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Characterization of almost $*$-conformal η-Ricci soliton on para-Kenmotsu manifolds

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Abstract. The goal of this research paper is to deliberate $*$-conformal η-Ricci soliton and gradient almost $*$-conformal η-Ricci soliton within the framework of para-Kenmotsu manifolds as a characterization of Einstein metrics. Here, we explore that a para-Kenmotsu metric as a $*$-conformal η-Ricci soliton is Einstein metric if the soliton vector field is contact and the vector field is strictly infinitesimal contact transformation. Next, we turn up the nature of the soliton and discover the scalar curvature when the manifold admitting $*$-conformal η-Ricci soliton on para-Kenmotsu manifold. After that, we have shown the characterization of the vector field when the manifold satisfies $*$-conformal η-Ricci soliton. Further, we have developed the nature of the potential vector field when the manifold admits gradient almost $*$-conformal η-Ricci soliton. Then, we have studied gradient $*$-conformal η-Ricci soliton to yield the nature of Riemannian curvature tensor and enactment of potential vector field on para-Kenmotsu manifold. Lastly, we give an example of conformal $*$-η-Ricci soliton and gradient almost conformal $*$-η-Ricci soliton on para-Kenmotsu manifold to prove our findings.

1. Introduction

The scientists and mathematicians across many disciplines have always been fascinated to study indefinite structures on manifolds. When a manifold is endowed with a geometric structure, we have more opportunities to explore its geometric properties. Consider a pseudo-Riemannian manifold (M, g). We say the metric g is a Ricci soliton if there exist a smooth vector field V and a constant λ such that

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0,$$

where L_V denotes the Lie derivative along V and S is the manifold's Ricci tensor. The vector field V is called potential and λ is the soliton constant. Taking V to be zero, or a Killing vector, the condition reduces to the Einstein equation, and the soliton is called trivial.

The Ricci soliton is a self-similar solution of the Hamilton's Ricci flow [19]: $\frac{\partial g(t)}{\partial t} = -2S(g(t))$, where $g(t)$ is a one-parameter family of metrics starting at $g(0) = g$. The potential V and the constant λ play a

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fundamental role in determining the soliton's nature. A soliton is called shrinking, steady or expanding according to whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

If the potential vector field V is the gradient of a smooth function f, denoted by Df then the soliton equation reduces to

$$Hess f + S + \lambda g = 0,$$

where $Hess f$ is Hessian of f. Perelman [26] proved that a Ricci soliton on a compact manifold is a gradient Ricci soliton.

In 2009, J. T. Cho and M. Kimura [7] introduced the concept of η-Ricci soliton which is another generalization of classical Ricci soliton and is given by

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where μ is a real constant, η is a 1-form defined as $\eta(X) = g(X, \xi)$ for any $X \in \chi(M)$. Clearly it can be noted that if $\mu = 0$ then the η-Ricci soliton reduces to Ricci soliton.

In 2014, Kaimakamis and Panagiotidou [20] modified the definition of Ricci soliton where they have used $*$-Ricci tensor S^* which was introduced by Tachibana [36], in place of Ricci tensor S. The $*$-Ricci tensor S^* is defined by

$$S^*(X, Y) = \frac{1}{2}(trace\{\phi \cdot R(X, \phi Y)\})$$

for all vector fields X and Y on M, where ϕ is a $(1, 1)$-tensor field. They have used the concept of $*$-Ricci soliton within the framework of real hypersurfaces of a complex space form.

In 2020, S. Dey et al. [10] defined $*$-η-Ricci soliton as

$$\mathcal{L}_\xi g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0.$$

As per the authors knowledge, the results concerning $*$-η-Ricci soliton were studied when the potential vector field V is the characteristic vector field ξ. Motivated from this, in [11] we generalize the definition by considering the potential vector field as arbitrary vector field V and define as:

$$\mathcal{L}_V g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where we considered the manifold as $(2n + 1)$-dimensional. Now if we consider the potential vector field V as the gradient of a smooth function f, then the $*$-η-Ricci soliton equation can be rewritten as

$$Hess f + S^* + \lambda g + \mu\eta \otimes \eta = 0.$$

In 2019, S. Roy et al. [28] defined $*$-conformal η-Ricci soliton as

$$\mathcal{L}_\xi g + 2S^* + \Big[2\lambda - \Big(p + \frac{2}{2n+1}\Big)\Big]g + 2\mu\eta \otimes \eta = 0.$$

As per the authors knowledge, the results concerning $*$-conformal η-Ricci soliton were studied when the potential vector field V is the characteristic vector field ξ. Recently, in [35] Sarkar et al. have generalized the definition by considering the potential vector field as arbitrary vector field V and define as:

$$\mathcal{L}_V g + 2S^* + \Big[2\lambda - \Big(p + \frac{2}{(2n+1)}\Big)\Big]g + 2\mu\eta \otimes \eta = 0, \tag{1}$$

where we considered the manifold as $(2n + 1)$-dimensional. Now if we consider the potential vector field V as the gradient of a smooth function f, then the $*$-conformal η-Ricci soliton equation can be rewritten as,

$$Hess f + S^* + \Big[\lambda - \Big(\frac{p}{2} + \frac{1}{(2n+1)}\Big)\Big]g + \mu\eta \otimes \eta = 0. \tag{2}$$

By gradient almost $*$-conformal η-Ricci soliton we mean gradient $*$-conformal η-Ricci soliton where we consider λ as a smooth function.

