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Extended multi-valued pseudocontractive mappings and extended Mann and Ishikawa iteration schemes for finite family of mappings

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Abstract. In this paper, new classes of *extended* multi-valued pseudocontractive mappings are introduced. It is established that the type-one subclass of the extended strictly pseudocontractive mappings is more closely related to the class of single-valued strictly pseudocontractive mappings in the sense that the possession of L-Lipschitzian and demiclosedness properties as well as closed and convex set of fixed points are guaranteed. Also, we introduce an extended Mann and an extended Ishikawa iteration schemes for approximating a common fixed point of a finite family of mappings. Furthermore, using the extended Mann and the extended Ishikawa iteration schemes, we prove weak and strong convergence theorems for our new classes of extended strictly pseudocontractive and pseudocontractive mappings, respectively. Numerical examples are also included to illustrate our results. The results obtained improve, complement and extend the results on multi-valued and single-valued mappings in the contemporary literature.

1. Introduction

Let X be a normed space. A subset C of X is called proximinal if for each $g \in X$ there exists $k \in C$ such that

$$\|g - k\| = \inf\{\|g - h\| : h \in C\} = d(g, C). \tag{1}$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote the collection of all nonempty closed and bounded subsets of X by $CB(X)$, the collection of all nonempty closed and convex subsets of X by $CC(X)$, the collection of all nonempty subsets of X by 2^X and the collection of all proximinal subsets of X by $P(X)$, for a nonempty set X.
Let X be a nonempty set and let $T : X \to X$ be a mapping. A point $g \in X$ is called a fixed point of T if $g = Tg$. If $T : X \to 2^X$ is a multi-valued mapping then g is a fixed point of T if $g \in Tg$. If $Tg = \{g\}$ then g is called a strict fixed point of T. The set $F(T) = \{g \in D(T) : g \in Tg\}$(respectively $F(T) = \{g \in D(T) : g = Tg\}$) is called the fixed

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point set of multi-valued(respectively single-valued) mapping T, while the set $F_s(T) = \{g \in D(T) : Tg = \{g\}\}$ is called the strict fixed point set of T.
Let X be a metric space, the Hausdorff metric induced on $CB(X)$ by the metric d on X is defined for every $A, B \in CB(X)$ by

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}. \tag{2}$$

Let X be a normed space and $T : X \to P(X)$ a multi-valued mapping. The mapping P_T is defined for each $x \in X$ by $P_T x = \{y \in Tx : \|x - y\| = d(x, Tx)\}$.
Recall that one of the applications of fixed point theory is to solve the problem that arises from initial value problems of differential inclusion of the form

$$0 \in \frac{dU}{dt} + A(t)U(t), \tag{3}$$

which describes an evolution system where A is in general nonlinear, set valued and accretive.
At equilibrium or stable state of the system, $\frac{dU}{dt} = 0$ and U(t) is constant. Therefore, (3) becomes

$$0 \in A(t). \tag{4}$$

Since A is in general nonlinear, there is no closed form solution of equation (4). The standard technique is to introduce an operator T defined by

$$T = I - A, \tag{5}$$

where I is the identity mapping on X. Such T is called a pseudocontraction (or pseudocontractive). Observe that if $g \in Tg$, then $g = g - w$ for some $w \in A(g)$. Consequently, we have that $w = 0 \in A(g)$.
If A is single-valued, then (3) and (4) become

$$\frac{dU}{dt} + A(t)U(t) = 0 \tag{6}$$

and

$$A(t) = 0 \tag{7}$$

respectively.
It then follows from (5) that if $g = Tg$ then $g = g - A(g)$. Consequently $A(g) = 0$ (i.e., any zero of A is a fixed point of T).
In real Banach spaces, authors have studied extensively the fixed point problems of single-valued classes of pseudocontractive mappings (see for example [7, 14, 31, 44] and references therein). Some of the single-valued mappings were introduced as follows.

Definition 1.1. *Let E be an arbitrary real Banach space. A mapping $T : D(T) \subseteq E \to E$ is said to be*
(i) Lipschitz continuous with constant $L \geq 0$ if

$$\|Tx - Ty\| \leq L\|x - y\| \tag{8}$$

for all $x, y \in D(T)$. If $L \in (0, 1)$, T is said to be a contraction. T is said to be nonexpansive if $L = 1$.
Nonepansive mappings are linked intimately with several other nonlinear mappings that are of interest in ordinary and partial differential equations (see for example [3, 24]). Bruck [8] remarked that the intimate connection between nonexpansive operators and accretive operators accounts partly for the importance of nonexpansive mappings. The class of nonexpansive mappings is one of the initial classes of operators for which fixed point results were obtained using the geometric structure of the underlying Banach space rather than the compactness property.
(ii) T is called quasi-nonexpansive if the fixed point set of T is nonempty and for all $x \in D(T)$, $p \in F(T)$,

$$\|Tx - p\| \leq \|x - p\|. \tag{9}$$

It is clear that every nonexpansive mapping with nonempty set of fixed points is quasi-nonexpansive.

Closely related to the class of single-valued nonexpansive and quasi-nonexpansive mappings are the classes of $k-$strictly psudocontractive and pseudocontractive mappings in the sense of Browder and Petryshyn [7], demicontractive mappings of Hicks and Kubicek [14] and Hemicontractive mapping of Naimpally and Singh [31]. These mappings are defined as follows.

Definition 1.2. *Let H be a real Hilbert space and K a closed convex subset of H. A mapping $T : K \to K$ is called*
(i) k-strictly pseudocontractive mapping if there exists $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||x - Tx - (y - Ty)||^2, \ \forall x, y \in K. \tag{10}$$

(ii) Pseudocontractive if

$$||Tx - Ty||^2 \leq ||x - y||^2 + ||x - Tx - (y - Ty)||^2, \ \forall x, y \in K. \tag{11}$$

(iii) Demicontractive if $F(T) \neq \emptyset$ and

$$||Tx - Tp||^2 \leq ||x - p||^2 + k||x - Tx||^2, \ \forall x \in K, \ p \in F(T). \tag{12}$$

(iv) Hemicontractive if

$$||Tx - Tp||^2 \leq ||x - p||^2 + ||x - Tx||^2, \ \forall x \in K, \ p \in F(T). \tag{13}$$

In recent years, authors have introduced and studied different multi-valued versions of the above single-valued pseudocontractive mappings using Hausdorff metric (see for example, [1, 9, 11, 17, 20, 23, 27, 37]). Also, significant achievement has be recorded in the area of developing iteration schemes for approximating their fixed points (see [37, 39]). Below are some of the different versions of the multi-valued mapping considered recently by authors and also some contributions in the developments of the iteration schemes for approximating fixed points.

Definition 1.3. *Let X be a normed space. Let $T : D(T) \subseteq X \to 2^X$ be a multi-valued mapping on X. T is called*
(i) $L-$Lipschitzian if there exists $L \geq 0$ such that for all $g, h \in D(T)$

$$H(Tg, Th) \leq L||g - h||. \tag{14}$$

In (14), if $L \in [0, 1)$, T is said to be a contraction while T is nonexpansive if $L = 1$. T is called quasi-nonexpansive if $F(T) = \{g \in D(T) : g \in Tg\} \neq \emptyset$ and for all $p \in F(T)$,

$$H(Tg, Tp) \leq ||g - p||. \tag{15}$$

Clearly every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive.
(ii) T is said to be k-strictly pseudocontractive mapping of Chidume et. al [9] if there exists $k \in (0, 1)$ such that

$$H^2(Tg, Th) \leq ||g - h||^2 + k||g - u - (h - v)||^2, \quad \text{for all } u \in Tg, v \in Th. \tag{16}$$

If $k = 1$ in (1.16), T is said to be pseudocontractive.
(iii) T is said to be k-strictly pseudocontractive-type of Isiogugu (see [17]) if there exist $k \in (0, 1)$ such that given any pair $g, h \in D(T)$ and $u \in Tg$, there exist $v \in Th$ satisfying $||u - v|| \leq H(Tg, Th)$ and

$$H^2(Tg, Th) \leq ||g - h||^2 + k||g - u - (h - v)||^2. \tag{17}$$

If $k = 1$ in (17), T is said to be pseudocontractive-type. It is easy to see that any proximinal, pseudocontractive mapping T of Chidume et. al [9] is pseudoconytractive-type in the sense of Isiogugu [17].
(iv) T is said to be demicontractive (see [20]) if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that given any pair $g \in D(T)$ and $p \in F(T)$,

$$H^2(Tg, Tp) \leq ||g - p||^2 + kd^2(g, Tg). \tag{18}$$

If $k = 1$ in (18), T is said to be hemicontractive. T is called a quasi-nonexpansive mapping if $k = 0$. It is easy to see that every k-strictly pseudocontractive (respectively, pseudocontractive) mapping T in the sense of Definition 1.3 (ii) with nonempty set of fixed points is demicontractive (respectively, hemicontractive).

(v) If a k-strictly pseudocontractive-type (respectively, pseudocontractive-type)mapping in Definition 1.3(i) has the property that its set of strict fixed points $F_s(T) \neq \emptyset$, then it is demicontractive-type (respectively, hemicontractive-type) with respect to its set of strict fixed points. That is, given any pair $g \in D(T)$ and $p \in F_s(T)$, and $u \in Tg$, $\|u - p\| \leq H(Tg, Tp)$ and

$$H^2(Tg, Tp) \leq \|g - p\|^2 + k\|g - u\|^2 (\text{respectively } \|g - p\|^2 + \|g - u\|^2). \tag{19}$$

Observe that a demicontractive (respectively, hemicontractive) in definition 1.3 (iv) is demicontractive-type (respectively, hemicontractive-type) with respect to its set of strict fixed points.

Remark 1.4. *The relationships between the classes of k-strictly pseudocontractive-type (respectively, pseudocontractive-type) mappings in definition 1.3 (iii) and demicontractive(respectively, hemicontractive) mappings in definition 1.3 (iv) has been established with respect to the set of fixed points.*

In 2005, Sastry and Babu [37] introduced Mann and Ishikawa iteration scheme for multi-valued nonexpansive mappings as follows:
Let $T : X \to P(X)$ and p be a fixed point of T. The sequence of Mann iterates is given for $x_0 \in X$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \quad \forall n \geq 0 \tag{20}$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(Tx_n, p)$ and α_n is a real sequence in (0,1), $\sum_{n=1}^{\infty} \alpha_n = \infty$.
The sequence of Ishikawa iterates is given by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n z_n \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{21}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ are such that $\|z_n - p\| = d(p, Tx_n)$, $\|u_n - p\| = d(Ty_n, p)$ and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences satisfying (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\lim_{n\to\infty} \beta_n = 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. Using the above iterative schemes, Panyanak [33], generalized the result proved in [37]. Song and Wang [41], observed that generating the Mann and Ishikawa sequences in [33] are in some sense dependent on the knowledge of the fixed point. Using the following Nadler's Lemma (see [30], Lemma 1.0 (1.1)).

Lemma 1.5. *let $A, B \in CB(X)$ and $a \in A$, if $\gamma > 0$ then there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \gamma, \tag{22}$$

they modified the iteration process due to Panyanak [33] and improved the results therein. They gave their iteration scheme as follows:

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n z_n \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{23}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ satisfy $\|z_n - u_n\| \leq H(Tx_n, Ty_n) + \gamma_n$, $\|z_{n+1} - u_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$ and $\{\alpha_n\}$,$\{\beta_n\}$ are real sequences in $[0, 1)$ satisfying $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. Using the above iteration, they proved the following theorem:

Theorem 1.6. *([41], Theorem 1) Let K be a nonempty compact convex subset of a uniformly convex Banach space X. Suppose that $T : K \to CB(K)$ is a multivalued nonexpansive mapping such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for all $p \in F(T)$. Then the Ishikawa sequence defined as above converges strongly to a fixed point of T.*

Shahzad and Zegeye [38] observed that if X is a normed space and $T : D(T) \subseteq E \to P(X)$ is a any multivalued mapping, then the mapping $P_T : D(T) \to P(X)$ defined for each x by

$$P_T(x) = \{y \in Tx : d(x, Tx) = \|x - y\|\}, \tag{24}$$

has the property that $P_T(q) = \{q\}$ for all $q \in F(T)$. Using this idea, they removed the strong condition "$T(p) = \{p\}$ for all $p \in F(T)$" introduced by Song and Wang [41].

