbitrary x and y in $\ell_2 = \ell_2(\mathbb{N})$,

1.11 $$\langle ax, y\rangle = \langle x, a^* y\rangle.$$

Restricted to ℓ_p and c_0, the shifts become bounded operators of norm one. Following Brown and Halmos [3, Theorem 6], we shall call the linear operator $a : X \to X$ a *Hankel* operator if

1.12 $$va = au,$$

and a *Toeplitz* operator if

1.13 $$vau = a.$$

In terms of the matrix representation (a_{ij}) of (1.6), the Toeplitz condition says that the entries are constant on diagonals, the Hankel condition that they are constant on skew diagonals. We use the same terminology on each of the Banach spaces ℓ_p, but most particularly on the Hilbert space ℓ_2. Evidently the Hankel operators form the null space of the *generalized inner derivation*

1.14 $$L_v - R_u : a \mapsto va - au,$$

while the Toeplitz operators form the null space of its multiplicative analogue

1.15 $$L_v R_u - I : a \mapsto vau - a.$$

The shifts u and v are each Toeplitz, while the rank one operator $1-uv$ is Hankel. More generally bounded Toeplitz operators on ℓ_2 have expansions

1.16 $$a=\alpha_0+\sum_{j=1}^{\infty}\alpha_j u^j+\sum_{j=1}^{\infty}\alpha_{-j}v^j,\ (\alpha_0,\alpha_j,\alpha_{-j}\in\mathbb{C})$$

while Hankel operators can be expanded

1.17 $$m=\sum_{j=1}^{\infty}\beta_j w_j,\ (\beta_j\in\mathbb{C})$$

where

$$w_1=1-uv, w_2=uw_1+w_1v, w_3=u^2w_1+uw_1v+w_1v^2,\ldots$$

i.e.

1.18 $$w_k=\sum_{i=0}^{k-1}u^{k-1-i}w_1v^i \text{ for } k\geq 1.$$

Thus

$$vw_1=0=w_1u, vw_2=w_1=w_2u, vw_3=w_2=w_3u,\ldots$$

i.e.

1.19 $$vw_k=w_{k-1}=w_ku \text{ for } k\geq 1 \text{ and } w_0=0.$$

Notice [7, Theorem 7] that Toeplitz operators a can be divided into *analytic* Toeplitz operators, which satisfy

1.20 $$au=ua,$$

and *co-analytic* Toeplitz operators,

1.21 $$av=va:$$

then every Toeplitz operator is the sum of an analytic and a co-analytic Toeplitz operator, uniquely to within a scalar. In the notation of (1.16), a is analytic iff $j<0\Longrightarrow\alpha_j=0$, co-analytic iff $j>0\Longrightarrow\alpha_j=0$. Stronger than the condition (1.12), we are tempted to call $a:X\to X$ *hyper Hankel* if it satisfies

1.22 $$ua=av\ ,$$

equivalent to equality

1.23 $$vav=a=uau\ .$$

Stronger than (1.13), the condition

1.24 $$uav=a$$

implies both $au=ua$ and $va=av$, which together imply that $a=\lambda$ is a scalar.

2. Laurent and bilateral Hankel operators

We recall the forward and backward "unilateral shifts" of (1.1) and (1.2), linear operators on $X = \mathbb{C}^{\mathbb{N}}$, which characterized (1.13) "Toeplitz" and (1.12) "Hankel" operators. Their "bilateral" equivalents, acting on $\mathbb{C}^{\mathbb{Z}}$, can be interpreted as linear operators on $X \times X$, and hence as 2×2 matrices of operators on $X \subseteq \mathbb{C}^{\mathbb{N}}$ which satisfy the conditions (1.10); note the isomorphism

2.1 $$w \in \mathbb{C}^{\mathbb{Z}} \leftrightarrow (x, y) \in \mathbb{C}^{\mathbb{N}} \times \mathbb{C}^{\mathbb{N}} \ : (n \in \mathbb{N} \Longrightarrow w_n = x_n, w_{1-n} = y_n) \ .$$

This of course holds in particular when $X = \ell_2 \subseteq \mathbb{C}^{\mathbb{N}}$. For example the *bilateral shifts*, forward and backward, are given by

2.2 $$U = \begin{pmatrix} u & 1 - uv \\ 0 & v \end{pmatrix} \ ; \ V = \begin{pmatrix} v & 0 \\ 1 - uv & u \end{pmatrix};$$

the difference is that now, in place of (1.4), we have

2.3 $$VU = I = UV.$$

There are bilateral analogues of the Toeplitz and Hankel conditions (cf [3, Theorem 2]):

Definition 2.1. *$A : X^2 \to X^2$ will be called a Laurent operator, or bilateral Toeplitz operator if*

2.4 $$AU = UA,$$

and a bilateral Hankel operator if

2.5 $$AU = VA.$$

There is here no distinction between "Laurent", "analytic Laurent", or "co-analytic Laurent" operators. The identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an example of a Laurent operator, while the matrix $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a bilateral Hankel operator. Evidently, $E^2 = I$, $EUE = V$, and each of the shifts U and V is Laurent.

