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On ϕ-biflatness-like properties of certain Banach algebras with applications

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Abstract. In this paper left ϕ-biflatness of abstract Segal algebras is investigated. For a locally compact group G, we show that any abstract Segal algebra with respect to $L^1(G)$ is left ϕ-biflat if and only if the underlying group G is amenable. We then prove that the Lipschitz algebras $\mathrm{Lip}_\alpha(\mathrm{X})$ and $\mathrm{lip}_\alpha(\mathrm{X})$ are left C-ϕ-biflat if and only if X is finite. Finally, we also study left ϕ-biflatness of lower triangular matrix algebras.

1. Introduction and preliminaries

The homological concept of biflatness for Banach algebras, introduced by Helemskii [5], has proved to be of great importance in Banach algebra theory. Left ϕ-biflatness which is a modification of biflatness was introduced in [13]. We recall the definition in the sequel. In the current paper we continue the investigation of this notion.

Given a Banach algebra A, we let $\pi_A : A \otimes_p A \to A$ denote the multiplication operator, i.e., $\pi_A(a \otimes b) = ab$ for all $a, b \in A$. It is known that the projective tensor product $A \otimes_p A$ becomes a Banach A-bimodule in a canonical way, turning π_A into a A-bimodule morphism. The character space of A is denoted by $\Delta(A)$, that is, the set of all non-zero multiplicative linear functionals on A.

Let A be a Banach algebra and let $\phi \in \Delta(A)$. We recall that A is *left ϕ-amenable* if there exists an element $m \in A^{**}$ such that $am = \phi(a)m$ and $\tilde{\phi}(m) = 1$ for all $a \in A$, where $\tilde{\phi}$ is the unique extension of ϕ to A^{**} given by $\tilde{\phi}(F) = F(\phi)$ for all $F \in A^{**}$. This concept of amenability as a generalization of left amenability of Lau algebras has been recently introduced and investigated by Kaniuth, Lau and Pym [10] under the name of ϕ-amenability; see also Monfared [11].

More recently, the authors in [13] introduced and studied the homological concept of left ϕ-biflatness of Banach algebras. Precisely, A is called *left ϕ-biflat* if there exists a bounded linear map $\rho : A \to (A \otimes_p A)^{**}$ such that $\rho(ab) = \phi(b)\rho(a) = a \cdot \rho(b)$ and $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$ for each $a, b \in A$. Also A is *left C-ϕ-biflat* if there exists $C > 0$ such that $\|\rho\| \leq C$. The reader may also see [12] for definition of φ-biflat Banach algebras.

The content of the paper is as follows. In Section 2, we investigate relations between left ϕ-biflatness and left ϕ-amenability of (abstract) Segal algebras. For a locally compact group G, we prove that an abstract

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Segal algebra with respect to $L^1(G)$ is left ϕ-biflat if and only if G is an amenable group. In Section 3, we study left ϕ-biflatness of some (concrete) Banach algebras including Lipschitz algebras $Lip_\alpha(X)$ and $\ell ip_\alpha(X)$, lower triangular matrices $LO(I,A)$ and also $C^1[0,1]$.

2. Left ϕ-biflatness of abstract Segal algebras

Let A be a Banach algebra with the norm $\|\cdot\|_A$. We recall that a Banach algebra B with the norm $\|\cdot\|_B$ is an *abstract Segal algebra* with respect to A if

(i) B is a dense left ideal in A,
(ii) there exists $M > 0$ such that $\|b\|_A \leq M\|b\|_B$ for every $b \in B$,
(iii) there exists $C > 0$ such that $\|ab\|_B \leq C\|a\|_A\|b\|_B$ for every $a \in A$ and $b \in B$.

It is known that $\Delta(B) = \{\phi|_B : \phi \in \Delta(A)\}$, [2, Lemma 2.2].

Two following lemmas will be needed.

Lemma 2.1. *([16, Lemma 2.2]) Let A be a Banach algebra and let $\phi \in \Delta(A)$. If A is left ϕ-amenable, then A is left ϕ-biflat.*

In the following example we show that the converse of Lemma 2.1 is not true necessarily.

