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Bisimulations for weighted networks with weights in a quantale

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Abstract. In this paper we introduce two types of simulations (forward and backward) and five types of bisimulations (forward, backward, forward-backward, backward-forward, and regular) for weighted networks with weights taken in a quntale, we describe their basic properties and provide procedures for testing the existence of a simulation or bisimulation of a given type and computing the greatest one, in cases when such simulations or bisimulations exist. We also describe basic properties of homogeneous simulations and bisimulations, which relate actors within the same weighted network, and establish relationships between heterogeneous bisimulations between two different networks and homogeneous bisimulations on these networks. In addition, we characterize the greatest forward, backward and regular bisimulations between weighted networks by means of multi-valued multimodal logics.

1. Introduction and preliminaries

A large number of natural and man-made systems are structured in the form of networks. Typical examples include technological infrastructures (the Internet, the World Wide Web, telephone networks, power grids), transportation infrastructures (road, rail, and airline routes, public transport networks), production systems (company or cross-company owned systems with geographically dispersed plants), distribution systems (oil and gas pipelines, water and sewerage lines, post office and package delivery routes), biological systems (gene and protein interaction networks), various social interaction structures, and others. Along with a complex topological structure, real networks display a large heterogeneity in the capacity and intensity of the connections. Weighted networks have been extensively studied because they can cope with this heterogeneity. In weighted networks, each connection has additional information called 'weight'. Weights are important for representing many interesting phenomena emerging from such networks. For example, in real weighted networks such as traffic networks, brain networks, and social networks, the weight represents the number of commuters between towns, the magnitudes of correlational interactions between brain regions, and the intimacy between humans.

Weights are usually taken to be real numbers (often only integers or natural numbers are used), and classical algebraic operations and/or functions on real numbers are used to manipulate weights. However, in many situations, it is more convenient to take weights from some other algebraic structures. Typical examples are classic unweighted networks, which can be treated as weighted networks that take Boolean values

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1 ("there is a connection") and 0 ("no connection") as weights. To handle such weights logical operations in the two-element Boolean algebra are used. Here we are dealing with weighted networks with weights taken from much more general algebraic structures called quantales. Quantales were introduced by C.J. Mulvey in 1986, in order to provide a general algebraic framework for studying the extension of the Gelfand-Naimark representation of commutative C^*-algebras to the non-commutative case. The motivating example of a quantale was the involutive quantale Max$A$ of all closed linear subspaces of a non-commutative C^*-algebra A, from which the algebra A can be reconstructed. However, quantales were present even earlier as lattices of ideals, subgroups, and other suitable substructures of algebras. Natural examples of quantales are lattices of relations on a set and lattices of linear relations on a Hilbert space. It is important to note that quantales are also applied in linear and other substructural logics and automata theory. In particular, quantales give a semantics for propositional linear logic in the same way as Boolean algebras give a semantics for classical propositional logic.

We use quantales as weight structures for weighted networks for several reasons. The main reason is that quantales possess residuation, which we need in solving systems of inequations with weighted relations we discuss in the paper. Another reason is that quantales include two extremely important algebraic structures, complete residuated lattices and max-plus quantales, and everything that is proven for weighted networks with weights in quantales also applies to weighted networks with weights taken in these structures. Complete residuated lattices represent the structure of truth values widely used in fuzzy logic and fuzzy set theory, research approaches that deal with problems relating to ambiguous, subjective, and imprecise judgments. Given that relationships between individuals are essentially vague and uncertain, weighted networks with weights taken in complete residuated lattices, commonly referred to as fuzzy networks, have a very prominent place in social network analysis. On the other hand, the max-plus quantale is an algebraic structure derived from the field of real numbers by replacing addition by max-operation and multiplication by addition, as well as adding minus and plus infinity as the smallest and greatest element. Max-plus quantales and related max-plus algebras are very successfully used in representing the behaviour of timed discrete event systems with synchronization of tasks and resource sharing, such us, for instance, production systems, railroad networks, urban traffic networks, queuing systems, array processors, and others.

In this paper, weighted networks are studied from the point of view of social network analysis. A weighted social network, which for simplicity we call only a weighted network, consists of a set of actors and ties between them that carry certain weights. As we have already said, weights are taken from quantales. The weighted networks we work with here are also multi-relational, which means that there can be many different types of ties, and are multi-attribute, which means that there can be many different types of attributes assigned to actors. The problems we deal with belong to the field of positional analysis of social networks, whose main task is to determine the positions or roles of actors in the network based on the interrelationships between actors. For example, in a terrorist or criminal network, we should infer the role of the actors on the basis of mutual communication, without insight into the content of the communication (which could be encrypted). The basic tools used in the positional analysis of classical unweighted networks are regular equivalences. Informally speaking, any equivalence class of a regular equivalence is composed of actors who have similar relations to members of other equivalence classes. This means that two actors can be considered regularly equivalent if they occupy the same position, that is, positions can be understood as equivalence classes of a regular equivalence. Regular equivalences were introduced by White and Reitz in 1983, and later they were studied in numerous articles. For more information about them we refer to [5, 18, 25, 36]. In the context of weighted networks, more precisely for fuzzy networks, weighted (fuzzy) versions of regular equivalences were studied by Fan, Liau, and Lin [23, 24], Ćirić and Bogdanović [9] (see also [28, 30]), and Fan and Liau [22].

Regular equivalences describe structural similarities between actors within the same network. In this paper, we explore structural similarities between actors from two different networks and look for the same positions in different networks. The main tools we use for this purpose are bisimulations. Bisimulations have been introduced in the early 1980s by Milner and Park, as a very powerful tool used in many areas of computer science to match moves and compare the behaviour of various systems, as well as to reduce the number of states of these systems. Roughly at the same time they have been also discovered in some areas of mathematics, e.g., in modal logic and set theory. The use of bisimulations has gained a long and rich history and their various forms have been defined and applied to different systems. They are employed today in the study of functional languages, object-oriented languages, types, datatypes, domains, databases, compiler optimizations, program analysis, verification tools, etc.

In social network analysis, in the context of unweighted networks, bisimulations first appeared in the paper of Marx and Masuch [32], and later they were studied in [6, 7]. In the context of weighted networks, bisimulations have not been studied so far, with the exception of the aforementioned weighted regular equivalences, which are a special case of bisimulations connecting actors within the same network. Motivated by concepts of simulations and bisimulations for fuzzy and weighted automata developed in [12–14], in Section 4 we introduce two types of simulations (forward and backward) and five types of bisimulations (forward, backward, forward-backward, backward-forward, and regular) for weighted networks, and we describe their basic properties (Theorems 4.1 and 4.2). Then in Section 5 we provide procedures for testing the existence of a simulation or bisimulation of a given type and computing the greatest one, in cases when such simulations or bisimulations exist (Theorems 5.1 and 5.2).

Simulations and bisimultions discussed in Sections 4 and 5 are heterogeneous, they connect actors from different weighted networks. In Section 6 we deal with homogeneous simulations and bisimulations, which relate actors within the same weighted network. We prove the existence of the greatest homogeneous simulations and bisimulations on a weighted network (Theorems 6.1 and 6.2), and besides, we show that the greatest homogeneous simulations are weighted quasi-orders (Theorem 6.1) and that the greatest homogeneous forward, backward and regular bisimulations are weighted eqivalences (Theorem 6.2). In Section 7 we establish certain relationships between heterogeneous bisimulations between two different networks and homogeneous bisimulations on these networks. Finally, in Section 8 we characterize the greatest forward, backward and regular bisimulations between weighted networks by means of multi-valued multimodal logics. Informally speaking, we show that forward bisimulations preserve compound attributes expressed by plus formulas, backward bisimulations preserve attributes expressed by minus formulas, whereas regular bisimulations preserve attributes expressed by arbitrary multimodal formulas.

In Sections 4 and 5 we deal with weighted networks with weights taken in an arbitrary quantale, while in Section 6 we require that quantale to be integral. Since in Section 7 we use results from [11] proved for weighted relations with weights in a complete residuated lattice, in that section we require weights to be taken in such a structure. Also, in Section 8 we use concepts and results from [37, 38] concerning multi-valued multimodal logics over a complete Heyting algebra, so weighted networks considered in that section also take weights in a complete Heyting algebra.

2. Basic notions and notation

2.1. Quantales, complete residuated lattices and Heyting algebras

A *quantale* is defined as an algebra $\mathscr{Q} = (Q, \wedge, \vee, \otimes, 0, 1)$ satisfying the following conditions:

(Q1) $(Q, \wedge, \vee, 0, 1)$ is a complete lattice in which 0 is the least element and 1 is the greatest element;

(Q2) $(Q, \otimes)$ is a semigroup, which does not have to be commutative;

(Q3) for arbitrary $a \in Q$ and $\{b_i\}_{i\in I} \subseteq Q$ the following infinite distributive laws hold

$$a \otimes \Big(\bigvee_{i\in I} b_i\Big) = \bigvee_{i\in I}(a \otimes b_i), \qquad \Big(\bigvee_{i\in I} b_i\Big) \otimes a = \bigvee_{i\in I}(b_i \otimes a). \tag{1}$$

The operation $\otimes$ is called a *multiplication*, and if it is a commutative operation, then $\mathscr{Q}$ is called a *commutative quantale*. If the semigroup $(Q, \otimes)$ has an identity, then $\mathscr{Q}$ is called a *unital quantale*, and if this identity coincides with the greatest element 1, then $\mathscr{Q}$ is called an *integral quantale*. One can easily check that the least element 0 is the zero of the semigroup $(Q, \otimes)$, i.e., $a \otimes 0 = 0 \otimes a = 0$, for every $a \in Q$.

If $\mathscr{Q} = (Q, \wedge, \vee, \otimes, 0, 1)$ is a quantale, then $\mathscr{Q}^s = (Q, \vee, \otimes, 0)$ is a semiring (with zero, but not necessarily with identity), which is called the *semiring reduct* of $\mathscr{Q}$.

For an arbitrary quantale $\mathscr{Q}$ we can define two more binary operations \ and / on its carrier Q as follows:

$$a\backslash b = \bigvee\{x \in Q \mid a \otimes x \leqslant b\}, \qquad b/a = \bigvee\{y \in Q \mid y \otimes a \leqslant b\}, \tag{2}$$

for any $a, b \in Q$. We call $a\backslash b$ the *right residual* of b by a, and b/a the *left residual* of b by a. The right residual $a\backslash b$ can be understood as what remains of b on the right when divided by a on the left, and similarly the left residual can be understood. Due to infinite distributivity, $a\backslash b$ and b/a belong to the sets $\{x \in Q \mid a \otimes x \leqslant b\}$ and $\{y \in Q \mid y \otimes a \leqslant b\}$, respectively, and they are their greatest elements.

The key relationship between multiplication and residuals in the quantale $\mathscr{Q}$ is given by the formula

$$a \otimes b \leqslant c \quad \Leftrightarrow \quad a \leqslant c/b \quad \Leftrightarrow \quad b \leqslant a\backslash c, \tag{3}$$

for all $a, b, c \in Q$, which is called the *residuation property*. According to (1) and (2) we get that for each $a \in Q$ the mappings $x \mapsto a \otimes x$, $x \mapsto x \otimes a$, $x \mapsto a\backslash x$ and $x \mapsto x/a$ are isotone (order-preserving) mappings of $\mathscr{Q}$ into itself, that is, for all $a, b, c \in Q$ we have that

$$b \leqslant c \;\Rightarrow\; a \otimes b \leqslant a \otimes c \;\&\; b \otimes a \leqslant c \otimes a \;\&\; a\backslash b \leqslant a\backslash c \;\&\; b/a \leqslant c/a. \tag{4}$$

For a quantale $\mathscr{Q} = (Q, \wedge, \vee, \otimes, 0, 1)$, the algebra $\mathscr{Q}^\sharp = (Q, \wedge, \vee, \otimes, \backslash, /)$ will be called the *extended quantale*.

A commutative integral quantale is called a *complete residuated lattice*. Due to the commutativity we have that $a\backslash b = b/a$, for all $a, b \in Q$, which means that the operations \ and / coincide, and in that case they are denoted by $\to$. Then the residuation property (3) becomes

$$a \otimes b \leqslant c \quad \Leftrightarrow \quad a \leqslant b \to c, \tag{5}$$

for all $a, b, c \in Q$.

In many sources, in the definition of a complete residuated lattice, infinite distributivity (1) is replaced by the residuation property (5), that is, a complete residuated lattice is defined as an algebra $(Q, \wedge, \vee, \otimes, \to, 0, 1)$ that satisfies (Q1), (Q2) (with the additional assumptions that the multiplication $\otimes$ is commutative and 1 is a multiplicative identity) and the residuation property (5) (cf. [3]).

