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On (L, N)-fuzzy betweenness relations

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Abstract. In this paper, aiming at the shortcomings of the definition of LRG-Galois connections in [51], we give a new definition of (strong) LRG-Galois connections, and we also introduce the notions of (L, N)-fuzzy betweenness relations. By using a strong LRG-Galois connection in our sense, it is shown that the category of (L, N)-fuzzy betweenness spaces and the category of (L, M)-fuzzy convex spaces are isomorphic. Moreover, it is proved that the lattice of (L, M)-fuzzy convex structures and the lattice of (L, N)-fuzzy betweenness relations are complete lattice isomorphic.

1. Introduction

Abstract convexity theory [30, 32] is one of the important branches of mathematics, it deals with set-theoretic structures which satisfies axioms similar to that usual convex sets fulfill. It plays an important role in various branches of mathematics. There are many different mathematical research fields that can be applied to axiomatic convexity, such as lattices, topological spaces, metric spaces and graphs (see, for example, [8, 13, 29, 31, 33, 34, 37, 44]).

Rosa [23, 24] first generalized convex structure to I-convex structure. Also, introduced fuzzy topology fuzzy convexity spaces and the notion of fuzzy local convexity. Subsequently, many scholars generalize convex structures to other fuzzy context from different viewpoints. Generally speaking, there are three approaches to extensions of convex structures to the fuzzy context, they are called L-convex structures(see, for example, [7, 9, 12, 15–17, 53]), M-fuzzifying convex structures (see, for example, [10, 14, 27, 36, 41, 42, 44, 51]) and (L, M)-fuzzy convex structures(see, for example, [25, 28]), respectively. Recently, there has been significant research on fuzzy convex structures (see, for example, [2, 11, 18–21, 26, 38–40, 43, 47–49, 52]). It should be stressed here that the notion of (L, M)-fuzzy convex structures was first introduced by Shi and Xiu in [28], and the concept of the product of (L, M)-fuzzy convex structures was presented and their fundamental properties were discussed. In particular, it was shown that that both L-convex structures and M-fuzzifying convex structures can be regarded as special cases of (L, M)-fuzzy convex structures.

When studying the relationship between convex structures and other structures, An order-reversing involution on lattice M plays an important role. An order-reversing involution can be regarded as a

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kind of generalization of complement. By using an order-reversing involution, Shi and Li [27] given the isomorphism between M-fuzzifying convex spaces and M-fuzzifying betweenness spaces. Yang and Pang[45] showed L-betweenness relations are categorically isomorphic to restricted L-hull operators and L-remotehood systems, respectively. Pang[21] introduced two types of fuzzy hull operators, which are called (L, M)-fuzzy hull operators and (L, M)-fuzzy restricted hull operators, respectively. It was shown that they can be used to characterize (L, M)-fuzzy convex structures. But an order-reversing involution can only transform each other on the same lattice. In view of this, Zhang[51] introduced a connection combined with the wedge-below relation, by using this tool, it was shown that the category of M-fuzzifying betweenness spaces and the category of M-fuzzifying convex spaces are isomorphic. It's worth mentioning that M-fuzzifying convex structures is a special kind of (L, M)-fuzzy convex structures. And there is a close relationship between M-fuzzifying betweenness space and M-fuzzifying hull spaces. This inspires us to define and study betweenness spaces in (L, N) (resp., (L, M))-fuzzy setting. We know that a pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ in [51] is called an LRG-Galois connection, if it satisfies the following conditions: For any $x \in M$ and $y \in N$,

(SG) $f(x) \geq y \Longleftrightarrow g(y) \geq x$.

(**LRG**) $f(x) \lhd^{op} y \Longleftrightarrow g(y) \lhd x$.

The idea of LRG-Galois connections proposed by Zhang is very good, because they can transform each other on two completely distributive lattices. However, if a pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ satisfies (**LRG**), then, for each $A \subseteq M$, $f(\bigwedge A) = \bigvee f(A)$ need not be ture, and there is no $f(1_M) = 0_N$ in general. Noticing this, in the current paper, we will redefine the concept of (strong) LRG-Galois connections on two completely distributive lattices to ensure that they have the function similar to order-reversing involutions.

This paper is organized as follows. In Section 2, we recall some necessary concepts which will be used in this paper. In Section 3, we will give a counterexamples to show that if a pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ satisfies (**LRG**), then f need not be an $\bigwedge - \bigvee$ mapping and there is no $f(1_M) = 0_N$ in general, then we will give a new defnition of (strong) LRG-Galois connections. In Section 4, as a generalization of L-betweenness spaces and N-fuzzifying betweenness spaces, we will introduce the concept of (L, N)-fuzzy betweenness spaces, then we will use a (strong) LRG-Galois connection to discuss the categorical relationship between (L, M)-fuzzy convex spaces and (L, N)-fuzzy betweenness spaces. Meanwhile, we also discuss the lattice relationship between (L, M)-fuzzy convex structures and (L, N)-fuzzy betweenness relations.

2. Preliminaries

Let X be a non-empty set, M (resp. N) be a complete lattice and 0_M (resp. 0_N) and 1_M (resp. 1_N) denote the least and the greatest elements in M, respectively. If a pair of mappings $f : M \longrightarrow N$ and $g : N \longrightarrow M$ ($M \underset{g}{\overset{f}{\rightleftarrows}} N$ for short) satisfies $f(x) \geq y$ if and only if $g(y) \geq x$ for any $x \in M$ and $y \in N$, then the pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ is called a connection between M and N (see[51]). We say that a is wedge below b in M, denoted by $a \lhd b$, if for every subset $C \subseteq M$, $\bigvee C \geqslant b$ implies $c \geqslant a$ for some $c \in C$ (see [6]). A complete lattice M is completely distributive lattice if and only if $b = \bigvee\{a \in M \mid a \lhd b\}$. $\beta(b)$ is the greatest minimal family of b. Further, the relation $\lhd^{op}$ in M is defined as follows: $b \lhd^{op} a$ iff for every subset $D \subseteq M$, $b \geqslant \bigwedge D$ implies $a \geqslant d$ for some $d \in D$. $\alpha(b) = \{a \in M \mid b \lhd^{op} a\}$ is the greatest maximal family of b. When M is a completely distributive lattice, each element b in M has the greatest minimal family (resp. the greatest maximal family), and $b = \bigvee \beta(b) = \bigwedge \alpha(b)$ for each $b \in M$(see [35]).

Let L be a complete lattice, we say that a is way below b in L, denoted by $a \ll b$, if for all directed subsets $E \subseteq L$, $\bigvee E \geqslant b$ implies $e \geqslant a$ for some $e \in E$. A complete lattice L is said to be continuous, if for all $c \in L$, $\Downarrow c$ is directed and $c = \bigvee \Downarrow c$, where $\Downarrow c = \{a \in L \mid a \ll c\}$ (see [4]). An L-fuzzy subset of X in [5] is a mapping $C : X \longrightarrow L$ and the family L^X denoted the set of all fuzzy subsets of a given X.

The operators on L can be translated onto L^X in a pointwise way. The greatest and the least elements in L^X are denoted by 1_X and 0_X, respectively. For each $\lambda \in L$, let $\underline{\lambda}$ denote the constant L-fuzzy subset of X with the value λ. x_λ is defined by $x_\lambda(y) = \lambda$ if $y = x$ and $x_\lambda(y) = 0$ otherwise, is called a fuzzy point. The set of all fuzzy points in L^X is denoted by $J(L^X)$. We say $\{B_i\}_{i\in I}$ is a directed subset of L^X, in symbols $\{B_i\}_{i\in I} \overset{dir}{\subseteq} L^X$ if for each $B_1, B_2 \in \{B_i\}_{i\in I}$, there exists $B_3 \in \{B_i\}_{i\in I}$ such that $B_1, B_2 \leq B_3$. We usually use the symbol $\bigvee^d_{i\in I} B_i$ to represent the supremum of a directed subset $\{B_i\}_{i\in I}$ of L^X. Let $L^{(X)}$ denote the family of all finite L-subsets of X, i.e., $L^{(X)} = \{B \in L^X \mid B$ is a finite L-subset of $X\}$, where B is finite means the support set of B, $\mathrm{Supp}B = \{x \in X \mid B(x) \neq 0_L\}$ is finite (see [21]). Some concepts related to category theorey can be found in [1].