Example 2.2. *Suppose that S is a left zero semigroup with $|S| \geq 2$, that is, a semigroup with action $st = s$ for every $s,t \in S$. This semigroup action induces a product on the related semigroup algebra $\ell^1(S)$. Indeed, we have $fg = \phi_S(g)f$, where ϕ_S is the augmentation character on $\ell^1(S)$ given by $\phi_S(\sum_{s\in S}\alpha_s\delta_s) = \sum_{s\in S}\alpha_s$, for all $f,g \in \ell^1(S)$.*
First we show that $\ell^1(S)$ is left ϕ_S-biflat. To see this, suppose that f_0 is an element in $\ell^1(S)$ such that $\phi_S(f_0) = 1$. Define $\rho : \ell^1(S) \to (\ell^1(S) \otimes_p \ell^1(S))^{**}$ by $\rho(f) = f \otimes f_0$ for all $f \in \ell^1(S)$. One can see that

$$f \cdot \rho(g) = \rho(fg), \quad \rho(fg) = \phi_S(g)\rho(f)$$

and

$$\tilde{\phi}_S \circ \pi_A^{**} \circ \rho(f) = \phi_S(f_0 f) = \phi_S(f)$$

for each $f,g \in \ell^1(S)$.
Now, we show that $\ell^1(S)$ is not left ϕ_S-amenable, whenever $|S| \geq 2$. We assume in contradiction and suppose that $\ell^1(S)$ is left ϕ_S-amenable. Then there exists a bounded net (f_α) in $\ell^1(S)$ such that

$$\phi_S(f_\alpha) = 1, \quad \phi_S(f_\alpha)f - \phi_S(f)f_\alpha = ff_\alpha - \phi_S(f)f_\alpha \to 0 \quad (f \in \ell^1(S)).$$

It gives that $f - \phi_S(f)f_\alpha \to 0$ for each $f \in \ell^1(S)$. Since S has at least two distinct elements s_1 and s_2, consider δ_{s_1} and δ_{s_2} and replace them in $f - \phi_S(f)f_\alpha \to 0$. It follows that $\delta_{s_1} = \delta_{s_2}$, so $s_1 = s_2$ which is impossible.

Lemma 2.3. *([13, Lemma 2.1]) Suppose that A is a left ϕ-biflat Banach algebra with $\overline{A\ker\phi}^{\|\cdot\|} = \ker\phi$. Then A is left ϕ-amenable.*

In the following example we show that the condition $\overline{A\ker\phi}^{\|\cdot\|} = \ker\phi$ is necessary in the above lemma.

Example 2.4. *Let $A = \left\{\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} : \alpha,\beta \in \mathbb{C}\right\}$ be a two–dimensional subspace of $\mathbb{M}_2(\mathbb{C})$ with the multiplication*

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\begin{bmatrix} \gamma & -\theta \\ \theta & \gamma \end{bmatrix} = \begin{bmatrix} \alpha\theta & -\beta\theta \\ \beta\theta & \alpha\theta \end{bmatrix}$$

and with the ℓ^1-norm. Consider a character $\phi : A \longrightarrow \mathbb{C}$ by

$$\phi\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\right) = \beta.$$

Then $\ker\phi = \mathbb{C}I$ *and so* $\overline{A\ker\phi} \neq \ker\phi$. *It is easy to verify that the map* $\rho : A \longrightarrow (A \otimes_p A)^{**}$ *define by*

$$\rho\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\right) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

implies that A *is left* ϕ*-biflat. But* A *is not left* ϕ*-amenable, since otherwise there exists a bounded net* $m_j = \begin{bmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{bmatrix} \in A$ *such that* $am_j - \phi(a)m_j \to 0$ *for all* $a \in A$ *and* $\phi(m_j) = 1$ *for all* j. *The second equation implies that* $\beta_j = 1$ *for all* j *and the first relation for* $a = I$, *the identity matrix, implies that* $Im_j - \phi(I)m_j = Im_j = I \to 0$ *that is a contradiction.*