Recently, Khan and Yildirim [25] introduced a new iteration scheme for multivalued nonexpansive mappings using the idea of the iteration scheme for single valued nearly asymptotically nonexpansive mapping introduced by Agarwal et.al [2] as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\lambda)v_n + \lambda u_n \\ y_n = (1-\eta)x_n + \eta v_n, \forall n \in \mathbb{N}. \end{cases} \tag{25}$$

where $v_n \in P_T(x_n)$, $u_n \in P_T(y_n)$ and $\lambda \in [0,1)$. Also, using a lemma in Schu [40], the idea of removal of the condition "$T(p) = \{p\}$ for all $p \in F(T)$" introduced by Shahzad and Zegeye [38] and method of direct construction of Cauchy sequence as indicated by Song and Cho [42], they stated the following theorem:

Theorem 1.7. *([25],Theorem 1) Let X be a uniformly convex Banach space satisfying Opial's condition and K a nonempty closed convex subset of X. Let $T : K \to P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and P_T is a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined in (25). Let $(I - P_T)$ be demiclosed with respect to zero, then $\{x_n\}$ converges weakly to a point of $F(T)$.*

In 2016, Isiogugu et al. [16], introduced the "type-one" condition which guarantees a weak convergence of the Mann iteration schemes to a fixed point of multi-valued quasi-nonexpansive mappings without imposing the condition that the fixed point set of T is strict in Hilbert spaces. They obtained the following results.

Proposition 1.8. *([16]) Let H be a real Hilbert space. Let C be a nonempty weakly closed subset of H. Let $T : C \subseteq H \to P(H)$ be a multi-valued mapping from C into the collection of all proximinal subsets of H. Suppose that T is a nonexpansive mapping which is of type-one. Then whenever $\{g_n\}_{n=1}^{\infty} \subseteq C$ is such that $\{g_n\}$ weakly converges to p and a sequence $\{h_n\}$ with $h_n \in Tg_n$ and $\|g_n - h_n\| = d(g_n, Tg_n)$ for all $n \in \mathbb{N}$ such that $\{g_n - h_n\}$ strongly converges to 0. Then $0 \in (I - T)p$ (i.e., $p = v$ for some $v \in Tp$).*

Theorem 1.9. *([16]) Let C be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $T : C \to P(C)$ is of type-one and nonexpansive mapping from C into the collection of all proximinal subsets of C such that $F(T) \neq \emptyset$. Then the Mann type sequence defined by*

$$g_{n+1} = (1 - \mu_n)g_n + \mu_n h_n, \tag{26}$$

weakly converges to $q \in F(T)$, where $h_n \in Tg_n$ with $\|g_n - h_n\| = d(g_n, Tg_n)$ and $\mu_n \subseteq (0,1)$ satisfying: $\mu_n \to \mu \in (0,1)$.

Theorem 1.10. *([16]) Let C be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $T : C \to P(C)$ is of type-one and quasi-nonexpansive mapping from C into the collection of all proximinal subsets of C. If $(I - T)$ satisfies Proposition 1.8, then the Mann type sequence defined by*

$$g_{n+1} = (1 - \mu_n)g_n + \mu_n h_n, \tag{27}$$

weakly converges to $q \in F(T)$, where $h_n \in Tg_n$ with $\|g_n - h_n\| = d(g_n, Tg_n)$ and $\mu_n \subseteq (0,1)$ satisfies: (i) $\mu_n \to \mu \in (0,1)$.

In [19], the above results were extended to the more general class of Multi-valued pseudocontactive mappings of Chidume et. al [9]. The following results were obtained:

Proposition 1.11. *([19]) Let H be a real Hilbert space. Let C be a nonempty weakly closed subset of H. Let $T : C \subseteq H \to P(H)$ be a multi-valued mapping from C into the collection of all nonempty proximinal subsets of H. Suppose that T is a k-strictly pseudocontractive mapping and of type-one. Then $(I - T)$ is demiclosed at zero (i.e., the graph of $I - T$ is closed at zero in $\sigma(H, H^*) \times (H, \|.\|)$ or weakly demiclosed at zero).*

Theorem 1.12. *([19]) Let C be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $T : C \to P(C)$ is k-strictly pseudocontractive mapping from C into the collection of all proximinal subsets of C with $k \in (0,1)$ such that $F(T) \neq \emptyset$. If T is of type-one, then the Mann-type sequence defined by*

$$g_{n+1} = (1 - \mu_n)g_n + \mu_n h_n, \tag{28}$$

weakly converges to $q \in F(T)$*, where* $h_n \in Tg_n$ *with* $\|g_n - h_n\| = d(g_n, Tg_n)$ *and* $\mu_n \subseteq (0,1)$ *satisfies: (i)* $\mu_n \to \mu < 1-k$*; (ii)* $\mu > 0$*; (iii)* $\sum_{n=1}^{\infty} \mu_n(1-\mu_n) = \infty$.

Theorem 1.13. *([19]) Let C be a nonempty closed and convex subset of a real Hilbert space X. Suppose that* $T : C \to P(C)$ *is of type-one and L-Lipschitzian pseudocontractive mapping from C into the collection of all proximinal subsets of C such that* $F(T) \neq \emptyset$*. Suppose T satisfies condition (1). Then the Ishikawa sequence defined by*

$$\begin{cases} h_n = (1-\xi_n)g_n + \xi_n u_n \\ g_{n+1} = (1-\mu_n)g_n + \mu_n w_n, \end{cases} \tag{29}$$

strongly converges to $p \in F(T)$*, where* $u_n \in Tg_n$ *with* $\|g_n - u_n\| = d(g_n, Tg_n)$*,* $w_n \in Th_n$ *with* $\|h_n - w_n\| = d(h_n, Th_n)$ *and* $\{\mu_n\}$ *and* $\{\xi_n\}$ *are real sequences satisfying (i)* $0 \leq \mu_n \leq \xi_n < 1$*; (ii)* $\liminf_{n\to\infty} \mu_n = \mu > 0$*; (iii)* $\sup_{n\geq 1} \xi_n \leq \xi \leq \frac{1}{\sqrt{1+L^2}+1}$.

Although authors have made significant improvement towards the study of multi-valued mappings and the iteration schemes for the approximation of their fixed points, they are still confronted with many unresolved challenges due to the complicated nature of multi-valued mappings. For instance, important properties such as Lipschitzian, demiclosedness, as well as the property of the set of fixed points being closed and convex could not be established for some of these classes of multi-valued mappings. Also, there is imposition of strict set of fixed points condition on the multi-valued mapping. The summary of the main challenges are given below.

$\mathbf{P_1}$. Type-one condition on the class of *k*-strictly pseudocontractive of Chidume et al. [9], guarantees the possession of Lipschitzian and demiclosedness properties as well as closed and convex set of fixed points (respectively, closed and convex set of strict fixed points) by the members of this class. Type-one condition also gurantees the convergence of the Mann iterations to the fixed points of the mappings without the imposition of strict set of fixed points. Unfortunately, it has not been established that this class of mappings properly contains the class of multi-valued nonexpansive mappings and the class of single-valued strictly pseudocontractive mappings of Browder and Petryshyn [7].

$\mathbf{P_2}$. The class of *k*-strictly pseudocontractive mappings in the sense of Isiogugu [17] properly contains the class of multi-valued nonexpansive mappings and the class of single-valued strictly pseudocontractive mappings of Browder and Petryshyn [7]. The mappings are Lipschitzian, type-one condition guarantees the possession of closed and convex set of strict fixed points but authors are yet to establish the possession of closed and convex set of fixed points as well as demiclosedness property by these mappings under type one condition.

Therefore, the purpose of this work is to first, introduce the new classes of extended multi-valued *pseudocontractive* mappings whose strictly pseudocontractive subclass of mappings will properly contain the classes of single valued strictly pseudocontractive mappings of Browder and Petryshyn [7] and multi-valued nonexpansive mappings. The type-one condition on the strictly pseudocontractive subclass will guarantee the possession of demiclosedness and Lipschitzian properties as well as closed and convex set of fixed point properties. Second, establish the relationship between these new classes of mappings and other classes of multi-valued pseudocontractive mappings which have been considered by authors. Third, established the possession of Lipschtizan and demiclosedness properties as well as closed and convex set of fixed points under type-one condition by the strictly pseudocontractive subclass. Fourth, introduce extended Mann and Ishikawa iteration schemes for approximating a common fixed point of a finite family mappings. Finally, prove some weak and strong convergence theorems for finite family of pseudocontractive and strictly pseudocontractive mappings using the extended Ishikawa and Mann iteration schemes, respectively. Thus, the results extend, complement and improve the results of the multi-valued and single-valued mappings in the contemporary literature.

2. Preliminaries

In the sequel, we shall need the following definitions and lemmas.

Definition 2.1. *(see e.g., [11, 12]) Let X be a Banach space. Let $T : D(T) \subseteq X \to 2^X$ be a multi-valued mapping. $I - T$ is said to be weakly demiclosed at zero if for any sequence $\{g_n\}_{n=1}^{\infty} \subseteq D(T)$ such that $\{g_n\}$ converges weakly to p and a sequence $\{h_n\}$ with $h_n \in Tg_n$ for all $n \in \mathbb{N}$ such that $\{g_n - h_n\}$ strongly converges to zero. Then $p \in Tp$ (i.e., $0 \in (I - T)p$).*

Definition 2.2. *A Banach X is said to satisfy Opial's condition [32], if whenever a sequence $\{g_n\}$ weakly converges to $g \in X$ then it is the case that*

$$\liminf \|g_n - g\| < \liminf \|g_n - h\|,$$

for all $h \in X$, $h \neq g$.

Definition 2.3. *([41]) A multi-valued mapping $T : C \to P(C)$ is said to satisfy condition (1) (see for example [41]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that*

$$d(g, Tg) \geq f(d(g, F(T))), \quad \forall g \in C.$$

Definition 2.4. *([16]) Let X be a normed space and $T : D(T) \subseteq X \to 2^X$ be a multi-valued mapping. T is said to be of type-one if given any pair $g, h \in D(T)$, then*

$$\|u - v\| \leq H(Tg, Th), \text{ for all } u \in P_T g, v \in P_T h.$$

Lemma 2.5. *([43]) Let $\{a_n\}$, $\{\xi_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 + \xi_n)a_n + \gamma_n, \quad n \geq n_0,$$

where n_0 is a nonnegative integer. If $\sum \xi_n < \infty$, $\sum \gamma_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.6. *Let H be a real Hilbert space and $\{g_n\}_{n=1}^{\infty}$ is a sequence in H which weakly converges to $z \in H$ then the following holds*

$$\limsup_{n\to\infty} \|g_n - h\|^2 = \limsup_{n\to\infty} \|g_n - z\|^2 + \|z - h\|^2, \ \forall\, h \in H.$$

Lemma 2.7. *[20] Let X be a metric space. If $A, B \in P(X)$ and $a \in A$. If $\gamma \geq 0$ then it is a simple consequence of the Hausdorff metric H that there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \gamma.$$

Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $\{T_i\}_{i=1}^{N}$, $N \geq 2$ is a finite family of mappings $T_i : K \to K$, in [21](see also [22]) the authors consider the horizontal iteration process generated from an arbitrary x_1 for the finite family of mappings $\{T^i\}_{i=1}^{N}$, using a finite family of the control sequences $\{\{\alpha_n^i\}_{n=1}^{\infty}\}_{i=1}^{N}$ as follows.