Definition 2.1. ([27]) A pair $(X, \mathcal{B})$ is called an M-fuzzifying betweeenness space, where $\mathcal{B} : X \times 2^{(X)} \longrightarrow M$ satisfies the following conditions:

(MB1) For each $x \in X$, $\mathcal{B}(x, \emptyset) = 0_M$.

(MB2) If $A \in 2^{(X)}$ and $x \in A$, then $\mathcal{B}(x, A) = 1_M$.

(MB3)For each $A, B \in 2^{(X)}$ and $x \in X$, $\mathcal{B}(x, A) \geq \mathcal{B}(x, B) \wedge \bigwedge_{y\in B} \mathcal{B}(y, A)$.

Where $2^{(X)}$ be the family of all finite subsets of X, i.e., $A \in 2^{(X)}$ indicates that A is finite.

Definition 2.2. ([45]) A pair $(X, \mathcal{B})$ is called an L-betweenness space, where $\mathcal{B} \subseteq L^{(X)} \times J(L^X)$ satisfies the following conditions:

(LB1) For each $x_\lambda \in J(L^X)$, $(0_X, x_\lambda) \notin \mathcal{B}$.

(LB2) If $x_\lambda \leq A$, then $(A, x_\lambda) \in \mathcal{B}$.

(LB3) For each $A, B \in L^{(X)}$ and $x_\lambda \in J(L^X)$, if $(A, x_\lambda) \in \mathcal{B}$ and $(B, y_\mu) \in \mathcal{B}$ for all $y_\mu \leq A$, then $(B, x_\lambda) \in \mathcal{B}$.

(LB4) $(A, x_\lambda) \in \mathcal{B}$ if and only if $\forall \mu \ll \lambda$, there exists $B \ll A$ such that $(B, x_\mu) \in \mathcal{B}$.

(LB5) $(A, x_{\bigvee\limits_{i\in I} \lambda_i}) \in \mathcal{B}$ if and only if $\forall i \in I, (A, x_{\lambda_i}) \in \mathcal{B}$.

Definition 2.3. ([3]) A mapping $C : L^X \longrightarrow M$ is called an (L, M)-fuzzy closure system on X if it satisfies the following axioms:

(LMC1) $C(0_X) = C(1_X) = 1_M$.

(LMC2) If $\{A_j : j \in J\} \subseteq L^X$ is nonempty, then $C(\bigwedge_{j\in J} A_j) \geq \bigwedge_{j\in J} C(A_j)$.

If $C : L^X \longrightarrow M$ is an (L, M)-fuzzy closure system on X, then the pair (X, C) is called an (L, M)-fuzzy closure space.

Definition 2.4. ([20, 21, 25, 28]) A closure system C is called an (L, M)-fuzzy convex structure, if one of the following conditions hold (the second then following as a consequence)

(LMC3) If $\{A_j : j \in J\} \subseteq L^X$ is nonempty and totally ordered by inclusion, then $C(\bigvee_{j\in J} A_j) \geq \bigwedge_{j\in J} C(A_j)$.

(LMC3) * If $\{A_k : k \in K\} \subseteq L^X$ is directed, then $C(\bigvee^d_{k\in K} A_k) \geq \bigwedge_{k\in K} C(A_k)$.

The pair (X, C) is called an (L, M)-fuzzy convex space. A mapping $h : X \longrightarrow Y$ from an (L, M)-fuzzy convex space (X, C) to another (L, M)-fuzzy convex space $(Y, \mathcal{D})$ is said to be (L, M)-fuzzy convexity preserving function ((L, M)-CP in short) if $C(h^\leftarrow(B)) \geq \mathcal{D}(B)$ for all $B \in L^Y$. The category of all (L, M)-fuzzy convex spaces as objects and all (L, M)-CPs as morphisms is denoted by (L, M)-**FC**. Obviously, (L, M)-fuzzy convex space can degenerate to M-fuzzifying convex space and L-convex space by restricting $L = \{0, 1\}$, and $M = \{0, 1\}$.

Definition 2.5. ([21]) A pair $(X, \mathcal{H})$ is called an (L, M)-fuzzy restricted hull space, where $\mathcal{H} : L^{(X)} \longrightarrow M^{J(L^X)}$ satisfies the following conditions:

(LMRH1) $\mathcal{H}(0_X)(x_\lambda) = 0_M$.

(LMRH2) $\mathcal{H}(F)(x_\lambda) = 1_M$ for each $x_\lambda \leq F$.

(LMRH3) $\mathcal{H}(F)(y_\mu) \geq \mathcal{H}(G)(y_\mu) \wedge \bigwedge_{x_\lambda \leq G} \mathcal{H}(F)(x_\lambda)$.

(LMRH4) $\mathcal{H}(F)(x_\lambda) = \bigwedge_{\mu \ll \lambda} \bigvee_{G \ll F} \mathcal{H}(G)(x_\mu)$.

(LMRH5) $\mathcal{H}(F)(x_{\bigvee_{i\in I}\lambda_i}) = \bigwedge_{i\in I}\mathcal{H}(F)(x_{\lambda_i})$.

A mapping $h : X \longrightarrow Y$ from an (L, M)-fuzzy restricted hull space $(X, \mathcal{H}_1)$ to another (L, M)-fuzzy restricted hull space $(Y, \mathcal{H}_2)$ is said to be (L, M)-restricted hull-preserving mapping ((L, M)-RHP in short) if $\mathcal{H}_2(h^{\rightarrow}(U))(h(x)_\lambda) \geq \mathcal{H}_1(U)(x_\lambda)$ for each $U \in L^{(X)}$ and each $x_\lambda \in J(L^X)$. The category of all (L, M)-fuzzy restricted hull spaces as objects and all (L, M)-RHPs as morphisms is denoted by (L, M)-**FRH**.

Definition 2.6. ([22, 46]) Let $h : X \longrightarrow Y$. Then the image $h^{\rightarrow}(A)$ of $A \in L^X$ and the preimage $h^{\leftarrow}(B)$ of $B \in L^Y$ are defined by: $h^{\rightarrow}(A)(y) = \bigvee\{A(x) : x \in X, h(x) = y\}$ and $h^{\leftarrow}(B) = B \circ h$, respectively. It can be verified that the pair $(h^{\rightarrow}, h^{\leftarrow})$ is a Galois connection on $(L^X, \leq)$ and $(L^Y, \leq)$.

Lemma 2.7. ([21]) *(L, M)-**FC** is isomorphic to (L, M)-**FGH**.*

In the following sections, if not emphasized, we always assume that both M and N are completely distributive lattices, and L is a continuous lattice.

3. A new definition of LRG-Galois connections

In [51], if a pair of mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ is a connection and it satisfies the following condition:

(**LRG**) $f(x) \lhd^{op} y \Longleftrightarrow g(y) \lhd x$ for any $x \in M$ and $y \in N$.

Then the pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ is called an LRG-Galois connection.