The basic properties of a Laurent operator A on X^2 with $X \subseteq \mathbb{C}^{\mathbb{N}}$ are very simple. The defining condition (2.4) says that A is in the null space of the inner derivation $L_U - R_U$, *commutant* of the bilateral shift U, therefore closed under the taking of inverses; the bilateral Hankel operators make up (2.5) the null space of $L_V - R_U$. When we restrict to X^2 with $X = \ell_2$ then the bilateral shifts become *normal* operators, and then *Fuglede's theorem* [7] says that the Laurent operators are also the *double commutant* of U, and also closed under the taking of adjoints. It follows that if A is Laurent then so is A^*, also A^n, and more generally $p(A)$ whenever p is polynomial. More generally still $f(A)$ is Laurent whenever f is a function analytic on some neighborhood of the *spectrum* $\sigma(A)$ of A. Collectively therefore the Laurent operators constitute a commutative C* algebra, isometrically isomorphic to $L_\infty(\mathbb{S})$. The bilateral Hankel operators do not form a subalgebra, but collectively are just the Laurent operators multiplied by one specific Hankel operator:

2.6 $$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Evidently

2.7 $$E^2 = I \ ; \ EU = VE \ ; \ EUE = V.$$

Theorem 2.2. *If $A \in B(X^2)$ then*

2.8 $$A \text{ bilateral Hankel} \Longleftrightarrow AE \text{ Laurent} \Longleftrightarrow EA \text{ Laurent}$$

and dually

2.9 $$A \text{ Laurent} \Longleftrightarrow AE \text{ bilateral Hankel} \Longleftrightarrow EAE \text{ Laurent}.$$

Rewriting this

2.10 $$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \text{ Laurent} \Longleftrightarrow \begin{pmatrix} b & n \\ m & a \end{pmatrix} \text{ Laurent}$$

and

2.11 $$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \text{ bilateral Hankel} \Longleftrightarrow \begin{pmatrix} m & a \\ b & n \end{pmatrix} \text{Laurent} \Longleftrightarrow \begin{pmatrix} n & b \\ a & m \end{pmatrix} \text{ Laurent.}$$

If $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ is either Laurent or bilateral Hankel, then (cf [15, Lemma 1]) each of a, b, m, n will be either Toeplitz or Hankel:

Theorem 2.3. *There is implication*

2.12 $$A = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \text{ Laurent} \Longrightarrow a, b \textit{ Toeplitz and } m, n \textit{ Hankel.}$$

Dually

2.13 $$A = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \text{bilateral Hankel} \Longrightarrow m, n \textit{ Toeplitz and } a, b \textit{ Hankel.}$$

Proof. By multiplication of 2×2 matrices, necessary and sufficient for (2.11) are the following four conditions:

2.14 $$au - ua = (1 - uv)n;$$

2.15 $$mv - um = (1 - uv)b - a(1 - uv);$$

2.16 $$nu - vn = 0;$$

2.17 $$vb - bv = n(1 - uv).$$

Multiplying (2.14) on the left by v, and (2.17) on the right by u, shows that a and b are both Toeplitz. (2.16) says that n is Hankel, and finally multiplying (2.15) left and right by v and u says that m is "hyper Hankel" in the sense (1.22). This establishes (2.11), which is converted to (2.12) by Theorem 2.2 □

Now we consider the reverse implications of Theorem 2.3.

Corollary 2.4. *(i)* $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ *is Laurent if and only if* a, b *Toeplitz and* m, n *Hankel and*

$$vb\delta_1 = n\delta_1, va^*\delta_1 = n^*\delta_1, \lambda a\delta_1 = \overline{\lambda} b^*\delta_1 = \delta_1, (\lambda \in \mathbb{C}).$$

(ii) $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ *is bilateral Hankel if and only if* m, n *Toeplitz and* a, b *Hankel and*

$$vn\delta_1 = b\delta_1, vm^*\delta_1 = b^*\delta_1, \lambda m\delta_1 = \overline{\lambda} n^*\delta_1 = \delta_1, (\lambda \in \mathbb{C}).$$

Proof. By Theorem 2.3, it suffices to show the converse implication. Suppose that a, b are Toeplitz (i.e., $vau = a, vbu = b$) and m, n are Hankel (i.e., $vn = nu, vm = mu$), and

$$vb\delta_1 = n\delta_1, va^*\delta_1 = n^*\delta_1, \lambda a\delta_1 = \overline{\lambda} b^*\delta_1 = \delta_1, (\lambda \in \mathbb{C}).$$

Then (2.16) always holds. Since $vb\delta_1 = n\delta_1$, it follows that

$$\begin{aligned} vb - bv - n(1-uv) &= vb - vbuv - n(1-uv) = vb(1-uv) - n(1-uv) \\ &= (vb-n)(1-uv) = (vb-n)\delta_1 \otimes \delta_1 = 0. \end{aligned}$$