A complete residuated lattice in which $\otimes = \wedge$ is called a *complete Heyting algebra*. If the underlying ordering $\leqslant$ is linear, we say that this Heyting algebra is *linearly ordered*.

For the main examples of quantales we refer to [29], while for the main examples of complete residuated lattices and Heyting algebras we refer to [3, 12, 37].

2.2. Weighted sets and relations

A *weighted subset* of a set A, with weights taken in a quantale $\mathscr{Q} = (Q, \wedge, \vee, \otimes, 0, 1)$, is defined as any mapping $f : A \to Q$. For $a \in A$, its image $f(a)$ is called the *weight* of a and can be understood as a measure of the membership of the element a to the weighted subset f. Ordinary subsets of A can be viewed as weighted subsets taking weights in the set $\{0, 1\} \subseteq Q$. The collection of all weighted subsets of a set A, with weights taken in $\mathscr{Q}$ is denoted by Q^A. A weighted set $f : A \to Q$ is *empty* if $f(a) = 0$, for all $a \in A$, and otherwise, it is *non-empty*. The *equality* of weighted subsets is defined as the equality of mappings, i.e., $f = g$ if and only if $f(a) = g(a)$, for each $a \in A$, and the *inclusion* $f \leqslant g$ is also defined pointwise: $f \leqslant g$ if and only if $f(a) \leqslant g(a)$, for every $a \in A$. The *intersection* $\bigwedge_{i\in I} f_i$ and the *union* $\bigvee_{i\in I} f_i$ of a collection $\{f_i\}_{i\in I}$ of weighted subsets of A are weighted subsets of A defined by

$$\Big(\bigwedge_{i\in I} f_i\Big)(x) = \bigwedge_{i\in I} f_i(x), \qquad \Big(\bigvee_{i\in I} f_i\Big)(x) = \bigvee_{i\in I} f_i(x).$$

A *weighted relation* between non-empty sets A and B (in this order) is defined as a weighted subset R of $A \times B$, i.e., a mapping $R : A \times B \to Q$. If $A = B$, we say that R is a weighted relation on A. The notions of equality, inclusion, intersection and union of weighted relations are obtained directly from the definitions of the corresponding notions concerning weighed subsets. The *inverse weighted relation* of $R : A \times B \to Q$ (which is also called a *converse weighted relation*) is a weighted relation $R^{-1} : B \times A \to Q$ defined by $R^{-1}(b,a) = R(a,b)$, for all $a \in A$ and $b \in B$. It is clear that $(R^{-1})^{-1} = R$.

The *product of weighted relations* $R : A\times B \to Q$ and $S : B\times C \to Q$ is a weighted relation $R\circ S : A\times C \to Q$ defined for any $a \in A$ and $c \in C$ by

$$(R \circ S)(a,c) = \bigvee_{b\in B} R(a,b) \otimes S(b,c).$$

On the other hand, for a weighted relation $R : A \times B \to C$ and weighted sets $f : A \to Q$ and $g : B \to Q$, the *weighted set-relation product* $f \circ R : B \to Q$ and *weighted relation-set product* $R \circ g : A \to Q$ are weighted sets defined for any $b \in B$ and $a \in A$ by

$$(f \circ R)(b) = \bigvee_{a'\in A} f(a') \otimes R(a',b), \qquad (R \circ g)(a) = \bigvee_{b'\in B} R(a,b') \otimes g(b').$$

It should be noted that if A and B are finite sets, A with m elements and B with n elements, then a weighted relation $R : A \times b \to Q$ can be viewed as an $m \times n$ *matrix* with entries in Q, while weighted sets $f : A \to Q$ and $g : B \to Q$ can be viewed as *vectors* from semilinear spaces Q^m and Q^n over the semiring reduct $\mathcal{Q}^s$ of $\mathcal{Q}$, respectively. In this case, the products $f \circ R$ and $R \circ g$ are the usual vector-matrix and matrix-vector products. Moreover, if A, B and C are finite sets, then the product $R\circ S$ of weighted relations $R : A\times B \to Q$ and $g : B \times C \to Q$ is the usual product of matrices over the semiring $\mathcal{Q}^s$.

For all $R, R_i \in Q^{A\times B}$ $(i \in I)$, $S, S_1, S_2, S_i \in Q^{B\times C}$ $(i \in I)$, $T \in Q^{C\times D}$, $f \in Q^A$, $g \in Q^B$ and $h \in Q^C$ we have that:

$$(R \circ S) \circ T = R \circ (S \circ T); \tag{6}$$

$$R \circ \Big(\bigvee_{i\in I} S_i\Big) = \bigvee_{i\in I}(R \circ S_i), \quad \Big(\bigvee_{i\in I} R_i\Big) \circ S = \bigvee_{i\in I}(R_i \circ S); \tag{7}$$

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}; \tag{8}$$

$$\Big(\bigvee_{i\in I} R_i\Big)^{-1} = \bigvee_{i\in I} R_i^{-1}; \tag{9}$$

$$f \circ \Big(\bigvee_{i\in I} R_i\Big) = \bigvee_{i\in I}(f \circ R_i), \quad \Big(\bigvee_{i\in I} R_i\Big) \circ g = \bigvee_{i\in I}(R_i \circ g); \tag{10}$$

$$S_1 \leqslant S_2 \;\Rightarrow\; S_1^{-1} \leqslant S_2^{-1} \;\&\; R \circ S_1 \leqslant R \circ S_2 \;\&\; S_1 \circ T \leqslant S_2 \circ T \;\&\; g \circ S_1 \leqslant g \circ S_2 \;\&\; S_1 \circ h \leqslant S_2 \circ h; \tag{11}$$

$$(f \circ R) \circ S = f \circ (R \circ S), \quad (f \circ R) \circ g = f \circ (R \circ g), \quad (R \circ S) \circ h = R \circ (S \circ h). \tag{12}$$

If, in addition, the multiplication $\otimes$ is commutative, then for any $R \in Q^{A\times B}$, $f \in Q^A$ and $g \in Q^B$ we have that

$$R^{-1} \circ f = f \circ R, \qquad g \circ R^{-1} = R \circ g. \tag{13}$$

It is evident that $(Q^{A\times B}, \wedge, \vee, \emptyset, A \times B)$ is a complete lattice, and if $\mathcal{Q}$ is a unital quantale with identity $1_\otimes$, then by (6) and (7) it follows that $(Q^{A\times A}, \wedge, \vee, \circ, \emptyset, A\times A)$ is also a unital quantale with the identity Δ_A, where Δ_A is the ordinary equality relation on A which is defined as a weighted relation by $\Delta_A(a,b) = 1_\otimes$, if $a = b$, and $\Delta_A(a,b) = 0$, if $a \neq b$. This means that we can define residuals for weighted relations on A using formulas (2). However, residuals of weighted relations can be defined in a more general context, as follows. For weighted relations $R \in Q^{A\times B}$, $S \in Q^{B\times C}$ and $T \in Q^{A\times C}$, where A, B and C are non-empty sets, we define weighted relations $R\backslash T \in Q^{B\times C}$ and $T/S \in Q^{A\times B}$ by

$$(R\backslash T)(b,c) = \bigwedge_{a'\in A} R(a',b)\backslash T(a',c), \qquad (T/S)(a,b) = \bigwedge_{c'\in C} S(b,c')/T(a,c'), \tag{14}$$

for all $a \in A$, $b \in B$ and $c \in C$ (on the right hand side, the residuals in the quantale $\mathcal{Q}$ are used). As in quantales, $R\backslash T$ is called the *right residual* of T by R, and T/S is called the *left residual* of T by S. It can be easily proven that

$$R\backslash T = \bigvee\{U \in Q^{B\times C} \mid R \circ U \leqslant T\}, \qquad T/S = \bigvee\{V \in Q^{A\times B} \mid V \circ S \leqslant T\},$$

and therefore, the following residuation property for weighted relations holds

$$R \circ S \leqslant T \quad \Leftrightarrow \quad R \leqslant T/S \quad \Leftrightarrow \quad S \leqslant R\backslash T. \tag{15}$$

In addition, for weighted sets $f \in Q^A$ and $g \in Q^B$, where A and B are non-empty sets, we define weighted relations $f\backslash g, f/g \in Q^{A\times B}$ by

$$(f\backslash g)(a,b) = f(a)\backslash g(b), \qquad (f/g)(a,b) = f(a)/g(b), \tag{16}$$

for all $a \in A$ and $b \in B$. We call $f\backslash g$ the *right residual* of g by f, and f/g the *left residual* of f by g. We have that

$$f\backslash g = \bigvee\{R \in Q^{A\times B} \mid f \circ R \leqslant g\}, \qquad f/g = \bigvee\{R \in Q^{A\times B} \mid R \circ g \leqslant f\},$$

whence it follows that for each $R \in Q^{A\times B}$ the following residuation property holds

$$f \circ R \leqslant g \quad \Leftrightarrow \quad R \leqslant f\backslash g, \qquad R \circ g \leqslant f \quad \Leftrightarrow \quad R \leqslant f/g. \tag{17}$$

According to (4), for all $R \in Q^{A\times B}$, $X, Y \in Q^{B\times C}$, and $S \in Q^{C\times D}$ we have that

$$X \leqslant Y \;\Rightarrow\; R \circ X \leqslant R \circ Y \;\&\; X \circ S \leqslant Y \circ S, \tag{18}$$

while for all $R \in Q^{A\times B}$, $X, Y \in Q^{A\times C}$, and $S \in Q^{B\times C}$ we have that

$$X \leqslant Y \;\Rightarrow\; R\backslash X \leqslant R\backslash Y \;\&\; X/S \leqslant Y/S. \tag{19}$$

2.3. *Other necessary notions and notation*

Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers (without zero).

Let $(P, \leqslant)$ be a partially ordered set. A sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of P is called a *decreasing sequence* if $a_{n+1} \leqslant a_n$, for each $n \in \mathbb{N}$, and it is called *strictly decreasing* if $a_{n+1} < a_n$, for each $n \in \mathbb{N}$. We say that $(P, \leqslant)$ satisfies the *descending chain condition*, briefly *DCC*, if for every descending sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of P there exists $m \in \mathbb{N}$ such that $a_m = a_{m+k}$, for every $k \in \mathbb{N}$. If $m \in \mathbb{N}$ is the smallest natural number for which this holds, we say that the sequence $\{a_n\}_{n\in\mathbb{N}}$ stabilizes after m steps.

3. Weighted networks

A *weighted network*, with weights taken in a quantale $\mathscr{Q}$, is a triple $\mathscr{N} = (A, \{R_i\}_{i\in I}, \{p_j\}_{j\in J})$, where A is a non-empty set, $\{R_i\}_{i\in I}$ is a collection of non-empty weighted relations on A, and $\{p_j\}_{j\in J}$ is a collection of non-empty weighted subsets of A. The members of the set A are called *actors* (when it comes to social networks) or *nodes*. A network can be understood as a system whose elements are somehow connected or interact with each other. The role of each of the weighted relations R_i is to establish certain connections among interacting elements from A. Depending on the context, these connections are called *ties*, *links* or *edges*. For arbitrary $a, b \in A$, the weight $R_i(a, b)$ is understood as the strength of the tie between a and b which corresponds to R_i. On the other hand, for any $a \in A$ the weight $p_j(a)$ is understood as the extent to which a has an attribute associated to p_j. In social network analysis weighted networks are also called *valued networks* [36, 39].

A weighted network $\mathscr{N} = (A, \{R_i\}_{i\in I}, \{p_j\}_{j\in J})$ in which A, I and J are finite sets is called a *finite weighted network*. In practice, we encounter only such weighted networks. However, here we still allow weighted networks to be infinite, because most of the results obtained are valid in such cases. The cases in which we assume that weighted networks are finite will be especially emphasized.

Two weighted networks $\mathscr{N}'$ and $\mathscr{N}''$ are said to be *weighted networks of the same type* if their collections of weighted relations and weighted subsets are indexed by the same sets I and J, i.e., $\mathscr{N}' = (A', \{R'_i\}_{i\in I}, \{p'_j\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R''_i\}_{i\in I}, \{p''_j\}_{j\in J})$. This means that for each $i \in I$ both R'_i and R''_i represent the same kind of ties, and for each $j \in J$ both p'_j and p''_j represent the same kind of attributes.