Remark 3.1. If a pair of mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ satisfies (**LRG**), then f need not be an $\bigwedge - \bigvee$ mapping and there is no $f(1_M) = 0_N$ in general, i.e., Corollary 3.4 in [51] need not be true. For example, let $M = \{0_M, p, q, 1_M\}$ be a diamond-type lattice (see Fig.1), and let $N = \{0_N, a, b, c, d, 1_N\}$ (see Fig.2). Define two mappings $f : M \longrightarrow N$ and $g : N \longrightarrow M$ as follows:

$$f(x) = \begin{cases} a, & \text{if } x = 1_M, \\ b, & \text{if } x = p, \\ c, & \text{if } x = q, \\ 1_N, & \text{otherwise,} \end{cases} \qquad g(y) = \begin{cases} 0_M, & \text{if } y = d, \\ 0_M, & \text{if } y = 1_N, \\ q, & \text{if } y = c, \\ p, & \text{if } y = b, \\ 1_M, & \text{otherwise.} \end{cases}$$

Notice that $0_M \ntriangleleft 0_M \lhd p \lhd p$ $(q \lhd q) \lhd 1_M \ntriangleleft 1_M$, and $0_N \lhd^{op} 0_N \lhd^{op} a \ntriangleleft^{op} a \lhd^{op} b \lhd^{op} b$ $(c \lhd^{op} c) \lhd^{op} d \lhd^{op} d \lhd^{op} 1_N \ntriangleleft^{op} 1_N$. We can verify that $f(x) \lhd^{op} y \Longleftrightarrow g(y) \lhd x$ for any $x \in M$ and $y \in N$. However, $f(p \wedge q) \neq f(p) \vee f(q)$, and $f(1_M) \neq 0_N$, i.e., the pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ satisfies (**LRG**). But f need not be an $\bigwedge - \bigvee$ mapping and there is no $f(1_M) = 0_N$.

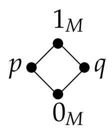

Fig.1 The structure of M

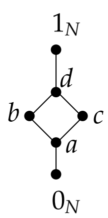

Fig.2 The structure of N

In order to give a reasonable definition with respect to LRG-Galois connections, let us recall that an order-reversing involution $'$ on M is a map $(-)' : M \longrightarrow M$ such that for any $c, d \in M$, the following conditions hold: (1) $c \leq d$ implies $d' \leq c'$. (2) $c'' = c$. The following properties hold for any subset $\{d_i : i \in J\} \in M$: (1) $\left(\bigvee_{i\in J} d_i\right)' = \bigwedge_{i\in J} d_i'$. (2) $\left(\bigwedge_{i\in J} d_i\right)' = \bigvee_{i\in J} d_i'$. In particular, $c' \geq d \Longleftrightarrow d' \geq c$, $1' = 0$ and $0' = 1$.

Notice that the properties of the order-reversing involution mentioned above, we redefine an LRG-Galois connection as follows.

Definition 3.2. A pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ is called an LRG-Galois connection, if $M \underset{g}{\overset{f}{\rightleftarrows}} N$ satisfies the following conditions: for any $x \in M$ and $y \in N$,

(SC) (Connection) $f(x) \geq y \Longleftrightarrow g(y) \geq x$.

(**LG**) ($\lhd^{op}$ connection on the left) $f(x) \lhd^{op} y \Longleftrightarrow g(y) \lhd x$.

(**RG**) ($\lhd^{op}$ connection on the right) $f(x) \lhd y \Longleftrightarrow g(y) \lhd^{op} x$.

(MJ) ($\bigwedge - \bigvee$ mapping) $f(\bigwedge A) = \bigvee f(A)$ for any $A \subseteq M$.

If a pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ is an LRG-Galois connection in our sense, and f is a bijection between M and N, then we say that the pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ is a strong LRG-Galois connection.

Remark 3.3. By **(SC)**, we obtain $f \circ g \geq id_N$ and $g \circ f \geq id_M$. By (**LG**), we obtain f is an antitone mapping, $f(0_M) = 1_N$, and $f(\bigvee A) = \bigwedge f(A)$ for any $A \subseteq M$(see [51]). Similarly, by (**RG**), we obtain g is also an antitone mapping, $g(0_N) = 1_M$ and $g(\bigvee B) = \bigwedge g(B)$ for any $B \subseteq N$.

Here we provide an example of an LRG-Galois connection in our sense.

Example 3.4. Let $M = \{0_M, c, d, e, 1_M\}$ (see Fig.3), and let $N = \{0_N, a, b, 1_N\}$ be a diamond-type lattice (see Fig.4). Define a pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ as follows:

$$f(x) = \begin{cases} a, & \text{if } x = c, \\ b, & \text{if } x = d, \\ 1_N, & \text{if } x = 0_M, \\ 0_N, & \text{otherwise,} \end{cases} \qquad g(y) = \begin{cases} 1_M, & \text{if } y = 0_N, \\ c, & \text{if } y = a, \\ d, & \text{if } y = b, \\ 0_M, & \text{otherwise.} \end{cases}$$

Then we can verify that the pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ is an LRG-Galois connection as defined in Definition 3.2.

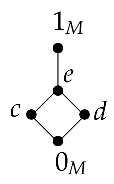

Fig.3 The structure of M

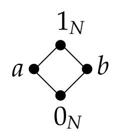

Fig.4 The structure of N

4. (L, N)-fuzzy betweenness relations

In this section, if not emphasized, we always assume that a pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ is an LRG-Galois connection as defined in Definition 3.2.

Definition 4.1. A mapping $\mathcal{B} : J(L^X) \times L^{(X)} \longrightarrow N$ is called an (L, N)-fuzzy betweenness relation on X if it satisfies the following conditions:

(LNB1) $\mathcal{B}(x_\lambda, 0_X) = 0_N$.

(LNB2) $\mathcal{B}(x_\lambda, F) = 1_N$ for each $x_\lambda \leq F$.

(LNB3)
$$\mathcal{B}(x_\lambda, G) \wedge \bigwedge_{y_\mu \leq G} \mathcal{B}(y_\mu, F) \leq \mathcal{B}(x_\lambda, F).$$

(LNB4)
$$\mathcal{B}(x_\lambda, F) = \bigwedge_{\mu \ll \lambda} \bigvee_{G \ll F} \mathcal{B}(x_\mu, G).$$

(LNB5)
$$\mathcal{B}(x_{\bigvee_{i \in I} \lambda_i}, F) = \bigwedge_{i \in I} \mathcal{B}(x_{\lambda_i}, F).$$

The pair $(X, \mathcal{B})$ is called an (L, N)-fuzzy betweenness space. A mapping $h : X \longrightarrow Y$ from an (L, N)-fuzzy betweenness space $(X, \mathcal{B}_1)$ to another (L, N)-fuzzy betweenness space $(Y, \mathcal{B}_2)$ is said to be (L, N)-fuzzy betweenness-preserving mapping ((L, N)-BP in short) if $\mathcal{B}_1(x_\lambda, A) \leq \mathcal{B}_2(h(x)_\lambda, h^{\rightarrow}(A))$ for each $A \in L^{(X)}$ and each $x_\lambda \in J(L^X)$. The category of all (L, N)-fuzzy betweenness spaces as objects and all (L, N)-BPs as morphisms is denoted by (L, N)-**FB**.

Remark 4.2. Obviously, if $L = \{0, 1\}$ and $N = M$, then (L, N)-fuzzy betweenness spaces can degenerate to M-fuzzifying betweenness spaces. If $N = \{0, 1\}$, then (L, N)-fuzzy betweenness spaces can degenerate to L-betweenness spaces. And, if $N = M$, then (L, M)-**FRH** is isomorphic to (L, M)-**FB**.

Theorem 4.3. *Let $(X, \mathcal{B})$ be an (L, N)-fuzzy betweenness space. Define $C_{\mathcal{B}} : L^X \longrightarrow M$ by*
$$\forall A \in L^X,\ C_{\mathcal{B}}(A) = \bigwedge_{x_\lambda \not\leq A} \bigwedge_{B \ll A} g(\mathcal{B}(x_\lambda, B)).$$

Then $(X, C_{\mathcal{B}})$ is an (L, M)-fuzzy convex space.