As follows in the literature, Ricci soliton on paracontact geometry studied by many authors ([5, 8, 25]). In particular, Calvaruso and Perrone [8] explicitly studied Ricci soliton on 3-dimensional almost para-contact manifolds. In 2018, Ghosh and Patra [15] first studied $*$-Ricci soliton on almost contact metric manifolds. Very recently, $*$-Ricci soliton and its generalizations have been investigated by Dey et al. [9, 10, 16, 17, 22, 23, 28–32]. The case of $*$-Ricci soliton in para-Sasakian mannifold was treated by Prakasha and Veeresha in [27].

Further, in 2019, V. Venkatesha et al. [37] considered the metric of η-Einstein para-Kenmotsu manifold as $*$-Ricci soliton and proved that the manifold is Einstein. I. K. Erken [14] in 2019 considered Yamabe solitons on 3-dimensional para-cosymplectic manifold and proved some vital results like the manifold is either η-Einstein or Ricci flat. So, several authors studied η-Ricci soliton and its abstraction on paracontact metric manifolds, for instance, Dey et al. [33] consider a paracontact metric as a conformal Ricci soliton and $*$-conformal Ricci soliton, Deshmukh et al. [13] studied some results on Ricci almost solitons, Sarkar et al. [34] examine conformal η-Ricci soliton on para-Sasakian manifold and Naik et al. [24] consider a para-Sasakian metric as η-Ricci soliton. In [38], Welyczko introduced notion of para-Kenmotsu manifold, which is the analogous of Kenmotsu manifold [21] in paracontact geometry and detailed studied by Zamkovoy [40]. Further, Blaga studied some aspects of η-Ricci solitons on para-Kenmotsu and Lorentzian para-Sasakian manifolds (see [2–4]). Recently, [12], authors deals with Kenmotsu manifolds admitting conformal η-Ricci soliton and gradient conformal η-Ricci soliton. Motivated by these results, we consider a para-Kenmotsu metric as conformal $*$-η-Einstein solitons and gradient conformal $*$-η-Einstein solitons.
Based on the above facts and discussions in the research of contact geometry, a natural **question** arises.
Are there paracontact metric almost manifolds, whose metrics are conformal $*$-η-Ricci soliton?
In later sections, we show that indeed the answer to this question is affirmative. The paper is organized as follows: in Section 2, the basic definitions and facts about para-Kenmotsu manifolds are given. In the next section, we discuss if the metric g represents a $*$-conformal η-Ricci soliton and if the soliton vector field V is contact, then V is strictly infinitesimal contact transformation and the manifold is Einstein. We have demonstrated some results where $*$-conformal η-Ricci soliton admitting para-Kenmotsu manifold and obtained the nature of soliton, Laplacian of the smooth function. We have also looked over that the manifold is η-Einstein when the manifold satisfies $*$-conformal η-Ricci soliton and the vector field is conformal Killing. We also provide some examples to support our findings in that section. In Section 4, we consider gradient almost $*$-conformal η-Ricci soliton and show that if the metric g represents a gradient almost $*$-conformal η-Ricci soliton then either M is Einstein or there exists an open set where the potential vector field V is pointwise collinear with the characteristic vector field ξ. Section 5 deals with gradient conformal $*$-η-Ricci soliton and we find out some beautiful result. In Section 6, we have constructed an example to illustrate the existence of $*$-conformal η-Ricci soliton on 3-dimensional para-Kenmotsu manifold.

2. Some preliminaries on para-Kenmotsu manifold

A $(2n+1)$-dimensional smooth manifold M is said to have an almost paracontact structure if it admits a vector field ξ, (1, 1)-tensor field ϕ and a 1-form η satisfying the following conditions:

$$i)\phi^2 = I - \eta \otimes \xi, \tag{3}$$

$$ii)\eta(\xi) = 1. \tag{4}$$

iii) ϕ induces on the $2n$-dimensional distribution $\mathcal{D} \equiv ker(\eta)$, an almost paracomplex structure $\mathcal{P}$ i.e., $\mathcal{P}^2 \equiv I_{\chi(M)}$ and the eigensubbundles $\mathcal{D}^+$ and $\mathcal{D}^-$, corresponding to the eigenvalues $1, -1$ of $\mathcal{P}$ respectively, have equal dimension n; hence $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^+$.

If a manifold with an almost paracontact structure (M, ϕ, ξ, η) admits a pseudo-Riemannian metric g of signature $(n+1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{5}$$

holds for any $X, Y \in \chi(M)$, then g is called compatible metric and the manifold (M, ϕ, ξ, η, g) is called almost paracontact metric manifold. If an almost paracontact metric manifold satisfies:

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{6}$$

then the manifold is called almost para-Kenmotsu manifold. The normality of an almost paracontact structure (M, ϕ, ξ, η) is equivalent to vanishing of the (1,2)-torsion tensor defined by $N_\phi(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi$, where $[\phi, \phi]$ is the Nijenhuis torsion tensor of ϕ and is defined by $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ for any $X, Y \in \chi(M)$. A normal almost para-Kenmotsu manifold is called para-Kenmotsu manifold.