Theorem 2.5. *Let* A *be a Banach algebra and let* $\phi \in \Delta(A)$. *Suppose that* B *is an abstract Segal algebra with respect to* A *which posses an approximate identity. Then the following statements are equivalent:*

(i) *A is left ϕ-biflat;*
(ii) *B is left $\phi|_B$-biflat;*
(iii) *B is left $\phi|_B$-amenable;*
(iv) *A is left ϕ-amenable.*

Proof. (i) ⇒ (ii) Suppose that A is left ϕ-biflat. Then there exists a bounded liner map $\Gamma : A \to (A \otimes_p A)^{**}$ such that $\Gamma(ab) = a \cdot \Gamma(b) = \phi(b)\Gamma(a)$ and $\tilde{\phi} \circ \pi_A^{**} \circ \Gamma(a) = \phi(a)$, for all $a, b \in A$. Since B is dense in A, we can choose i_0 in B such that $\phi(i_0) = 1$. Define $R_{i_0} : A \to B$ by $R_{i_0}(a) = ai_0$, for each $a \in A$. Clearly R_{i_0} is a bounded linear map. Set

$$\rho := (R_{i_0} \otimes R_{i_0})^{**} \circ \Gamma|_B : B \to (B \otimes_p B)^{**}.$$

One can see that ρ is a bounded linear map such that

$$\rho(b_1 b_2) = b_1\rho(b_2) = \phi(b_2)\rho(b_1), \quad (b_1, b_2 \in B),$$

and

$$\tilde{\phi|_B} \circ \pi_B^{**} \circ \rho(b_1) = \tilde{\phi|_B} \circ \pi_B^{**} \circ (R_{i_0} \otimes R_{i_0})^{**} \circ \Gamma|_B(b_1) = \tilde{\phi} \circ \pi_A^{**} \circ \Gamma(b_1) = \phi(b_1).$$

It follows that B is left $\phi|_B$-biflat.

(ii) ⇒ (iii) It is immediate by Lemma 2.3.

(iii) ⇒ (iv) See [2, Proposition 2.3].

(iv) ⇒ (i) This is Lemma 2.1. □

Inspired by the argument in [13, Lemma 2.1] we give the following result.

Theorem 2.6. *Let* A *be a Banach algebra with a left approximate identity and let* $\phi \in \Delta(A)$. *Suppose that* B *is an abstract Segal algebra with respect to* A. *Then* B *is left* $\phi|_B$*-biflat if and only if* B *is left* $\phi|_B$*-amenable.*

Proof. Suppose that B is left ϕ-biflat. Then there exists a bounded linear map $\rho : B \to (B \otimes_p B)^{**}$ such that $\tilde{\phi} \circ \pi_B^{**} \circ \rho(b) = \phi(b)$ for all $b \in B$. Let i_0 and R_{i_0} be as in the proof of Theorem 2.5. We denote $\imath$ for the inclusion map from B into A. Set

$$\lambda := (\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0} : A \to (A \otimes_p A)^{**}.$$

It is easy to see that λ is a bounded linear map such that

$$\tilde{\phi} \circ \pi_A^{**} \circ \lambda(a) = \tilde{\phi} \circ \pi_A^{**} \circ (\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0}(a) = \tilde{\phi} \circ \pi_B^{**} \circ \rho \circ R_{i_0}(a) = \phi(a),$$

and

$$\begin{aligned} b_1 \cdot \lambda(b_2) &= b_1 \cdot (\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0}(b_2) \\ &= (\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0}(b_1 b_2) \\ &= \phi(b_2)(\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0}(b_1) \\ &= \phi(b_2)\lambda(b_1), \quad (b_1, b_2 \in B). \end{aligned}$$