Proof. It suffices to verify that $C_{\mathcal{B}}$ satisfies (LMC1), (LMC2) and (LMC3)*. Indeed,

(LMC1)
$$C_{\mathcal{B}}(0_X) = \bigwedge_{x_\lambda \not\leq 0_X} \bigwedge_{B \ll 0_X} g(\mathcal{B}(x_\lambda, B)) = \bigwedge_{x_\lambda \not\leq 0_X} g\Big(\bigvee_{B \ll 0_X} \mathcal{B}(x_\lambda, B)\Big) = \bigwedge_{x_\lambda \not\leq 0_X} g(0_N) = 1_M,$$

and
$$C_{\mathcal{B}}(1_X) = \bigwedge_{x_\lambda \not\leq 1_X} \bigwedge_{B \ll 1_X} g(\mathcal{B}(x_\lambda, B)) = \bigwedge_{B \ll 1_X} g\Big(\bigvee_{x_\lambda \not\leq 1_X} \mathcal{B}(x_\lambda, B)\Big) = \bigwedge_{B \ll 1_X} g(\bigvee \emptyset) = 1_M.$$

(LMC2) If $\{A_j : j \in J\} \subseteq L^X$ is nonempty. Let $\alpha \in M$ and $C_{\mathcal{B}}(\bigwedge_{j \in J} A_j) \lhd^{op} \alpha$, then
$$g\Big(\bigvee_{x_\lambda \not\leq \bigwedge_{j \in J} A_j} \bigvee_{B \ll \bigwedge_{j \in J} A_j} \mathcal{B}(x_\lambda, B)\Big) = \bigwedge_{x_\lambda \not\leq \bigwedge_{j \in J} A_j} \bigwedge_{B \ll \bigwedge_{j \in J} A_j} g(\mathcal{B}(x_\lambda, B)) \lhd^{op} \alpha.$$

By **(RG)**, we have

$$f(\alpha) \lhd \bigvee_{x_\lambda \not\leq \bigwedge_{j\in J} A_j} \bigvee_{B \ll \bigwedge_{j\in J} A_j} \mathcal{B}(x_\lambda, B).$$

There exists $x_\lambda \in J(L^X)$ and $B_0 \in L^{(X)}$ such that $x_\lambda \not\leq \bigwedge_{j\in J} A_j, B_0 \ll \bigwedge_{j\in J} A_j$, and $f(a) \lhd \mathcal{B}(x_\lambda, B_0)$. Further, there exists $j_0 \in J$ such that $x_\lambda \not\leq A_{j_0}, B_0 \ll A_{j_0}$, and $g(\mathcal{B}(x_\lambda, B_0)) \lhd^{op} \alpha$. So, we have

$$\bigwedge_{j\in J} C_{\mathcal{B}}(A_j) = \bigwedge_{j\in J} \bigwedge_{y_\mu \not\leq A_j} \bigwedge_{C \ll A_j} g(\mathcal{B}(y_\mu, C)) \leq \bigwedge_{y_\mu \not\leq A_{j_0}} \bigwedge_{C \ll A_{j_0}} g(\mathcal{B}(y_\mu, C)) \leq g(\mathcal{B}(x_\lambda, B_0)) \lhd^{op} \alpha.$$

It follows that $\bigwedge_{j\in J} C_{\mathcal{B}}(A_j) \lhd^{op} \alpha$. Hence, we obtain

$$C_{\mathcal{B}}(\bigwedge_{j\in J} A_j) \geq \bigwedge_{j\in J} C_{\mathcal{B}}(A_j)$$

as desired.

(LMC3) $\star$ If $\{A_k : k \in K\} \subseteq L^X$ is directed. Let $\alpha \in M$ and $C_{\mathcal{B}}(\bigvee^d_{k\in K} A_k) \lhd^{op} \alpha$, then

$$g\Big(\bigvee_{x_\lambda \not\leq \bigvee^d_{k\in K} A_k} \bigvee_{B \ll \bigvee^d_{k\in K} A_k} \mathcal{B}(x_\lambda, B)\Big) = \bigwedge_{x_\lambda \not\leq \bigvee^d_{k\in K} A_k} \bigwedge_{B \ll \bigvee^d_{k\in K} A_k} g(\mathcal{B}(x_\lambda, B)) \lhd^{op} \alpha.$$

By **(RG)**, we have

$$f(a) \lhd \bigvee_{x_\lambda \not\leq \bigvee^d_{k\in K} A_k} \bigvee_{B \ll \bigvee^d_{k\in K} A_k} \mathcal{B}(x_\lambda, B).$$

There exists $x_\lambda \in J(L^X)$ and $B_0 \in L^{(X)}$ such that

$$x_\lambda \not\leq \bigvee^d_{k\in K} A_k, B_0 \ll \bigvee^d_{k\in K} A_k \text{ and } f(a) \lhd \mathcal{B}(x_\lambda, B_0).$$

Further, there exists $k_0 \in K$ such that $B_0 \ll A_{k_0}$ and $g(\mathcal{B}(x_\lambda, B_0)) \lhd^{op} \alpha$. And, if $x_\lambda \not\leq \bigvee^d_{k\in K} A_k$, then $x_\lambda \not\leq A_k$ for each $k \in K$. So, we have

$$\bigwedge_{k\in K} C_{\mathcal{B}}(A_k) = \bigwedge_{k\in K} \bigwedge_{y_\mu \not\leq A_k} \bigwedge_{C \ll A_k} g(\mathcal{B}(y_\mu, C)) \leq \bigwedge_{y_\mu \not\leq A_{k_0}} \bigwedge_{C \ll A_{k_0}} g(\mathcal{B}(y_\mu, C)) \leq g(\mathcal{B}(x_\lambda, B_0)) \lhd^{op} \alpha.$$

It follows that $\bigwedge_{k\in K} C_{\mathcal{B}}(A_k) \lhd^{op} \alpha$. Hence, we obtain

$$C_{\mathcal{B}}(\bigvee^d_{k\in K} A_k) \geq \bigwedge_{k\in K} C_{\mathcal{B}}(A_k)$$

as desired. $\square$

Theorem 4.4. *Let (X, C) be an (L, M)-fuzzy convex space. Define $\mathcal{B}_C : J(L^X) \times L^{(X)} \longrightarrow N$ by*

$$\forall F \in L^{(X)},\ \forall x_\lambda \in J(L^X),\ \mathcal{B}_C(x_\lambda, F) = \bigwedge_{x_\lambda \not\leq G \geq F} f(C(G)).$$

Then $(X, \mathcal{B}_C)$ is an (L, N)-fuzzy betweenness space.

Proof. It suffices to verify that $\mathcal{B}_C$ satisfies (LNB1)-(LNB5). Indeed,

(LNB1) By **(MJ)**, we have

$$\mathcal{B}_C(x_\lambda, 0_X) = \bigwedge_{x_\lambda \not\leq G \geq 0_X} f(C(G)) \leq f(C(0_X)) = 0_N.$$

(LNB2) For each $x_\lambda \leq F$, we have

$$\mathcal{B}_C(x_\lambda, F) = \bigwedge_{x_\lambda \not\leq G \geq F} f(C(G)) = \bigwedge \emptyset = 1_N.$$

(LNB3) Let $\alpha \in N$, and

$$\alpha \lhd \mathcal{B}_C(x_\lambda, G) \wedge \bigwedge_{y_\mu \leq G} \mathcal{B}_C(y_\mu, F),$$

then $\alpha \leq \mathcal{B}_C(x_\lambda, G)$, and $\alpha \leq \mathcal{B}_C(y_\mu, F)$ for each $y_\mu \leq G$. Let

$$D_0 = \bigvee \{z_w \in J(L^X) \mid \alpha \leq \mathcal{B}_C(z_w, F)\},$$

then $G \leq D_0$, and

$$\alpha \leq \mathcal{B}_C(x_\lambda, G) = \bigwedge_{x_\lambda \not\leq H \geq G} f(C(H)) \leq \bigwedge_{x_\lambda \not\leq H \geq D_0} f(C(D_0)) = \mathcal{B}_C(x_\lambda, D_0).$$