On a $(2n+1)$-dimensional para-Kenmotsu manifold the following properties hold:

$$\begin{aligned}
\phi(\xi) &= 0, && (7)\\
\eta \circ \phi &= 0, && (8)\\
\nabla_X \xi &= X - \eta(X)\xi, && (9)\\
(\nabla_X \eta)Y &= g(X, Y) - \eta(X)\eta(Y), && (10)\\
Q\xi &= -2n\xi, && (11)\\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, && (12)\\
R(X, \xi)Y &= g(X, Y)\xi - \eta(Y)X, && (13)\\
(\mathcal{L}_\xi g)(X, Y) &= 2[g(X, Y) - \eta(X)\eta(Y)] && (14)
\end{aligned}$$

for any $X, Y \in \chi(M)$ where, $\mathcal{L}$ and ∇ are the operators of Lie differentiation and covariant differentiation of g respectively. Q denotes the Ricci operator associated with the Ricci tensor S defined by $S(X, Y) = g(QX, Y)$ and R denotes the Riemannian curvature tensor.
A $(2n+1)$-dimensional para-Kenmotsu manifold is said to be a η-Einstein para-Kenmotsu manifold if there exists two smooth functions a and b which satisfies the following relation

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{15}$$

for all $X, Y \in \chi(M)$. Clearly if $b = 0$ then η-Einstein manifold reduces to Einstein manifold. Now considering $X = \xi$ in the last equation and using (12) we have, $a + b = -2n$. Contracting (15) over X and Y we get, $r = (2n+1)a + b$, where r denotes the scalar curvature of the manifold. Solving these two we have, $a = (1 + \frac{r}{2n})$ and $b = -(2n + 1 + \frac{r}{2n})$. Using these values we can rewrite (15) as

$$S(X, Y) = \Big(1 + \frac{r}{2n}\Big)g(X, Y) - \Big(2n + 1 + \frac{r}{2n}\Big)\eta(X)\eta(Y). \tag{16}$$

3. $*$-Conformal η-Ricci soliton on Para-Kenmotsu manifold

In this section we consider that the metric g of a $(2n+1)$-dimensional para-Kenmotsu manifold represents a $*$-conformal η-Ricci soliton. We recall some important lemmas relevant to our results.

Lemma 3.1. *[33] The Ricci operator Q on a $(2n+1)$-dimensional para-Kenmotsu manifold satisfies*

$$\begin{aligned}
(\nabla_X Q)\xi &= -QX - 2nX, && (17)\\
(\nabla_\xi Q)X &= -2QX - 4nX && (18)
\end{aligned}$$

for arbitrary vector field X on the manifold.

Lemma 3.2. *Venkatesha et al. [37] have deduced the expression of ∗-Ricci tensor for para-Kenmotsu manifold as*

$$S^*(X,Y) = -S(X,Y) - (2n-1)g(X,Y) - \eta(X)\eta(Y). \tag{19}$$

Also, we plug $X = e_i$, $Y = e_i$ in the above equation, where e_i's are a local orthonormal frame and summing over $i = 1, 2, \cdots, (2n+1)$ to arrive

$$r^* = -r - 4n^2, \tag{20}$$

where r^* is the ∗-scalar curvature of M.

Theorem 3.3. *Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If the metric g represents a ∗-conformal η-Ricci soliton and if the soliton vector field V is contact, then V is strictly infinitesimal contact transformation and the manifold is Einstein.*

Proof. Since the metric g of the para-Kenmotsu manifold represents a ∗-conformal η-Ricci soliton so both of the equations (1) and (19) are satisfied. Combining these two we have

$$\begin{aligned}(\mathcal{L}_V g)(X,Y) &= 2S(X,Y) - \Big[2\lambda - \Big(p + \frac{2}{(2n+1)}\Big) - 4n + 2\Big]g(X,Y) \\ &\quad -2(\mu-1)\eta(X)\eta(Y).\end{aligned} \tag{21}$$

Taking covariant derivative w.r.t. arbitrary vector field Z and using (10), we obtain

$$\begin{aligned}(\nabla_Z \mathcal{L}_V g)(X,Y) &= 2(\nabla_Z S)(X,Y) - 2(\mu-1)\{g(X,Z)\eta(Y) \\ &\quad + g(Y,Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}\end{aligned} \tag{22}$$

for all $X, Y, Z \in \chi(M)$. Again from Yano [39], we have the following commutation formula

$$\begin{aligned}(\mathcal{L}_V \nabla_Z g - \nabla_Z \mathcal{L}_V g - \nabla_{[V,Z]} g)(X,Y) &= -g((\mathcal{L}_V \nabla)(X,Z), Y) \\ &\quad -g((\mathcal{L}_V \nabla)(Y,Z), X),\end{aligned}$$

where g is the metric connection i.e., $\nabla g = 0$. So the above equation reduces to

$$(\nabla_Z \mathcal{L}_V g)(X,Y) = g((\mathcal{L}_V \nabla)(X,Z), Y) + g((\mathcal{L}_V \nabla)(Y,Z), X) \tag{23}$$