Suppose that $K = \ker\phi$ (in A). We denote $\mathrm{id_A}$ for the identity map and $q : A \to \frac{A}{K}$ for the quotient map. Let ζ be the bounded linear map specified by

$$\zeta := (\mathrm{id}_A \otimes q)^{**} \circ \lambda : A \to (A \otimes_p \frac{A}{K})^{**}.$$

Since A has a left approximate identity, $\overline{AK}^{\|\cdot\|} = K$. Thus for each $k \in K$ we have

$$\zeta(k) = (\mathrm{id}_A \otimes q)^{**} \circ \lambda(k) = (\mathrm{id}_A \otimes q)^{**} \circ \lambda(\lim_n a_n k_n) = \lim_n \phi(k_n)(\mathrm{id}_A \otimes q)^{**} \circ \lambda(a_n) = 0,$$

for some sequences (a_n) in A and (k_n) in K. So ζ induces a map on $\frac{A}{K}$ which still is denoted by ζ. Since $\frac{A}{K} \cong \mathbb{C}$, we have $A \otimes_p \frac{A}{K} \cong A$. So we can assume that $m = \zeta(i_0 + K) \in A^{**}$. Consider

$$bm = b\zeta(i_0 + K) = \zeta(bi_0 + K) = \zeta(\phi(b)i_0 + K) = \phi(b)m, \quad (b \in B) \tag{1}$$

also

$$(\phi \otimes \overline{\phi})^{**} \circ \lambda(b) = \tilde{\phi} \circ \pi_B^{**} \circ \rho(b) = \phi(b), \quad (b \in B)$$

and $\tilde{\phi} \circ (\mathrm{id}_A \otimes \overline{\phi})^{**} = (\phi \otimes \overline{\phi})^{**}$, where $\overline{\phi}$ is a character on $\frac{A}{K}$ given by $\overline{\phi}(a + K) = \phi(a)$ for each $a \in A$. These facts follow that

$$\begin{aligned}\tilde{\phi}(m) = \tilde{\phi} \circ \zeta(i_0 + K) &= \tilde{\phi} \circ (\mathrm{id}_A \otimes q)^{**} \circ \lambda(i_0)\\ &= (\phi \otimes \overline{\phi})^{**} \circ \lambda(i_0)\\ &= \tilde{\phi} \circ \pi_B^{**} \circ \rho(i_0)\\ &= \phi(i_0) = 1.\end{aligned} \tag{2}$$

Since B is dense in A, by (1) $am = \phi(a)m$ for all $a \in A$. It follows that A is left ϕ-amenable. Replacing m with mi_0, we can assume that $m \in B^{**}$. So B is left $\phi|_B$-amenable. The converse is valid by Lemma 2.1. □

A Banach algebra A with $\phi \in \Delta(A)$ is called *ϕ-inner amenable* if there exists a bounded net (a_α) in A such that $aa_\alpha - a_\alpha a \to 0$ and $\phi(a_\alpha) = 1$ for all $a \in A$, [8].

Lemma 2.7. *Let A be a Banach algebra and let $\phi \in \Delta(A)$. Suppose that A is left ϕ-biflat and ϕ-inner amenable. Then A is left ϕ-amenable.*

Proof. Suppose that A is left ϕ-biflat. Then there exists a bounded linear map $\rho : A \to (A \otimes_p A)^{**}$ such that $\rho(ab) = a \cdot \rho(b) = \phi(b)\rho(a)$ and $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$, for all $a, b \in A$. Since A is ϕ-inner amenable, there exists a bounded linear net (a_α) in A such that $aa_\alpha - a_\alpha a \to 0$ and $\phi(a_\alpha) = 1$, for all $a \in A$. Define $m_\alpha = \rho(a_\alpha)$. It is easy to see that (m_α) is a bounded net in $(A \otimes_p A)^{**}$ such that

$$a \cdot m_\alpha - \phi(a)m_\alpha \to 0, \quad \tilde{\phi} \circ \pi_A^{**}(m_\alpha) \to 1, \qquad (a \in A).$$