Thus (2.17) holds. Moreover, since $vm = mu$ implies $v[mv - um]u = vmvu - vumu = vm - mu = 0$, it follows that $mv = um$ and so

$$\begin{aligned} mv - um - (1-uv)b + a(1-uv) &= -(1-uv)b + a(1-uv) = -(\delta_1 \otimes b^*\delta_1) + a\delta_1 \otimes \delta_1 \\ &= -(\delta_1 \otimes \frac{1}{\lambda}\delta_1) + \frac{1}{\lambda}\delta_1 \otimes \delta_1 = 0 \end{aligned}$$

because the last equality follows from $\lambda a\delta_1 = \overline{\lambda} b^*\delta_1 = \delta_1, (|\lambda| = 1)$. Thus (2.15) holds. On the other hand, since $va^*\delta_1 = n^*\delta_1$, we have

$$\begin{aligned} au - ua - (1-uv)n &= au - uvau - (1-uv)n = (1-uv)au - (1-uv)n \\ &= (1-uv)(au-n) = (\delta_1 \otimes (au-n)^*\delta_1) \\ &= (\delta_1 \otimes (va^* - n^*)\delta_1) = 0 \end{aligned}$$

and (2.14) holds.

(ii) The proof follows from (2.11) and similar arguments of (i). □

The "compression process" which converts the Laurent operator A to the Toeplitz operator a is effected by the projection

2.18 $$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} :$$

2.19 $$A = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \Longrightarrow PAP = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Corollary 2.5. *Let $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ on X^2 where a, b, m, n are nonzero. Then the following statements hold;*
(i) If m, n are Hankel, a is an analytic Toeplitz, and $b = a^*$ where $a\delta_1 = \lambda\delta_1$ for some $\lambda \in \mathbb{C}$, then A is not Laurent.
(ii) If a, b are Hankel, m is an analytic Toeplitz, and $n = m^*$ where $m\delta_1 = \lambda\delta_1$ for some $\lambda \in \mathbb{C}$, then A is not bilateral Hankel.

Proof. (i) If m, n are Hankel, a is analytic Toeplitz, and $b = a^*$ with $a\delta_1 = \lambda\delta_1$ for some $\lambda \in \mathbb{C}$, then

$$a(1-uv) - (1-uv)b = a(1-uv) - (1-uv)a^* = (a\delta_1 \otimes \delta_1) - (\delta_1 \otimes a\delta_1) = 0$$

and $au = ua$, but $(1-uv)n \neq 0$. Hence, by (2.17) from the proof of Theorem 2.3, A is not Laurent.

(ii) Theorem 2.2 converts (i) into (ii). Notice that $(1-uv)(X) = \mathbb{C}\delta_1$ is one dimensional, so that

$$(a-\lambda)(1-uv) = 0 \Longleftrightarrow a\delta_1 = \lambda\delta_1 .$$

□

Example 2.6. *(i) Let* $A = \begin{pmatrix} a & m \\ n & a \end{pmatrix}$ *where n is Hankel, a is Toeplitz, and for all $\lambda \in \mathbb{C}$,*

2.19 $$a\delta_1 \neq \lambda\delta_1 \text{ or } a^*\delta_1 \neq \overline{\lambda}\delta_1.$$

Note that [1, Theorem 1.2.28] implies

2.20 $$a(1-uv) = (1-uv)a = a\delta_1 \otimes \delta_1 = \delta_1 \otimes a^*\delta_1$$

is the negation of (2.19). It follows that $a(1-uv) - (1-uv)a \neq 0$. By the hypothesis, we know from (2.15) that m is not Hankel, and hence that A is not Laurent.
(ii) Let $A = \begin{pmatrix} a & m \\ m & b \end{pmatrix}$ *where b is Hankel, m is Toeplitz, and for all $\lambda \in \mathbb{C}$,*

2.21 $$m\delta_1 \neq \lambda\delta_1 \text{ or } m^*\delta_1 \neq \overline{\lambda}\delta_1.$$

Since

2.22 $$m(1-uv) - (1-uv)m = m\delta_1 \otimes \delta_1 - \delta_1 \otimes m^*\delta_1 \neq 0$$

it follows that a is not Hankel from (2.17). Therefore, A is not bilateral Hankel.

There are converses to Theorem 2.3:

Theorem 2.7. *(i) If $a = vau$ is a Toeplitz operator, there exist b, m, and n for which*

$$A = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \text{ is Laurent.}$$

(ii) If a is a Hankel operator, there exist b, m, and n for which

$$A = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \text{ is bilateral Hankel.}$$

Proof. (i) Recalling the notation of (1.16) and (1.17), we have inductively

$$U^j = \begin{pmatrix} u^j & w_j \\ 0 & v^j \end{pmatrix} \text{ and } V^j = \begin{pmatrix} v^j & 0 \\ w_j & u^j \end{pmatrix}$$

where $w_j = \sum_{i=0}^{j-1} u^{j-1-i} w_1 v^i$ for $j \geq 1$. Indeed, if $U = \begin{pmatrix} u & w_1 \\ 0 & v \end{pmatrix}$, then (1.17) implies

$$U^2 = \begin{pmatrix} u & w_1 \\ 0 & v \end{pmatrix}\begin{pmatrix} u & w_1 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u^2 & uw_1 + w_1 v \\ 0 & v^2 \end{pmatrix} = \begin{pmatrix} u^2 & w_2 \\ 0 & v^2 \end{pmatrix}.$$