for all vector fields X, Y, Z on M. Combining (22) and (23) and by a straightforward combinatorial computation and applying the symmetry of $\mathcal{L}_V \nabla$ the foregoing equation yields

$$\begin{aligned}g((\mathcal{L}_V \nabla)(X,Y), Z) &= (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(Z,X) \\ &\quad -2(\mu-1)\{g(X,Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\}\end{aligned} \tag{24}$$

for arbitrary vector fields X, Y and Z on M. Using (17) and (18), the foregoing equation yields

$$(\mathcal{L}_V \nabla)(X, \xi) = 2QX + 4nX \tag{25}$$

for all $X \in \chi(M)$. Now differentiating covariantly this with respect to arbitrary vector field Y, we achieve

$$(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) = 2(\nabla_Y Q)X - (\mathcal{L}_V \nabla)(X,Y) + \eta(Y)(2QX + 4nX). \tag{26}$$

We know that, $(\mathcal{L}_V R)(X,Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y,Z) - (\nabla_Y \mathcal{L}_V \nabla)(X,Z)$. In view of (26) in the previous relation we acquire

$$\begin{aligned}(\mathcal{L}_V R)(X,Y)\xi &= 2\{(\nabla_X Q)Y - (\nabla_Y Q)X\} + 2\eta(X)(QY + 2nY) \\ &\quad -2\eta(Y)(QX + 2nX)\end{aligned} \tag{27}$$

for arbitrary vector fields X and Y on M. Setting $Y = \xi$ in the aforementioned equation and using (12), (17) and (18) we get

$$(\mathcal{L}_V R)(X, \xi)\xi = 0. \tag{28}$$

Now, taking (21) in account, the Lie derivative of $g(\xi,\xi) = 1$ along the potential vector field V yields

$$\eta(\mathcal{L}_V\xi) = \lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) + \mu. \tag{29}$$

Plugging $Y = \xi$ and noting that (4), the equation (21) provides

$$(\mathcal{L}_V\eta)X - g(X, \mathcal{L}_V\xi) = -\Big[2\lambda - \Big(p + \frac{2}{(2n+1)}\Big) + 2\mu\Big]\eta(X), \tag{30}$$

which holds for arbitrary vector field X on M. From (12) we compute, $R(X,\xi)\xi = \eta(X)\xi - X$. Taking Lie derivative along the potential vector field V and inserting (29) and (30) in account, this reduces to

$$(\mathcal{L}_V R)(X,\xi)\xi = 2\Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) + \mu\Big](X - \eta(X)\xi) \tag{31}$$

for all $X \in \chi(M)$. Finally comparing (28) and (31) we have, $2(\lambda + \mu)(X - \eta(X)\xi) = 0$. Since this holds for arbitrary $X \in \chi(M)$ so, we infer

$$\lambda = -\mu + \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big). \tag{32}$$

Invoking the relation (32) in (29), we easily obtain $\eta(\mathcal{L}_V\xi) = 0$. Since we have considered the potential vector field V as contact vector field so there must exists a smooth function f such that $\mathcal{L}_V\xi = f\xi$. Making use of this in (29) we get $f = \lambda - \frac{1}{2}\big(p + \frac{2}{(2n+1)}\big) + \mu$. Therefore by using the relation (32), we get $f = 0$ and thus $\mathcal{L}_V\xi = 0$. Finally the equation (30) reduces to

$$\mathcal{L}_V\eta = 0. \tag{33}$$

So, V is strictly infinitesimal contact transformation.

We know the well-known formula from Yano [39] that $(\mathcal{L}_V\nabla)(X,Y) = \mathcal{L}_V\nabla_X Y - \nabla_X\mathcal{L}_V Y - \nabla_{[V,X]}Y$. Inserting $Y = \xi$ and using (9), $\mathcal{L}_V\xi = 0$ and (33) yields, $(\mathcal{L}_V\nabla)(X,\xi) = 0$. Substituting this in (25), we deduce $QX = -2nX$ for all $X \in \chi(M)$, which settles our claim. □

$*$-conformal η-Ricci soliton is a generalisation of $*$-conformal Ricci soliton, where we consider $\mu = 0$ in (1) to get $*$-conformal Ricci soliton equation. We can rewrite the above theorem as:

Corollary 3.4. *Let $M^{(2n+1)}(\phi,\xi,\eta,g)$ be a para-Kenmotsu manifold. If the metric g represents a $*$-conformal Ricci soliton and if the soliton vector field V is contact, then V is strictly infinitesimal contact transformation and the manifold is Einstein.*

Theorem 3.5. *If the metric g of a $(2n+1)$-dimensional para-Kenmotsu manifold satisfies the $*$-conformal η-Ricci soliton (g,ξ,λ,μ), where ξ is the Reeb vector field, then the soliton constants λ and μ are related by $\lambda = -\mu + \frac{1}{2}\big(p + \frac{2}{(2n+1)}\big)$.*

Proof. Let M be a $(2n+1)$ dimensional para-Kenmotsu manifold. Consider $V = \xi$ in the equation of $*$-conformal η-Ricci soliton (1) on M, we obtain:

$$(\mathcal{L}_\xi g)(X,Y) + 2S^*(X,Y) + \Big[2\lambda - \Big(p + \frac{2}{(2n+1)}\Big)\Big]g(X,Y) + 2\mu\eta(X)\eta(Y) = 0 \tag{34}$$

for all vector fields $X, Y, \in \chi(M)$.