For N=2,

$$x_{n+1} = \alpha_n^1 x_n + (1 - \alpha_n^1)[\alpha_n^2 T_1 x_n + (1 - \alpha_n^2)T_2 x_n].$$

For N=3,

$$x_{n+1} = \alpha_n^1 x_n + (1 - \alpha_n^1)[\alpha_n^2 T_1 x_n + (1 - \alpha_n^2)[\alpha_n^3 T_2 x_n + (1 - \alpha_n^3)T_3 x_n]].$$

For an arbitrary but finite N≥ 2,

$$\begin{aligned} x_{n+1} &= \alpha_n^1 x_n + (1 - \alpha_n^1)[\alpha_n^2 T_1 x_n + (1 - \alpha_n^2)[\alpha_n^3 T_2 x_n \\ &\quad +(1 - \alpha_n^3)[...[\alpha_N T_{N-1} x_n + (1 - \alpha_N)T_N x_n]...]]] \\ &= \alpha_n^1 x_n + \sum_{i=2}^{N} \alpha_n^i \prod_{j=1}^{i-1}(1 - \alpha_n^j)T_{i-1} x_n + \prod_{j=1}^{N}(1 - \alpha_n^j)T_N x_n, \ n \geq 1. \end{aligned}$$

The proofs of the following lemmas (Lemmas 2.7, 2.9 and 2.10) are given in [18], however, we reproduce the proofs here for avoidance of doubt.

Lemma 2.8. *Let $\{\alpha_i\}_{i=1}^N$ be a countable subset of the set of real numbers $\mathbb{R}$, where $N \geq 2$ is an arbitrary integer. Then, the following holds.*

$$\alpha_1 + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \prod_{j=1}^{N}(1-\alpha_j) = 1. \tag{30}$$

Proof. . For $N = 2$,

$$\begin{aligned}
\alpha_1 + \sum_{i=2}^{2} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \prod_{j=1}^{2}(1-\alpha_j) &= \alpha_1 + \alpha_2(1-\alpha_1) + (1-\alpha_1)(1-\alpha_2) \\
&= \alpha_1 + (1-\alpha_1)[\alpha_2 + (1-\alpha_2)] \\
&= \alpha_1 + (1-\alpha_1) = 1.
\end{aligned}$$

We assume it is true for N and prove for N+1.

$$\begin{aligned}
\alpha_1 + \sum_{i=2}^{N+1} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \prod_{j=1}^{N+1}(1-\alpha_j) &= \alpha_1 + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \alpha_{N+1}\prod_{j=1}^{N}(1-\alpha_j) + \prod_{j=1}^{N+1}(1-\alpha_j), \\
&= \alpha_1 + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \prod_{j=1}^{N}(1-\alpha_j)[\alpha_{N+1} + (1-\alpha_{N+1})], \\
&= \alpha_1 + \sum_{i=2}^{N} \alpha_i \prod_{j=1}^{i-1}(1-\alpha_j) + \prod_{j=1}^{N}(1-\alpha_j) \\
&= 1.
\end{aligned}$$

□

Remark 2.9. *Lemma 2.8 holds if $\{\alpha_i\}_{i=1}^N$ is replaced with $\{\alpha_i\}_{i=0}^N$, and $N \geq 2$ is replaced with $N \geq 1$.*

Lemma 2.10. *Let $\{\alpha_i\}_{i=k}^N$ be a countable subset of the set of real numbers $\mathbb{R}$, where k is a fixed non negative integer and $N \in \mathbb{N}$ is any integer with $k+1 \leq N$. Then the following holds,*

$$\alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j) + \prod_{j=k}^{N}(1-\alpha_j) = 1. \tag{31}$$

Proof. . For $k = 0$ and $k = 1$, the proofs follow from Remark 2.9 and Lemma 2.8, respectively. We assume it is true for k and N. Now for k and $N+1$,

$$\begin{aligned}
\alpha_k + \sum_{i=k+1}^{N+1} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j) + \prod_{j=k}^{N+1}(1-\alpha_j) &= \alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j) + \alpha_{N+1}\prod_{j=k}^{N}(1-\alpha_j) + \prod_{j=k}^{N+1}(1-\alpha_j), \\
&= \alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j) + \prod_{j=k}^{N}(1-\alpha_j)[\alpha_{N+1} + (1-\alpha_{N+1})], \\
&= \alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j) + \prod_{j=k}^{N}(1-\alpha_j) = 1.
\end{aligned}$$

□

Lemma 2.11. *Let t, u and v be arbitrary elements of a real Hilbert space H. Let k be a fixed non-negative integer and $N \in \mathbb{N}$ be such that $k+1 \leq N$. Let $\{v_i\}_{i=k}^{N-1} \subseteq H$ and $\{\alpha_i\}_{i=k}^{N} \subseteq [0,1]$ be a countable finite subset of H and $\mathbb{R}$, respectively. Define*

$$y = \alpha_k t + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=k}^{N}(1-\alpha_j)v.$$

Then,

$$\begin{aligned}
\|y-u\|^2 \;=\;& \alpha_k\|t-u\|^2 + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2 + \prod_{j=k}^{N}(1-\alpha_j)\|v-u\|^2 \\
& -\alpha_k\Big[\sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2 + \prod_{j=k}^{N}(1-\alpha_j)\|t-v\|^2\Big] \\
& -(1-\alpha_k)\Big[\sum_{i=k+1}^{N-1} \alpha_i \prod_{j=k}^{i}(1-\alpha_j)\|v_{i-1} - [\alpha_{i+1}v_i + w_{i+1}]\|^2 \\
& +\alpha_N \prod_{j=k}^{N}(1-\alpha_j)\|v - v_{N-1}\|^2\Big],
\end{aligned}$$

where $w_k = \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=k}^{N}(1-\alpha_j)v$, $k = 1, 2, ..., N-1$ and $w_N = (1-\alpha_N)v$.

Proof. . Observe that for $k \leq N-1$, $w_k = (1-\alpha_k)[\alpha_{k+1}v_k + w_{k+1}]$. Consequently, using the well known identity:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,$$

which holds for all $x, y \in H$ and for all $t \in [0,1]$, we obtain that

$$\begin{aligned}
\|y-u\|^2 \;=\;& \|\alpha_k t + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=k}^{N}(1-\alpha_j)v - u\|^2, \\
=\;& \|\alpha_k t + w_k - u\|^2 \\
=\;& \|\alpha_k t + (1-\alpha_k)[\alpha_{k+1}v_k + w_{k+1}] - u]\|^2 \\
=\;& \alpha_k\|t-u\|^2 + (1-\alpha_k)\|\alpha_{k+1}v_k + w_{k+1} - u\|^2 \\
& -\alpha_k(1-\alpha_k)\|t - [\alpha_{k+1}v_k + w_{k+1}]\|^2 \\
=\;& \alpha_k\|t-u\|^2 + (1-\alpha_k)[\alpha_{k+1}\|v_k - u\|^2 \\
& +(1-\alpha_{k+1})\|\alpha_{k+2}v_{k+1} + w_{k+2} - u\|^2 \\
& -\alpha_{k+1}(1-\alpha_{k+1})\|v_k - [\alpha_{k+2}v_{k+1} + w_{k+2}]\|^2] \\
& -\alpha_k(1-\alpha_k)[\alpha_{k+1}\|t-v_k\|^2 + (1-\alpha_{k+1})\|t - [\alpha_{k+2}v_{k+1} + w_{k+2}]\|^2 \\
& -\alpha_{k+1}(1-\alpha_{k+1})\|v_k - [\alpha_{k+2}v_{k+1} + w_{k+2}]\|^2] \\
=\;& \alpha_k\|t-u\|^2 + (1-\alpha_k)\alpha_{k+1}\|v_k - u\|^2 \\
& +(1-\alpha_k)(1-\alpha_{k+1})\|\alpha_{k+2}v_{k+1} + w_{k+2} - u\|^2 \\
& -(1-\alpha_k)\alpha_{k+1}(1-\alpha_{k+1})\|v_k - [\alpha_{k+2}v_{k+1} + w_{k+2}]\|^2 \\
& -\alpha_k(1-\alpha_k)\alpha_{k+1}\|t-v_k\|^2 \\
& -\alpha_k(1-\alpha_k)(1-\alpha_{k+1})\|t - [\alpha_{k+2}v_{k+1} + w_{k+2}]\|^2 \\
& +\alpha_k(1-\alpha_k)\alpha_{k+1}(1-\alpha_{k+1})\|v_k - [\alpha_{k+2}v_{k+1} + w_{k+2}]\|^2]
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_k\|t-u\|^2 + (1-\alpha_k)\alpha_{k+1}\|v_k-u\|^2 - \alpha_k(1-\alpha_k)\alpha_{k+1}\|t-v_k\|^2 \\
&\quad +(1-\alpha_k)(1-\alpha_{k+1})\|\alpha_{k+2}v_{k+1}+w_{k+2}-u\|^2 \\
&\quad -\alpha_k(1-\alpha_k)(1-\alpha_{k+1})\|t-[\alpha_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&\quad -\alpha_{k+1}(1-\alpha_k)(1-\alpha_{k+1})(1-\alpha_k)\|v_k-[\alpha_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&= \alpha_k\|t-u\|^2 + (1-\alpha_k)\alpha_{k+1}\|v_k-u\|^2 - \alpha_k(1-\alpha_k)\alpha_{k+1}\|t-v_k\|^2 \\
&\quad -\alpha_{k+1}(1-\alpha_k)(1-\alpha_{k+1})(1-\alpha_k)\|v_k-[\alpha_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&\quad +(1-\alpha_k)(1-\alpha_{k+1})\|\alpha_{k+2}v_{k+1}+w_{k+2}-u\|^2 \\
&\quad -\alpha_k(1-\alpha_k)(1-\alpha_{k+1})\|t-[\alpha_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&= \alpha_k\|t-u\|^2 + (1-\alpha_k)\alpha_{k+1}\|v_k-u\|^2 - \alpha_k(1-\alpha_k)\alpha_{k+1}\|t-v_k\|^2 \\
&\quad -\alpha_{k+1}(1-\alpha_k)(1-\alpha_{k+1})(1-\alpha_k)\|v_k-[\alpha_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&\quad +(1-\alpha_k)(1-\alpha_{k+1})\|\alpha_{k+2}v_{k+1}+(1-\alpha_{k+2})[\alpha_{k+3}v_{k+2}+w_{k+3}]-u\|^2 \\
&\quad -\alpha_k(1-\alpha_k)(1-\alpha_{k+1})\|\alpha_{k+2}v_{k+1}+(1-\alpha_{k+2})[\alpha_{k+3}v_{k+2}+w_{k+3}]-t\|^2 \\
&= \alpha_k\|t-u\|^2 + (1-\alpha_n^1)\alpha_{k+1}\|v_k-u\|^2 - \alpha_n^1(1-\alpha_n^1)\alpha_{k+1}\|t-v_k\|^2 \\
&\quad +(1-\alpha_k)(1-\alpha_{k+1})\alpha_{k+2}\|v_{k+1}-u\|^2 \\
&\quad +(1-\alpha_k)(1-\alpha_{k+1})(1-\alpha_{k+2})\|\alpha_{k+3}v_{k+2}+w_{k+3}-u\|^2 \\
&\quad -(1-\alpha_k)(1-\alpha_{k+1})\alpha_{k+2}(1-\alpha_{k+2})\|v_{k+1}-[\alpha_{k+3}v_{k+2}+w_{k+3}]\|^2 \\
&\quad -\alpha_k(1-\alpha_k)(1-\alpha_{k+1})\alpha_{k+2}\|v_{k+1}-t\|^2 \\
&\quad -\alpha_k(1-\alpha_k)(1-\alpha_{k+1})(1-\alpha_{k+2})\|\alpha_{k+3}v_{k+2}+w_{k+3}-t\|^2 \\
&\quad +\alpha_k(1-\alpha_k)(1-\alpha_{k+1})\alpha_{k+2}(1-\alpha_{k+2}\|v_{k+1}-\alpha_{k+3}v_{k+2}+w_{k+3}\|^2 \\
&\quad -\alpha_{k+1}(1-\alpha_k)(1-\alpha_{k+1})(1-\alpha_k)\|v_k-[\alpha_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&= \alpha_k\|t-u\|^2 + (1-\alpha_k)\alpha_{k+1}\|v_k-u\|^2 - \alpha_k(1-\alpha_k)\alpha_{k+1}\|t-v_k\|^2 \\
&\quad +(1-\alpha_k)(1-\alpha_{k+1})\alpha_{k+2}\|v_{k+1}-u\|^2 \\
&\quad +(1-\alpha_k)(1-\alpha_{k+1})(1-\alpha_{k+2})\|\alpha_{k+3}v_{k+2}+w_{k+3}-u\|^2 \\
&\quad -\alpha_k(1-\alpha_k)(1-\alpha_{k+1})\alpha_{k+2}\|v_{k+1}-t\|^2 \\
&\quad -\alpha_k(1-\alpha_k)(1-\alpha_{k+1})(1-\alpha_{k+2})\|\alpha_{k+3}v_{k+2}+w_{k+3}-t\|^2 \\
&\quad -\alpha_{k+1}(1-\alpha_k)(1-\alpha_{k+1})(1-\alpha_k)\|v_k-[\alpha_{k+2}v_{k+1}+w_{k+2}]\|^2 \\
&\quad -(1-\alpha_k)^2(1-\alpha_{k+1})\alpha_{k+2}(1-\alpha_{k+2})\|v_{k+1}-[\alpha_{k+3}v_{k+2}+w_{k+3}]\|^2 \\
&\;\vdots \\
&= \alpha_k\|t-u\|^2 + \sum_{i=k+1}^{k+2}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2 \\
&\quad -\alpha_k\Big[\sum_{i=k+1}^{k+2}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2\Big] \\
&\quad -(1-\alpha_k)\Big[\sum_{i=k+1}^{k+2}\alpha_i\prod_{j=k}^{i}(1-\alpha_j)\|v_{i-1}-[\alpha_{i+1}v_i+w_{i+1}]\|^2 \\
&\quad +\prod_{j=k}^{k+2}(1-\alpha_j)\|[\alpha_{k+3}v_{k+2}+w_{k+3}]-u\|^2 \\
&\quad -\alpha_k\prod_{j=k}^{k+2}(1-\alpha_j)\|t-[\alpha_{k+3}v_{k+2}+w_{k+3}]\|^2
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \alpha_k\|t-u\|^2 + \sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2 + \prod_{j=k}^{N}(1-\alpha_j)\|v-u\|^2 \\
& -\alpha_k\Big[\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2 + \prod_{j=k}^{N}(1-\alpha_j)\|t-v\|^2\Big] \\
& -(1-\alpha_k)\Big[\sum_{i=k+1}^{N-1}\alpha_i\prod_{j=k}^{i}(1-\alpha_j)\|v_{i-1}-[\alpha_{i+1}v_i + w_{i+1}]\|^2 \\
& +\alpha_N\prod_{j=k}^{N}(1-\alpha_j)\|v-v_{N-1}\|^2\Big].
\end{aligned}
$$