If $U^j = \begin{pmatrix} u^j & \sum_{i=0}^{j-1} u^{j-1-i} w_1 v^i \\ 0 & v^j \end{pmatrix} = \begin{pmatrix} u^j & w_j \\ 0 & v^j \end{pmatrix}$ holds, then

$$\begin{aligned} U^{j+1} &= \begin{pmatrix} u^j & w_j \\ 0 & v^j \end{pmatrix}\begin{pmatrix} u & w_1 \\ 0 & v \end{pmatrix} \\ &= \begin{pmatrix} u^{j+1} & u^j w_1 + w_j v \\ 0 & v^{j+1} \end{pmatrix} \\ &= \begin{pmatrix} u^{j+1} & \sum_{i=0}^{j} u^{j-i} w_1 v^i \\ 0 & v^{j+1} \end{pmatrix} = \begin{pmatrix} u^{j+1} & w_{j+1} \\ 0 & v^{j+1} \end{pmatrix}. \end{aligned}$$

Thus, if $a = vau$ is given by expansion (1.16), we can find b, m, and n for which

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \alpha_0 I + \sum_{j=1}^{\infty} \alpha_j U^j + \sum_{j=1}^{\infty} \alpha_{-j} V^j,$$

that is,

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} \alpha_0 + \sum_{j=1}^{\infty} \alpha_j u^j + \sum_{j=1}^{\infty} \alpha_{-j} v^j & \sum_{j=1}^{\infty} \alpha_j w_j \\ \sum_{j=1}^{\infty} \alpha_{-j} w_j & \alpha_0 + \sum_{j=1}^{\infty} \alpha_j v^j + \sum_{j=1}^{\infty} \alpha_{-j} u^j \end{pmatrix}.$$

Therefore, since $UV = VU = I$, it follows that

$$\begin{aligned} & AU - UA \\ = \;& (\alpha_0 I + \sum_{j=1}^{\infty} \alpha_j U^j + \sum_{j=1}^{\infty} \alpha_{-j} V^j) U - U(\alpha_0 I + \sum_{j=1}^{\infty} \alpha_j U^j + \sum_{j=1}^{\infty} \alpha_{-j} V^j) \\ = \;& \sum_{j=1}^{\infty} \alpha_{-j} V^{j-1} - \sum_{j=1}^{\infty} \alpha_{-j} V^{j-1} = 0, \end{aligned}$$

which means that A is Laurent.

(ii) If m is a Hankel operator with an expansion of the form (1.17), we can find a, b, and n for which

2.23
$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \sum_{j=1}^{\infty} \beta_j U^j,$$

that is,

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{\infty} \beta_j u^j & \sum_{j=1}^{\infty} \beta_j w_j \\ 0 & \sum_{j=1}^{\infty} \beta_j v^j \end{pmatrix}.$$

If instead a is a Hankel operator, then (2.23) applies to the matrix $AE = \begin{pmatrix} m & a \\ b & n \end{pmatrix}$. Then AE is clearly Laurent. Hence A is bilateral Hankel from Theorem 2.2. □

Remark 2.8. *In Theorem 2.7, if m is a Hankel operator with an expansion of the form (1.17), we can find a, b and n for which*

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \sum_{j=1}^{\infty} \beta_j U^j,$$

that is,

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{\infty} \beta_j u^j & \sum_{j=1}^{\infty} \beta_j w_j \\ 0 & \sum_{j=1}^{\infty} \beta_j v^j \end{pmatrix}.$$

But, since $UV = VU = I$, in this case, we know that

$$\begin{aligned} UA - AV &= U(\sum_{j=1}^{\infty} \beta_j U^j) - (\sum_{j=1}^{\infty} \beta_j U^j) V \\ &= \sum_{j=1}^{\infty} \beta_j U^{j+1} - \sum_{j=1}^{\infty} \beta_j U^{j-1} \\ &= \sum_{j=1}^{\infty} \beta_j (U^{j+1} - U^{j-1}) \neq 0. \end{aligned}$$

Hence A is not bilateral Hankel.

When a is an analytic Toeplitz operator, then it is part of a Laurent operator $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ with $n = 0$, while conversely a is analytic when $n = 0$ and a is co-analytic when $m = 0$.

3. Normality and positivity

Recall the Fourier series isomorphism

3.1 $$x \mapsto x^{\sim} : \ell_2 \equiv \ell_2(\mathbb{Z}) \longrightarrow L_2 \equiv L_2(\mathbb{S}),$$

where $\mathbb{S} \subseteq \mathbb{C}$ is the unit circle, and then introduce the "symbol" $\varphi \in L_\infty(\mathbb{S})$ of the Laurent operator L_φ acting on $L_2(\mathbb{S})$. This reveals that the algebra of Laurent operators is indistinguishable from the C* algebra L_∞, and of course, with the help of *Fuglede's theorem* [7], coincides with its own commutant among the bounded operators on L_2. The bilateral Hankel operators are not closed under multiplication, but form a two-sided module over the Laurent operators; the set of all sums of Laurent and bilateral Hankel operators is however [13] again a C* algebra. The Toeplitz operator