We fetch the above equation into the identities (14), (19) and (20) to yield

$$\Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) - 2n + 2\Big]g(X,Y) - S(X,Y) + (\mu - 2)\eta(X)\eta(Y) = 0. \tag{35}$$

Now, we feed $Y = \xi$ in the previous equation and use the identity (11) to infer

$$\lambda = -\mu + \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big), \tag{36}$$

since $\eta(X) \neq 0$, which finishes the proof. □

Also, we see that if $\mu = 0$, then (36) gives $\lambda = \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big)$. So, we can state

Corollary 3.6. *If the metric g of a $(2n+1)$-dimensional para-Kenmotsu manifold satisfies the $*$-conformal Ricci soliton (g, ξ, λ), where ξ is the Reeb vector field, then the soliton is shrinking if $\Big(p + \frac{2}{(2n+1)}\Big) < 0$, steady if $\Big(p + \frac{2}{(2n+1)}\Big) = 0$ and expanding if $\Big(p + \frac{2}{(2n+1)}\Big) > 0$.*

Theorem 3.7. *If the metric g of a $(2n + 1)$-dimensional para-Kenmotsu manifold satisfies the $*$-conformal η-Ricci soliton (g, V, λ, μ), where V is the gradient of a smooth function f, then the Laplacian (or Poisson's) equation satisfied by f is*

$$\Delta(f) = (r + 4n^2) - \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big)\Big](2n+1) - \mu.$$

Proof. Now, we consider a $*$-conformal η-Ricci soliton (g, V, λ, μ) on M as:

$$(\mathcal{L}_V g)(X, Y) + 2S^*(X, Y) + \Big[2\lambda - \Big(p + \frac{2}{(2n+1)}\Big)\Big]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0 \tag{37}$$

for all vector fields $X, Y \in \chi(M)$.
We set $X = e_i$, $Y = e_i$, in the above equation, where e_i's are a local orthonormal frame and summing over $i = 1, 2, ..., (2n + 1)$ and using (20) to obtain

$$divV - (r + 4n^2) + \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big)\Big](2n+1) + \mu = 0. \tag{38}$$

If we take the vector field V is of gradient type i.e., $V = Df$, for f is a smooth function on M, then the equation (38) provides

$$\Delta(f) = (r + 4n^2) - \lambda(2n+1) - \mu, \tag{39}$$

where $\Delta(f)$ is the Laplacian equation satisfied by f. This completes the proof. □

If we replace the value of μ from (36) into the identity (38), λ takes the form

$$\lambda = \frac{(r + 4n^2) - divV}{2n} + \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big). \tag{40}$$

In view of (40), (36) becomes

$$\mu = \frac{divV - (r + 4n^2)}{2n}. \tag{41}$$

So, we can state

Corollary 3.8. *If the metric of an $(2n+1)$-dimensional para-Kenmotsu manifold admits a $*$-conformal η-Ricci soliton (g, V, λ, μ), where V is the gradient of a smooth function f, Then the soliton constants λ and μ takes the form of $\lambda = \frac{(r+4n^2)-divV}{2n} + \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big)$ and $\mu = \frac{divV-(r+4n^2)}{2n}$, respectively, where $div\xi$ is the divergence of the vector field ξ.*

Theorem 3.9. *Let the metric g of a $(2n + 1)$-dimensional para-Kenmotsu manifold satisfy the $*$-conformal η-Ricci soliton (g, V, λ, μ). Then the vector field V is solenoidal if and only if $\lambda = \frac{r+4n^2}{2n} + \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big)$ and $\mu = -\frac{(r+4n^2)}{2n}$.*

Proof. We consider the vector field V as solenoidal i.e., $divV = 0$, then (40) and (41) provides

$$\lambda = \frac{r + 4n^2}{2n} + \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) \tag{42}$$

and

$$\mu = -\frac{r + 4n^2}{2n}. \tag{43}$$

Again if we insert the value of λ and μ into the identity (38) to yield

$$divV = 0, \tag{44}$$

i.e., V is solenoidal, which ends our proof. □

A vector field V is said to be a conformal Killing vector field if and only if the following relation holds:

$$(\mathcal{L}_V g)(X, Y) = 2\Omega g(X, Y), \tag{45}$$

where Ω is some function of the coordinates (conformal scalar). Moreover if Ω is not constant the conformal Killing vector field V is said to be proper. Also when Ω is constant, V is called homothetic vector field and when the constant Ω becomes non zero, V is said to be proper homothetic vector field. If $\Omega = 0$ in the above equation, then V is called Killing vector field.