□

3. Main Results

We now present the following results:

Proposition 3.1. *Let H be a real Hilbert space and $T : D(T) \subseteq H \to P(H)$ be a multi-valued $L-$Lipschtizian mapping, then, fixed point set of T is closed.*

Proof. . Let $\{x_n\}_{n=1}^{\infty} \subseteq F(T)$ such that $x_n \to x^*$. Then,

$$
\begin{aligned}
d^2(x^*, Tx^*) &\leq d(x^*, x_n) + d(x_n, Tx_n) + H(Tx_n, Tx^*) \\
&= \|x^* - x_n\| + H(Tx_n, Tx^*) \\
&\leq (1+L)\|x_n - x^*\| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Therefore, $d(x^*, Tx^*) = 0$. Since T is proximinal, there exist $v \in Tx^*$ such that $\|x^* - v\| = d(x^*, Tx^*) = 0$. Consequently, $x^* \in Tx^*$. □

Definition 3.2. *Let $\{T_j\}_{j=1}^{N}$ be a finite collection of mappings such that $\bigcap_{j=1}^{N} F(T_j) \neq \emptyset$. $T_1, T2, ..., T_j, T_{j+1}, ..., T_N$ are said to satisfy condition 1 uniformly, if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that*

$$d(g, T_j g) \geq f(d(g, \bigcap_{j=1}^{N} F(T_j)), \quad \forall g \in C.$$

Definition 3.3. *Let $(X, \|.\|)$ be a normed inner product space and $T : D(T) \subseteq X \to P(X)$ be a multi-valued mapping. T is said to be an extended k-strictly pseudocontactive mapping if there exists $k \in [0,1)$ such that given any pair $g, h \in D(T)$, then for all $u \in P_T g, v \in P_T h$, we have*

$$H^2(Tg, Th) \leq \|g-h\|^2 + k\|g - u - (h - v)\|^2. \tag{32}$$

If $k = 1$ in (32), then T is called an extended pseudocontactive mapping. T is called extended nonexpansive if $k = 0$.

Clearly,
(i) every multi-valued nonexpansive mapping is an extended k-strictly pseudocontactive.
(ii) every single-valued k-strictly pseudocontactive (respectively, pseudocontractive) mapping of [7](respectively, [14]) is an extended k-strictly pseudocontractive (respectively, pseudocontractive) mapping.
(iii) every multi-valued k-strictly pseudocontactive (respectively pseudocontractive) mapping of [9] is an extended k-strictly pseudocontactive (respectively pseudocontractive) mapping.

The following example shows that the class of extended multi-valued k-strictly pseudocontactive (respectively pseudocontractive) mappings properly generalises the class of multi-valued nonexpansive mappings, single-valued k-strictly pseudocontactive mappings of [7] and multi-valued k-strictly pseudocontactive mappings of [9], while extended multi-valued pseudocontactive properly extends the classes of single-valued pseudocontactive mappings of [14] and multi-valued pseudocontactive mappings of [9].

Example 3.4. *([20], Example 3.4) Let $X = \mathbb{R}$ (the reals with usual metric). Define $T : [0, \infty) \subseteq (\mathbb{R}) \to P(\mathbb{R})$ by*

$$Tg = [-\frac{5g}{2}, -2g]. \tag{33}$$

Then given any pair $g, h \in D(T)$

$$\begin{aligned} H^2(Tg, Th) &= \frac{25}{4}|g-h|^2 \leq |g-h|^2 + \frac{7}{12}|g-u-(h-v)|^2 \\ &\leq |g-h|^2 + |g-u-(h-v)|^2, \end{aligned}$$

for all $u \in P_T g, v \in P_T h$. Hence, T is not a multi-valued nonexpansive mapping but it is an extended multi-valued k-strictly pseudocontactive mapping, consiquently, an extended pseudocontractive mapping. However, it was shown in [20] that for $x = 3$, $y = 2$, if we choose $u = -6 \in Tx$ and $v = -5 \in Ty$ then $H^2(Tx, Ty) = \frac{25}{4}$ and $|x-y|^2 + |x-u-(y-v)|^2 = 5$. Consequently, $H^2(Tx, Ty) > |x-y|^2 + |x-u-(y-v)|^2$. Thus, T is not pseudocontactive of [9]. Therefore, T is neither k-strictly pseudocontractive nor pseudocontractive mapping which were considered in [9].

The following example shows that extended pseudocontractive condition on a multi-valued mapping T does not imply that the fixed point set of T is strict.

Example 3.5. *Let $X = \mathbb{R}$ (the reals with usual metric). Define $T : [0, \infty) \subseteq (\mathbb{R}) \to P(\mathbb{R})$ by*

$$Tg = [-\frac{5g}{2} - 1, -2g]. \tag{34}$$

Clearly, T is an extended pseudocontractive-type mapping whose set of fixed points $F(T)$ is not empty, but its set of strict fixed points $F_s(T)$ is empty. Observe also that T has type-one property.

Remark 3.6. *(i) The relationship between the class of extended multi-valued pseudocontractive mappings and the class of multi-valued pseudocontractive-type mappings are yet to be established.*

Definition 3.7. *Let $(X, \|.\|)$ be a normed space and $T : D(T) \subseteq X \to P(X)$ be a multi-valued mapping. T is said to be an extended demicontactive mapping if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that given any pair $g \in D(T)$ and $p \in F(T)$ then for all $u \in P_T g$, we have*

$$H^2(Tg, Tp) \leq \|g-p\|^2 + k\|g-u\|^2. \tag{35}$$

If $k = 1$ in (35), then T is called an extended hemicontactive mapping. T is called extended quasi-nonexpansive if $k = 0$.

Clearly, if $F(T) \neq \emptyset$, for a extended strictly pseudocontractive (respectively, pseudocontractive) mapping T, then T is extended demicontractive (respectively, hemicontractive) mappings. Also, demicontractive (respectively, hemicontractive) mappings in the sense of [8] is an extended demicontractive (respectively, hemicontractive) mapping because,

$$\begin{aligned} H^2(Tg, Tp) &\leq \|g-p\|^2 + kd^2(g, u), \ \forall\, u \in Tg,\ p \in F(T) \\ &\leq \|g-p\|^2 + k\|g-u\|^2, \ \forall\, u \in P_T g,\ p \in F(T). \end{aligned} \tag{36}$$

Obviously, Examples 3.4 and 3.5 are both extended demicontractive and hemicontractive mappings.
We now obtain a demiclosedness property in the sense that if $\{g_n\}_{n=1}^{\infty} \subseteq C$ is such that $\{g_n\}$ weakly converges to p and a sequence $\{h_n\}$ with $d(g_n, h_n) = d(g_n, Tg_n)$ for all $n \in \mathbb{N}$ such that $\{g_n - h_n\}$ strongly converges to 0, then $0 \in (I - T)p$ (i.e., $p = v$ for some $v \in Tp$).