3.2 $$\mathrm{T}_\varphi = \mathbf{P} \circ L_\varphi : f \mapsto \mathbf{P}(\varphi \cdot f)$$

with symbol φ is the *truncation* of the Laurent operator $L_\varphi : f \mapsto \varphi \cdot f$: we write

3.3 $$H_2(\mathbb{S}) = \mathbf{P}L_2(\mathbb{S}),$$

and $\mathbf{P} : L_2 \to L_2$ is given by setting, for arbitrary $w \in \ell_2(\mathbb{Z})$,

3.4 $$\mathbf{P}\Big(\sum_{n=-\infty}^{\infty} w_n e^{in\theta}\Big) = \sum_{n=0}^{\infty} w_n e^{in\theta}.$$

Observe that

3.5 $$\begin{pmatrix} a & m \\ n & b \end{pmatrix}^* = \begin{pmatrix} a^* & n^* \\ m^* & b^* \end{pmatrix}.$$

A is described as *self-adjoint* whenever $A^* = A$, *normal* if $A^*A = AA^*$, and *unitary* if $A^*A = AA^* = I$. We remark that it is necessary and sufficient, for $A \in L(X)$ to be self-adjoint that it have real *numerical range*

3.6 $$\{\langle Ax; x\rangle : \|x\| = 1\} \subseteq \mathbb{R}.$$

We shall further further describe $A \in L(X)$ as *positive*, written $A \geq 0$, if it is self-adjoint with positive numerical range:

3.7 $$A \geq 0 \Longleftrightarrow \{\langle Ax; x\rangle : \|x\| \leq 1\} \subseteq [0, \infty) \subseteq \mathbb{R}.$$

Finally, A is said to be *hyponormal* if $A^*A - AA^* \geq 0$.

Lemma 3.1. *Let* $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ *on* X^2. *Then the following statements hold.*
(i) For $[R, S] := RS - SR$, *we have*

3.8 $$[A^*, A] = \begin{pmatrix} a^*a + n^*n - aa^* - mm^* & a^*m + n^*b - an^* - mb^* \\ m^*a + b^*n - na^* - bm^* & m^*m + b^*b - nn^* - bb^* \end{pmatrix}$$

(ii) A is normal if and only if

$$\begin{cases} a^*a + n^*n - aa^* - mm^* = 0, \\ a^*m + n^*b - an^* - mb^* = 0, \\ m^*m + b^*b - nn^* - bb^* = 0. \end{cases}$$

(iii) A is positive if and only if, for all $x, y \in X$,

$$\begin{cases} a \geq 0, \\ b \geq 0, \\ n = m^*, \\ |\langle my, x\rangle|^2 \leq \langle ax, x\rangle \cdot \langle by, y\rangle. \end{cases}$$

(iv) A is hyponormal if and only if , for all $x, y \in X$,

$$\begin{aligned} & a^*a + n^*n - aa^* - mm^* \geq 0, \\ & m^*m + b^*b - nn^* - bb^* \geq 0, \\ & |\langle (a^*m + n^*b - an^* - mb^*)y, x\rangle|^2 \\ \leq\ & \langle (a^*a + n^*n - aa^* - mm^*)x, x\rangle \cdot \langle (m^*m + b^*b - nn^* - bb^*)y, y\rangle. \end{aligned}$$

Proof. (i) Let $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$. Then $A^* = \begin{pmatrix} a^* & n^* \\ m^* & b^* \end{pmatrix}$ so that

$$A^*A = \begin{pmatrix} a^*a + n^*n & a^*m + n^*b \\ m^*a + b^*n & m^*m + b^*b \end{pmatrix}$$

and

$$AA^* = \begin{pmatrix} aa^* + mm^* & an^* + mb^* \\ na^* + bm^* & nn^* + bb^* \end{pmatrix}.$$

Therefore the equation (3.8) holds.

(ii) The proof follows from the equation (3.8).

(iii) The proof follows from (3.8) and [5, Lemma 1.4]. □

To decide whether or not the Laurent operator

$$A = \begin{pmatrix} a & m \\ b & n \end{pmatrix} \cong L_\varphi$$

is self-adjoint, normal, or hyponormal, we introduce, for each $\varphi \in L_\infty$ ([12]) the subset

3.9 $$\triangle(\varphi) = \{k \in H_\infty : \|k\|_\infty \leq 1 \text{ and } \varphi - k\overline{\varphi} \in H_\infty\} \text{ for } \varphi \in L_\infty .$$

Using Lemma 3.1, we obtain the following results.

Proposition 3.2. *Let $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ be Laurent. Then the following statements hold.*

(i) A is self-adjoint if and only if $a^\sim$ and $b^\sim$ are real-valued functions almost everywhere and $m^\sim - (n^*)^\sim \in e^{i\theta}H_2$.

(ii) If $a^\sim$ and $b^\sim$ are linear combination of a real-valued function and the identity, and m and n are scalar multiples of self-adjoint operators, $an = nb$, $ma = bm$, and $n^*n = mm^*$, then A is normal.

(iii) If $a^\sim$ and $b^\sim$ belong to the set $\Delta(\varphi)$ of (3.9) and m and n are scalar multiples of self-adjoint operators, $an = nb$, $ma = bm$, and $n^*n = mm^*$, then A is hyponormal.