Weighted networks $\mathscr{N}' = (A', \{R'_i\}_{i\in I}, \{p'_j\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R''_i\}_{i\in I}, \{p''_j\}_{j\in J})$ of the same type are said to be *isomorphic* if there exists a bijective mapping $\psi : A' \to A''$ such that the following holds: $p'_j(a') = p''_j(\psi(a'))$, for all $a' \in A'$ and $j \in J$, and $R'_i(a'_1, a'_2) = R''_i(\psi(a'_1), \psi(a'_2))$, for all $a'_1, a'_2 \in A'$ and $i \in I$.

4. Simulations and bisimulations between weighted networks

Hereinafter, let $\mathscr{N}' = (A', \{R'_i\}_{i\in I}, \{p'_j\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R''_i\}_{i\in I}, \{p''_j\}_{j\in J})$ be two weighted networks of the same type and let X be a non-empty weighted relation between A' and A''. If X satisfies the conditions

$$p'_j \leqslant p''_j \circ X^{-1}, \qquad \text{for every } j \in J, \tag{fs-1}$$
$$X^{-1} \circ R'_i \leqslant R''_i \circ X^{-1}, \qquad \text{for every } i \in I, \tag{fs-2}$$
$$X^{-1} \circ p'_j \leqslant p''_j, \qquad \text{for every } j \in J, \tag{fs-3}$$

then it is called a *forward simulation* between $\mathscr{N}'$ and $\mathscr{N}''$, and if X satisfies

$$p'_j \leqslant X \circ p''_j, \qquad \text{for every } j \in J, \tag{bs-1}$$
$$R'_i \circ X \leqslant X \circ R''_i, \qquad \text{for every } i \in I, \tag{bs-2}$$
$$p'_j \circ X \leqslant p''_j, \qquad \text{for every } j \in J, \tag{bs-3}$$

then it is called a *backward simulation* between $\mathscr{N}'$ and $\mathscr{N}''$. According to Lemma **??**, if $\mathscr{Q}$ is a commutative quantale, then (fs-1) coincides with (bs-1) and (fs-3) coincides with (bs-3).

Before we continue with the definitions of bisimulations, we will explain the meaning of simulations. It is most convenient to explain their meaning in the case of unweighted networks, where R_i's and p_j's are ordinary relations and sets. This case is obtained when a quantale $\mathscr{Q}$ is taken to be the two-element Boolean algebra $(\{0,1\}, \wedge, \vee, 0, 1)$ (assuming that the multiplication $\otimes$ coincides with the conjunction $\wedge$). Under these circumstances, let us consider a tie between actors a' and b' in the network $\mathscr{N}'$. The condition (fs-1) says that if a' has a certain attribute associated to p_j, then a' is simulated by some $a'' \in A''$ with the same attribute, after which (fs-3) says that a'' has all the attributes that a' has. Then condition (fs-2) says that a'' is tied to some $b'' \in A''$ which simulates b', and again, (fs-3) says that b'' has all the attributes that b' has (see Fig. 1 (a)). This means that the tie between a'' and b'' consistently simulates the tie between a' and b' .

Conditions (bs-1)–(bs-3) are used in an opposite way. Let us start again from a tie between actors a' and b' in the network $\mathscr{N}'$. According to (bs-1), if b' has a certain attribute associated to p_j, then b' is simulated by some $b'' \in A''$ with the same attribute, after which (bs-3) says that b'' has all the attributes that b' has. Then, according to (bs-2), there is some $a'' \in A''$, tied to b'', which simulates a', and finnaly, (bs-3) says that a'' has all the attributes that a' has (see Fig. 1 (b)). And again we have that the tie between a'' and b'' consistently simulates the tie between a' and b'.

As we have considered here the case when the underlying quantale is the two-element Boolean algebra, which is a commutative quantale, the condition (bs-1) has the same meaning as (fs-1) and (bs-3) has the same meaning as (fs-3), but for non-commutative quantales the meanings of these conditions differ.

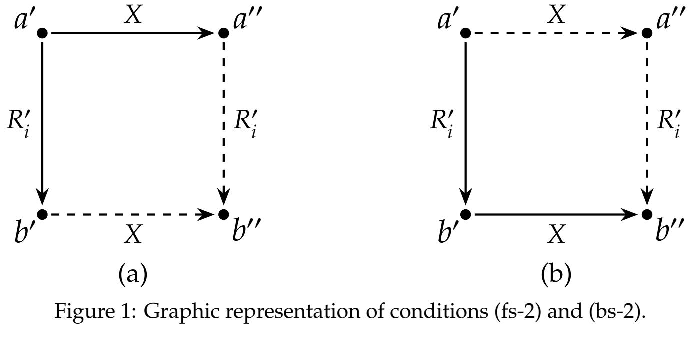

Figure 1: Graphic representation of conditions (fs-2) and (bs-2).

We continue with weighted networks $\mathscr{N}' = (A', \{R'_i\}_{i\in I}, \{p'_j\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R''_i\}_{i\in I}, \{p''_j\}_{j\in J})$ and a weighted relation X between A' and A''. If both X and X^{-1} are forward simulations, i.e., if they satisfy

$$p'_j \leqslant p''_j \circ X^{-1}, \qquad p''_j \leqslant p'_j \circ X, \qquad \text{for every } j \in J, \tag{fb-1}$$

$$X^{-1} \circ R'_i \leqslant R''_i \circ X^{-1}, \qquad X \circ R''_i \leqslant R'_i \circ X, \qquad \text{for every } i \in I, \tag{fb-2}$$

$$X^{-1} \circ p'_j \leqslant p''_j, \qquad X \circ p''_j \leqslant p'_j, \qquad \text{for every } j \in J, \tag{fb-3}$$

then X is called a *forward bisimulation* between $\mathscr{N}'$ and $\mathscr{N}''$, and if both X and X^{-1} are backward simulations, i.e., if they satisfy

$$p'_j \leqslant X \circ p''_j, \qquad p''_j \leqslant X^{-1} \circ p'_j, \qquad \text{for every } j \in J, \tag{bb-1}$$

$$R'_i \circ X \leqslant X \circ R''_i, \qquad R''_i \circ X^{-1} \leqslant X^{-1} \circ R'_i, \qquad \text{for every } i \in I, \tag{bb-2}$$

$$p'_j \circ X \leqslant p''_j, \qquad p''_j \circ X^{-1} \leqslant p'_j, \qquad \text{for every } j \in J, \tag{bb-3}$$

then X is called a *backward bisimulation* between $\mathscr{N}'$ and $\mathscr{N}''$.

On the other hand, if X is a forward simulation and X^{-1} is a backward simulation, i.e., if

$$p'_j \leqslant p''_j \circ X^{-1}, \qquad p''_j \leqslant X^{-1} \circ p'_j, \qquad \text{for every } j \in J, \tag{fbb-1}$$

$$X^{-1} \circ R'_i \leqslant R''_i \circ X^{-1}, \qquad R''_i \circ X^{-1} \leqslant X^{-1} \circ R'_i, \qquad \text{for every } i \in I, \tag{fbb-2}$$

$$X^{-1} \circ p'_j \leqslant p''_j, \qquad p''_j \circ X^{-1} \leqslant p'_j, \qquad \text{for every } j \in J, \tag{fbb-3}$$

then it is called a *forward-backward bisimulation* between $\mathscr{N}'$ and $\mathscr{N}''$, and if X is a backward simulation and X^{-1} is a forward simulation, i.e., if

$$p'_j \leqslant X \circ p''_j, \qquad p''_j \leqslant p'_j \circ X, \qquad \text{for every } j \in J, \tag{bfb-1}$$

$$R'_i \circ X \leqslant X \circ R''_i, \qquad X \circ R''_i \leqslant R'_i \circ X, \qquad \text{for every } i \in I, \tag{bfb-2}$$

$$p'_j \circ X \leqslant p''_j, \qquad X \circ p''_j \leqslant p'_j, \qquad \text{for every } j \in J, \tag{bfb-3}$$

then it is called a *backward-forward bisimulation* between $\mathscr{N}'$ and $\mathscr{N}''$.

Note that (fbb-2) can be written in a simpler way as $X^{-1} \circ R'_i = R''_i \circ X^{-1}$, whereas (bfb-2) can be written as $R'_i \circ X = X \circ R''_i$. In addition, inequalities that form (fbb-1) and (fbb-3) can be combined to give $p'_j = p''_j \circ X^{-1}$ and $p''_j = X^{-1} \circ p'_j$, and inequalities from (bfb-1) and (bfb-3) can be combined to give $p'_j = X \circ p''_j$ and $p''_j = p'_j \circ X$. However, for methodological reasons, the conditions that define forward-backward and backward-forward bisimulations are written as above.

Finally, X is called a *regular bisimulation* if it is both a forward and backward bisimulation. In other words, X is a regular bisimulation if the following assertions are true:

$$p'_j \leqslant p''_j \circ X^{-1}, \quad p''_j \leqslant p'_j \circ X, \quad p'_j \leqslant X \circ p''_j, \quad p''_j \leqslant X^{-1} \circ p'_j, \quad \text{for every } j \in J, \tag{rb-1}$$

$$X^{-1} \circ R'_i \leqslant R''_i \circ X^{-1}, \quad X \circ R''_i \leqslant R'_i \circ X, \quad R'_i \circ X \leqslant X \circ R''_i, \quad R''_i \circ X^{-1} \leqslant X^{-1} \circ R'_i, \quad \text{for every } i \in I, \tag{rb-2}$$

$$X^{-1} \circ p'_j \leqslant p''_j, \quad X \circ p''_j \leqslant p'_j, \quad p'_j \circ X \leqslant p''_j, \quad p''_j \circ X^{-1} \leqslant p'_j, \quad \text{for every } j \in J, \tag{rb-3}$$

It is clear that a regular bisimulation is also a forward-backward and backward-forward bisimulation. The name regular bisimulation is chosen because of its connection with *regular equivalences*, a very important concept of social network analysis. That connection will be discussed later.

For $\theta \in \{\text{fs}, \text{bs}\}$, a weighted relation satisfying (θ-2) and (θ-3) will be called a *presimulation of type θ*, or more simply a *θ-presimulation*, and a weighted relation satisfying all three conditions (θ-1), (θ-2) and (θ-3) will be called a *simulation of type* θ, or shortly a θ-*simulation*. The set of all simulations of type θ between weighted networks $\mathscr{N}'$ and $\mathscr{N}''$ will be denoted by $\mathscr{S}^\theta(\mathscr{N}',\mathscr{N}'')$, and the set of all presimulations of type θ between $\mathscr{N}'$ and $\mathscr{N}''$ will be denoted by $\mathscr{S}^\theta_*(\mathscr{N}',\mathscr{N}'')$.

Similarly, for $\theta \in \{\mathrm{fb},\mathrm{bb},\mathrm{fbb},\mathrm{bfb},\mathrm{rb}\}$, a weighted relation satisfying (θ-2) and (θ-3) will be called a *prebisimulation of type* θ or a θ-*prebisimulation*, and a weighted relation satisfying all three conditions (θ-1), (θ-2) and (θ-3) will be called a *bisimulation of type* θ or a θ-*bisimulation*. The set of all bisimulations of type θ between weighted networks $\mathscr{N}'$ and $\mathscr{N}''$ will be denoted by $\mathscr{B}^\theta(\mathscr{N}',\mathscr{N}'')$, and the set of all prebisimulations of type θ between $\mathscr{N}'$ and $\mathscr{N}''$ will be denoted by $\mathscr{B}^\theta_*(\mathscr{N}',\mathscr{N}'')$.

Simulations and bisimulations between different weighted networks will be called *heterogeneous*, while simulations and bisimulations between a weighted network and itself will be called *homogeneous*.

In the continuation of this section, we describe the basic properties of simulations and bisimulations.

Theorem 4.1. *Let* $\theta \in \{\mathrm{fs},\mathrm{bs}\}$ *and let* $\mathscr{N}'$ *and* $\mathscr{N}''$ *be weighted networks. If* $\mathscr{S}^\theta_*(\mathscr{N}',\mathscr{N}'')$ *is non-empty, then the following assertions are true:*

(a) $\mathscr{S}^\theta_*(\mathscr{N}',\mathscr{N}'')$ *is closed under unions and products of weighted relations;*

(b) $\mathscr{S}^\theta_*(\mathscr{N}',\mathscr{N}'')$ *has the greatest element* X*;*

(c) *If* X *satisfies* (θ-1)*, then it is also the greatest element of* $\mathscr{S}^\theta(\mathscr{N}',\mathscr{N}'')$*;*

(d) *If* X *does not satisfy* (θ-1)*, then* $\mathscr{S}^\theta(\mathscr{N}',\mathscr{N}'')$ *is empty.*

Proof. (a) Closedness under unions follows directly from (7), (9) and (10), while closedness under products follows from (6), (8), (11) and (12).