Proposition 3.8. *Let H be a real Hilbert space. Let C be a nonempty weakly closed subset of H. Let $T : C \subseteq H \to P(H)$ be a multi-valued mapping from C into the collection of all nonempty proximinal subsets of H. Suppose that T is an extended k-strictly pseudocontractive mapping with type-one property. Then $(I - T)$ is demiclosed at zero (i.e., the graph of $I - T$ is closed at zero in $\sigma(H, H^*) \times (H, \|.\|)$ or weakly demiclosed at zero), where I denotes the identity on H, $\sigma(H, H^*)$ the weak topology, $(H, \|.\|)$ the norm (or strong) topology.*

Proof. . Let $\{g_n\}_{n=1}^{\infty} \subseteq C$ be such that $\{g_n\}$ weakly converges to p and a sequence $\{h_n\}$ with $\|g_n - h_n\| = d(g_n, Tg_n)$ for all $n \in \mathbb{N}$ such that $\{g_n - h_n\}$ strongly converges to 0. We prove that $0 \in (I - T)p$ (i.e., $p = v$ for some $v \in Tp$). Since $\{g_n\}_{n=1}^{\infty}$ converges weakly, it is bounded. Let $q \in Tp$ with $\|p - q\| = d(p, Tp)$. From type-one property and the definition of extended k-strictly pseudocontractive, for each $n \in \mathbb{N}$, we have that

$$\|h_n - q\| \leq H(Tg_n, Tp), \tag{37}$$

and

$$H^2(Tg_n, Tp) \leq \|g_n - p\|^2 + k\|g_n - h_n - (p - q)\|^2. \tag{38}$$

Thus, for each $g \in H$ define $f : H \to [0, \infty)$ by

$$f(g) := \limsup_{n\to\infty} \|g_n - g\|^2.$$

Then from Lemma 2.6, we obtain

$$f(g) = \limsup_{n\to\infty} \|g_n - p\|^2 + \|p - g\|^2 \ \forall g \in H.$$

Thus

$$f(g) = f(p) + \|p - g\|^2 \ \ \forall g \in H.$$

Therefore,

$$f(q) = f(p) + \|p - q\|^2. \tag{39}$$

Observe also that

$$\begin{aligned} f(q) &= \limsup_{n\to\infty} \|g_n - q\|^2 \\ &= \limsup_{n\to\infty} \|g_n - h_n + (h_n - q)\|^2 \\ &= \limsup_{n\to\infty} \|h_n - q\|^2 \\ &\leq \limsup_{n\to\infty} H^2(Tg_n, Tp) \\ &\leq \limsup_{n\to\infty}\Big[\|g_n - p\|^2 + k\|g_n - h_n - (p - q)\|^2\Big] \\ &= \limsup_{n\to\infty} \|g_n - p\|^2 + k\|(p - q)\|^2 \\ &= f(p) + k\|p - q\|^2. \end{aligned} \tag{40}$$

Hence, it follows from (39) and (40) that $(1 - k)\|p - q\|^2 = 0$. Therefore, $p = q \in Tp$. □

Proposition 3.9. *Let C be a nonempty subset of a real Hilbert space H. And let $T : C \to P(C)$ be an extended k-strictly pseudocontractive mapping with type-one property such that $F(T)$ is nonempty. Then*
(a) $F(T)$ is closed ; (b) $F(T)$ is convex.

Proof. . (a) Let $\{g_n\}_{n=1}^{\infty} \subseteq F(T)$ such that $\{g_n\}_{n=1}^{\infty}$ converges to $g \in C$. We show that $g \in F(T)$. Let $u \in P_T g$ be arbitrary,

$$\begin{aligned} \|g-u\| &\leq \|g-g_n\| + \|g_n - u\| \\ &\leq \|g-g_n\| + H(Tg_n, Tg) \\ &\leq \|g-g_n\| + \|g-g_n\| + \sqrt{k}\|g-u\|. \end{aligned}$$

Taking limits as $n \to \infty$, we have that $\|g-u\| \leq \sqrt{k}\|g-u\|$. Hence $g \in Tg$
(b) We now prove that $F(T)$ is convex. Let $p_1, p_2 \in F(T)$ and $z = \alpha p_1 + (1-\alpha)p_2$ then $z - p_1 = (1-\alpha)(p_2 - p_1)$ and $z - p_2 = \alpha(p_1 - p_2)$.

$$\begin{aligned} d^2(z, Tz) &\leq \|z-u\|^2, \ \ \forall u \in Tz \\ &= \|\alpha p_1 + (1-\alpha)p_2 - u\|^2 \\ &= \alpha\|p_1 - u\|^2 + (1-\alpha)\|p_2 - u\|^2 - \alpha(1-\alpha)\|p_2 - p_1\|^2. \end{aligned}$$

In particular, for $u \in P_T z$, extended k-strictly pseudocontractive condition on T implies that

$$\begin{aligned} d^2(z, Tz) &\leq \alpha H^2(Tz, Tp_1) + (1-\alpha)H^2(Tz, Tp_2) - \alpha(1-\alpha)\|p_1 - p_2\|^2 \\ &\leq \alpha[\|z-p_1\|^2 + k\|z-u\|^2] + (1-\alpha)[\|z-p_2\|^2 + k\|z-u\|^2] \\ &\quad -\alpha(1-\alpha)\|p_1 - p_2\|^2 \\ &= \alpha[\|z-p_1\|^2 + kd^2(z, Tz)] + (1-\alpha)[\|z-p_2\|^2 + kd^2(z, Tz)] \\ &\quad -\alpha(1-\alpha)\|p_1 - p_2\|^2 \\ &= \|\alpha p_1 + (1-\alpha)p_2 - z\|^2 + kd(z, Tz) = kd(z, Tz). \end{aligned}$$

Hence, $z \in Tz$. □

Proposition 3.10. *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \subseteq H \to CC(C)$ be an extended multi-valued, $L-$Lipschtizian pseudocontractive mapping which is also of type-one. If the set of fixed points $F(T)$ of T is nonempty, then, it is convex.*

Proof. . Let $p_1, p_2 \in F(T)$, we prove that $p = \lambda p_1 + (1-\lambda)p_2 \in F(T)$. For each $x \in D(T)$, let $T_\beta x = T[(1-\beta)x + \beta u_x]$, where $u_x \in Tx$ with $d(x, Tx) = \|x - u_x\|$ (i.e, $u_x \in P_T x$) and $\beta \in (0, \frac{1}{\sqrt{1+L^2}+1})$. Clearly, $T_\beta x$ is well defined since u_x is unique (because Tx is closed and convex subset of a real Hilbert space) and C is convex. Also if $p^* \in F(T)$, then, $T_\beta p^* = Tp^*$. Observe that for any $u_{\beta x} \in T_\beta x = T[(1-\beta)x + \beta u_x]$, given any $p^* \in F(T)$, then $H(T[(1-\beta)x + \beta u_x], Tp^*) = H(T_\beta x, Tp^*)$. Hence, by type-one condition and the definition of extended pseudocontractive mapping, we have

$$\begin{aligned} \|u_{\beta x} - p^*\|^2 &\leq H^2(T[(1-\beta)x + \beta u_x], Tp^*) = H^2(T_\beta x, Tp^*) \\ &\leq \|(1-\beta)x + \beta u_x - p^*\|^2 + \|((1-\beta)x + \beta u_x) - u_{\beta x}\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \|u_x - p^*\|^2 &\leq H^2(Tx, Tp^*) \\ &\leq \|x - p^*\|^2 + \|x - u_x\|^2. \end{aligned}$$

It follows that for the pair $p, (1-\beta)p + \beta u_p$ and $u_p \in P_T p$, $u_{\beta p} \in P_{T_\beta} p = P_T[(1-\beta)p + \beta u_p]$, we have that $\|u_p - u_{\beta p}\| \leq H(Tp, T_\beta p)$. Now,

$$\begin{aligned} d^2(p, T_\beta p) \leq \|p - u_{\beta p}\|^2 &= \|\lambda p_1 + (1-\lambda)p_2 - u_{\beta p}\|^2 \\ &= \|\lambda[p_1 - u_{\beta p}] + (1-\lambda)[p_2 - u_{\beta p}]\|^2 \\ &= \lambda\|p_1 - u_{\beta p}\|^2 + (1-\lambda)\|p_2 - u_{\beta p}\|^2 - \lambda(1-\lambda)\|p_1 - p_2\|^2. \end{aligned}$$

Also,

$$\begin{aligned}
d^2(p_1, T_\beta p) &\leq \|p_1 - u_{\beta p}\|^2 \leq H^2(Tp_1, T_\beta p)\\
&\leq \|[(1-\beta)p + \beta u_p] - p_1\|^2 + \|[(1-\beta)p + \beta u_p] - u_{\beta p}\|^2\\
&= \|(1-\beta)[p - p_1] + \beta[u_p - p_1]\|^2 + \|(1-\beta)[p - u_{\beta p}] + \beta[u_p - u_{\beta p}]\|^2\\
&= (1-\beta)\|p - p_1\|^2 + \beta\|u_p - p_1\|^2 - \beta(1-\beta)\|p - u_p\|^2\\
&\quad +(1-\beta)\|p - u_{\beta p}\|^2 + \beta\|u_p - u_{\beta p}\|^2 - \beta(1-\beta)\|p - u_p\|^2\\
&\leq (1-\beta)\|p - p_1\|^2 + \beta H^2(Tp, Tp_1) - \beta(1-\beta)\|p - u_p\|^2\\
&\quad +(1-\beta)\|p - u_{\beta p}\|^2 + \beta H^2(Tp, T_\beta p\|^2 - \beta(1-\beta)\|p - u_p\|^2\\
&\leq (1-\beta)\|p - p_1\|^2 + \beta[\|p - p_1\|^2 + \|p - u_p\|^2] - \beta(1-\beta)\|p - u_p\|^2.\\
&\quad +(1-\beta)\|p - u_{\beta p}\|^2 + \beta L^2\|p - [(1-\beta)p + \beta u_p]\|^2 - \beta(1-\beta)\|p - u_p\|^2\\
&\leq (1-\beta)\|p - p_1\|^2 + \beta[\|p - p_1\|^2 + \|p - u_p\|^2] - \beta(1-\beta)\|p - u_p\|^2\\
&\quad +(1-\beta)\|p - u_{\beta p}\|^2 + \beta L^2\beta^2\|p - u_p\|^2 - \beta(1-\beta)\|p - u_p\|^2\\
&= \|p - p_1\|^2 - \beta[1 - 2\beta - L^2\beta^2]\|p - u_p\|^2 + (1-\beta)\|p - u_{\beta p}\|^2\\
&\leq \|p - p_1\|^2 + (1-\beta)\|p - u_{\beta p}\|^2.
\end{aligned}$$

Similarly,

$$d^2(p_2, T_\beta p) \leq \|p_2 - u_{\beta p}\|^2 \leq \|p - p_2\|^2 + (1-\beta)\|p - u_{\beta p}\|^2.$$

Hence,

$$\begin{aligned}
\|p - u_{\beta p}\|^2 &\leq \lambda[\|p - p_1\|^2 + (1-\beta)\|p - u_{\beta p}\|^2]\\
&\quad +(1-\lambda)[\|p - p_2\|^2 + (1-\beta)\|p - u_{\beta p}\|^2]\\
&\quad -\lambda(1-\lambda)\|p_1 - p_2\|^2\\
&= \|\lambda p_1 + (1-\lambda)p_2 - p\|^2 + (1-\beta)\|p - u_{\beta p}\|^2\\
&= +(1-\beta)\|p - u_{\beta p}\|^2.
\end{aligned}$$

This implies that $0 \leq \beta\|p - u_{\beta p}\| \leq 0$. Since $\beta \in (0, \frac{1}{\sqrt{1+L^2}+1})$, we have that $\|p - u_{\beta p}\| = 0$. Observe that $d(p, T_\beta p) \leq \|p - u_{\beta p}\| = 0 \leq d(p, T_\beta p)$, therefore, $d(p, T_\beta p) = \|p - u_{\beta p}\| = 0$ and $p = u_{\beta p} \in T_\beta p$.

$$\begin{aligned}
d(p, Tp) \leq d(p, T_\beta p) + H(T_\beta p, Tp) &\leq L\|(1-\beta)p + \beta u_p - p\|\\
&= L\beta d(p, Tp).
\end{aligned}$$

Thus, $0 \leq (1 - \beta L)d(p, T_P) \leq 0$. Consequently, $d(p, Tp) = 0$ and proximinal property of T (because Tx is a closed and convex subset of a Hilbert space H) guarantees the existence of $u \in Tp$ such that $\|u - p\| = 0$. Hence, $p \in Tp$. □

Remark 3.11. *If $\alpha_k = \alpha$, $1 \leq k \leq N$, equation* (31) *in Lemma 2.10 becomes*

$$\alpha + \sum_{i=k+1}^{N} \alpha(1-\alpha)^{i-1} + (1-\alpha)^N = 1. \tag{41}$$

Consequently, given any sequence $\{\mu_n\}_{n=1}^{\infty}$, for each n, we have

$$\mu_n + \sum_{i=k+1}^{N} \mu_n(1-\mu_n)^{i-1} + (1-\mu_n)^N = 1. \tag{42}$$

Setting $g_{n+1} = y$, $g_n = t$, $p = u$, $k = 1$, $\{\alpha_{n,i}\}_{i=1}^N = \{\mu_n\}$ for all $i = 1, 2, ..., N$ and $h_{n,N} \in T_N g_n = v$ in Lemma 2.11 and interchanging the roll of μ and $(1-\mu)$, we obtain the Extended Mann Sequence for finite family of mappings $\{T_i\}_{i=1}^N$ defined by

$$g_{n+1} = (1-\mu_n)g_n + \sum_{i=2}^{N} \mu_n^{i-1}(1-\mu_n)h_{n,i-1} + \mu_n^N h_{n,N}. \tag{43}$$

Theorem 3.12. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Suppose for each $j = 1, 2, ..., N$, $T_j : C \to P(C)$ is an extended λ_j-strictly pseudocontractive mapping from C into the collection of all proximinal subsets of C with $\lambda_j \in (0,1)$. Assume that $\bigcap_{j=1}^{N} F(T_j) \neq \emptyset$, then, the extended Mann sequence for finite family of mappings defined by*

$$g_{n+1} = (1-\mu_n)g_n + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)h_{n,j-1} + \mu_n^N h_{n,N},$$

weakly converges to $q \in \bigcap_{j=1}^{N} F(T_j)$, where $h_{n,j} \in T_j g_n$ with $\|g_n - h_{n,j}\| = d(g_n, T_j g_n)$ and $\mu_n \subseteq (0,1)$ satisfies:
(i) $\mu_n \to \mu < 1 - \max \lambda_j$, $j = 1, 2, ..., N$; (ii) $1 > \mu > 0$; (iii) $(1-\mu_n) > \max\{\lambda_j\}_{j=1}^N$.