Lemma 3.10. *If the metric g of a (2n+1) dimensional para-Kenmotsu manifold satisfies the $*$-conformal η-Ricci soliton (g, V, λ, μ), where V is a conformal Killing vector field, then the manifold becomes η-Einstein.*

Proof. Let (g, V, λ, μ) be a $*$-conformal η-Ricci soliton on a $(2n + 1)$-dimensional para-Kenmotsu manifold M, where V is a conformal Killing vector field. Then from (1), (19), (20) and (45), we achieve

$$S(X, Y) = \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) + \Omega - 2n + 1\Big]g(X, Y) + (\mu - 1)\eta(X)\eta(Y), \tag{46}$$

which leads to the fact that the manifold is η-Einstein. □

Lemma 3.11. *Let the metric g of a $(2n + 1)$-dimensional para-Kenmotsu manifold satisfy the $*$-conformal η-Ricci soliton (g, V, λ, μ), where V is a conformal Killing vector field. Then V is one of the following cases:*

(i) proper vector field if $-\Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) + \mu\Big]$ *is not constant.*

(ii) homothetic vector field if $-\Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) + \mu\Big]$ *is constant.*

(iii) proper homothetic vector field if $-\Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) + \mu\Big]$ *is non-zero constant.*

(iv) Killing vector field if $\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) + \mu = 0.$

Proof. We fetch $Y = \xi$ into identity (46) and using (11), (20) to finally arrive

$$\Omega = -\Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) + \mu\Big], \tag{47}$$

since $\eta(X) \neq 0$. Now, using the properties of conformal Killing vector field, we obtain our result. □

4. Gradient almost $*$-conformal η-Ricci soliton on para-Kenmotsu manifolds

In this section, we will study gradient almost $*$-conformal η-Ricci soliton on para-Kenmotsu manifolds.

Theorem 4.1. *Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If the metric g represents a gradient almost $*$-conformal η-Ricci soliton then either M is Einstein or there exists an open set where the potential vector field V is pointwise collinear with the characteristic vector field ξ.*

Proof. In view of (19) in the definition of gradient almost $*$-conformal η-Ricci soliton give by equation (2), we acquire

$$\nabla_X Df = QX - \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) - 2n + 1\Big]X - (\mu - 1)\eta(X)\xi \tag{48}$$

for any vector field X on M. Taking covariant derivative along arbitrary vector Y and using (9), (10) yields

$$\begin{aligned} \nabla_Y \nabla_X Df &= (\nabla_Y Q)X + Q(\nabla_Y X) - Y(\lambda)X - (\lambda + 2n - 1)(\nabla_Y X) \\ &\quad -(\mu - 1)\{g(X, Y)\xi - 2\eta(X)\eta(Y)\xi \\ &\quad +\eta(\nabla_Y X)\xi + \eta(X)Y\}. \end{aligned} \tag{49}$$

Applying this in the expression of Riemannian curvature tensor we obtain

$$\begin{aligned} R(X, Y)Df &= (\nabla_X Q)Y - (\nabla_Y Q)X + Y(\lambda)X - X(\lambda)Y \\ &\quad -(\mu - 1)\{\eta(Y)X - \eta(X)Y\}. \end{aligned} \tag{50}$$

Moreover an inner product w.r.t. ξ and use of (17) and (18) yields

$$g(R(X, Y)Df, \xi) = Y(\lambda)\eta(X) - X(\lambda)\eta(Y) \tag{51}$$

for $X, Y \in \chi(M)$. Furthermore the inner product of (12) with the potential vector field Df provides

$$g(R(X, Y)Df, \xi) = \eta(Y)X(f) - \eta(X)Y(f) \tag{52}$$

for arbitrary X and Y on M. Comparing (51) and (52) and plugging $Y = \xi$, we have $X(f + \lambda) = \xi(f + \lambda)\eta(X)$. From this we achieve

$$d(f + \lambda) = \xi(f + \lambda)\eta. \tag{53}$$

So, $(f + \lambda)$ is invariant along the distribution $Ker(\eta)$ i.e., if $X \in Ker(\eta)$ then $X(f + \lambda) = d(f + \lambda)X = 0$. Now, if we take inner product w.r.t. arbitrary vector field Z after plugging $X = \xi$ in (50) we get

$$\begin{aligned} g(R(\xi, Y)Df, Z) &= S(Y, Z) + (2n - \xi(\lambda) + \mu - 1)g(Y, Z) + Y(\lambda)\eta(Z) \\ &\quad -(\mu - 1)\eta(Y)\eta(Z). \end{aligned} \tag{54}$$

Again noting that from (12), we can easily deduce for arbitrary vector fields Y and Z on M

$$g(R(\xi, Y)Df, Z) = \xi(f)g(Y, Z) - Y(f)\eta(Z). \tag{55}$$

We compare the equations (54) and (55) and applying (53) to obtain

$$S(Y, Z) = \{\xi(f + \lambda) - \mu - 2n - 1\}g(Y, Z) - \{\xi(f + \lambda) - \mu - 1\}\eta(Y)\eta(Z). \tag{56}$$