This implies that $\alpha \leq \mathcal{B}_C(x_\lambda, D_0)$. By the definition of $\mathcal{B}_C$, we have

$$\begin{aligned}
\mathcal{B}_C(x_\lambda, F) = \bigwedge_{x_\lambda \not\leq G \geq F} f(C(G)) = f\Big(\bigvee_{x_\lambda \not\leq G \geq F} C(G)\Big) &= f\Big(\bigvee_{x_\lambda \not\leq G \geq F} \bigwedge_{z_\delta \not\leq G} C(G)\Big) \\
&\geq f\Big(\bigvee_{x_\lambda \not\leq G \geq F} \bigwedge_{z_\delta \not\leq G} \bigvee_{z_\delta \not\leq H \geq G} C(H)\Big) \\
&= \bigwedge_{x_\lambda \not\leq G \geq F} f\Big(\bigwedge_{z_\delta \not\leq G} \bigvee_{z_\delta \not\leq H \geq G} C(H)\Big) \\
&\geq \bigwedge_{x_\lambda \not\leq G \geq F} f\Big(\bigvee_{x_\lambda \not\leq H \geq G} C(G)\Big) \\
&\geq f\Big(\bigvee_{x_\lambda \not\leq H \geq F} C(H)\Big) \\
&= \mathcal{B}_C(x_\lambda, F).
\end{aligned}$$

Hence,

$$\mathcal{B}_C(x_\lambda, F) = \bigwedge_{x_\lambda \not\leq G \geq F} f\Big(\bigwedge_{z_\delta \not\leq G} \bigvee_{z_\delta \not\leq H \geq G} C(H)\Big).$$

Now, we only need to show that $\alpha \leq \mathcal{B}_C(x_\lambda, F)$. If not, there exists $G_0 \in L^{(X)}$ such that $x_\lambda \not\leq G_0 \geq F$, and

$$\alpha \not\leq f\Big(\bigwedge_{z_\delta \not\leq G_0} \bigvee_{z_\delta \not\leq H \geq G_0} C(H)\Big) \geq \bigvee_{z_\delta \not\leq G_0} f\Big(\bigvee_{z_\delta \not\leq H \geq G_0} C(H)\Big).$$

Hence,

$$\alpha \not\leq \bigvee_{z_\delta \not\leq G_0} f\Big(\bigvee_{z_\delta \not\leq H \geq G_0} C(H)\Big) = \bigvee_{z_\delta \not\leq G_0} \mathcal{B}_C(z_\delta, G_0). \tag{1}$$

It implies that $\alpha \not\leq \mathcal{B}_C(z_\delta, G_0)$ for each $z_\delta \not\leq G_0$, i.e., if $z_\delta \in \{x_\gamma \in J(L^X) \mid \alpha \leq \mathcal{B}_C(x_\gamma, G_0)\}$, then $z_\delta \in \{x_\gamma \in J(L^X) \mid x_\gamma \leq G_0\}$. Let $H_0 = \bigvee \{x_\gamma \in J(L^X) \mid \alpha \leq \mathcal{B}_C(x_\gamma, G_0)\}$, then $x_\lambda \not\leq G_0 \geq H_0 \geq D_0$. By (1), we have

$$\alpha \not\leq \bigvee_{z_\delta \not\leq G_0} f\Big(\bigvee_{z_\delta \not\leq H \geq G_0} C(H)\Big) \geq f\Big(\bigvee_{x_\lambda \not\leq H \geq D_0} C(H)\Big) = \mathcal{B}_C(x_\lambda, D_0).$$

So, we have $\alpha \nleq \mathcal{B}_C(x_\lambda, D_0)$. Which is a contradiction. Therefore, $\alpha \leq \mathcal{B}_C(x_\lambda, F)$. Hence,

$$\mathcal{B}_C(x_\lambda, G) \wedge \bigwedge_{y_\mu \leq G} \mathcal{B}_C(y_\mu, F) \leq \mathcal{B}_C(x_\lambda, F).$$

(LNB4) Let $\beta \in N$ and $\mathcal{B}_C(x_\lambda, F) \lhd^{op} \beta$. By **(LG)**, we have

$$f\Big(\bigvee_{x_\lambda \nleq G \geq F} C(G)\Big) \lhd^{op} \beta \Longleftrightarrow g(\beta) \lhd \bigvee_{x_\lambda \nleq G \geq F} C(G)),$$

there exists $G_0 \in L^X$ such that $\bigvee_{\mu \ll \lambda} x_\mu = x_\lambda \nleq G_0 \geq F$ and $g(\beta) \lhd C(G_0)$. Further, there exists $\mu_0 \ll \lambda$ such that $x_{\mu_0} \nleq G_0 \geq F$, and $g(\beta) \lhd C(G_0)$. Notice that

$$\bigvee_{\mu \ll \lambda} \bigwedge_{G \ll F} \bigvee_{x_\mu \nleq H \geq G} C(H) \geq \bigvee_{\mu \ll \lambda} \bigvee_{x_\mu \nleq H \geq F} C(H) \geq \bigvee_{x_{\mu_0} \nleq H \geq F} C(H) \geq C(G_0).$$

So, we have

$$g(\beta) \lhd \bigvee_{\mu \ll \lambda} \bigwedge_{G \ll F} \bigvee_{x_\mu \nleq H \geq G} C(H).$$

By **(LG)**, we have

$$\bigwedge_{\mu \ll \lambda} f\Big(\bigwedge_{G \ll F} \bigvee_{x_\mu \nleq H \geq G} C(H)\Big) = f\Big(\bigvee_{\mu \ll \lambda} \bigwedge_{G \ll F} \bigvee_{x_\mu \nleq H \geq G} C(H)\Big) \lhd^{op} \beta.$$

Notice that

$$\bigwedge_{\mu \ll \lambda} \bigvee_{G \ll F} \mathcal{B}_C(x_\mu, G) = \bigwedge_{\mu \ll \lambda} \bigvee_{G \ll F} f\Big(\bigvee_{x_\mu \nleq H \geq G} C(H)\Big) \leq \bigwedge_{\mu \ll \lambda} f\Big(\bigwedge_{G \ll F} \bigvee_{x_\mu \nleq H \geq G} C(H)\Big).$$

So, we have

$$\bigwedge_{\mu \ll \lambda} \bigvee_{G \ll F} \mathcal{B}_C(x_\mu, G) \lhd^{op} \beta.$$

Hence,

$$\mathcal{B}_C(x_\lambda, F) \geq \bigwedge_{\mu \ll \lambda} \bigvee_{G \ll F} \mathcal{B}_C(x_\mu, G).$$

Conversely, let $\beta \in N$ and $\bigwedge_{\mu \ll \lambda} \bigvee_{G \ll F} \mathcal{B}_C(x_\mu, G) \lhd^{op} \beta$, then

$$\bigwedge_{\mu \ll \lambda} \bigvee_{G \ll F} \bigwedge_{x_\mu \nleq H \geq G} f\big(C(H)\big) \lhd^{op} \beta.$$

So, we have

$$f\Big(\bigvee_{\mu \ll \lambda} \bigwedge_{G \ll F} \bigvee_{x_\mu \nleq H \geq G} C(H)\Big) \lhd^{op} \beta \Longleftrightarrow g(\beta) \lhd \bigvee_{\mu \ll \lambda} \bigwedge_{G \ll F} \bigvee_{x_\mu \nleq H \geq G} C(H).$$

Notice that

$$\bigvee_{\mu \ll \lambda} \bigwedge_{G \ll F} \bigvee_{x_\mu \nleq H \geq G} C(H) \leq \bigvee_{x_\lambda \nleq H \geq F} C(H).$$