Proof. . Applying Remark 3.11 in Lemma 2.11

$$\begin{aligned}
\|g_{n+1} - p\|^2 &\leq (1-\mu_n)\|g_n - p\|^2 + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)\|h_{n,(j-1)} - p\|^2 \\
&\quad + \mu_n^N \|h_{n,N} - p\|^2 \\
&\quad -(1-\mu_n)\Big[\sum_{j=2}^{N} \mu^{j-1}(1-\mu_n)\|g_n - h_{n,j-1}\|^2 + (1-\mu_n)^N \|g_n - h_{n,N}\|^2\Big] \\
&= (1-\mu_n)\|g_n - p\|^2 + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)H^2(T_{j-1}g_n, T_{j-1}p) \\
&\quad + \mu_n^N H^2(T_N g_n, T_N p) \\
&\quad (1-\mu_n)\Big[\sum_{j=2}^{N} \mu^{j-1}(1-\mu_n)\|g_n - h_{n,j-1}\|^2 + (1-\mu_n)^N \|g_n - h_{n,N}\|^2\Big].
\end{aligned}$$

Applying extended λ_j-strictly pseudocontractive condition on each T_j, we obtain

$$\begin{aligned}
\|g_{n+1} - p\|^2 &\leq (1-\mu_n)\|g_n - p\|^2 + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)\Big[\|g_n - p\|^2 + \lambda_{j-1}\|g_n - h_{n,(j-1)}\|^2\Big] \\
&\quad + \mu_n^N \Big[\|g_n - p\|^2 + \lambda_N \|g_n - h_{n,N}\|^2\Big] \\
&\quad -(1-\mu_n)\Big[\sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)\|g_n - h_{n,j-1}\|^2 + \mu_n^N \|g_n - h_{n,N}\|^2\Big],
\end{aligned}$$

$$
\begin{aligned}
&= \Big[(1-\mu_n) + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_{n,j}) + \mu_n^N\Big]\|g_n - p\|^2 \\
&\quad - \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)[(1-\mu_n) - \lambda_{j-1}]\|g_n - h_{n,j-1}\|^2 \\
&\quad - \mu_n^N[(1-\mu_n) - \lambda_N]\|g_n - h_{n,N}\|^2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|g_{n+1} - p\|^2 &\leq \|g_n - p\|^2 - \Big[\sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)[(1-\mu_n) - \lambda_{j-1}]\|g_n - h_{n,j-1}\|^2 \\
&\quad + \mu_n^N[(1-\mu_n) - \lambda_N]\|g_n - h_{n,N}\|^2\Big].
\end{aligned}
$$

It then follows that $\lim_{n\to\infty}\|g_n - p\|$ exists and hence $\{g_n\}$ is bounded. Also, $\sum_{n=1}^{\infty} \mu_n^j(1-\mu_n)[(1-\mu_n)-\lambda_j]\|g_n - h_{n,j}\|^2 < \infty$, $j = 1, 2, ..., N-1$ and $\sum_{n=1}^{\infty} \mu_n^N[(1-\mu_n) - \lambda_N]\|g_n - h_{n,N}\|^2 < \infty$. Since $1 > \mu > 0$ from (ii), we have that $\lim_{n\to\infty}\|g_n - h_{n,j}\| = 0$, for all $j = 1, 2, ..., N$. Also since C is closed and convex and $\{g_n\} \subseteq C$ with $\{g_n\}$ bounded, there exist a subsequence $\{g_{n_t}\} \subseteq \{g_n\}$ such that $\{g_{n_t}\}$ weakly converges to some $q \in C$. Also $\lim_{n\to\infty}\|g_n - h_{n,j}\| = 0$ implies that $\lim_{n\to\infty}\|g_{n_t,j} - h_{n_t,j}\| = 0$. Since $(I - T_j)$ is weakly demiclosed at *zero* for each j, we have that $q \in T_j q$, for all $j = 1, 2, ..., N$. Since H satisfies Opial's condition [32], we have that $\{g_n\}$ weakly converges to $q \in \bigcap_{j=1}^{N} F(T_j)$. □

Theorem 3.13. *Let C be a nonempty closed and convex subset of a real Hilbert space X. Suppose that for each $j = 1, 2, ..., N$, $T_j : C \to CC(C)$ is an L_j-Lipschitzian extended pseudocontractive mapping from C into the collection of all closed and convex subsets of C such that $\bigcap_{j=1}^{N} F(T_j) \neq \emptyset$. Suppose T_j is of type-one, for each j and $T_1, T_2, ..., T_j, ..., T_N$ satisfies condition (1) uniformly. Then the extended Ishikawa sequence defined by*

$$
\begin{cases}
h_{n,j} = (1-\xi_{n,j})g_n + \xi_{n,j}u_{n,j} \\
g_{n+1} = (1-\mu_n)g_n + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)w_{n,j-1} + \mu_n^N w_{n,N}.
\end{cases}
\tag{44}
$$

strongly converges to $p \in F(T)$, where $u_{n,j} \in T_j g_n$ with $\|g_n - u_{n,j}\| = d(g_n, T_j g_n)$, $w_{n,j} \in T_j h_{n,j}$ with $\|h_{n,j} - w_{n,j}\| = d(h_{n,j}, T_j h_{n,j})$, $\|u_{n,j} - w_{n,j}\| \leq H^2(T_j g_n, T_j h_{n,j})$, and $\{\mu_n\}$ and $\{\xi_{n,j}\}$ are real sequences satisfying (i) $0 \leq \mu_n \leq \xi_{n,j} < 1$ for each $j = 1, 2, ..., N$;(ii) $\liminf_{n\to\infty}\mu_n = \mu > 0$; (iii) $\sup_{n\geq 1}\xi_{n,j} \leq \xi_j \leq \frac{1}{\sqrt{1+L_j^2}+1}$.

Proof. . From Lemma 2.11 and Remark 3.11, we have

$$
\begin{aligned}
\|g_{n+1} - p\|^2 &\leq (1-\mu_n)\|g_n - p\|^2 + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)\|w_{n,(j-1)} - p\|^2 \\
&\quad + \mu_n^N\|w_{n,N} - p\|^2 \\
&\quad - (1-\mu_n)\Big[\sum_{j=2}^{N} \mu^{j-1}(1-\mu_n)\|g_n - w_{n,j-1}\|^2 + (1-\mu_n)^N\|g_n - w_{n,N}\|^2\Big].
\end{aligned}
$$

Type-one property of the mappings gives

$$\begin{aligned} \|g_{n+1}-p\|^2 \leq\ & (1-\mu_n)\|g_n-p\|^2 + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)H^2(T_{j-1}h_{n,j-1}, T_{j-1}p) \\ & +\mu_n^N H^2(T_N h_{n,N}, T_N p) \\ & -(1-\mu_n)\Big[\sum_{j=2}^{N} \mu^{j-1}(1-\mu_n)\|g_n - w_{n,j-1}\|^2 \\ & +(1-\mu_n)^N\|g_n - w_{n,N}\|^2\Big]. \end{aligned}$$

$w_{n,j} \in P_{T_j}h_{n,j}$ and $p \in P_{T_j}p$, for each j, it then follows from extended pseudocontractive mapping definition that

$$\begin{aligned} \|g_{n+1}-p\|^2 \leq\ & (1-\mu_n)\|g_n-p\|^2 + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)\Big[\|h_{n,j-1}-p)\|^2 + \|h_{n,j-1}-w_{n,j-1}\|^2\Big] \\ & +\mu_n^N\Big[\|h_{n,N}-p\|^2 + \|h_{n,N}-w_{n,N}\|^2\Big] \\ & -(1-\mu_n)\Big[\sum_{j=2}^{N} \mu^{j-1}(1-\mu_n)\|g_n - w_{n,j-1}\|^2 \\ & +(1-\mu_n)^N\|g_n - w_{n,N}\|^2\Big]. \end{aligned} \tag{45}$$

Also,

$$\begin{aligned} \|h_{n,j}-w_{n,j}\|^2 &= \|(1-\xi_{n,j})g_n + \xi_{n,j}u_{n,j} - w_{n,j}\|^2 \\ &= \|(1-\xi_{n,j})(g_n - w_{n,j}) + \xi_{n,j}(u_{n,j} - w_{n,j})\|^2 \\ &= (1-\xi_{n,j})\|g_n - w_{n,j}\|^2 + \xi_{n,j}\|u_{n,j} - w_{n,j}\|^2 - \xi_{n,j}(1-\xi_{n,j})\|g_n - u_{n,j}\|^2. \end{aligned} \tag{46}$$

(45) and (46) imply that

$$\begin{aligned} \|g_{n+1}-p\|^2 \leq\ & (1-\mu_n)\|g_n-p\|^2 + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)\Big[\|h_{n,i-1}-p)\|^2 \\ & +(1-\xi_{n,j-1})\|g_n - w_{n,j-1}\|^2 + \xi_{n,j-1}\|u_{n,j-1} - w_{n,j-1}\|^2 \\ & -\xi_{n,j-1}(1-\xi_{n,j-1})\|g_n - u_{n,j-1}\|^2\Big] \\ & +\mu_n^N\Big[\|h_{n,N}-p\|^2 + (1-\xi_{n,N})\|g_n - w_{n,N}\|^2 + \xi_{n,N}\|u_{n,N} - w_{n,N}\|^2 \\ & -\xi_{n,N}(1-\xi_{n,N})\|g_n - u_{n,N}\|^2\Big] \\ & -(1-\mu_n)\Big[\sum_{j=2}^{N} \mu^{j-1}(1-\mu_n)\|g_n - w_{n,j-1}\|^2 \\ & +(1-\mu_n)^N\|g_n - w_{n,N}\|^2\Big]. \end{aligned} \tag{47}$$

$$
\begin{aligned}
\|h_{n,j}-p\|^2 &= \|(1-\xi_{n,j})g_n+\xi_{n,j}u_{n,j}-p\|^2\\
&= \|(1-\xi_{n,j})(g_n-p)+\xi_{n,j}(u_{n,j}-p)\|^2\\
&= (1-\xi_{n,j})\|g_n-p\|^2+\xi_{n,j}\|u_{n,j}-p\|^2-\xi_{n,j}(1-\xi_{n,j})\|g_n-u_{n,j}\|^2\\
&\leq (1-\xi_{n,j})\|g_n-p\|^2+\xi_{n,j}H^2(T_jg_n,T_jp)-\xi_{n,j}(1-\xi_{n,j})\|g_n-u_{n,j}\|^2\\
&\leq (1-\xi_{n,j})\|g_n-p\|^2+\xi_{n,j}\Big[\|g_n-p\|^2+\|g_n-u_{n,j}\|^2\Big]-\xi_{n,j}(1-\xi_{n,j})\|g_n-u_{n,j}\|^2\\
&= \|g_n-p\|^2+\xi_{n,j}^2\|g_n-u_{n,j}\|^2. \qquad (48)
\end{aligned}
$$