Since the above equation holds good for arbitrary Y and Z, so the manifold is η-Einstein. Now contracting (56), we infer

$$\xi(f + \lambda) = \frac{r}{2n} + \mu + 2n + 2. \tag{57}$$

Now, we plug this in (56) to acquire

$$S(Y,Z) = \Big(\frac{r}{2n}+1\Big)g(Y,Z) - \Big(\frac{r}{2n}+2n+1\Big)\eta(Y)\eta(Z)$$

for arbitrary vector fields Y and Z on M which is exactly same as (16). Now contracting (50) w.r.t. X reduces to

$$S(Y,Df) = \frac{1}{2}Y(r) + 2nY(\lambda) - 2n(\mu+1)\eta(Y), \tag{58}$$

which holds for any $Y \in \chi(M)$. Now, taking into with (16), we compute

$$\begin{aligned}&(r+2n)Y(f) - (r+2n(2n+1))\eta(Y)\xi(f) - nY(r)\\ &-4n^2Y(\lambda) + 4n^2(\mu+1)\eta(y) = 0\end{aligned} \tag{59}$$

for all $Y \in \chi(M)$. Now, setting $Y = \xi$ and then in view of (57), we easily derive the relation

$$\xi(r) = -2(r+2n(2n+1)). \tag{60}$$

Since $d^2 = 0$ and $d\eta = 0$, from (53) it follows $dr \wedge \eta = 0$ i.e., $dr(X)\eta(Y) - dr(Y)\eta(X) = 0$ for arbitrary $X, Y \in \chi(M)$. After inserting $Y = \xi$ and applying (60) it reduces to $X(r) = -2(r+2n(2n+1))\xi$. Since X is an arbitrary vector field so we conclude that

$$Dr = -2(r+2n(2n+1))\xi. \tag{61}$$

Let X be a vector field of the distribution $Ker(\eta)$. Then, (59) provides

$$(r+2n)X(f) - 4n^2X(\lambda) = 0.$$

Invoking (53) and (57) we obtain, $(r+2n(2n+1))X(f) = 0$. From here we conclude

$$(r+2n(2n+1))(Df - \xi(f)\xi) = 0.$$

If $r = -2n(2n+1)$, then from (16) we acquire that the manifold is Einstein with Einstein constant $-2n$.

If $r \neq -2n(2n+1)$ on some open set O of M, then $Df = \xi(f)\xi$ on that open set that is, the potential vector field is pointwise collinear with the characteristic vector field ξ, which finishes the proof. □

5. Gradient ∗-conformal η-Ricci soliton on para-Kenmotsu manifolds

This section is devoted to the study of para-Kenmotsu manifolds admitting gradient ∗-conformal η-Ricci soliton and we try to characterize the potential vector field of the soliton. First, we prove the following important lemma.

Lemma 5.1. *If (g, V, λ, μ) is a gradient ∗-conformal η-Ricci soliton on a $(2n+1)$-dimensional para-Kenmotsu manifold (M, g, ϕ, ξ, η), then the Riemannian curvature tensor R satisfies*

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X + (\mu-1)[\eta(X)Y - \eta(Y)X].$$

Proof. Since the metric is gradient ∗-conformal η-Ricci soliton, so using (2), (19) and (20), we can write

$$\begin{aligned}Hess f(X,Y) &= S(X,Y) - \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) - (2n-1)\Big]g(X,Y)\\ &- (\mu-1)\eta(X)\eta(Y)\end{aligned} \tag{62}$$

for all $X, Y \in \chi(M)$.
Now, the foregoing equation can be rewritten as

$$\nabla_X Df = QX - \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) - (2n-1)\Big]X - (\mu-1)\eta(X)\xi. \tag{63}$$

Covariantly diffrentiating the previous equation along an arbitrary vector field Y and using (9), we obtain

$$\begin{aligned} \nabla_Y \nabla_X Df &= \nabla_Y QX - \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) - (2n-1)\Big]\nabla_Y X \\ &- (\mu - 1)[\nabla_Y \eta(X)\xi + (Y - \eta(Y)\xi)\eta(X)]. \end{aligned} \tag{64}$$

Now, we replace X and Y into the identity (64) to yield

$$\begin{aligned} \nabla_X \nabla_Y Df &= \nabla_X QY - \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) - (2n-1)\Big]\nabla_X Y \\ &- (\mu - 1)[\nabla_X \eta(Y)\xi + (X - \eta(X)\xi)\eta(Y)]. \end{aligned} \tag{65}$$

Also in view of (63), we acquire

$$\begin{aligned} \nabla_{[X,Y]} Df &= Q(\nabla_X Y - \nabla_Y X) - \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{(2n+1)}\Big) - (2n-1)\Big](\nabla_X Y \\ &- \nabla_Y X) - (\mu - 1)\eta(\nabla_X Y - \nabla_Y X)\xi. \end{aligned} \tag{66}$$

Now, we plug the values of (64), (65) and (66) into the very well known Riemannian curvature formula

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

to achieve

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X + (\mu - 1)[\eta(X)Y - \eta(Y)X]. \tag{67}$$

This completes the proof. □

Now, we prove our main theorem using the above lemma.