Thus, we obtain $g(\beta) \lhd \bigvee_{x_\lambda \nleq H \geq F} C(H)$. By **(LG)**, we have

$$\mathcal{B}_C(x_\lambda, F) = f\Big(\bigvee_{x_\lambda \nleq H \geq F} C(H)\Big) \lhd^{op} \beta.$$

Hence,

$$\mathcal{B}_C(x_\lambda, F) \leq \bigwedge_{\mu \ll \lambda} \bigvee_{G \ll F} \mathcal{B}_C(x_\mu, G).$$

(LNB5) Let $\beta \in N$ and $\mathcal{B}_C(x_{\bigvee_{i\in I}\lambda_i}, F) \lhd^{op} \beta$. By **(LG)**, we have

$$f\Big(\bigvee_{x_{\bigvee_{i\in I}\lambda_i} \nleq G \geq F} C(G))\Big) \lhd^{op} \beta \iff g(\beta) \lhd \bigvee_{x_{\bigvee_{i\in I}\lambda_i} \nleq G \geq F} C(G)) = \bigvee_{i\in I} \bigvee_{x_{\lambda_i} \nleq G \geq F} C(G)$$

$$\iff f\Big(\bigvee_{i\in I} \bigvee_{x_{\lambda_i} \nleq G \geq F} C(G)\Big) = \bigwedge_{i\in I} \mathcal{B}_C(x_{\lambda_i}, F) \lhd^{op} \beta.$$

Hence, $\mathcal{B}_C(x_{\bigvee_{i\in I}\lambda_i}, F) = \bigwedge_{i\in I} \mathcal{B}_C(x_{\lambda_i}, F)$. $\square$

Proposition 4.5. *(1) If $h : (X, C_X) \longrightarrow (Y, C_Y)$ is an (L, M)-CP, then $g : (X, \mathcal{B}_{C_X}) \longrightarrow (Y, \mathcal{B}_{C_Y})$ is an (L, N)-BP.*
(2) If $h : (X, \mathcal{B}_X) \longrightarrow (Y, \mathcal{B}_Y)$ is an (L, N)-BP, then $h : (X, C_{\mathcal{B}_X}) \longrightarrow (Y, C_{\mathcal{B}_Y})$ is an (L, M)-CP.

Proof. The proof is straightforward. So, we omit it. $\square$

Theorem 4.6. *If (X, C) is an (L, M)-fuzzy convex space, and $(X, \mathcal{B})$ an (L, N)-fuzzy betweenness space. Then,*
(1) $\forall A \in L^X$, $C_{\mathcal{B}_C}(A) \geq C(A)$.
(2) $\forall F \in L^{(X)}, \forall x_\lambda \in J(L^X)$, $\mathcal{B}_{C_{\mathcal{B}}}(x_\lambda, F) \geq \mathcal{B}(x_\lambda, F)$.

Proof. For (1), $\forall A \in L^X$. By **(SC)**, we have

$$\begin{aligned}
C_{\mathcal{B}_C}(A) &= \bigwedge_{x_\lambda \nleq A} \bigwedge_{B \ll A} g(\mathcal{B}_C(x_\lambda, B)) \\
&= \bigwedge_{x_\lambda \nleq A} \bigwedge_{B \ll A} g\Big(\bigwedge_{x_\lambda \nleq G \geq B} f\big(C(G))\big)\Big) \\
&= \bigwedge_{x_\lambda \nleq A} \bigwedge_{B \ll A} g\Big(f\big(\bigvee_{x_\lambda \nleq G \geq B} C(G))\big)\Big) \\
&\geq \bigwedge_{x_\lambda \nleq A} \bigwedge_{B \ll A} \bigvee_{x_\lambda \nleq G \geq B} C(G)) \\
&= \bigwedge_{B \ll A} \bigwedge_{x_\lambda \nleq A} \bigvee_{x_\lambda \nleq G \geq B} C(G)) \\
&\geq \bigwedge_{B \ll A} \bigwedge_{x_\lambda \nleq A} \bigvee_{x_\lambda \nleq G \geq A} C(G)) \\
&\geq \bigwedge_{B \ll A} \bigwedge_{x_\lambda \nleq A} C(A) = C(A).
\end{aligned}$$

For (2), $\forall F \in L^{(X)}, \forall x_\lambda \in J(L^X)$, we have

$$\begin{aligned}
\mathcal{B}_{C_{\mathcal{B}}}(x_\lambda, F) &= \bigwedge_{x_\lambda \nleq G \geq F} f(C_{\mathcal{B}}(G)) \\
&= f\Big(\bigvee_{x_\lambda \nleq G \geq F} \bigwedge_{y_\mu \nleq G} \bigwedge_{H \ll G} g(\mathcal{B}(y_\mu, H))\Big) \\
&= f\Big(\bigvee_{x_\lambda \nleq G \geq F} \bigwedge_{H \ll G} \bigwedge_{y_\mu \nleq G} g(\mathcal{B}(y_\mu, H))\Big) \\
&= f\Big(\bigvee_{\omega \ll \lambda} \bigvee_{x_\omega \nleq G \geq F} \bigwedge_{H \ll G} \bigwedge_{y_\mu \nleq G} g(\mathcal{B}(y_\mu, H))\Big).
\end{aligned}$$

Let $\alpha \in N$ and $\mathcal{B}_{C_{\mathcal{B}}}(x_\lambda, F) \lhd^{op} \alpha$, then

$$f\Big(\bigvee_{\omega \ll \lambda} \bigvee_{x_\omega \nleq G \geq F} \bigwedge_{H \ll G} \bigwedge_{y_\mu \nleq G} g(\mathcal{B}(y_\mu, H))\Big) \lhd^{op} \alpha.$$

By **(LG)**, we obtain

$$\begin{aligned} g(\alpha) &\lhd \bigvee_{\omega\ll\lambda}\bigvee_{x_\omega\nleq G\geq F}\bigwedge_{H\ll G}\bigwedge_{y_\mu\nleq G} g(\mathcal{B}(y_\mu,H)) \\ &\leq \bigvee_{\omega\ll\lambda}\bigvee_{x_\omega\nleq G\geq F}\bigwedge_{H\ll G} g(\mathcal{B}(x_\omega,H)) \\ &\leq \bigvee_{\omega\ll\lambda}\bigvee_{x_\omega\nleq G\geq F}\bigwedge_{H\ll F} g(\mathcal{B}(x_\omega,H)) \\ &= \bigvee_{\omega\ll\lambda}\bigwedge_{H\ll F} g(\mathcal{B}(x_\omega,H)) \\ &= \bigvee_{\omega\ll\lambda} g\Big(\bigvee_{H\ll F}\mathcal{B}(x_\omega,H)\Big), \end{aligned}$$

there exists $\omega_0 \ll \lambda$ such that $g(\alpha) \lhd g\Big(\bigvee_{H\ll F}\mathcal{B}(x_{\omega_0},H)\Big)$, which is equivalent to that

$$f\Big(g\Big(\bigvee_{H\ll F}\mathcal{B}(x_{\omega_0},H)\Big)\Big) \lhd^{op} \alpha.$$

Notice that

$$\mathcal{B}(x_\lambda,F) = \bigwedge_{\mu\ll\lambda}\bigvee_{H\ll F}\mathcal{B}(x_\mu,H) \leq \bigvee_{H\ll F}\mathcal{B}(x_{\omega_0},H) \leq f\Big(g\Big(\bigvee_{H\ll F}\mathcal{B}(x_{\omega_0},H)\Big)\Big).$$

So, we have $\mathcal{B}(x_\lambda,F) \lhd^{op} \alpha$. It follows that $\mathcal{B}_{C_\mathcal{B}}(x_\lambda,F) \geq \mathcal{B}(x_\lambda,F)$. □

By Theorem 4.3 and Proposition 4.5, we obtain a concrete functor $\Theta : (L,N)$-**FB**$\longrightarrow (L,M)$-**FC** by

$$\Theta : (X,\mathcal{B}) \mapsto (X, C_\mathcal{B}) \text{ and } h \mapsto h.$$

Similarly, by Theorem 4.4 and Proposition 4.5, we obtain a concrete functor $\Psi : (L,M)$-**FC**$\longrightarrow (L,N)$-**FB** by

$$\Psi : (X,C) \mapsto (X,\mathcal{B}_C) \text{ and } h \mapsto h.$$

Next, let us prove that if a pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ is a strong LRG-Galois connection, then Θ and Ψ are isomorphic functors.