(b) Due to closedness under unions, the union X of all members of the set $\mathscr{S}^\theta_*(\mathscr{N}',\mathscr{N}'')$ is contained in this set and it is its greatest element.

(c) If X satisfies (θ-1), then it belongs to $\mathscr{S}^\theta(\mathscr{N}',\mathscr{N}'')$, and since $\mathscr{S}^\theta(\mathscr{N}',\mathscr{N}'') \subseteq \mathscr{S}^\theta_*(\mathscr{N}',\mathscr{N}'')$, we have that X is greater than all the elements from $\mathscr{S}^\theta(\mathscr{N}',\mathscr{N}'')$. Hence, X is the greatest element of $\mathscr{S}^\theta(\mathscr{N}',\mathscr{N}'')$.

(d) Assume that X does not satisfy (θ-1) and suppose that $\mathscr{S}^\theta(\mathscr{N}',\mathscr{N}'')$ is non-empty, i.e., that there exists $Y \in \mathscr{S}^\theta(\mathscr{N}',\mathscr{N}'')$. Then $Y \in \mathscr{S}^\theta_*(\mathscr{N}',\mathscr{N}'')$, so $Y \leqslant X$. Now, according to (11) and the assumption that Y satisfies (θ-1), for any $j \in J$ we obtain that

$$p'_j \leqslant p''_j \circ Y^{-1} \leqslant p''_j \circ X^{-1},$$

if $\theta = \mathrm{fs}$, and

$$p'_j \leqslant Y \circ p''_j \leqslant X \circ p''_j,$$

if $\theta = \mathrm{bs}$. Therefore, in both cases we have obtained that X satisfies (θ-1), which is in contradiction with the starting assumption. Consequently, we conclude that if X does not satisfy (θ-1), then $\mathscr{S}^\theta(\mathscr{N}',\mathscr{N}'')$ must be empty. □

Theorem 4.2. *Let* $\theta \in \{\mathrm{fb},\mathrm{bb},\mathrm{fbb},\mathrm{bfb},\mathrm{rb}\}$ *and let* $\mathscr{N}'$ *and* $\mathscr{N}''$ *be weighted networks. If* $\mathscr{B}^\theta_*(\mathscr{N}',\mathscr{N}'')$ *is non-empty, then the following assertions are true:*

(a) $\mathscr{B}^\theta_*(\mathscr{N}',\mathscr{N}'')$ *is closed under unions and products of weighted relations;*

(b) $\mathscr{B}^\theta_*(\mathscr{N}',\mathscr{N}'')$ *has the greatest element* X*;*

(c) *If* $\theta \in \{\mathrm{fb},\mathrm{bb},\mathrm{rb}\}$*, then* X *satisfies* $X \circ X^{-1} \circ X \leqslant X$*;*

(d) *If* X *satisfies* (θ-1)*, then it is also the greatest element of* $\mathscr{B}^\theta(\mathscr{N}',\mathscr{N}'')$*;*

(e) *If* X *does not satisfy* (θ-1)*, then* $\mathscr{B}^\theta(\mathscr{N}',\mathscr{N}'')$ *is empty.*

Proof. Except the condition (c), everything else is proved in the same way as in the previous theorem.

(c) Let us prove $X \circ X^{-1} \circ X \leqslant X$ assuming that X is the greatest element of $\mathscr{B}^\theta_*(\mathscr{N}',\mathscr{N}'')$. Starting from the fact that X satisfies (θ-2) and (θ-3), and using (11), a straightforward verification yields that $X \circ X^{-1} \circ X$ satisfies (θ-2) and (θ-3), which means that $X \circ X^{-1} \circ X \in \mathscr{B}^\theta_*(\mathscr{N}',\mathscr{N}'')$, and therefore, $X \circ X^{-1} \circ X \leqslant X$. □

Let $\mathscr{N}'$ and $\mathscr{N}''$ be weighted networks and let $\theta \in \{\mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}, \mathrm{rb}\}$. If $\mathscr{B}^{\theta}_{*}(\mathscr{N}', \mathscr{N}'')$ has the greatest element, it is denoted by X^{θ} and called a *θ-bisimilarity* between $\mathscr{N}'$ and $\mathscr{N}''$. In particular, X^{fb} is the *forward bisimilarity*, X^{bb} is the *backward bisimilarity*, and X^{rb} is the *regular bisimilarity*.

5. Testing the existence and computing the greatest simulations and bisimulations

Let $\mathscr{N}' = (A', \{R'_i\}_{i\in I}, \{p'_j\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R''_i\}_{i\in I}, \{p''_j\}_{j\in J})$ be two weighted networks of the same type. For any $\theta \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}, \mathrm{rb}\}$ we define a mapping $\phi^{\theta} : Q^{A'\times A''} \to Q^{A'\times A''}$ as follows:

$$\phi^{\mathrm{fs}}(X) = \bigwedge_{i\in I} [(R''_i \circ X^{-1})/R'_i]^{-1}, \qquad \phi^{\mathrm{bs}}(X) = \bigwedge_{i\in I} [R'_i \backslash (X \circ R''_i)],$$

$$\phi^{\mathrm{fb}}(X) = \bigwedge_{i\in I} [(R''_i \circ X^{-1})/R'_i]^{-1} \wedge [(R'_i \circ X)/R''_i], \qquad \phi^{\mathrm{bb}}(X) = \bigwedge_{i\in I} [R'_i \backslash (X \circ R''_i)] \wedge [R''_i \backslash (X^{-1} \circ R'_i)]^{-1},$$

$$\phi^{\mathrm{fbb}}(X) = \bigwedge_{i\in I} [(R''_i \circ X^{-1})/R'_i]^{-1} \wedge [R''_i \backslash (X^{-1} \circ R'_i)]^{-1}, \qquad \phi^{\mathrm{bfb}}(X) = \bigwedge_{i\in I} [R'_i \backslash (X \circ R''_i)] \wedge [(R'_i \circ X)/R''_i],$$

$$\phi^{\mathrm{rb}}(X) = \bigwedge_{i\in I} [(R''_i \circ X^{-1})/R'_i]^{-1} \wedge [(R'_i \circ X)/R''_i] \wedge [R'_i \backslash (X \circ R''_i)] \wedge [R''_i \backslash (X^{-1} \circ R'_i)]^{-1},$$

for each $X \in Q^{A'\times A''}$. In addition, for each $\theta \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}, \mathrm{rb}\}$ we also define a weighted relation $\pi^{\theta} \in Q^{A'\times A''}$ as follows:

$$\pi^{\mathrm{fs}} = \bigwedge_{j\in J} (p''_j/p'_j)^{-1}, \qquad \pi^{\mathrm{bs}} = \bigwedge_{j\in J} p'_j \backslash p''_j,$$

$$\pi^{\mathrm{fb}} = \bigwedge_{j\in J} (p''_j/p'_j)^{-1} \wedge (p'_j/p''_j), \qquad \pi^{\mathrm{bb}} = \bigwedge_{j\in J} (p'_j \backslash p''_j) \wedge (p''_j \backslash p'_j)^{-1},$$

$$\pi^{\mathrm{fbb}} = \bigwedge_{j\in J} (p''_j/p'_j)^{-1} \wedge (p''_j \backslash p'_j)^{-1}, \qquad \pi^{\mathrm{bfb}} = \bigwedge_{j\in J} (p'_j \backslash p''_j) \wedge (p'_j/p''_j),$$

$$\pi^{\mathrm{rb}} = \bigwedge_{j\in J} (p''_j/p'_j)^{-1} \wedge (p'_j/p''_j) \wedge (p'_j \backslash p''_j) \wedge (p''_j \backslash p'_j)^{-1}.$$

Theorem 5.1. *Let $\mathscr{N}' = (A', \{R'_i\}_{i\in I}, \{p'_j\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R''_i\}_{i\in I}, \{p''_j\}_{j\in J})$ be weighted networks. For an arbitrary $\theta \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}, \mathrm{rb}\}$ the following statements are true:*

(a) *ϕ^{θ} is an isotone mapping of a complete lattice $Q^{A'\times A''}$ into itself;*

(b) *for any non-empty $X \in Q^{A'\times A''}$ we have that X satisfies (θ-2) and (θ-3) if an only it it satisfies*

$$X \leqslant \phi^{\theta}(X), \qquad X \leqslant \pi^{\theta}. \tag{θ-4}$$

Proof. We will provide a proof for the case $\theta = \mathrm{fb}$. Similar proofs can be provided in all other cases.

(a) Let $X, Y \in Q^{A'\times A''}$ such that $X \leqslant Y$. According to (11), (18), and (19), for an arbitrary $i \in I$ we have that $[(R''_i \circ X^{-1})/R'_i]^{-1} \leqslant [(R''_i \circ Y^{-1})/R'_i]^{-1}$ and $(R'_i \circ X)/R''_i \leqslant (R'_i \circ Y)/R''_i$, which implies that

$$\phi^{\mathrm{fb}}(X) = \bigwedge_{i\in I} [(R''_i \circ X^{-1})/R'_i]^{-1} \wedge [(R'_i \circ X)/R''_i] \leqslant \bigwedge_{i\in I} [(R''_i \circ Y^{-1})/R'_i]^{-1} \wedge [(R'_i \circ Y)/R''_i] = \phi^{\mathrm{fb}}(Y).$$

Therefore, ϕ^{fb} is an isotone mapping.

(b) For every $i \in I$ have that

$$X^{-1} \circ R'_i \leqslant R''_i \circ X^{-1} \;\Leftrightarrow\; X^{-1} \leqslant R'_i \backslash (R''_i \circ X^{-1}) \;\Leftrightarrow\; X \leqslant [R'_i \backslash (R''_i \circ X^{-1})]^{-1},$$

and, on the other hand,

$$X \circ R''_i \leqslant R'_i \circ X \;\Leftrightarrow\; X \leqslant (R'_i \circ X)/R''_i.$$

Hence, X satisfies (fb-2) if an only if $X \leqslant [R_i' \backslash (R_i'' \circ X^{-1})]^{-1} \wedge (R_i' \circ X)/R_i''$, for each $i \in I$, which is equivalent to

$$X \leqslant \bigwedge_{i\in I} [R_i' \backslash (R_i'' \circ X^{-1})]^{-1} \wedge (R_i' \circ X)/R_i'' = \phi^{\mathrm{fb}}(X).$$

Moreover, for every $j \in J$ we have that

$$X^{-1} \circ p_J' \leqslant p_j'' \quad\Leftrightarrow\quad X^{-1} \leqslant p_j''/p_j' \quad\Leftrightarrow\quad X \leqslant (p_j''/p_j')^{-1}, \qquad\qquad X \circ p_j'' \leqslant p_j' \quad\Leftrightarrow\quad X \leqslant p_j'/p_j'',$$

so (fb-3) is equivalent to

$$X \leqslant \bigwedge_{j\in J} (p_j''/p_j')^{-1} \wedge p_j'/p_j'' = \pi^{\mathrm{fb}}.$$

This completes the proof of the theorem. □

For weighted networks $\mathscr{N}' = (A', \{R_i'\}_{i\in I}, \{p_j'\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R_i''\}_{i\in I}, \{p_j''\}_{j\in J})$, by $\mathscr{Q}^\sharp(\mathscr{N}', \mathscr{N}'')$ we denote the subalgebra of $\mathscr{Q}^\sharp$ generated by all values from Q taken by weighted relations R_i' and R_i'' ($i \in I$), and weighted sets p_j' and p_j'' ($j \in J$).

Theorem 5.2. *Let $\mathscr{N}' = (A', \{R_i'\}_{i\in I}, \{p_j'\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R_i''\}_{i\in I}, \{p_j''\}_{j\in J})$ be weighted networks. For an arbitrary $\theta \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}, \mathrm{rb}\}$ let us define a sequence $\{X_n\}_{n\in\mathbb{N}}$ of weighted relations between A' and A'' as follows:*

$$X_1 = \pi^\theta, \qquad X_{n+1} = X_n \wedge \phi^\theta(X_n), \quad \text{for each } n \in \mathbb{N}. \tag{20}$$

Then the following statements are true:

(a) *the sequence $\{X_n\}_{n\in\mathbb{N}}$ is decreasing;*
(b) *for any $n \in \mathbb{N}$, X_n satisfies (θ-4) if and only if $X_n = X_{n+1}$;*
(c) *if there exists $n \in \mathbb{N}$ such that $X_n = X_{n+1}$, then X_n is the greatest weighted relation that satisfies (θ-4);*
(d) *if A', A'', I and J are finite sets and $\mathscr{Q}^\sharp(\mathscr{N}', \mathscr{N}'')$ satisfies DCC, then the sequence $\{X_n\}_{n\in\mathbb{N}}$ is finite and there exists $n \in \mathbb{N}$ such that $X_n = X_{n+1}$.*

Proof. (a) It follows directly from (20) that $X_{n+1} \leqslant X_n$, for each $n \in \mathbb{N}$, so $\{X_n\}_{n\in\mathbb{N}}$ is indeed a descending sequence.