Theorem 5.2. *Let (g, V, λ, μ) be a gradient $*$-conformal η-Ricci soliton on a (2n+1)-dimensional para-Kenmotsu manifold. Then the potential vector field V is pointwise collinear with the characteristic vector field ξ.*

Proof. We take an inner product of the identity (12) with Df to achieve

$$g(R(X,Y)\xi, Df) = (Yf)\eta(X) - (Xf)\eta(Y). \tag{68}$$

Also, we know well known formula

$$g(R(X,Y)\xi, Df) = -g(R(X,Y)Df, \xi)$$

and using this into (68) to yield

$$g(R(X,Y)Df, \xi) = (Xf)\eta(Y) - (Yf)\eta(X). \tag{69}$$

Now taking inner product of (67) with ξ, we obtain

$$g(R(X,Y)Df, \xi) = 0. \tag{70}$$

Using the combination of (69) and (70), we get

$$\eta(Y)(Xf) = \eta(X)(Yf). \tag{71}$$

We insert $Y = \xi$ in the previous equation to yield

$$(Xf) = \eta(X)(\xi f), \tag{72}$$

which implies

$$g(X, Df) = g(X, (\xi f)\xi)$$

for all vector fields X on M. So, we conclude that

$$V = D(f) = \xi(f)\xi,$$

which settles our claim. □

6. Example of 3-dimensional para-Kenmotsu manifold admitting $*$-conformal η-Ricci soliton

In this section, we provide an example to verify our outcomes.

Example 6.1. *We consider the manifold as $M = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields defined bellow:*

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent at each point on M. The metric g is defined by:

$$g(e_1, e_1) = g(e_3, e_3) = 1, g(e_2, e_2) = -1,$$
$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let $\xi = e_3$. Then the 1-form η is defined by $\eta(X) = g(X, e_3)$, for arbitrary $X \in \chi(M)$, then we have the following relations:

$$\eta(e_1) = \eta(e_2) = 0, \qquad \eta(e_3) = 1.$$

Let us define the (1,1)-tensor field ϕ as

$$\phi e_2 = e_1, \qquad \phi e_1 = e_2, \qquad \phi e_3 = 0,$$

then it satisfies

$$\begin{aligned} \phi^2(X) &= X - \eta(X)e_3, \\ g(\phi X, \phi Y) &= -g(X, Y) + \eta(X)\eta(Y) \end{aligned}$$

for arbitrary $X, Y \in \chi(M)$.

Thus (ϕ, ξ, η, g) defines an almost paracontact metric structure on M. We can now easily conclude:

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = e_2, \qquad [e_1, e_3] = e_1.$$

Let ∇ be the Levi-Civita connection of g. Then from Koszul's formula for arbitrary $X, Y, Z \in \chi(M)$ given by:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &- g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

we can have:

$$\nabla_{e_1} e_3 = e_1 \quad \nabla_{e_1} e_1 = -\nabla_{e_2} e_2 = -e_3 \quad \nabla_{e_2} e_3 = e_2. \tag{73}$$

From here, we can easily verify that the relation (6) is satisfied. Hence the considered manifold is para-Kenmotsu manifold. The components of Riemannian curvature tensor are given by,

$$\begin{aligned} &R(e_1, e_2)e_1 = e_2, && R(e_1, e_2)e_2 = e_1, && R(e_1, e_2)e_3 = 0, \\ &R(e_1, e_3)e_1 = e_3, && R(e_1, e_3)e_2 = 0, && R(e_1, e_3)e_3 = -e_1, \\ &R(e_2, e_3)e_1 = 0, && R(e_2, e_3)e_2 = -e_3, && R(e_2, e_3)e_3 = -e_2. \end{aligned}$$

And the components of Ricci tensor and $*$-Ricci tensor are given by:

$$\begin{aligned} &S(e_1, e_1) = -2, && S(e_2, e_2) = 2, && S(e_3, e_3) = -2, \\ &S^*(e_1, e_1) = 1, && S^*(e_2, e_2) = -1, && S^*(e_3, e_3) = 0. \end{aligned}$$

From here, we can easily deduce that the scalar curvature of the manifold $r = -6$ *and* $S(X, Y) = -2g(X, Y)$ *for all* $X, Y \in \chi(M)$. *Again using* (20) *we get,* $r^* = -r - 4 = 2$ *(as* $r = -6$*) and*

$$S^*(X, Y) = g(X, Y) - \eta(X)\eta(Y) \tag{74}$$

for all $X, Y \in \chi(M)$. *Hence* (M^3, g) *is* η*-Einstein with constant* $*$*-scalar curvature* $r^* = 2n(2n + 1) - 4n^2$ *for* $n = 1$.
Let us define a vector field by

$$V = (x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Then we can obtain:

$$(\mathcal{L}_V g)(e_1, e_1) = 2, \qquad (\mathcal{L}_V g)(e_2, e_2) = -2, \qquad (\mathcal{L}_V g)(e_3, e_3) = 0.$$

Then using (73), *we obtain*

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) = g(X, Y) - \eta(X)\eta(Y) \tag{75}$$

for all X, $Y \in \chi(M^3)$. *If we choose the potential function* $f(x, y, z) = \frac{(x-1)^2}{2} + \frac{(y-1)^2}{2} + z$, *then from* (74) *and* (75), *we can conclude that the metric* g *is a gradient* $*$*-conformal* η*-Ricci soliton with constants* $\lambda = \frac{p}{2} - \frac{5}{3}$ *and* $\mu = 2$. *Also it satisfies the relation* (36).

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