Theorem 4.7. *If a pair mappings $M \underset{g}{\overset{f}{\rightleftarrows}} N$ is a strong LRG-Galois connection, then (L,M)-**FC** is isomorphic to (L,N)-**FB**.*

Proof. We only need show that the following results: if (X,C) is an (L,M)-fuzzy convex space, and $(X,\mathcal{B})$ an (L,N)-fuzzy betweenness space. Then,

(1) $\forall A \in L^X$, $C_{\mathcal{B}_C}(A) \leq C(A)$.

(2) $\forall F \in L^{(X)}, \forall x_\lambda \in J(L^X)$, $\mathcal{B}_{C_\mathcal{B}}(x_\lambda,F) \leq \mathcal{B}(x_\lambda,F)$.

Indeed, for (1), since

$$\forall A \in L^X,\ C(A) \leq \bigwedge_{x_\lambda\nleq A}\bigvee_{x_\lambda\nleq G\geq A} C(G) = \bigvee_{h\in\prod\limits_{x_\lambda\nleq A}\mathfrak{D}_{x_\lambda}}\bigwedge_{x_\lambda\nleq A} C(h(x_\lambda)) \leq \bigvee_{h\in\prod\limits_{x_\lambda\nleq A}\mathfrak{D}_{x_\lambda}} C\Big(\bigwedge_{x_\lambda\nleq A} h(x_\lambda)\Big) = C(A),$$

where $\mathfrak{D}_{x_\lambda} = \{G \in L^X \mid x_\lambda \nleq G \geq A\}$. This implies that $\mathcal{C}(A) = \bigwedge_{x_\lambda \nleq A} \bigvee_{x_\lambda \nleq G \geq A} \mathcal{C}(G)$. By **(MJ)**, we have

$$\begin{aligned}
\mathcal{C}_{\mathcal{B}_{\mathcal{C}}}(A) &= \bigwedge_{x_\lambda \nleq A} \bigvee_{x_\lambda \nleq G \geq A} \mathcal{C}_{\mathcal{B}_{\mathcal{C}}}(G) \\
&= \bigwedge_{x_\lambda \nleq A} \bigvee_{x_\lambda \nleq G \geq A} \bigwedge_{y_\omega \nleq G} \bigwedge_{H \ll G} g(\mathcal{B}_{\mathcal{C}}(y_\omega, H)) \\
&= \bigwedge_{x_\lambda \nleq A} \bigvee_{x_\lambda \nleq G \geq A} \bigwedge_{y_\omega \nleq G} \bigwedge_{H \ll G} g\Big(\bigwedge_{y_\omega \nleq W \geq H} f\big(\mathcal{C}(W)\big)\Big) \\
&= \bigwedge_{x_\lambda \nleq A} \bigvee_{x_\lambda \nleq G \geq A} \bigwedge_{y_\omega \nleq G} \bigwedge_{H \ll G} \bigvee_{y_\omega \nleq W \geq H} \mathcal{C}(W) \\
&= \bigwedge_{x_\lambda \nleq A} \bigvee_{\mu \ll \lambda} \bigvee_{x_\mu \nleq G \geq A} \bigwedge_{y_\omega \nleq G} \bigwedge_{H \ll G} \bigvee_{y_\omega \nleq W \geq H} \mathcal{C}(W) \\
&\leq \bigwedge_{x_\lambda \nleq A} \bigvee_{\mu \ll \lambda} \bigvee_{x_\mu \nleq G \geq A} \bigwedge_{H \ll G} \bigvee_{x_\mu \nleq W \geq H} \mathcal{C}(W) \\
&\leq \bigwedge_{x_\lambda \nleq A} \bigvee_{\mu \ll \lambda} \bigvee_{x_\mu \nleq G \geq A} \bigwedge_{H \ll A} \bigvee_{x_\mu \nleq W \geq H} \mathcal{C}(W) \\
&= \bigwedge_{x_\lambda \nleq A} \bigvee_{\mu \ll \lambda} \bigwedge_{H \ll A} \bigvee_{x_\mu \nleq W \geq H} \mathcal{C}(W) \\
&= \bigwedge_{x_\lambda \nleq A} g\Big(\bigwedge_{\mu \ll \lambda} \bigvee_{H \ll A} \bigwedge_{x_\mu \nleq W \geq H} f\big(\mathcal{C}(W)\big)\Big) \\
&= \bigwedge_{x_\lambda \nleq A} g\Big(\bigwedge_{\mu \ll \lambda} \bigvee_{H \ll A} \mathcal{B}_{\mathcal{C}}(x_\mu, H)\Big) \\
&= g\Big(\bigvee_{x_\lambda \nleq A} \mathcal{B}_{\mathcal{C}}(x_\lambda, A)\Big) \\
&= g\Big(f\Big(\bigwedge_{x_\lambda \nleq A} \bigvee_{x_\lambda \nleq G \geq A} \mathcal{C}(G)\Big)\Big) \\
&= \bigwedge_{x_\lambda \nleq A} \bigvee_{x_\lambda \nleq G \geq A} \mathcal{C}(G) \\
&= \mathcal{C}(A).
\end{aligned}$$

For (2), by **(MJ)**, we have

$$\begin{aligned}
\mathcal{B}_{\mathcal{C}_{\mathcal{B}}}(x_\lambda, F) &= \bigwedge_{x_\lambda \nleq G \geq F} f(\mathcal{C}_{\mathcal{B}}(G)) \\
&= \bigwedge_{x_\lambda \nleq G \geq F} f\Big(\bigwedge_{z_\omega \nleq G} \bigwedge_{H \ll G} g(\mathcal{B}(z_\omega, H))\Big) \\
&= \bigwedge_{x_\lambda \nleq G \geq F} \bigvee_{z_\omega \nleq G} \bigvee_{H \ll G} f\big(g(\mathcal{B}(z_\omega, H))\big) \\
&= \bigwedge_{x_\lambda \nleq G \geq F} \bigvee_{z_\omega \nleq G} \bigvee_{H \ll G} \mathcal{B}(z_\omega, H).
\end{aligned}$$

Let $a \in N$, and $a \lhd \mathcal{B}_{\mathcal{C}_{\mathcal{B}}}(x_\lambda, F)$, then for each $G \in \{G \in L^X \mid x_\lambda \nleq G \geq F\}$, there exists $z_\omega \in J(L^X)$ and $H \in L^X$, such that $z_\omega \nleq G$, $H \ll G$ and $a \leq \mathcal{B}(z_\omega, H)$. Now, we only need to show that $a \leq \mathcal{B}(x_\lambda, F)$. If not, let $G_0 = \bigvee\{y_\mu \in J(L^X) \mid \mathcal{B}(y_\mu, F) \geq a\}$, then by (LNB2) and (LNB5), we have $x_\lambda \nleq G_0 \geq F$, i.e., $G_0 \in \{G \in L^X \mid x_\lambda \nleq G \geq F\}$. So, there exists $(z_0)_\omega \in J(L^X)$ and $H_0 \in L^X$, such that $(z_0)_\omega \nleq G_0$, $H_0 \ll G_0$ and $a \leq \mathcal{B}((z_0)_\omega, H_0)$. By (LNB3) and (LNB5), we obtain $a \leq \mathcal{B}((z_0)_\omega, H_0) \wedge \bigwedge_{y_\mu \leq H_0} \mathcal{B}(y_\mu, F) \leq \mathcal{B}((z_0)_\omega, F)$. This implies that $(z_0)_\omega \leq G_0$. Which is a contradiction. Hence, $a \leq \mathcal{B}(x_\lambda, F)$. By the arbitrariness of a, we obtain $\mathcal{B}_{C_\mathcal{B}}(x_\lambda, F) \leq \mathcal{B}(x_\lambda, F)$ as desired. □