Using Banach-Alaoglu theorem (m_α) has a w^*-cluster point in $(A \otimes_p A)^{**}$, say M. One can show that

$$a \cdot M = \phi(a)M, \quad \tilde{\phi} \circ \pi_A^{**}(M) = 1, \qquad (a \in A).$$

So

$$a\pi_A^{**}(M) = \phi(a)\pi_A^{**}(M), \quad \tilde{\phi} \circ \pi_A^{**}(M) = 1, \qquad (a \in A).$$

It follows that A is left ϕ-amenable. □

Proposition 2.8. *Let A be an ϕ-inner amenable Banach algebra and $\phi \in \Delta(A)$. Suppose that B is an abstract Segal algebra with respect to A. Then B is left $\phi|_B$-biflat if and only if B is left $\phi|_B$-amenable.*

Proof. Suppose that B is left ϕ-biflat. Let R_{i_0}, $\imath$, q, id_A, ρ, λ, and ζ are the same as in the proof of Theorem 2.5. Since A is ϕ-inner amenable, there exists a bounded net (a_α) in A such that $aa_\alpha - a_\alpha a \to 0$ and $\phi(a_\alpha) = 1$, for all $a \in A$. Define $m_\alpha = \zeta(a_\alpha) \in (A \otimes \frac{A}{K})^{**} \cong A^{**}$. Clearly (m_α) is a bounded net in A^{**}. Consider

$$\begin{aligned} &bm_\alpha - \phi(b)m_\alpha \\ &= b(\mathrm{id}_A \otimes q)^{**} \circ (\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0}(a_\alpha) - \phi(b)(\mathrm{id}_A \otimes q)^{**} \circ (\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0}(a_\alpha) \\ &= (\mathrm{id}_A \otimes q)^{**} \circ (\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0}(ba_\alpha) - (\mathrm{id}_A \otimes q)^{**} \circ (\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0}(a_\alpha b) \\ &= (\mathrm{id}_A \otimes q)^{**} \circ (\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0}(ba_\alpha - a_\alpha b) \to 0, \quad (b \in B). \end{aligned}$$

Also

$$\tilde{\phi}(m_\alpha) = \tilde{\phi} \circ (\mathrm{id}_A \otimes q)^{**} \circ (\imath \otimes \imath)^{**} \circ \rho \circ R_{i_0}(a_\alpha) = \tilde{\phi} \circ \pi_B \circ \rho \circ R_{i_0}(a_\alpha) = \phi(a_\alpha) = 1.$$

Thus we found a bounded net (m_α) in A^{**} such that $bm_\alpha - \phi(b)m_\alpha \to 0$ and $\tilde{\phi}(m_\alpha) = 1$, for all $b \in B$. Since (m_α) is a bounded net in A^{**}, Banach-Alaoglu theorem yields (m_α) has a w^*-limit point, say M. Thus $bM = \phi(b)M$ and $\tilde{\phi}(M) = 1$ for all $b \in B$. Since B is dense in A, $aM = \phi(a)M$ and $\tilde{\phi}(M) = 1$, for all $a \in A$. It follows that A is left ϕ-amenable. So by [2, Proposition 2.3], B is left $\phi|_B$-amenable. The converse is true by Lemma 2.1. □

Let $L^1(G)$ be the group algebra of a locally compact group G with the convolution product defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy \qquad (x \in G)$$

for $f, g \in L^1(G)$ and with the norm $\|\cdot\|_1$. Let $\widehat{G}$ denote the dual group of G consisting of all continuous homomorphisms ν from G into the unit circle $\mathbb{T}$. Define the character $\phi_\nu \in \Delta(L^1(G))$ by

$$\phi_\nu(h) = \int_G \overline{\nu(x)}h(x)dx \quad (h \in L^1(G)).$$

It is known that

$$\Delta(L^1(G)) = \{\phi_\nu : \nu \in \widehat{G}\};$$

see, for example [6, Theorem 23.7].