(b) Consider an arbitrary $n \in \mathbb{N}$. If X_n satisfies (θ-4), then $X_n \leqslant \phi^\theta(X_n)$, whence $X_{n+1} = X_n \wedge \phi^\theta(X_n) = X_n$. Conversely, if $X_n = X_{n+1}$, then

$$X_n = X_{n+1} = X_n \wedge \phi^\theta(X_n) \leqslant \phi^\theta(X_n),$$

which means that X_n satisfies the first inequality in (θ-4). As it is clear that $X_n \leqslant X_1 = \pi^\theta$, we conclude that X_n satisfies (θ-4).

(c) Let X be an arbitrary weighted relation between A' and A'' that satisfies (θ-4). We will prove by induction that $X \leqslant X_n$, for every $n \in \mathbb{N}$. It is clear that $X \leqslant \pi^\theta = X_1$. Suppose that $X \leqslant X_n$, for some $n \in \mathbb{N}$. Since X satisfies (θ-4) and ϕ^θ is isotone, we get that $X \leqslant \phi^\theta(X) \leqslant \phi^\theta(X_n)$, whence

$$X \leqslant X_n \wedge \phi^\theta(X_n) = X_{n+1}.$$

This completes our prof by induction.

Now, assume that there exists $n \in \mathbb{N}$ such that $X_n = X_{n+1}$. According to (b) we get that X_n satisfies (θ-4), and as we have just proved, for every weighted relation X between A' and A'' which satisfies (θ-4) we have that $X \leqslant X_n$. Therefore, X_n is the greatest weighted relation between A' and A'' that satisfies (θ-4).

(d) Let A', A'', I and J be finite sets, and let $\mathscr{Q}^\sharp(\mathscr{N}', \mathscr{N}'')$ satisfy DCC.

For every pair $(a', a'') \in A' \times A''$, $\{X_n(a', a'')\}_{n\in\mathbb{N}}$ is a decreasing sequence of elements of $\mathscr{Q}^\sharp(\mathscr{N}', \mathscr{N}'')$, so it stabilizes after $n(a', a'')$ steps. Let n be the greatest element of the set $\{n(a', a'') \mid (a', a'') \in A' \times A''\}$. This greatest element exists because A' and A'' are finite sets. Then $X_n(a', a'') = X_{n+1}(a', a'')$, for all $(a', a'') \in A' \times A''$, so we conclude that $X_n = X_{n+1}$, which had to be proved. □

6. Homogeneous simulations and bisimulations

In this section we require the underlying quantale $\mathscr{Q}$ to be an integral quantale, that is, we require that the greatest element 1 of $\mathscr{Q}$ is a multiplicative identity. A weighted relation $R \in Q^{A\times A}$ is called a *reflexive weighted relation* if $R(a,a) = 1$, for all $a \in A$. The least reflexive weighted relation on A is the equality relation Δ_A, and moreover, $R \in Q^{A\times A}$ is a reflexive weighted relation if and only if $\Delta_A \leqslant R$.

We also say that $R \in Q^{A\times A}$ is a *symmetric weighted relation* if $R(a,b) = R(b,a)$, for all $a, b \in A$, and that it is a *transitive weighted relation* if $R(a,b)\otimes R(b,c) \leqslant R(a,c)$, for all $a,b,c \in A$. It is evident that a weighted relation R is symmetric if and only if $R^{-1} = R$, and it easy to verify that R is transitive if and only if $R\circ R \leqslant R$. A reflexive and transitive weighted relation is called a weighted quasi-order, and a reflexive, symmetric and transitive weighted relation is called a weighted equivalence. It is important to note that $R \circ R = R$ for each weighted quasi-order R (and therefore for each weighted equivalence).

Theorem 6.1. *Let $\mathscr{N}$ be a weighted network and $\theta \in \{\mathrm{fs}, \mathrm{bs}\}$. Then the following statements are true:*

(a) *$\mathscr{S}_*^{\theta}(\mathscr{N})$ is non-empty and it is closed under unions and products of weighted relations;*

(b) *$\mathscr{S}_*^{\theta}(\mathscr{N})$ has the greatest element X which is a weighted quasi-order;*

(c) *X is also the greatest element of $\mathscr{S}^{\theta}(\mathscr{N})$.*

Proof. Let $\mathscr{N} = (A, \{R_i\}_{i\in I}, \{p_j\}_{j\in J})$.

(a) It is clear that Δ_A satisfies (θ-2) and (θ-3), which means that $\Delta_A \in \mathscr{S}_*^{\theta}(\mathscr{N})$, and therefore, $\mathscr{S}_*^{\theta}(\mathscr{N})$ is non-empty. According to Theorem 4.1, $\mathscr{S}_*^{\theta}(\mathscr{N})$ is closed under unions and products of weighted relations.

(b) Again according to Theorem 4.1, $\mathscr{S}_*^{\theta}(\mathscr{N})$ has the greatest element X. Let us prove that X is a weighted quasi-order. Since X is the greatest element of $\mathscr{S}_*^{\theta}(\mathscr{N})$ and $\Delta_A \in \mathscr{S}_*^{\theta}(\mathscr{N})$, we have that $\Delta_A \leqslant X$, so X is a reflexive weighted relation.

Assume that $\theta = \mathrm{fs}$. Then according to (fs-2) and (fs-3) we get that

$$(X\circ X)^{-1}\circ R_i = X^{-1}\circ X^{-1}\circ R_i \leqslant X^{-1}\circ R_i\circ X^{-1} \leqslant R_i\circ X^{-1}\circ X^{-1} = R_i\circ (X\circ X)^{-1},$$
$$(X\circ X)^{-1}\circ p_j = X^{-1}\circ X^{-1}\circ p_j \leqslant X^{-1}\circ p_j \leqslant p_j,$$

for all $i \in I$ and $j \in J$, whence it follows that $X\circ X$ satisfies (fs-2) and (fs-3), that is, $X\circ X \in \mathscr{S}_*^{\mathrm{fs}}(\mathscr{N})$. Since X is the greatest element of $\mathscr{S}_*^{\mathrm{fs}}(\mathscr{N})$, we conclude that $X\circ X \leqslant X$, which means that X is a transitive weighted relation. As we have already proved that X is reflexive, we get that X is a weighted quasi-order.

In the same way we prove that X is a weighted quasi-order in the case when $\theta = \mathrm{bs}$.

(c) Due to the reflexivity of the weighted relations X and X^{-1} we have that

$$p_j = p_j\circ\Delta_A \leqslant p_j\circ X^{-1} \quad \text{and} \quad p_j = \Delta_A\circ p_j \leqslant X\circ p_j,$$

for each $j \in J$, whence it follows that X satisfies (θ-2) and (θ-3). This means that $X \in \mathscr{S}^{\theta}(\mathscr{N})$, and clearly, X is the greatest element of $\mathscr{S}^{\theta}(\mathscr{N})$. □

Theorem 6.2. *Let $\mathscr{N}$ be a weighted network and $\theta \in \{\mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}, \mathrm{rb}\}$. Then the following statements are true:*

(a) *$\mathscr{B}_*^{\theta}(\mathscr{N})$ is non-empty and it is closed under unions and products of weighted relations;*

(b) *$\mathscr{B}_*^{\theta}(\mathscr{N})$ has the greatest element X;*

(c) *If $\theta \in \{\mathrm{fb}, \mathrm{bb}, \mathrm{rb}\}$, then X is a weighted equivalence;*

(d) *X is also the greatest element of $\mathscr{B}^{\theta}(\mathscr{N})$.*

Proof. We will prove only the statement (c). Everything else is proved in a similar way as in the proof of the previous theorem.

(c) In the same way as in the previous theorem we get that X is a reflexive and transitive weighted relation. It remains to prove that X is symmetric.

Indeed, it follows directly from (θ-2) and (θ-3) that X^{-1} also satisfies (θ-2) and (θ-3), that is, $X^{-1} \in \mathscr{B}_*^{\theta}(\mathscr{N})$, which implies that $X^{-1} \leqslant X$, and this means that X is symmetric. □

Let $E \in Q^{A\times A}$ be a weighted equivalence. For an arbitrary $a \in A$ we define a weighted set $E_a \in Q^A$ by $E_a(x) = E(a,x)$, for each $x \in A$. We call E_a the *equivalence class* of E determined by a. One can simply check that $E_{a_1} = E_{a_2}$ if and only if $E(a_1,a_2) = 1$, for all $a_1,a_2 \in A$. The set of all equivalence classes of E is denoted by A/E.

Further, let $\mathscr{N} = (A,\{R_i\}_{i\in I},\{p_j\}_{j\in J})$ be a weighted network and let $E \in Q^{A\times A}$ be a weighted equivalence. For all $i \in I$ and $j \in J$ we define a weighted relation $\bar{R}_i \in Q^{(A/E)\times(A/E)}$ and a weighted set $\bar{p}_j \in Q^{A/E}$ as follows:

$$\bar{R}_i(E_{a_1},E_{a_2}) = (E \circ R_i \circ E)(a_1,a_2) = E_{a_1} \circ R_i \circ E_{a_2}, \tag{21}$$

$$\bar{p}_j(E_a) = (E \circ p_j)(a) = E_a \circ p_j, \tag{22}$$

for arbitrary $a,a_1,a_2 \in A$. Then $\bar{R}_i$ and $\bar{p}_j$ are well-defined, i.e., they do not depend on the choice of representatives of the equivalence classes of E, and $\mathscr{N}/E = (A/E,\{\bar{R}_i\}_{i\in I},\{\bar{p}_j\}_{j\in J})$ is a weighted network which is called the *quotient network* of $\mathscr{N}$ with respect to E.

7. Relationship between heterogeneous and homogeneous bisimulations

In investigating the relationship between heterogeneous and homogeneous bisimulations, which will be conducted in this section, we will use the results from [11, 12] concerning uniform weighted relations. As these results were proven for uniform weighted relations over a complete residuated lattice (which were called uniform fuzzy relations there), throughout this section the unederlying quantale $\mathscr{Q}$ will be a complete residuated lattice. Weighted sets and weighted relations with weights taken in a complete residuated lattice are usually called *fuzzy sets* and *fuzzy relations*. However, in order to achieve consistency in terminology throughout the paper, we will retain the names weighted sets and weighted relations.

Let A and B be non-empty sets. A weighted relation $X \in Q^{A\times B}$ is a *weighted mapping* or a *weighted function* if for every $a \in A$ there exists $b \in B$ such that $X(a,b) = 1$. It can be easily shown that X is a weighted mapping if and only if there exists an ordinary mapping $\psi : A \to B$ such that $X(a,\psi(a)) = 1$, for every $a \in A$ (cf. [15, 16]). A mapping ψ with such a property is called a *crisp description* of X. The set of all crisp descriptions of X is denoted by $CR(X)$. A weighted relation $X \in Q^{A\times B}$ is called *surjective* if for every $b \in B$ there exists $a \in A$ such that $X(a,b) = 1$.

Recall that the greatest bisimulation of type θ between two weighted networks (where $\theta \in \{\mathrm{fb},\mathrm{bb},\mathrm{rb}\}$), if it exists, satisfies $X\circ X^{-1}\circ X \leqslant X$. A weighted relation between two sets which satisfies $X\circ X^{-1}\circ X \leqslant X$ and it is a surjective weighted mapping is called a *uniform weighted relation*. Such weighted relations were studied in detail in [10–12] (under the name *uniform fuzzy relations*). It is important to note that a uniform weighted relation X also satisfies $X \circ X^{-1} \circ X = X$ (cf. [12]).

Theorem 7.1. *Let $\mathscr{N}' = (A',\{R'_i\}_{i\in I},\{p'_j\}_{j\in J})$ and $\mathscr{N}'' = (A'',\{R''_i\}_{i\in I},\{p''_j\}_{j\in J})$ be weighted networks and $X \in Q^{A'\times A''}$. Then the following statements are true:*

(a) *$X \circ X^{-1}$ and $X^{-1} \circ X$ are symmetric.*
(b) *If $X \circ X^{-1} \circ X \leqslant X$, then $X \circ X^{-1}$ and $X^{-1} \circ X$ are transitive.*
(c) *If X is a surjective weighted mapping then $X \circ X^{-1}$ and $X^{-1} \circ X$ are reflexive.*
(d) *If $\theta \in \{\mathrm{fb},\mathrm{bb},\mathrm{rb}\}$ and $X \in \mathscr{B}^\theta(\mathscr{N}',\mathscr{N}'')$, then $X \circ X^{-1} \in \mathscr{B}^\theta(\mathscr{N}')$ and $X^{-1} \circ X \in \mathscr{B}^\theta(\mathscr{N}'')$.*

Proof. (a) This statement follows directly from (8) and the fact that $(X^{-1})^{-1} = X$.