(47) and (48) imply that

$$
\begin{aligned}
\|g_{n+1}-p\|^2 \leq\;& (1-\mu_n)\|g_n-p\|^2+\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)[\|g_n-p\|^2+\xi_{n,j-1}^2\|g_n-u_{n,j-1}\|^2\\
&+(1-\xi_{n,j-1})\|g_n-w_{n,j-1}\|^2+\xi_{n,j-1}\|u_{n,j-1}-w_{n,j-1}\|^2\\
&-\xi_{n,j-1}(1-\xi_{n,j-1})\|g_n-u_{n,j-1}\|^2]\\
&+\mu_n^N[\|g_n-p\|^2+\xi_{n,N}^2\|g_n-u_{n,N}\|^2+(1-\xi_{n,N})\|g_n-w_{n,N}\|^2\\
&+\xi_{n,N}\|u_{n,N}-w_{n,N}\|^2-\xi_{n,N}(1-\xi_{n,N})\|g_n-u_{n,N}\|^2]\\
&-(1-\mu_n)\Big[\sum_{j=2}^{N}\mu^{j-1}(1-\mu_n)\xi_{n,j-1}\|g_n-w_{n,j-1}\|^2\\
&+(1-\mu_n)^N\xi_{n,N}\|g_n-w_{n,N}\|^2\Big]\\
\leq\;& (1-\mu_n)\|g_n-p\|^2+\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)\Big[\|g_n-p\|^2+\xi_{n,j-1}^2\|g_n-u_{n,j-1}\|^2\\
&+(1-\xi_{n,j-1})\|g_n-w_{n,j-1}\|^2+\xi_{n,j-1}H^2(T_{j-1}g_n,T_{j-1}h_{n,j-1})\\
&-\xi_{n,j-1}(1-\xi_{n,j-1})\|g_n-u_{n,j-1}\|^2\Big]\\
&+\mu_n^N\Big[\|g_n-p\|^2+\xi_{n,N}^2\|g_n-u_{n,N}\|^2+(1-\xi_{n,N})\|g_n-w_{n,N}\|^2\\
&+\xi_{n,N}H^2(T_Ng_n,T_Nh_{n,N})-\xi_{n,N}(1-\xi_{n,N})\|g_n-u_{n,N}\|^2\Big]\\
&-(1-\mu_n)\Big[\sum_{j=2}^{N}\mu^{j-1}(1-\mu_n)\xi_{n,j-1}\|g_n-w_{n,j-1}\|^2\\
&+(1-\mu_n)^N\xi_{n,N}\|g_n-w_{n,N}\|^2\Big]\\
\leq\;& (1-\mu_n)\|g_n-p\|^2+\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)\Big[\|g_n-p\|^2+\xi_{n,j-1}^2\|g_n-u_{n,j-1}\|^2\\
&+(1-\xi_{n,j-1})\|g_n-w_{n,j-1}\|^2+L_{j-1}^2\xi_{n,j-1}^3\|g_n-u_{n,j-1}\|^2\\
&-\xi_{n,j-1}(1-\xi_{n,j-1})\|g_n-u_{n,j-1}\|^2\Big]\\
&+\mu_n^N\Big[\|g_n-p\|^2+\xi_{n,N}^2\|g_n-u_{n,N}\|^2+(1-\xi_{n,N})\|g_n-w_{n,N}\|^2\\
&+\xi_{n,N}^3L_N^2\|g_n-u_{n,N}\|^2-\xi_{n,N}(1-\xi_{n,N})\|g_n-u_{n,N}\|^2\Big]\\
&-(1-\mu_n)\Big[\sum_{j=2}^{N}\mu^{j-1}(1-\mu_n)\xi_{n,j-1}\|g_n-w_{n,j-1}\|^2\\
&+(1-\mu_n)^N\xi_{n,N}\|g_n-w_{n,N}\|^2\Big]. \qquad (49)
\end{aligned}
$$

From (49), we have that

$$\begin{aligned}
\|g_{n+1}-p\|^2 \leq\ & (1-\mu_n)\|g_n-p\|^2+\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)\|g_n-p\|^2+\mu_n^N\|g_n-p\|^2 \\
&+\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)\xi_{n,j-1}^2\|g_n-u_{n,j-1}\|^2 \\
&+\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)(1-\xi_{n,j-1})\|g_n-w_{n,j-1}\|^2 \\
&+\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)\xi_{n,j-1}^3L_{j-1}^2\|g_n-u_{n,j-1}\|^2 \\
&-\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)\xi_{n,j-1}(1-\xi_{n,j-1})\|g_n-u_{n,j-1}\|^2] \\
&+\mu_n^N\xi_{n,N}^2\|g_n-u_{n,N}\|^2+\mu_n^N(1-\xi_{n,N})\|g_n-w_{n,N}\|^2 \\
&+\mu_n^N\xi_{n,N}^3L_N^2\|g_n-u_{n,N}\|^2-\mu_n^N\xi_{n,N}(1-\xi_{n,N})\|g_n-u_{n,N}\|^2 \\
&-(1-\mu_n)\Big[\sum_{j=2}^{N}\mu^{j-1}(1-\mu_n)\xi_{n,j-1}\|g_n-w_{n,j-1}\|^2 \\
&+(1-\mu_n)^N\xi_{n,N}\|g_n-w_{n,N}\|^2\Big]. \qquad (50)
\end{aligned}$$

Hence,

$$\begin{aligned}
\|g_{n+1}-p\|^2 \leq\ & \Big[(1-\mu_n)+\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)+\mu_n^N\Big]\|g_n-p\|^2 \\
&+\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)\Big[\xi_{n,j-1}^2+\xi_{n,j-1}^3L_{j-1}^2-\xi_{n,j-1}(1-\xi_{n,j-1})\Big]\|g_n-u_{n,j-1}\|^2 \\
&+\mu_n^N\xi_{n,N}\Big[\xi_{n,N}+\xi_{n,N}^2L_N^2-(1-\xi_{n,N})\Big]\|g_n-u_{n,N}\|^2 \\
&+\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)\Big[(1-\xi_{n,j-1})-(1-\mu_n)\xi_{n,j-1}\Big]\|g_n-w_{n,j-1}\|^2 \\
&+\mu_n^N\Big[(1-\xi_{n,N})-(1-\mu_n)\xi_{n,N}\Big]\|g_n-w_{n,N}\|^2. \qquad (51)
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|g_{n+1}-p\|^2 \leq\ & \|g_n-p\|^2 \\
&-\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)\xi_{n,j-1}\Big[1-[2\xi_{n,j-1}+\xi_{n,j-1}^2L_{j-1}^2]\Big]\|g_n-u_{n,j-1}\|^2 \\
&-\mu_n^N\xi_{n,N}\Big[1-[2\xi_{n,N}+\xi_{n,N}^2L_N^2]\Big]\|g_n-u_{n,N}\|^2 \\
&-\sum_{j=2}^{N}\mu_n^{j-1}(1-\mu_n)\Big[\xi_{n,j-1}-\mu_n))]\Big]\|g_n-w_{n,j-1}\|^2 \\
&-\mu_n^N\Big[(\xi_{n,N})-\mu_n)\Big]\|g_n-w_{n,N}\|^2. \qquad (52)
\end{aligned}$$

Consequently, it then follows from Lemma 2.5 that $\lim_{n\to\infty}\|g_n - p\|$ exists. Hence $\{g_n\}$ is bounded so also are $\{u_{n,j}\}_{j=1}^{\infty}$.

We then have from (52), (ii) and (iii) that

$$\begin{aligned}\sum_{n=1}^{\infty}\mu^j(1-\mu)\xi_j[1-2\xi_j - L_j^2\xi_j^2]\|g_n - u_{n,j}\|^2 &\leq \sum_{n=1}^{\infty}\mu_n^j(1-\mu)\xi_{n,j}\Big[1-2\xi_{n,j}\\ &\quad -L_j^2\xi_{n,j}^2\Big]\|g_n - u_{n,j}\|^2\\ &\leq \sum_{n=0}^{\infty}\Big[\|g_n-p\|^2 - \|g_{n+1}-p\|^2\Big]\\ &\leq \|g_0 - p\|^2 < \infty,\ j = 1,2,...,N-1.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n=0}^{\infty}\mu^N\xi_N[1-2\xi_N - L_N^2\xi_N^2]\|g_n - u_{n,N}\|^2 &\leq \sum_{n=0}^{\infty}\mu_n^N\xi_{n,N}\Big[1-2\xi_{n,N}\\ &\quad -L_N^2\xi_{n,N}^2\Big]\|g_n - u_{n,N}\|^2\\ &\leq \sum_{n=0}^{\infty}\Big[\|g_n-p\|^2 - \|g_{n+1}-p\|^2\Big]\\ &\leq \|g_0 - p\|^2 < \infty,\ j = N.\end{aligned}$$

It then follows that $\lim_{n\to\infty}\|g_n - u_{n,j}\| = 0$, for all $j = 1,2,...,N$. Since $u_{n,j} \in T_jg_n$ we have that $d(g_n, T_jg_n) \leq \|g_n - u_{n,j}\| \to 0$ as $n \to \infty$. Since $T_1, T_2, ..., T_j, ..., T_N$ satisfy condition (1), uniformly, $\lim_{n\to\infty} d(g_n, \bigcap_{j=1}^{N} F(T_j)) = 0$. Thus there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $\|g_{n_k} - p_k\| \leq \frac{1}{2^k}$ for some $\{p_k\} \subseteq \bigcap_{j=1}^{N} F(T_j)$.

From (52)

$$\|g_{n_{k+1}} - p_k\| \leq \|g_{n_k} - p_k\|.$$

We now show that $\{p_k\}$ is a Cauchy sequence in $\bigcap_{j=1}^{N} F(T_j)$.

$$\begin{aligned}\|p_{k+1} - p_k\| &\leq \|p_{k+1} - g_{n_{k+1}}\| + \|g_{n_{k+1}} - p_k\|\\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k}\\ &= \frac{1}{2^{k-1}}.\end{aligned}$$

Therefore $\{p_k\}$ is a Cauchy sequence and converges to some $q \in \bigcap_{j=1}^{N} F(T_j)$ because $\bigcap_{j=1}^{N} F(T_j)$ is closed. Now,

$$\|g_{n_k} - q\| \leq \|g_{n_k} - p_k\| + \|p_k - q\|.$$

Hence $g_{n_k} \to q$ as $k \to \infty$.

$$\begin{aligned}d(q, T_jq) &\leq \|q - p_k\| + \|p_k - g_{n_k}\| + d(g_{n_k}, T_jg_{n_k}) + H(T_jg_{n_k}, T_jq)\\ &\leq \|q - p_k\| + \|p_k - g_{n_k}\| + d(g_{n_k}, T_jg_{n_k}) + L_j\|g_{n_k} - q\|.\end{aligned}$$

Hence, $q \in T_jq$ for all $j = 1,2,...,N$ and $\{g_{n_k}\}$ strongly converges to q. Since $\lim\|g_n - q\|$ exists we have that g_n strongly converges to $q \in F(T)$. □

We now state the following Theorems which are the extensions of Theorems 3.12 and 3.13, respectively.