Corollary 2.9. *Let G be a locally compact group and let $\phi \in \Delta(L^1(G))$. Then the following statements are equivalent:*

(*i*) *$L^1(G)$ is left ϕ-biflat.*
(*ii*) *Each abstract Segal algebra with respect to $L^1(G)$ is left ϕ-biflat.*
(*iii*) *There exists a left ϕ-biflat abstract Segal algebra with respect to $L^1(G)$.*
(*iv*) *G is amenable.*

Proof. (*i*) ⇒ (*ii*) Suppose that $L^1(G)$ is left ϕ-biflat. By Lemma 2.3 $L^1(G)$ is left ϕ-amenable, since $L^1(G)$ has a bounded approximate identity. From [2, Proposition 2.3] it follows that each abstract Segal algebra with respect to $L^1(G)$ is left ϕ-amenable. Then by Lemma 2.1, each abstract Segal algebra with respect to $L^1(G)$ is left ϕ-biflat.

(*ii*) ⇒ (*iii*) It is clear.

(*iii*) ⇒ (*iv*) Suppose that an abstract Segal algebra B with respect to $L^1(G)$ is left $\phi|_B$-biflat. Since $L^1(G)$ has a bounded approximate identity, $L^1(G)$ is ϕ-inner amenable. By Proposition 2.8, B is left $\phi|_B$-amenable. It then follows from [2, Corollary 3.4] that G is amenable.

(*iv*) ⇒ (*i*) Since G is amenable, $L^1(G)$ is left ϕ-amenable by [2, Corollary 3.4]. Now $L^1(G)$ is left ϕ-biflat by Lemma 2.1. □

Remark 2.10. *Let G be a locally compact group and $L^\infty(G)$ be the usual Lebesgue space as defined in [6] equipped with the essential supremum norm $\|\cdot\|_\infty$ and the convolution product. Since G is compact so $L^\infty(G) \subseteq L^1(G)$ and then $L^1(G)$ has a bounded approximate identity (e_i) such that it is an approximate identity for $L^\infty(G)$. Also as $L^1(G) * L^\infty(G) * L^1(G) \subseteq L^\infty(G)$ with $\max\{\|f * g\|_\infty, \|g * f\|_\infty\} \leq \|f\|_1 \|g\|_\infty$ for $f \in L^1(G)$ and $g \in L^\infty(G)$, we conclude that the convolution Banach algebra $L^\infty(G)$ is an abstract Segal algebra with respect to $L^1(G)$. Moreover, $\Delta(L^\infty(G)) = \{\phi_\nu : \nu \in \widehat{G}\}$, where*

$$\phi_\nu(h) = \int_G \overline{\nu(x)} h(x) dx \quad (h \in L^\infty(G)),$$

and so by Corollary 2.9 $L^\infty(G)$ is left ϕ_ν-biflat and thus by Theorem 2.5 it is left ϕ_ν-biflat for all $\phi_\nu \in \Delta(L^\infty(G))$.

Corollary 2.11. *Let A be a Banach algebra and let $\phi \in \Delta(A)$. Suppose that B is an abstract Segal algebra with respect to A which is $\phi|_B$-inner amenable. Then B is left $\phi|_B$-biflat if and only if A is left ϕ-amenable.*

Proof. If B is left $\phi|_B$-biflat, then B is left $\phi|_B$-amenable, by Lemma 2.7. Thus A is left ϕ-amenable, by [2, Proposition 2.3].

Conversely, suppose that A be left ϕ-amenable. Then B is left ϕ-amenable, by [2, Proposition 2.3]. Now Lemma 2.1 gives us the result. □

3. Applications to some specified Banach algebras

Let (X, d) be a compact metric space and $\alpha > 0$. Set

$$Lip_\alpha(X) = \{f : X \to \mathbb{C} : p_\alpha(f) < \infty\},$$

where

$$p_\alpha(f) = \sup\{\frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y\}$$

and also

$$\ell ip_\alpha(X) = \{f \in Lip_\alpha(X) : \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \to 0 \quad \text{as} \quad d(x, y) \to 0\}.$$