(b) According to (11), it follows from $X \circ X^{-1} \circ X \leqslant X$ that $(X \circ X^{-1})^2 \leqslant X \circ X^{-1}$ and $(X^{-1} \circ X)^2 \leqslant X^{-1} \circ X$, which means that $X \circ X^{-1}$ and $X^{-1} \circ X$ are transitive.

(c) Let X be a weighted mapping. Then for an arbitrary $a' \in A'$ there exists $a'' \in A''$ such that $X(a',a'') = 1$, whence it follows that

$$(X \circ X^{-1})(a',a') = \bigvee_{b''\in A''} X(a',b'') \otimes X^{-1}(b'',a') \geqslant X(a',a'') \otimes X^{-1}(a'',a') = X(a',a'') \otimes X(a',a'') = 1,$$

and we conclude that $(X \circ X^{-1})(a',a') = 1$, for each $a' \in A'$. This means that $X \circ X^{-1}$ is reflexive.

Let X be a surjective weighted relation. Then in the same way we get that $(X^{-1} \circ X)(a'', a'') = 1$, for every $a'' \in A''$, which means that $X^{-1} \circ X$ is reflexive.

(d) We will prove the case when $\theta = \mathrm{fb}$. Other cases are proven similarly.

Let X satisfy (fb-1), (fb-2) and (fb-3). Then according to (11), for arbitrary $j \in J$ and $i \in I$ we get that

$$p'_j \leqslant p''_j \circ X^{-1} \leqslant p'_j \circ X \circ X^{-1},$$
$$X \circ X^{-1} \circ R'_i \leqslant X \circ R''_i \circ X^{-1} \leqslant R'_i \circ X \circ X^{-1},$$
$$X \circ X^{-1} \circ p'_j \leqslant X \circ p''_j \leqslant p'_j,$$

which means that $X \circ X^{-1} \in \mathscr{B}^{\mathrm{fb}}(\mathscr{N}')$. In the same way we get that $X^{-1} \circ X \in \mathscr{B}^{\mathrm{fb}}(\mathscr{N}'')$. □

Let A' and A'' be non-empty sets and let $X \in Q^{A' \times A''}$ be a uniform weighted relation. According to Theorem 7.1, $X \circ X^{-1}$ and $X^{-1} \circ X$ are weighted equivalences on A' and A'', respectively. For the sake of simplicity, we will use notation $E'_X = X \circ X^{-1}$ and $E''_X = X^{-1} \circ X$. In [11, 12] it has been proven that

$$E'_X(a'_1, a'_2) = (X \circ X^{-1})(a'_1, a'_2) = \bigvee_{a'' \in A''} X(a'_1, a'') \otimes X(a'_2, a'') = \bigwedge_{a'' \in A''} X(a'_1, a'') \leftrightarrow X(a'_2, a''), \tag{23}$$

for all $a'_1, a'_2 \in A'$, and

$$E''_X(a''_1, a''_2) = (X^{-1} \circ X)(a''_1, a''_2) = \bigvee_{a' \in A'} X(a', a''_1) \otimes X(a', a''_2) = \bigwedge_{a' \in A'} X(a', a''_1) \leftrightarrow X(a', a''_2), \tag{24}$$

for all $a''_1, a''_2 \in A''$. In cases when it is clear that we deal with weighted equivalences that correspond to the weighted relation X, the subscript X in E'_X and E''_X will be omitted, and we will simply write E' and E''.

A mapping $\Phi_X : A'/E' \to A''/E''$ given by

$$\Phi_X(E'_{a'}) = E''_{\psi(a')} \quad \text{for any } a' \in A' \text{ and } \psi \in CR(X), \tag{25}$$

is well-defined (it does not depend on the choice of $a' \in A'$ and $\psi \in CR(X)$), and it is a bijective mapping which satisfies $\Phi_X^{-1} = \Phi_{X^{-1}}$ (cf. Lemma 4.5 [12]).

The next lemma is a part of Theorem 3.2 [11] (see also Theorem 4.2 [12]).

Lemma 7.2. *Let $X \in Q^{A' \times A''}$ be a uniform weighted relation. Then the following statements are true:*

(a) $X(a'_1, \psi(a'_2)) = E'(a'_1, a'_2)$, *for all $a'_1, a'_2 \in A'$ and $\psi \in CR(X)$.*

(b) $X(a', a'') = E''(\psi(a'), a'')$, *for all $a' \in A'$, $a'' \in A''$ and $\psi \in CR(X)$.*

Theorem 7.3. *Let $\mathscr{N}' = (A', \{R'_i\}_{i \in I}, \{p'_j\}_{j \in J})$ and $\mathscr{N}'' = (A'', \{R''_i\}_{i \in I}, \{p''_j\}_{j \in J})$ be weighted networks, let $X \in Q^{A' \times A''}$ be a uniform weighted relation, and let $\theta \in \{\mathrm{fb}, \mathrm{bb}, \mathrm{rb}\}$. Then $X \in \mathscr{B}^{\theta}(\mathscr{N}', \mathscr{N}'')$ if and only if the following conditions are satisfied:*

(i) $E'_X \in \mathscr{B}^{\theta}(\mathscr{N}')$;

(ii) $E''_X \in \mathscr{B}^{\theta}(\mathscr{N}'')$;

(iii) Φ_X *is an isomorphism of quotient networks $\mathscr{N}'/E'_X$ and $\mathscr{N}''/E''_X$.*

Proof. Let $X \in \mathscr{B}^{\theta}(\mathscr{N}', \mathscr{N}'')$. According to the previous theorem we have that $E' = E'_X = X \circ X^{-1} \in \mathscr{B}^{\theta}(\mathscr{N}')$ and $E'' = E''_X = X^{-1} \circ X \in \mathscr{B}^{\theta}(\mathscr{N}'')$. Thus, it remains to prove that (iii) holds.

As we noted earlier, Φ_X is a bijective mapping. To prove that Φ_X is an isomorphism, we first consider an arbitrary $j \in J$. According to (fb-3), (fb-1) and (13) we get that

$$E' \circ p'_j = X \circ X^{-1} \circ p'_j \leqslant X \circ p''_j \leqslant X \circ p'_j \circ X = X \circ X^{-1} \circ p'_j = E' \circ p'_j,$$

and hence, $E' \circ p'_j = X \circ p''_j$. Besides, according to Lemma 7.2 (b), for any $a' \in A'$ and $\psi \in CR(X)$ we have that

$$\begin{aligned}\bar{p}''_j(\Phi_X(E'_{a'}) &= \bar{p}''_j(E''_{\psi(a')}) = (E'' \circ p''_j)(\psi(a')) = \bigvee_{a'' \in A''} E''(\psi(a'), a'') \otimes p''_j(a'') = \bigvee_{a'' \in A''} X(a', a'') \otimes p''_j(a'') \\ &= (X \circ p''_j)(a') = (E' \circ p'_j)(a') = \bar{p}'_j(E'_{a'}).\end{aligned}$$

Next, consider an arbitrary $i \in I$. According to (fb-2) and (11) we have that

$$\begin{aligned}E'' \circ R''_i \circ E'' &= X^{-1} \circ X \circ R''_i \circ X^{-1} \circ X \leqslant X^{-1} \circ R'_i \circ X \circ X^{-1} \circ X = X^{-1} \circ R'_i \circ X \\ &= X^{-1} \circ X \circ X^{-1} \circ R'_i \circ X \leqslant X^{-1} \circ X \circ R''_i \circ X^{-1} \circ X = E'' \circ R''_i \circ E'',\end{aligned}$$

and thus, $E'' \circ R''_i \circ E'' = X^{-1} \circ R'_i \circ X$. According to Lemma 7.2 (a), for all $a'_1, a'_2 \in A'$ and $\psi \in CR(X)$ the following is true:

$$\begin{aligned}\bar{R}''_i(\Phi_X(E'_{a'_1}), \Phi_X(E'_{a'_2})) &= \bar{R}''_i(E''_{\psi(a'_1)}, E''_{\psi(a'_2)}) = (E'' \circ R''_i \circ E'')(\psi(a'_1), \psi(a'_2)) = (X^{-1} \circ R'_i \circ X)(\psi(a'_1), \psi(a'_2)) \\ &= \bigvee_{a'_3, a'_4 \in A'} X^{-1}(\psi(a'_1), a'_3) \otimes R'_i(a'_3, a'_4) \otimes X(a'_4, \psi(a'_2)) = \bigvee_{a'_3, a'_4 \in A'} X(a'_3, \psi(a'_1)) \otimes R'_i(a'_3, a'_4) \otimes X(a'_4, \psi(a'_2)) \\ &= \bigvee_{a'_3, a'_4 \in A'} E'(a'_3, a'_1) \otimes R'_i(a'_3, a'_4) \otimes E'(a'_4, a'_2) = \bigvee_{a'_3, a'_4 \in A'} E'(a'_1, a'_3) \otimes R'_i(a'_3, a'_4) \otimes E'(a'_4, a'_2) \\ &= (E' \circ R'_i \circ E')(a'_1, a'_2) = \bar{R}'_i(E'_{a'_1}, E'_{a'_2}).\end{aligned}$$

Therefore, we have proved that Φ_X is an isomorphism of weighted networks $\mathscr{N}'/E'_X$ and $\mathscr{N}''/E''_X$.

In the converse, we will prove only the case when $\theta = \text{fb}$. The remaining cases are proven in a similar manner. So, let the conditions (i), (ii) and (iii) be satisfied for $\theta = \text{fb}$.

Consider an arbitrary $j \in J$. Due to (i) we have that $p'_j = E' \circ p'_j$, and according to Lemma 7.2 (b), for an arbitrary $a' \in A'$ we get that

$$\begin{aligned}p'_j(a') &= (E' \circ p'_j)(a') = \bar{p}'_j(E'_{a'}) = \bar{p}''_j(\Phi_X(E'_{a'})) = \bar{p}''_j(E''_{\psi(a')}) = (E'' \circ p''_j)(\psi(a')) \\ &= \bigvee_{a'' \in A''} E''(\psi(a'), a'') \otimes p''_j(a'') = \bigvee_{a'' \in A''} X(a', a'') \otimes p''_j(a'') = (X \circ p''_j)(a').\end{aligned}$$

Together with (13), this implies that $p'_j = X \circ p''_j = p''_j \circ X^{-1}$.

In the same way, using the isomorphism $\Phi_x^{-1} = \Phi_{X^{-1}}$ instead of the isomorphism Φ_X and an arbitrary $\xi \in CR(X^{-1})$ instead of $\psi \in CR(X)$ (and also using Lemma 4.5 [12]), we obtain that $p''_j = X^{-1} \circ p'_j = p'_j \circ X$.

Therefore, we have proved that X satisfies (fb-1) and (fb-3).

In order to prove that X satisfies (fb-2) consider an arbitrary $i \in I$. According to (ii), E'' satisfies (fb-2), that is, $E'' \circ R''_i \leqslant R''_i \circ E''$, whence $E'' \circ R''_i \circ E'' \leqslant R''_i \circ E'' \circ E'' = R''_i \circ E''$, and since the opposite inequality holds due to the reflexivity of E'', we conclude that $E'' \circ R''_i \circ E' = R''_i \circ E''$. Due to the reflexivity of E' we also have that $E' \circ R'_i \leqslant E' \circ R'_i \circ E'$.