Theorem 3.14. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Suppose for each $j = 1, 2, ..., N$, $T_j : C \to P(C)$ is an extended demicontractive mapping from C into the collection of all proximinal subsets of C with $\lambda_j \in (0, 1)$. Assume that $\bigcap_{j=1}^{N} F(T_j) \neq \emptyset$ and $(I - T_j)$ is demiclosed at zero for each j. Then, the extended Mann sequence for finite family of mappings defined by*

$$g_{n+1} = (1 - \mu_n)g_n + \sum_{i=2}^{N} \mu_n^{i-1}(1 - \mu_n)h_{n,i-1} + \mu_n^N h_{n,N},$$

weakly converges to $q \in \bigcap_{j=1}^{N} F(T_j)$, where $h_{n,j} \in T_j g_n$ with $\|g_n - h_{n,j}\| = d(g_n, T_j g_n)$ and $\mu_n \subseteq (0, 1)$ satisfies: (i) $\mu_n \to \mu < 1 - \max \lambda_j$, $j = 1, 2, ..., N$; (ii) $1 > \mu > 0$; (iii) $(1 - \mu_n) > \max\{\lambda_j\}_{j=1}^N$.

Proof. . The method of proof is similar to that of Theorem 3.12, hence, it is omitted. □

Theorem 3.15. *Let C be a nonempty closed and convex subset of a real Hilbert space X. Suppose that for each $j = 1, 2, ..., N$, $T_j : C \to CC(C)$ is an extended L_j-Lipschitzian hemicontractive mapping from C into the collection of all closed and convex subsets of C such that $\bigcap_{j=1}^{N} F(T_j) \neq \emptyset$. Suppose T_j is of type-one, for each j and $T_1, T_2, ..., T_j, ..., T_N$ satisfies condition (1) uniformly. Then the extended Ishikawa sequence defined by*

$$\begin{cases} h_{n,j} = (1 - \xi_{n,j})g_n + \xi_{n,j}u_{n,j} \\ g_{n+1} = (1 - \mu_n)g_n + \sum\limits_{j=2}^{N} \mu_n^{j-1}(1 - \mu_n)w_{n,j-1} + \mu_n^N w_{n,N}. \end{cases} \tag{53}$$

strongly converges to $p \in F(T)$, where $u_{n,j} \in T_j g_n$ with $\|g_n - u_{n,j}\| = d(g_n, T_j g_n)$, $w_{n,j} \in T_j h_{n,j}$ with $\|h_{n,j} - w_{n,j}\| = d(h_{n,j}, T_j h_{n,j})$ and $\{\mu_n\}$ and $\{\xi_{n,j}\}$ are real sequences satisfying (i) $0 \leq \mu_n \leq \xi_{n,j} < 1$ for each $j = 1, 2, ..., N$;(ii) $\liminf_{n\to\infty} \mu_n = \mu > 0$; (iii) $\sup_{n\geq 1} \xi_{n,j} \leq \xi_j \leq \frac{1}{\sqrt{1+L_j^2}+1}$.

Proof. . The proof is similar to that of Theorem 3.13, therefore, it is omitted □

Theorems 1.12 and 1.13 are corollaries which follow from Theorems 3.14 and 3.15, respectively.

4. Example

Example 4.1. *Let $H = \mathbb{R}$ (the reals with the usual norm), $j = 1, 2$, and $C = \mathbb{R}$. Then for each j, we define:*

(i) $T_j : \mathbb{R} \to CC(\mathbb{R})$ by

$$T_j g = \begin{cases} [-\sqrt{10j}g, -2jg], & g \in [0, \infty) \\ \{-\sqrt{10j}g\}, & g \in [-\infty, 0). \end{cases}$$

Obviously, $F(T_1) = \{0\}$, $F(T_2) = \{0\}$, $T_1 0 = \{0\}$, $T_2 0 = \{0\}$. Also, T_1 and T_2, satisfy condition 1 uniformly since $d(g, \bigcap_{j=1}^{2} F(T_j)) = d(g, \{0\}) = |g - 0| = |g|$, while

$$\begin{aligned} d(g, T_j g) &= \begin{cases} d(g, [-\sqrt{10j}g, -2jg]), & g \in [0, \infty) \\ d(g, \{-\sqrt{10j}g\}), & g \in (-\infty, 0). \end{cases} \\ &= \begin{cases} |g - (-2jg)|, & g \in [0, \infty) \\ |g - (-\sqrt{10j}g)|, & g \in (-\infty, 0). \end{cases} \\ &\geq |g| = f(d(g, \bigcap_{j=1}^{2} F(T_j)), \ \forall\ j. \end{aligned}$$

Where $f : [0.\infty) \to [0, \infty)$ *is defined by f(r)=r.*

Now, given any pair $g, h \in [0, \infty)$*,*

$$H(T_j g, T_j h) = \sqrt{10j}|g - h|.$$

Also, given any $u \in P_{T_j} g = \{-2jg\}$ *and* $v \in P_{T_j} h = \{-2jh\}$*, we have that*

$$|u - v| = 2j|g - h| \leq H(T_j g, T_j h).$$

Similarly, given any pair $g, h \in (-\infty, 0)$*,*

$$H(T_j g, T_j h) = \sqrt{10j}|g - h|.$$

Also, given any $u \in P_{T_j} g = \{-\sqrt{10j}g\}$ *and* $v \in P_{T_j} h = \{-\sqrt{10j}h\}$*, we have that*

$$|u - v| = \sqrt{10j}|g - h| = H(T_j g, T_j h).$$

Furthermore, given $g \in [0, \infty)$*,* $h \in (-\infty, 0)$

$$H(T_j g, T_j h) = \sqrt{10j}|g - h|.$$

Also, given any $u \in P_{T_j} g = \{-2jg\}$ *and* $v \in P_{T_j} h = \{-\sqrt{10j}h\}$*, we obtain*

$$|u - v| = |2jg - \sqrt{10j}h| \leq |\sqrt{10j}g - \sqrt{10j}h| = H(T_j g, T_j h).$$

Observe that given any $g \in [0, \infty)$ *and* $u \in P_{T_j} g = \{-2jg\}$*,* $|g - u|^2 = (1 + 2j)^2 |g|^2$*. It then follows that*

$$\begin{aligned} H^2(T_j g, T_j 0) &= 10j|g - 0|^2 = |g - 0|^2 + \frac{10j - 1}{(1 + 2j)^2}|g - u|^2 \\ &\leq |g - p|^2 + |g - u|^2, \forall j = 1, 2. \end{aligned}$$

Similarly, for any $g \in (-\infty, 0)$ *and* $u \in P_{T_j} g = \{-\sqrt{10j}g\}$*,* $|g - u|^2 = (1 + \sqrt{10j})^2 |g|^2$*.*

$$\begin{aligned} H^2(T_j g, T_j 0) &= 10j|g - 0|^2 = |g - 0|^2 + \frac{10j - 1}{(1 + \sqrt{10j})^2}|g - u|^2 \\ &\leq |g - p|^2 + |g - u|^2, \forall j = 1, 2. \end{aligned}$$

Furthermore, for $j = 1$ *and* $g \in [0, \infty)$*, we obtain from the above that*

$$\begin{aligned} H^2(T_1 g, T_1 0) &= |g - 0|^2 + |g - u|^2 \\ &> |g - 0|^2 + k|g - u|^2, \forall k \in [0, 1). \end{aligned}$$

Hence, T_1 is not extended demicontractive mapping. Therefore, T_j is an extended L_j-Lipschitzian hemicontractive mapping for each $j = 1, 2$, with $L_j = \sqrt{10j}$ such that T_1, T_2 satisfy condition 1 uniformly. It then follows that:

(ii) $u_{n,j} = \begin{cases} -2jg_n, & g_n \in [0,\infty) \\ -\sqrt{10j}g_n, & g_n \in (-\infty, 0). \end{cases}$

(iii) $\{\mu_n\}_{n=1}^{\infty} = \frac{10n-(n+1)(\sqrt{1+10}+1)}{10n(\sqrt{1+10}+1)}$.

(iv) $\{\xi_{n,j}\}_{n=1}^{\infty} = \frac{12nj-(n+1)(\sqrt{1+10j}+1)}{12nj(\sqrt{1+10j}+1)}$.

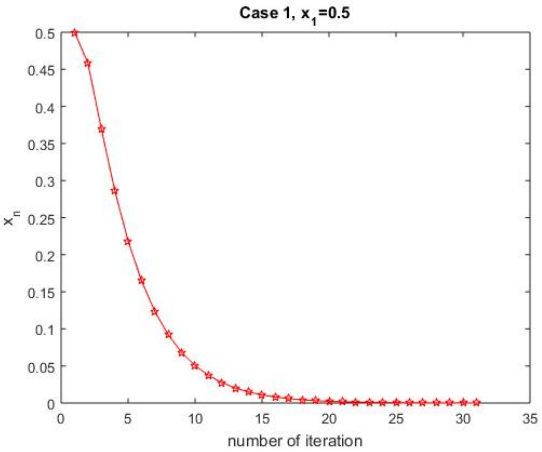
(v) $h_{n,j} = (1-\xi_{n,j})g_n + \xi_{n,j}u_{n,j}$.

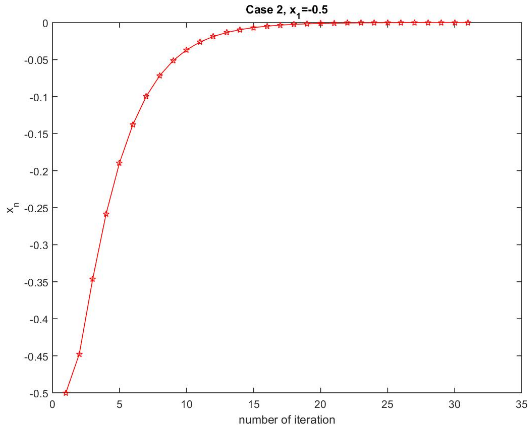
(vi) $w_{n,j} = \begin{cases} -2jh_{n,j}, & h_{n,j} \in [0,\infty) \\ -\sqrt{10j}h_{n,j}, & h_{n,j} \in (-\infty, 0). \end{cases}$

(vii) $g_{n+1} = (1-\mu_n)g_n + \sum_{j=2}^{N} \mu_n^{j-1}(1-\mu_n)w_{n,j-1} + \mu_n^N w_{n,N}$.

Table 1.

Case 1	$x_1 = 0.5$		Case 2	$x_1 = -0.5$
n	x_n		n	x_n
1	0.5		1	-0.5
2	0.458533123		2	-0.44763782
3	0.369895634		3	-0.346489435
4	0.287022833		4	-0.258293596
5	0.218650314		5	-0.189422417
6	0.164791887		6	-0.137672294
7	0.123336315		7	-0.099498106
8	0.091857781		8	-0.071632881
9	0.068165192		9	-0.051428005
10	0.050442139		10	-0.036844362
11	0.037244238		11	-0.026352744
12	0.027449847		12	-0.018823704
13	0.020200877		13	-0.013431116
14	0.014847474		14	-0.009574693
15	0.010901018		15	-0.0068203
16	0.007996086		16	-0.00485507
17	0.005860508		17	-0.003454131
18	0.004292228		18	-0.002456204
19	0.003141632		19	-0.001745814
20	0.002298169		20	-0.001240397
21	0.001680305		21	-0.00088099
22	0.00122799		22	-0.000625523
23	0.000897059		23	-0.000444009
24	0.000655062		24	-0.000315085
25	0.000478181		25	-0.000223542
26	0.000348951		26	-0.000158562
27	0.000254571		27	-0.000112448
28	0.000185667		28	-0.00007973
29	0.000135379		29	-0.000056523
30	0.000098689		30	-0.000040064
31	0.000071926		31	-0.000028393





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