Define

$$\|f\|_\alpha = \|f\|_\infty + p_\alpha(f),$$

where

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

With the pointwise multiplication and the norm $\|\cdot\|_\alpha$, $Lip_\alpha(X)$ and $\ell ip_\alpha(X)$ become Banach algebras, called Lipschitz algebra of order α and little Lipschitz algebra of order α, respectively. It is well-known [14, Lemma 3.2] that each nonzero multiplicative linear functional on $Lip_\alpha(X)$ or $\ell ip_\alpha(X)$ has a form ϕ_x, where $\phi_x(f) = f(x)$ for every $x \in X$. It is worthwile to mention that if X is not compact, then $Lip_\alpha(X)$ is always a Banach algebra, assuming that $Lip_\alpha(X)$ contains all bounded functions f (i.e. $\|f\|_\infty < \infty$) such that $p_\alpha(f) < \infty$. In this case and if $Lip_\alpha(X)$ separates the points of X, the set $\{\phi_x : x \in X\}$ is dense in $Lip_\alpha(X)$, in the Gelfand topology. This result is actually a consequence of the general theory of function algebras and holds for any algebra of functions on a set that is self-adjoint, inverse-closed and separates the points of X. For further information about Lipschitz algebras see [3], [14] and [15]. Hu, Monfared and Traynor in [7] studied character amenability of Lipschitz algebras. Recently C-character amenability of Lipschitz algebras have been investigated in [4].

Theorem 3.1. *Let X be a compact metric space and let A be either $Lip_\alpha(X)$ or $\ell ip_\alpha(X)$ and $x \in X$. Then the following statements are equivalent:*

(*i*) *A is left C-ϕ_x-biflat;*
(*ii*) *X is finite.*

Proof. $(i) \Rightarrow (ii)$ Suppose that A is left C-ϕ_x-biflat. Since A is unital so by Lemma 2.7, left C-ϕ_x-biflatness of A implies that $C \geq 1$ and A is left C-ϕ_x-amenable. It follows from [4, Proposition 2.1] that $\|\phi_x - \phi_y\| \geq C^{-1}$ for each distinct elements $x, y \in X$. On the other hand

$$\|\phi_x - \phi_y\| = \sup_{\|f\|_\alpha \leq 1} |\phi_x(f) - \phi_y(f)| = \sup_{\|f\|_\alpha \leq 1} |f(x) - f(y)| \leq d(x,y)^\alpha.$$

Hence $d(x,y)^\alpha \geq C^{-1}$, whence X is uniformly discrete. So X is finite.

$(ii) \Rightarrow (i)$ It is clear. □

In the sequel we study left ϕ-biflatness of lower triangular Banach algebras.
Suppose that I is a totally ordered set which has a smallest element. $LO(I,A)$ is denoted for the set of all lower triangular matrices which the entries come from A. With usual matrix operations and also with the finite ℓ^1-norm, one can see that $LO(I,A)$ is a Banach algebra. Let i_0 be the smallest element of I and also $\phi \in \Delta(A)$. Define $\psi_{i_0} : LO(I,A) \to \mathbb{C}$ by $\psi_{i_0}([a_{ij}]) = \phi(a_{i_0 i_0})$, for each $[a_{ij}] \in LO(I,A)$. We can see that ψ_{i_0} is a non-zero character on $LO(I,A)$.

We recall that a Banach algebra A with $\phi \in \Delta(A)$ is *approximately left ϕ-amenable* if there exists a (not necessarily bounded) net (m_α) in A such that $am_\alpha - \phi(a)m_\alpha \to 0$ and $\phi(m_\alpha) = 1$ for all $a \in A$, [1].