Now, for arbitrary $a' \in A'$, $a'' \in A''$, $\xi \in CR(X^{-1})$ and $\psi \in CR(X)$ we have that

$$\begin{aligned}(X^{-1} \circ R'_i)(a'', a') &= \bigvee_{a'_1 \in A'} X^{-1}(a'', a'_1) \otimes R'_i(a'_1, a') = \bigvee_{a'_1 \in A'} E'(\xi(a''), a'_1) \otimes R'_i(a'_1, a') = (E' \circ R'_i)(\xi(a''), a') \\ &\leqslant (E' \circ R'_i \circ E')(\xi(a''), a') = \bar{R}'_i(E'_{\xi(a'')}, E'_{a'}) = \bar{R}''_i(\Phi_X(E'_{\xi(a'')}), \Phi_X(E'_{a'})) = \bar{R}''_i(\Phi_X(\Phi_{X^{-1}}(E''_{a''})), \Phi_X(E'_{a'})) \\ &= \bar{R}''_i(E''_{a''}, E''_{\psi(a')}) = (E'' \circ R''_i \circ E'')(a'', \psi(a')) = (R''_i \circ E'')(a'', \psi(a')) \\ &= \bigvee_{a''_1 \in A''} R''_i(a'', a''_1) \otimes E''(a''_1, \psi(a')) = \bigvee_{a''_1 \in A''} R''_i(a'', a''_1) \otimes E''(\psi(a'), a''_1) = \bigvee_{a''_1 \in A''} R''_i(a'', a''_1) \otimes X(a', a''_1) \\ &= \bigvee_{a''_1 \in A''} R''_i(a'', a''_1) \otimes X^{-1}(a''_1, a') = (R''_i \circ X^{-1})(a'', a'),\end{aligned}$$

which implies $X^{-1} \circ R_i' \leqslant R_i'' \circ X^{-1}$. In the same way we prove that X satisfies the second inequality in (fb-2).

Therefore, we have proved that X is a forward bisimulation. This completes the proof of the theorem. □

The previous theorem showed that for two weighted networks $\mathscr{N}'$ and $\mathscr{N}''$ and $X \in \mathscr{B}^\theta(\mathscr{N}', \mathscr{N}'')$ we have that $E_X' \in \mathscr{B}^\theta(\mathscr{N}')$ and $E_X'' \in \mathscr{B}^\theta(\mathscr{N}'')$. The following question naturally arises: if weighted equivalences $E' \in \mathscr{B}^\theta(\mathscr{N}')$ and $E'' \in \mathscr{B}^\theta(\mathscr{N}'')$ are given, under what conditions does $X \in \mathscr{B}^\theta(\mathscr{N}', \mathscr{N}'')$ exists so that $E' = E_X'$ and $E'' = E_X''$. The answer to this question is given by the following theorem.

Theorem 7.4. *Let weighted networks $\mathscr{N}' = (A', \{R_i'\}_{i\in I}, \{p_j'\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R_i''\}_{i\in I}, \{p_j''\}_{j\in J})$ and weighted equivalences $E' \in \mathscr{B}^\theta(\mathscr{N}')$ and $E'' \in \mathscr{B}^\theta(\mathscr{N}'')$ be given, where $\theta \in \{\mathrm{fb}, \mathrm{bb}, \mathrm{rb}\}$.*

Then there exists $X \in \mathscr{B}^\theta(\mathscr{N}', \mathscr{N}'')$ which is a uniform weighted relation and satisfies $E_X' = E'$ and $E_X'' = E''$ if and only if there exists an isomorphism $\Phi : \mathscr{N}'/E' \to \mathscr{N}''/E''$ such that for all $a_1', a_2' \in A'$ the following is true

$$\widetilde{E}'(E'_{a_1'}, E'_{a_2'}) = \widetilde{E}''(\Phi(E'_{a_1'}), \Phi(E'_{a_2'})). \tag{26}$$

Proof. Let there exists $X \in \mathscr{B}^\theta(\mathscr{N}', \mathscr{N}'')$ which is a uniform relation and satisfies $E_X' = E'$ and $E_X'' = E''$. Let us consider an arbitrary $\psi \in CR(X)$. Then, according to the definitions of the relations $\widetilde{E}'$ and $\widetilde{E}''$ and the mapping Φ_X, we get that

$$\widetilde{E}'(E'_{a_1'}, E'_{a_2'}) = E'(a_1', a_2') = E''(\psi(a_1'), \psi(a_2')) = \widetilde{E}''(E''_{\psi(a_1')}, E''_{\psi(a_2')}) = \widetilde{E}''(\Phi_X(E'_{a_1'}), \Phi_X(E'_{a_2'})),$$

and by Theorem 7.3 it follows that Φ_X is an isomorphism between $\mathscr{N}'/E'$ and $\mathscr{N}''/E''$ which satisfies (26).

Conversely, let there exists an isomorphism $\Phi : \mathscr{N}'/E' \to \mathscr{N}''/E''$ such that (26) holds. Define mappings $\Phi' : A' \to A'/E'$, $\Phi'' : A''/E'' \to A''$ and $\psi : A' \to A''$ as follows:

(i) For each $a' \in A'$ we put $\Phi'(a') = E'_{a'}$;

(ii) For each $\alpha \in A''/E''$ we put $\Phi''(\alpha) = a''$, where $a'' \in A''$ is an arbitrary element such that $\alpha = E''_{a''}$;

(iii) $\psi = \Phi' \circ \Phi \circ \Phi''$.

In addition, define a weighted relation $X \in Q^{A'\times A''}$ by

$$X(a', a'') = E''(\psi(a'), a''),$$

for all $a' \in A'$ and $a'' \in A''$. According to the proof of Theorem 3.4 [11], we have that X is a uniform weighted relation such that $E_X' = E'$, $E_X'' = E''$ and $\psi \in CR(X)$.

For an arbitrary $a' \in A'$ let $\Phi(E'_{a'}) = E''_{a''}$, for some $a'' \in A''$. Then we have that

$$\psi(a') = \Phi''(\Phi(\Phi'(a'))) = \Phi''(\Phi(E'_{a'})) = \Phi''(E''_{a''}) = a_1'',$$

where $a_1'' \in A''$ is an element such that $E''_{a_1''} = E''_{a''}$, whence

$$\Phi_x(E'_{a'}) = E''_{\psi(a')} = E''_{a_1''} = E''_{a''} = \Phi(E'_{a'}),$$

and hence, $\Phi = \Phi_x$. Now, according to Theorem 7.3, we conclude that $X \in \mathscr{B}^\theta(\mathscr{N}', \mathscr{N}'')$. □

The previous theorems dealt with θ-simulations in the case when $\theta \in \{\mathrm{fb}, \mathrm{bb}, \mathrm{rb}\}$. The remaining cases, when $\theta \in \{\mathrm{fbb}, \mathrm{bfb}\}$, are considered in the next theorems.

Theorem 7.5. *Let weighted networks $\mathscr{N}' = (A', \{R_i'\}_{i\in I}, \{p_j'\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R_i''\}_{i\in I}, \{p_j''\}_{j\in J})$ and a uniform weighted relation $X \in Q^{A'\times A''}$ be given. Then the following statements are true:*

(a) *If $X \in \mathscr{B}^{\mathrm{bfb}}(\mathscr{N}', \mathscr{N}'')$, then $X \circ X^{-1} \in \mathscr{B}^{\mathrm{fb}}(\mathscr{N}')$ and $X^{-1} \circ X \in \mathscr{B}^{\mathrm{bb}}(\mathscr{N}'')$.*

(b) *If $X \in \mathscr{B}^{\mathrm{fbb}}(\mathscr{N}', \mathscr{N}'')$, then $X \circ X^{-1} \in \mathscr{B}^{\mathrm{bb}}(\mathscr{N}')$ and $X^{-1} \circ X \in \mathscr{B}^{\mathrm{fb}}(\mathscr{N}'')$.*

Proof. We will prove only (a). The statement (b) can be proved in a similar way.

(a) Let X satisfy (bfb-1), (bfb-2) and (bfb-3). Due to the reflexivity of $X \circ X^{-1}$ we get that $X \circ X^{-1}$ satisfies (fb-1). Further, according to (11) and (bfb-2), the reflexivity of $X \circ X^{-1}$, and the fact that $X \circ X^{-1} \circ X \leqslant X$, we obtain that

$$X \circ X^{-1} \circ R'_i \leqslant X \circ X^{-1} \circ R'_i \circ X \circ X^{-1} \leqslant X \circ X^{-1} \circ X \circ R''_i \circ X^{-1} \leqslant X \circ R''_i \circ X^{-1} \leqslant R'_i \circ X \circ X^{-1},$$

for each $i \in I$, which means that $X \circ X^{-1}$ satisfies (fb-2).

Finally, according to (13) and (bfb-3), we have that

$$X \circ X^{-1} \circ p'_j \leqslant (X \circ X^{-1})^{-1} \circ p'_j = p'_j \circ X \circ X^{-1} \leqslant p''_j \circ X^{-1} = X \circ p''_j \leqslant p'_j,$$

for every $j \in J$, so we conclude that $X \circ X^{-1}$ satisfies (fb-3). Thus, $X \circ X^{-1}$ is a forward bisimulation on $\mathscr{N}'$.

In the same way we prove that $X^{-1} \circ X$ is a backward bisimulation on $\mathscr{N}''$. □

The next two theorems can be proved in a similar way as Theorem 7.3, so their proofs will be omitted.

Theorem 7.6. *Let $\mathscr{N}' = (A', \{R'_i\}_{i\in I}, \{p'_j\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R''_i\}_{i\in I}, \{p''_j\}_{j\in J})$ be weighted networks, and $X \in Q^{A'\times A''}$ be a uniform weighted relation. Then $X \in \mathscr{B}^{\mathrm{bfb}}(\mathscr{N}', \mathscr{N}'')$ if and only if the following conditions are satisfied:*

(i) $E'_X \in \mathscr{B}^{\mathrm{fb}}(\mathscr{N}')$;

(ii) $E''_X \in \mathscr{B}^{\mathrm{bb}}(\mathscr{N}'')$;

(iii) Φ_X *is an isomorphism of quotient networks* $\mathscr{N}'/E'_X$ *and* $\mathscr{N}''/E''_X$.

Theorem 7.7. *Let $\mathscr{N}' = (A', \{R'_i\}_{i\in I}, \{p'_j\}_{j\in J})$ and $\mathscr{N}'' = (A'', \{R''_i\}_{i\in I}, \{p''_j\}_{j\in J})$ be weighted networks, and $X \in Q^{A'\times A''}$ be a uniform weighted relation. Then $X \in \mathscr{B}^{\mathrm{fbb}}(\mathscr{N}', \mathscr{N}'')$ if and only if the following conditions are satisfied:*

(i) $E'_X \in \mathscr{B}^{\mathrm{bb}}(\mathscr{N}')$;

(ii) $E''_X \in \mathscr{B}^{\mathrm{fb}}(\mathscr{N}'')$;

(iii) Φ_X *is an isomorphism of quotient networks* $\mathscr{N}'/E'_X$ *and* $\mathscr{N}''/E''_X$.

It should be noted that theorems analogous to Theorem 7.4 can also be stated and proved for backward-forward and forward-backward bisimulations. In order to save space, we will not state these theorems.

8. Weighted networks and many-valued multimodal logics

In this section, we deal with the connections between weighted networks and many-valued multimodal logics. For this purpose, we will use the results from [37, 38] concerning many-valued multimodal logics over complete Heyting algebras. Consequently, in this section, the underlying quantale $\mathscr{Q}$ will be a complete Heyting algebra, that is, the multiplication $\otimes$ coincides with the meet operation $\wedge$. In the final part of the section we will also assume that $\mathscr{Q}$ is linearly ordered.

A many-valued multimodal logic over $\mathscr{Q}$ is defined as follows. For a non-empty set I of indices, an alphabet of a many-valued multimodal logic $\mathscr{Q}(\{\Box_i, \Diamond_i, \Box_i^-, \Diamond_i^-\}_{i\in I})$ consists of

– a countable set of *propositional symbols* PV,
– a set of *truth constants* $\overline{Q} = \{\overline{q} \mid q \in Q\}$,
– *logical connectives* $\wedge$ (*conjunction*) and $\rightarrow$ (*implication*), and
– four families of *modal operators*: $\{\Box_i\}_{i\in I}$ and $\{\Box_i^-\}_{i\in I}$ (*necessity operators*), $\{\Diamond_i\}_{i\in I}$ and $\{\Diamond_i^-\}_{i\in I}$ (*possibility operators*).

The set of formulas $\Phi_{I,\mathscr{Q}}$ of a many-valued modal logic is the smallest set containing propositional symbols and truth constants, and is closed under logical connectives and modal operators:

$$A := \overline{q} \mid p \mid A \wedge B \mid A \rightarrow B \mid \Box_i A \mid \Diamond_i A \mid \Box_i^- A \mid \Diamond_i^- A$$

where $q \in Q$, $p \in PV$, $i \in I$, and A and B are formulas from $\Phi_{I,\mathscr{Q}}$. We use the following well-known abbreviations:

$\neg A \equiv A \to \overline{0}$ (*negation*),

$A \leftrightarrow B \equiv (A \to B) \wedge (B \to A)$ (*equivalence*),

$A \vee B \equiv ((A \to B) \to B) \wedge ((B \to A) \to A)$ (*disjunction*),

$\overline{1} \equiv \overline{0} \to \overline{0}$ (*top*).