Proof. Suppose that $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ is Laurent. Then it follows from Theorem 2.3 that a, b are Toeplitz, and m, n are Hankel.

(i) Since A is self-adjoint, $a = a^*$, $b = b^*$, and $m = n^*$. Hence by [1, Theorems 3.2.15 and 4.1.4], $a^\sim$ and $b^\sim$ are real-valued functions almost everywhere and $m^\sim - (n^*)^\sim \in e^{i\theta}H_2 \subseteq L_2$ where

$$H_2 := \{f \in L_2 : \langle f, e_n\rangle = 0,\ \text{for } n < 0\}.$$

The reverse implication holds by a similar way.

(ii) Since $a^{\sim}$ and $b^{\sim}$ are linear combination of a real-valued function and the identity and m and n are scalar multiples of self-adjoint Hankel operators, it follows that a and b are normal and m and n are normal by [3] (or [1, Corollary 3.2.16]) and [1, Corollary 4.4.9], respectively. If m, n are normal, $an = nb$ and $ma = bm$, then by the Fuglede-Putnam theorem, then $n^*b = an^*$ and $a^*m = mb^*$ and so $a^*m + n^*b = an^* + mb^*$. These, in turn, give that $A^*A = AA^*$ by Lemma 3.1. Hence A is normal.

(iii) Since $a^{\sim}$ and $b^{\sim}$ belong to the subset $\Delta(\varphi)$ of (3.9), it follows that a and b are hyponormal from [6, Theorem 1] (or [14]). By the similar method of the proof of (ii), we know that A is hyponormal. □

Corollary 3.3. *Let* $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ *be bilateral Hankel. Then the following statements hold.*

(i) A is self-adjoint if and only if $m^{\sim}$ *and* $n^{\sim}$ *are real-valued functions almost everywhere and* $a^{\sim} - (b^*)^{\sim} \in e^{i\theta}H_2$.

(ii) If $m^{\sim}$ *and* $n^{\sim}$ *are linear combination of a real-valued function and the identity, and a and b are scalar multiples of self-adjoint operators,* $mb = bn$, $am = na$, *and* $b^*b = aa^*$, *then A is normal.*

(iii) If $m^{\sim}$ *and* $n^{\sim}$ *belong to a subset* $\Delta(\varphi)(\neq \emptyset)$ *of (3.9) and a and b are scalar multiples of self-adjoint operators,* $mb = bn$, $am = na$, *and* $b^*b = aa^*$, *then A is hyponormal.*

Proof. Let $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ be bilateral Hankel. Then $AE = \begin{pmatrix} m & a \\ b & n \end{pmatrix}$ is Laurent by Theorem 2.2. Hence these results follow from Proposition 3.2. □

Proposition 3.4. *Let* $A = \begin{pmatrix} 0 & m \\ n & 0 \end{pmatrix}$ *be Laurent on* X^2. *Then the following properties hold.*

(i) If A is hyponormal and n is a unitary operator, then m is a hyponormal Hankel operator. In this case, m is normal.

(ii) If A is normal and n is a unitary operator, then m is a unitary Hankel operator.

Proof. Suppose that $A = \begin{pmatrix} 0 & m \\ n & 0 \end{pmatrix}$ is Laurent. By Theorem 2.3, we obtain that m, n are Hankel.

(i) If A is hyponormal and n is a unitary operator, then by Lemma 3.1, $n^*n - mm^* = 1 - mm^* \geq 0$ and $m^*m - nn^* = m^*m - 1 \geq 0$. Thus $m^*m \geq mm^*$. Hence m is a hyponormal Hankel operator. So m is normal by [1, Theorem 4.4.11].

(ii) If A is normal and n is a unitary operator, then by Lemma 3.1, $n^*n - mm^* = 1 - mm^* = 0$ and $m^*m - nn^* = m^*m - 1 = 0$. Thus $m^*m = mm^* = 1$. Hence m is a unitary Hankel operator. □

Corollary 3.5. *Let* $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ *be bilateral Hankel on* X^2. *Then the following properties hold.*

(i) If A is hyponormal and b is a unitary operator, then a is a hyponormal Hankel operator. In this case, a is normal.

(ii) If A is normal and b is a unitary operator, then a is a unitary Hankel operator.

Proof. If $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is bilateral Hankel, then $AE = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ is Laurent by Theorem 2.2. Hence these results follow from Proposition 3.4. □

Proposition 3.6. *Let* $A = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ *be Laurent and hyponormal on* X^2. *If m is hyponormal with* $ma = bm$, *and* $a^{\sim}$ *is linear combination of a real-valued function and the identity, then b is a hyponormal Toeplitz operator.*

Proof. By Theorem 2.3, a, b are Toeplitz, and m is Hankel. Since m is a hyponormal Hankel operator, it follows that m is normal from [1, Theorem 4.4.11]. Moreover, since $ma = bm$, it follows from Fuglede-Putnam theorem that $m^*a = bm^*$. By Lemma 3.1, A is hyponormal if and only if

$$\begin{cases} a^*a - aa^* - mm^* \geq 0, \\ m^*m + b^*b - bb^* \geq 0. \end{cases}$$