Theorem 3.2. *Let A be a Banach algebra and let $\phi \in \Delta(A)$. Suppose that A has an element a_0 such that $aa_0 = a_0a$ and $\phi(a_0) = 1$. Let I be a totally ordered set with smallest element. Then $LO(I,A)$ is left ψ_{i_0}-biflat if and only if $|I| = 1$ and A is left ϕ-biflat.*

Proof. Suppose that $LO(I,A)$ is left ψ_{i_0}-biflat. We denote $F(I)$ for the collection of all finite subsets of I. It is known that by inclusion $F(I)$ is an ordered set. For each $\gamma \in F(I)$, put $e_\gamma = [a_{ij}]_{i,j \in I}$, with $a_{ij} = a_0$ whenever $i = j \in \gamma$ otherwise $a_{ij} = 0$. It is easy to see that $ae_\gamma - e_\gamma a \to 0$ and $\psi_{i_0}(e_\gamma) = 1$, for each $a \in LO(I,A)$. By similar arguments as in Lemma 2.7 it is easy to see that left ψ_{i_0}-biflatness of $LO(I,A)$ gives that $LO(I,A)$ is approximately left ψ_{i_0}-amenable. So there exists a net (a_α) in $LO(I,A)$ such that $aa_\alpha - \psi_{i_0}(a)a_\alpha \to 0$ and $\psi_{i_0}(a_\alpha) = 1$, for all $a \in LO(I,A)$. Set

$$L = \{[a_{ij}] \in LO(I,A) : a_{ij} = 0, \text{ whenever } \quad j \neq i_0\}.$$

It is easy to see that L is a closed ideal of $LO(I,A)$ with $\psi_{i_0}|_L \neq 0$. Suppose that i_1 is an element of L such that $\psi_{i_0}(i_1) = 1$. Replacing the net (a_α) with $(a_\alpha i_1)$, we can assume that $a_\alpha \in L$ such that $aa_\alpha - \psi_{i_0}(a)a_\alpha \to 0$ and $\psi_{i_0}(a_\alpha) = 1$, for all $a \in L$. We claim that $|I| = 1$. Suppose conversely that $|I| > 1$. Set

$$a_\alpha = \begin{bmatrix} a^\alpha_{i_0 i_0} & 0 & \cdots & 0 & \cdots \\ a^\alpha_{kk'} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a^\alpha_{ss'} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \quad l = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots \\ a_0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where $a^\alpha_{i_0 i_0}$ is an element of A such that $\phi(a^\alpha_{i_0 i_0}) = 1$. Thus $la_\alpha - \psi_{i_0}(l)a_\alpha \to 0$, follows that $a_0 a^\alpha_{i_0 i_0} \to 0$. Take ϕ on this equation gives that $\phi(a_0 a^\alpha_{i_0 i_0}) = \phi(a_0)\phi(a^\alpha_{i_0 i_0}) = \phi(a^\alpha_{i_0 i_0}) \to 0$. But $\phi(a^\alpha_{i_0 i_0}) = \psi_{i_0}(a_\alpha) = 1$, which is a contradiction. So $|I| = 1$ and A is left ϕ-biflat.
The converse is clear. □

At the end we illustrate an example of a Banach algebra which is neither left ϕ-biflat nor left ϕ-amenable.

Example 3.3. *Let $A = C^1[0,1]$, the space of all complex-valued differentiable maps on $[0,1]$ with continuous derivative. With the pointwise multiplication and $\|f\|_{C^1[0,1]} = \|f\|_\infty + \|f'\|_\infty$, A becomes a Banach algebra. We know from [9, Example 2.2.9] that the character space of A is*

$$\Delta(A) = \{\phi_t : \ \phi_t(f) = f(t) \text{ for each } t \in [0,1]\}.$$

Clearly A is commutative, so A is ϕ_t-inner amenable. The map $D : A \to \mathbb{C}$ given by $D(f) = f'(t)$, is a non-zero point derivation at ϕ_t for arbitrary $t \in [0,1]$. Thus by [10, Remark 2.4] A is not left ϕ_t-amenable for each $t \in [0,1]$. Next, we claim that A is not left ϕ_t-biflat for each $t \in [0,1]$. For if A is left ϕ_t-biflat for some $t \in [0,1]$, then it must be left ϕ_t-amenable by Lemma 2.7, which is not the case.

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