Recall that 0 is the least element in $\mathcal{Q}$ and $\overline{0}$ is the corresponding truth constant. The set of all formulas over the alphabet $\mathcal{Q}(\{\Box_i, \Diamond_i\}_{i\in I})$, i.e., the set of those formulas from $\Phi_{I,\mathcal{Q}}$ that do not contain any of the modal operators $\Box_i^-$ and $\Diamond_i^-$, $i \in I$, will be denoted by $\Phi^+_{I,\mathcal{Q}}$. Similarly, the set of all formulas over the alphabet $\mathscr{H}(\{\Box_i^-, \Diamond_i^-\}_{i\in I})$, i.e., the set of those formulas from $\Phi_{I,\mathcal{Q}}$ that do not contain any of the modal operators $\Box_i$ and $\Diamond_i$, $i \in I$, will be denoted by $\Phi^-_{I,\mathcal{Q}}$. For the sake of simplicity, formulas from $\Phi^+_{I,\mathcal{Q}}$ will be called *plus-formulas*, and formulas from $\Phi^-_{I,\mathcal{Q}}$ will be called *minus-formulas*.

A *weighted Kripke frame* is a structure $\mathfrak{F} = (W, \{R_i\}_{i\in I})$ where W is a nonempty set of possible *worlds* (or *states* or *points*) and $R_i \in \mathscr{F}(W \times W)$ is a weighted relation on W, for every $i \in I$, called the *accessibility weighted relation* of the frame. It is usually assumed that I is a finite set with n elements, and then $\mathfrak{F}$ is called a *weighted Kripke n-frame*.

A *weighted Kripke model for* $\Phi_{I,\mathcal{Q}}$ is a structure $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$ such that $(W, \{R_i\}_{i\in I})$ is a weighted Kripke frame and $V : W \times (PV \cup \overline{Q}) \to Q$ is a truth assignment function, called the *evaluation of the model*, which assigns an Q-truth value to propositional variables (and truth constants) in each world, such that $V(w, \overline{q}) = q$, for every $w \in W$ and $q \in Q$. It is usually assumed that I is a finite set with n elements, and then $\mathfrak{M}$ is called a *weighted Kripke n-model*.

The truth assignment function V can be inductively extended to a function $V : W \times \Phi_{I,\mathcal{Q}} \to Q$ by:

$$V(w, A \wedge B) = V(w, A) \wedge V(w, B); \tag{V1}$$

$$V(w, A \to B) = V(w, A) \to V(w, B); \tag{V2}$$

$$V(w, \Box_i A) = \bigwedge_{u\in W} R_i(w, u) \to V(u, A), \text{ for every } i \in I; \tag{V3}$$

$$V(w, \Diamond_i A) = \bigvee_{u\in W} R_i(w, u) \wedge V(u, A), \text{ for every } i \in I; \tag{V4}$$

$$V(w, \Box_i^- A) = \bigwedge_{u\in W} R_i(u, w) \to V(u, A), \text{ for every } i \in I; \tag{V5}$$

$$V(w, \Diamond_i^- A) = \bigvee_{u\in W} R_i(u, w) \wedge V(u, A), \text{ for every } i \in I. \tag{V6}$$

Note that the same symbols are used for $\wedge$ and $\to$ in both sides of formulas (V1)–(V6). The meaning is clear from the context, so we keep the notation simple. For each world $w \in W$ the truth assignment V determines a function $V_w : \Phi_{I,\mathcal{Q}} \to Q$ given by $V_w(A) = V(w, A)$, for every $A \in \Phi_{I,\mathcal{Q}}$, and vice versa, for each $A \in \Phi_{I,\mathcal{Q}}$ the truth assignment V determines a function $V_A : W \to Q$ given by $V_A(w) = V(w, A)$, for every $w \in W$.

As we use the letter A here to denote modal formulas, the sets of actors of the considered weighted networks will be denoted in this section by W, W', W'', etc. The key point in this section is the observation that a weighted network $\mathscr{N} = (W, \{R_i\}_{i\in I}, \{p_j\}_{j\in J})$ can be treated as a weighted Kripke model $\mathfrak{M} = (W, \{R_i\}_{i\in I}, V)$, where $PV = \{p_j\}_{j\in J}$ and the evaluation V is defined so that $V(a, p_j) = p_j(a)$, for all $a \in W$ and $j \in J$. Here we assume that J is a countable set. We call $\mathfrak{M}$ a *weighted Kripke model* corresponding to the weighted network $\mathscr{N}$. Such an approach allows consideration of complex attributes that actors may have, which are defined by modal formulas from $\Phi_{I,\mathcal{Q}}$. Namely, for each formula $A \in \Phi_{I,\mathcal{Q}}$ and any $a \in W$ the value $V_A(a) = V(a, A)$, i.e., the extent to which a satisfies the modal formula A, is also understood as the extent to which a has an attribute associated to A.

Let $\mathscr{N}' = (W', \{R_i'\}_{i\in I}, \{p_j'\}_{j\in J})$ and $\mathscr{N}'' = (W'', \{R_i'\}_{i\in I}, \{p_j'\}_{j\in J})$ be weighted networks of the same type, and let $\mathfrak{M}' = (W', \{R_i'\}_{i\in I}, V')$ and $\mathfrak{M}'' = (W'', \{R_i''\}_{i\in I}, V'')$ be the corresponding weighted Kripke models. For a set of formulas $\Phi \subseteq \Phi_{I,\mathcal{Q}}$, the degree of similarity of the actors $a' \in W'$ and $a'' \in W''$ with respect to formulas from

Φ, denoted by $S_\Phi(a', a'')$, is defined by

$$S_\Phi(a', a'') = \bigwedge_{A\in\Phi} V'_A(a') \leftrightarrow V''_A(a'').$$

In other words, $S_\Phi(a', a'')$ can be understood as the extent to which a' and a'' have the same complex attributes associated with the formulas from Φ.

If $S_\Phi(a', a'') = 1$, i.e., if $V'_A(a') = V''_A(a'')$, for every $A \in \Phi$, then we say that a' and a'' are Φ-*equivalent actors*. Moreover, $\mathscr{N}'$ and $\mathscr{N}''$ are said to be Φ-*equivalent weighted networks* if each $a' \in W'$ is Φ-equivalent to some $a'' \in W''$, and vice versa, each $a'' \in W''$ is Φ-equivalent to some $a' \in W'$.

As we have said in Section 3, although in practical applications we encounter only finite weighted networks, here we still allow weighted networks to be infinite. However, in the following considerations, we will require weighted networks to satisfy certain properties weaker than finiteness. Namely, a weighted network $\mathscr{N} = (W, \{R_i\}_{i\in I}, \{p_j\}_{j\in J})$ is called *image-finite* if for all $a \in W$ and $i \in I$ the set $\{b \in W \mid R_i(a, b) > 0\}$ is finite, it is called *domain-finite* if for all $b \in W$ and $i \in I$ the set $\{a \in W \mid R_i(a, b) > 0\}$ is finite, and it is called *degree-finite* if it is both image-finite and domain-finite.

In the case when the underlying Heyting algebra $\mathscr{Q}$ is linearly ordered, the following can be proved:

Theorem 8.1. *Let $\mathscr{N}' = (W', \{R'_i\}_{i\in I}, \{p'_j\}_{j\in J})$ and $\mathscr{N}'' = (W'', \{R'_i\}_{i\in I}, \{p'_j\}_{j\in J})$ be weighted networks of the same type over a linearly ordered Heyting algebra $\mathscr{Q}$, and let $\mathfrak{M}' = (W', \{R'_i\}_{i\in I}, V')$ and $\mathfrak{M}'' = (W'', \{R''_i\}_{i\in I}, V'')$ be the corresponding weighted Kripke models. Then the following statements are true:*

(a) *If $\mathscr{N}'$ and $\mathscr{N}''$ are image-finite, and if there exists the greatest forward bisimulation $X^{\rm fb}$ between $\mathscr{N}'$ and $\mathscr{N}''$, then for arbitrary $a' \in A'$ and $a'' \in A''$ we have that*

$$X^{\rm fb}(a', a'') = S_{\Phi^{+}_{I,\mathscr{Q}}}(a', a'') = \bigwedge_{A\in\Phi^{+}_{I,\mathscr{Q}}} V'_A(a') \leftrightarrow V''_A(a''); \tag{27}$$

(b) *If $\mathscr{N}'$ and $\mathscr{N}''$ are domain-finite, and if there exists the greatest backward bisimulation $X^{\rm bb}$ between $\mathscr{N}'$ and $\mathscr{N}''$, then for arbitrary $a' \in A'$ and $a'' \in A''$ we have that*

$$X^{\rm bb}(a', a'') = S_{\Phi^{-}_{I,\mathscr{Q}}}(a', a'') = \bigwedge_{A\in\Phi^{-}_{I,\mathscr{Q}}} V'_A(a') \leftrightarrow V''_A(a''); \tag{28}$$

(c) *If $\mathscr{N}'$ and $\mathscr{N}''$ are degree-finite, and if there exists the greatest regular bisimulation $X^{\rm rb}$ between $\mathscr{N}'$ and $\mathscr{N}''$, then for arbitrary $a' \in A'$ and $a'' \in A''$ we have that*

$$X^{\rm rb}(a', a'') = S_{\Phi_{I,\mathscr{Q}}}(a', a'') = \bigwedge_{A\in\Phi_{I,\mathscr{Q}}} V'_A(a') \leftrightarrow V''_A(a''). \tag{29}$$

The proofs of these statements follow immediately from Theorems 2, 3 and 4 from [38] and they will be omitted. We will only provide their interpretation. Namely, the previous theorem says that the forward bisimilarity, if it exists, expresses the degree of similarity of actors with respect to plus formulas, while the backward bisimilarity expresses the degree of similarity with respect to minus formulas, and the regular bisimilarity expresses the degree of similarity with respect to all modal formulas from $\Phi_{I,\mathscr{Q}}$. In the unweighted case we can say that the forward bisimilarity preserves plus formulas, backward bisimilarity preserves minus formulas, and regular bisimilarity preserves all modal formulas.

Theorem 8.2. *Let $\theta \in \{\rm fb, bb, rb\}$, let $\mathscr{N} = (W, \{R_i\}_{i\in I}, \{p_j\}_{j\in J})$ be a weighted network, let $E \in Q^{W\times W}$ be a weighted equivalence, and let $\mathscr{N}/E = (A/E, \{\bar{R}_i\}_{i\in I}, \{\bar{p}_j\}_{j\in J})$ be the quotient network of $\mathscr{N}$ with respect to E.*

Let us define a weighted relation $X \in Q^{W\times(W/E)}$ by

$$X(a, E_b) = E(a, b), \qquad \textit{for all } a, b \in W. \tag{30}$$

Then the following statement is true

(a) *$E \in \mathscr{B}^{\theta}(\mathscr{N})$ if and only if $X \in \mathscr{B}^{\theta}(\mathscr{N}, \mathscr{N}/E)$.*

In addition, if $\mathscr{Q}$ is a linearly ordered Heyting algebra, then the following statements are also true:

(b) *If $E \in \mathscr{B}^{\rm fb}(\mathscr{N})$ and $\mathscr{N}$ is image-finite, then $\mathscr{N}$ and $\mathscr{N}/E$ are $\Phi^{+}_{I,\mathscr{Q}}$ -equivalent weighted networks;*

(c) *If $E \in \mathscr{B}^{\rm bb}(\mathscr{N})$ and $\mathscr{N}$ is domain-finite, then $\mathscr{N}$ and $\mathscr{N}/E$ are $\Phi^{-}_{I,\mathscr{Q}}$ -equivalent weighted networks;*

(d) *If $E \in \mathscr{B}^{\rm rb}(\mathscr{N})$ and $\mathscr{N}$ is degree-finite, then $\mathscr{N}$ and $\mathscr{N}/E$ are $\Phi_{I,\mathscr{Q}}$-equivalent weighted networks.*

The proofs of these statements follow from Theorems 7.6, 7.7, 7.8 and 7.9 from [37]. The statements (b), (c) and (d) say that the quotient weighted network with respect to a forward bisimulation equivalence inherits from the original weighted network those complex attributes expressed by plus formulas, and similarly, the quotient weighted network with respect to a backward bisimulation equivalence inherits those attributes expressed by minus formulas, and the quotient weighted network with respect to a regular bisimulation equivalence inherits those attributes expressed by all formulas from $\Phi_{I,\mathscr{Q}}$.

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