Now, let (L, M)-**FCS(X)** be the family of all (L, M)-fuzzy convex structures on X, define a relation $\leq$ on (L, M)-**FCS(X)** as follows: $C_1 \leq C_2$ if and only if $C_1(A) \leq C_2(A)$ for all $A \in L^X$, then we easily verify that $((L, M)$-**FCS(X)**$, \leq)$ is a poset. Further, define $C^1 : L^X \longrightarrow M$ as follows: $\forall A \in L^X$, $C^1(A) = 1_M$, then C^1 is the greatest element in $((L, M)$-**FCS(X)**$, \leq)$, and $\forall \{C_j\}_{j \in J} \subseteq (L, M)$-**FCS(X)**, we easily show that $C : L^X \longrightarrow M$ defined by $C(A) = \bigwedge_{j \in J} C_j(A)$ is the infimum of $\{C_j\}_{j \in J}$. So, $((L, M)$-**FCS(X)**$, \leq)$ is a complete lattice (see [50]). Similarly, let (L, N)-**FB(X)** be the family of all (L, N)-fuzzy betweenness relations on X, define a relation $\leq$ on (L, N)-**FB(X)** as follows: $\mathcal{B}_1 \leq \mathcal{B}_2$ if and only if $\mathcal{B}_1(x_\lambda, A) \geq \mathcal{B}_2(x_\lambda, A)$ for all $A \in L^{(X)}$ and $x_\lambda \in J(L^X)$, then we easily verify that $((L, N)$-**FB(X)**$, \leq)$ is also a poset.

Theorem 4.8. *Suppose a pair mappings* $M \underset{g}{\overset{f}{\rightleftarrows}} N$ *is a strong LRG-Galois connection. Define a mapping* $\mathfrak{F} : ((L, M)$-**FCS(X)**$, \leq) \longrightarrow ((L, N)$-**FB(X)**$, \leq)$ *as follows:* $\forall F \in L^{(X)}$, $\forall x_\lambda \in J(L^X)$,

$$\mathfrak{F}(C)(x_\lambda, F) = \mathcal{B}_C(x_\lambda, F) = \bigwedge_{x_\lambda \not\leq G \geq F} f(C(G)),$$

and define a mapping $\mathfrak{G} : ((L, N)$-**FB(X)**$, \leq) \longrightarrow ((L, M)$-**FCS(X)**$, \leq)$ *as follows:*

$$\forall A \in L^X, \mathfrak{G}(\mathcal{B})(A) = C_\mathcal{B}(A) = \bigwedge_{x_\lambda \not\leq A} \bigwedge_{B \ll A} g(\mathcal{B}(x_\lambda, B)).$$

Then,

(1) $\mathfrak{F}$ *is a bijection. And, both* $\mathfrak{F}$ *and* $\mathfrak{F}^{-1}$ *are order preserving mappings.*

(2) $((L, M)$-**FCS(X)**$, \leq)$ *and* $((L, N)$-**FB(X)**$, \leq)$ *are complete lattice isomorphic.*

Proof. (1) By Theorem 4.7, we easily obtain $\mathfrak{F}$ is a bijection. And, both $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ are order preserving mappings.

(2) For all $F \in L^{(X)}$ and $x_\lambda \in J(L^X)$. If $x_\lambda \leq F$, then $\mathfrak{F}(C^1)(x_\lambda, F) = \mathcal{B}_{C^1}(x_\lambda, F) = 1_N$; if $x_\lambda \not\leq F$, then

$$\mathfrak{F}(C^1)(x_\lambda, F) = \mathcal{B}_{C^1}(x_\lambda, F) = \bigwedge_{x_\lambda \not\leq G \geq F} f(C^1(G)) \leq f(C^1(F)) = f(1_M) = 0_N.$$

So, we easily obtain $\mathfrak{F}(C^1)$ is the greatest element in $((L, N)$-**FB(X)**$, \leq)$. Now, we only need to prove that it's closed for non-empty intersection operation in $((L, N)$-**FB(X)**$, \leq)$. Indeed, for any $\mathbb{B} \subseteq (L, N)$-**FB(X)** and $\mathbb{B} \neq \emptyset$, since $\mathfrak{F}$ is a bijection, there exists $\mathbb{C} \subseteq (L, M)$-**FCS(X)** such that $\mathfrak{F}(\mathbb{C}) = \mathbb{B}$. Thus, we obtain $\bigwedge \mathbb{B} = \bigwedge \mathfrak{F}(\mathbb{C})$. Now, we will prove that $\bigwedge \mathfrak{F}(\mathbb{C}) = \mathfrak{F}(\bigwedge \mathbb{C})$. Notice that $((L, M)$-**FCS(X)**$, \leq)$ is a complete lattice. So, $\bigwedge \mathbb{C} \in (L, M)$-**FCS(X)**, and $\mathfrak{F}(\bigwedge \mathbb{C}) \in (L, N)$-**FB(X)**. By (1), we have $\mathfrak{F}(\bigwedge \mathbb{C}) \leq \bigwedge_{C \in \mathbb{C}} \mathfrak{F}(C) = \bigwedge \mathfrak{F}(\mathbb{C})$. It implies that $\mathfrak{F}(\bigwedge \mathbb{C})$ is a lower bound of $\{\mathfrak{F}(C)\}_{C \in \mathbb{C}}$. Let $\mathcal{B}^\star$ is another element of (L, N)-**FB(X)** and $\mathcal{B}^\star \leq \mathfrak{F}(C)$ for each $C \in \mathbb{C}$. By (1), for each $C \in \mathbb{C}$, we have $\mathfrak{G}(\mathcal{B}^\star) \leq \mathfrak{G}(\mathfrak{F}(C)) = C$. It follows that $\mathfrak{G}(\mathcal{B}^\star) \leq \bigwedge \mathbb{C}$. Further, by (1), we have $\mathcal{B}^\star = \mathfrak{F}(\mathfrak{G}(\mathcal{B}^\star)) \leq \mathfrak{F}(\bigwedge \mathbb{C})$. So, we have $\bigwedge \mathbb{B} = \bigwedge \mathfrak{F}(\mathbb{C}) = \bigwedge_{C \in \mathbb{C}} \mathfrak{F}(C) = \mathfrak{F}(\bigwedge \mathbb{C})$. Hence, $\mathfrak{F}(\bigwedge \mathbb{C})$ is the infimum of $\mathbb{B}$, i.e., it's closed for non-empty intersection operation in $((L, N)$-**FB(X)**$, \leq)$. It follows that $((L, N)$-**FB(X)**$, \leq)$ is a complete lattice. Further, $((L, M)$-**FCS(X)**$, \leq)$ and $((L, N)$-**FB(X)**$, \leq)$ are complete lattice isomorphic. □

5. Conclusion

In this study, we gave a reasonable definition with respect to (strong) LRG-Galois connections. With the help of this tool, it is proved not only that (L, M)-fuzzy convex spaces and (L, N)-fuzzy betweenness spaces are categorically isomorphic, but also that (L, M)-fuzzy convex structures and (L, N)-fuzzy betweenness relations are complete lattice isomorphic. This tool can effectively transform on two completely distributive lattices, and has the function of order-reversing involutions. This provides a new idea for us to study the relationship between convex structures and other structures in the future.

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