Since a is normal, it follows that $m = 0$. Hence b is a hyponormal Toeplitz operator. □

4. Spectral Theory

For a bounded linear operator T on a Banach space X, or more generally a Banach algebra element, we write $\sigma(T)$, $\tau(T)$ and $\pi(T)$ for the *spectrum*, *approximate point spectrum* and *point spectrum*, respectively. Notice that the spectrum $\sigma(T)$ and approximate point spectrum $\tau(T)$ are compact and nonempty, although not necessarily the point spectrum $\pi(T)$(see [7] or [9]). For example

4.1 $$T = L_\varphi \Longrightarrow \sigma(T) = \tau(T) = \sigma(\varphi) = \varphi_{ess}(\mathbb{S}),$$

coincides with the *essential range* of the *symbol*:

4.2 $$A \in L(\ell_2) \cong L_\varphi \in L(L_2) \Longrightarrow \text{symbol}(A) = \varphi \in L_\infty.$$

For example

4.3 $$\text{symbol}(U) = z : \lambda \mapsto \lambda \ ; \ \text{symbol}(V) = z^* : \lambda \mapsto \overline{\lambda}.$$

The spectrum of the sum of a Laurent and a bilateral Hankel operator is (Walsh [15], Murphy [13]) interesting, and given by the spectrum of a 2×2 matrix of symbol functions:

4.4 $$A \cong L_\varphi, B \cong L_\psi \Longrightarrow \sigma(A + BE) = \sigma\begin{pmatrix} \varphi & \psi \\ \psi^\vee & \varphi^\vee \end{pmatrix}.$$

We are tempted to write

4.5 $$\text{symbol}(A + BE) = \begin{pmatrix} \varphi & \psi \\ \psi^\vee & \varphi^\vee \end{pmatrix},$$

where

4.6 $$\lambda \in \mathbb{S} \Longrightarrow \varphi^\vee(\lambda) = \varphi(-\lambda).$$

Thus the sum $A + BE$ of a Laurent and a bilateral Hankel operator has in effect ([Harte, Hernandez [10]) an *adjugate* and a *determinant*:

4.7 $$\text{adj}(A + BE) = EAE - BE \in L(\ell_2) \ ; \ \det(A + BE) = AEAE - BEBE \in L_\infty.$$

The spectrum of the Laurent operator $A \cong L_\varphi$ is given by its symbol as in (4.2), while for a bilateral Hankel operator we have

4.8 $$\sigma((BE)^2) = \sigma(L_\psi L_{\psi^\vee}) = \sigma(\psi\psi^\vee).$$

The spectrum of the sum of a Laurent and a bilateral Hankel is the same as that of its (vector valued) symbol:

4.9 $$\lambda \notin \sigma(A + BE) \Longleftrightarrow \det(A + BE - \lambda I) \in L_\infty(\mathbb{S})^{-1}.$$

The spectral theory of Toeplitz operators is much more complicated than that of Laurent operators, and we can say nothing about sums of Toeplitz and Hankel operators.

The Coburn alternative (see [7, Proposition 7.24] or [1, Theorem 3.3.10]) says that for a Toeplitz operator a whose symbol is a nonzero function in L_∞, $\pi(a) = \emptyset$ or $\pi(a^*) = \emptyset$. It is also true [2, Theorem 5] that the spectrum, and the Fredholm essential spectrum, of a Toeplitz operator coincide, and are a connected subset of $\mathbb{C}$:

4.10 $$\sigma(\mathrm{T}_\varphi) = \sigma_{ess}(\mathrm{T}_\varphi).$$

Theorem 4.1. *If* $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ *is Laurent where* $ab = mn$, $bn = nb$, *and* n *is invertible, then the following statements hold;*
(i) $\sigma(A) \setminus \{0\}$ *is connected. Moreover, if* ess ran $(a+b)^{\sim}$ *is convex, then* $\sigma(A) \setminus \{0\} =$ess ran $(a+b)^{\sim}$.
(ii) If $\pi(a^*) \neq \emptyset$ *and* $\pi(a^* + b^*) \neq \emptyset$, *then* $\pi(A) = \{0\}$.

Proof. (i) Suppose $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ is Laurent where $ab = mn$, $bn = nb$, and n is invertible. Then a, b are Toeplitz from Theorem 2.3 and so $a+b$ is also Toeplitz. We first show $\sigma(A) \setminus \{0\} = \sigma(a+b) \setminus \{0\}$. By (4.9), cf [8], Problem 71 that $A - \lambda$ is invertible if and only if $(a-\lambda)(b-\lambda) - mn$ is invertible. Since $ab = mn$, it follows that A is not invertible if and only if $a + b - \lambda$ is not invertible. Therefore $\sigma(A) \setminus \{0\} = \sigma(a+b) \setminus \{0\}$ from [2, Theorem 5]. Hence $\sigma(A) \setminus \{0\}$ is connected. The last statement holds from [1, Corollary 3.3.7].

(ii) We first claim that if $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ is Laurent where $ab = mn$, $bn = nb$, and n is invertible, then $\tau(A) \setminus \{0\} \subset \tau(a) \cup \tau(a+b)$. Indeed, if $\lambda \in \tau(A) \setminus \{0\}$, then there exists a sequence $\{x_k \oplus y_k\} \subset X \oplus X$ with $||x_k||^2 + ||y_k||^2 = 1$ for all k such that

$$\lim_{k\to\infty} ||(A-\lambda)(x_k \oplus y_k)|| = 0,$$

which yields that

4.11
$$\begin{cases} \lim_{k\to\infty} ||(a-\lambda)x_k + my_k|| = 0 \\ \lim_{k\to\infty} ||nx_k + (b-\lambda)y_k|| = 0. \end{cases}$$

Multiply the first equation of (4.11) with n and the second equation of (4.11) with a, respectively, we have

4.12
$$\begin{cases} \lim_{k\to\infty} ||(an-\lambda n)x_k + mny_k|| = 0 \\ \lim_{k\to\infty} ||anx_k + (ab-\lambda a)y_k|| = 0. \end{cases}$$

Since $ab = mn$ and λ is nonzero, we have $\lim_{k\to\infty} ||ay_k - nx_k|| = 0$ from (4.12). Combining this with the second equation of (4.11), we get that

$$\lim_{k\to\infty} ||(a+b-\lambda)y_k|| = 0.$$

If $\lim_{k\to\infty} ||y_k|| \neq 0$, then $\lambda \in \tau(a+b)$. Otherwise, i.e., $\lim_{k\to\infty} ||y_k|| = 0$, we obtain from (4.11) that $\lim_{k\to\infty} ||(a-\lambda)x_k|| = 0$. If $\lim_{k\to\infty} ||x_k|| \neq 0$, then $\lambda \in \tau(a)$. Therefore $\tau(A) \setminus \{0\} \subset \tau(a) \cup \tau(a+b)$.

By Coburn Alternative in [7], we know that a and $a+b$ are injective whose symbols are nonzero function in L_∞. Then $\pi(a)$ and $\pi(a+b)$ are empty sets. Since $\pi(A) \setminus \{0\} \subset \pi(a) \cup \pi(a+b) = \emptyset$ by the previous note, it follows that $\pi(A) = \{0\}$. □

Corollary 4.2. *If* $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ *is bilateral Hankel with* $ab = mn$, $bn = nb$, *and* n *is invertible, then the following statements hold;*
(i) $\sigma(A) \setminus \{0\}$ *is connected. Moreover, if* ess ran $(m+n)^{\sim}$ *is convex, then* $\sigma(A) \setminus \{0\} =$ess ran $(m+n)^{\sim}$.
(ii) If $\pi(m^*) \neq \emptyset$ *and* $\pi(m^* + n^*) \neq \emptyset$, *then* $\pi(A) = \{0\}$.

Proof. Suppose that A is bilateral Hankel. Then $AE = \begin{pmatrix} m & a \\ b & n \end{pmatrix}$ is Laurent by Theorem 2.2. Hence these results follow from Theorem 4.1. □

Proposition 4.3. *If* $A = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ *is Laurent, then the following properties hold;*
(i) If $\sigma(a) \cap \sigma(b) \neq \emptyset$, *then* $\sigma(A)$ *is connected.*
(ii) $\tau(A) \subset \tau(a) \cup \tau(b) \subset \sigma(a) \cup \sigma(b)$.
(iii) If $\pi(a^*) \neq \emptyset$ *and* $\pi(b^*) \neq \emptyset$, *then* A *is injective.*
(iv) If a^* *and* b *are analytic, and* b *is invertible, then* A *is invertible if and only if* ab *is an invertible Toeplitz operator.*

Proof. Suppose $A = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ is Laurent. Then a, b are Toeplitz from Theorem 2.3.

(i) Since a, b are Toeplitz, it follows from [2, Theorem 5] that $\sigma(a)$ and $\sigma(b)$ are connected. Moreover, since $\sigma(a) \cap \sigma(b) \neq \emptyset$, $\sigma(A) \subset \sigma(a) \cup \sigma(b)$, and the closure of a connected subset is connected, we conclude that $\sigma(A)$ is connected.

(ii) The proof is clear.

(iii) By Coburn Alternative in [7], we know that a and b are injective whose symbols are nonzero functions in L_∞. Then $\pi(a)$ and $\pi(b)$ are empty sets. Since $\pi(A) \subset \pi(a) \cup \pi(b) = \emptyset$, it follows that $\pi(A) = \emptyset$. Hence A is injective.

(iv) If b is invertible, then we know from [8] that A is invertible if and only if ab is invertible. Since a^* and b are analytic, it follows from [3, Theorem 8] that ab is a Toeplitz operator. Hence this result holds. □

Corollary 4.4. *Let* $A = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ *be Laurent. If a and b have the single-valued extension property, then Weyl's theorem holds for A.*

Proof. If $A = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ is Laurent, then a, b are Toeplitz from Theorem 2.3. Since a and b satisfy Weyl's theorem and are isoloid by [4, Theorem 4.1], and have the single-valued extension property, it follows from [11, Corollary 11] that Weyl's theorem holds for A. □

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