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Remarks on n-power quasinormal operators

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Abstract. In this paper, we study properties and structures of n-power quasinormal operators. In particular, we show that every n-power quasinormal operator satisfies some local spectral properties. Finally, we consider the n-power quasinormality of operator matrices.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$ and $\sigma_{ap}(T)$ for the spectrum and the approximate point spectrum of T, respectively, while $r(T)$ denotes the spectral radius of T.

A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is an *invariant subspace* under the operator A if $A\mathcal{M} \subseteq \mathcal{M}$. In addition, if both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are invariant subspaces for A, then we say $\mathcal{M}$ is a *reducing subspace* for A. The collection of all subspaces of $\mathcal{H}$ invariant under A is denoted by $LatA$. A *hyperinvariant subspace* for A is a closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that $\mathcal{S}\mathcal{M} \subseteq \mathcal{M}$ for every operator S which commutes with A. The collection of all subspaces of $\mathcal{H}$ hyperinvariant under A is denoted by $HLatA$.

An operator T in $\mathcal{L}(\mathcal{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $ker(U) = ker(|T|) = ker(T)$ and $ker(U^*) = ker(T^*)$. Associated with T is a related operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ called the *Aluthge transform* of T, denoted throughout this paper by $\tilde{T}$. In many cases, the Aluthge transforms of T have the better properties than T (see [12] for more details). The Duggal transform of T, denoted by $\tilde{T}^D$, is given by $\tilde{T}^D = |T|U$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if T and T^* commute, *quasinormal* if T and T^*T commute, respectively. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a *p-hyponormal* operator if $(T^*T)^p \geq (TT^*)^p$, where $0 < p < \infty$. Especially, if $p = 1$, T is called *hyponormal*.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *n-power normal* if and only if $T^nT^* = T^*T^n$ for some $n \in \mathbb{N}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *n-power quasinormal* if and only if $[T^n, T^*]T = 0$ for some $n \in \mathbb{N}$ where $[A, B] := AB - BA$.

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It is clear that every nilpotent operator of order $n+1$ is n-power quasinormal. However, every n-power quasinormal operator is not necessary to be normal, hyponormal, or p-hyponormal (see Example 3.9).

In this paper, we study properties and structures of n-power quasinormal operators. In particular, we show that every n-power quasinormal operator satisfies some local spectral properties. Finally, we consider the n-power quasinormality of operator matrices.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ has the *single valued extension property* (*i.e., SVEP*) at $\lambda_0 \in \mathbb{C}$ if for every open neighborhood U of λ_0 the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation $(T-\lambda)f(\lambda) \equiv 0$ is the constant function $f \equiv 0$ on U. The operator T is said to have the single valued extension property if T has the single valued extension property at every $\lambda \in \mathbb{C}$. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for a vector $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{H})$ by $\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $\mathcal{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on G such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, we get that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G An operator $T \in \mathcal{L}(H)$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\mathcal{H} = \mathcal{X} + \mathcal{Y}, \sigma(T|_{\mathcal{X}}) \subset \overline{U}, \text{ and } \sigma(T|_{\mathcal{Y}}) \subset \overline{V}.$$

It is well known that

Bishop's property $(\beta) \Rightarrow$ Dunford's property $(C) \Rightarrow$ SVEP.

Any of the converse implications does not hold, in general (see [16] for more details).

3. Main results

In this section, we investigate several properties of n-power quasinormal operators. We start with the following lemma.

Lemma 3.1. *If $T \in \mathcal{L}(\mathcal{H})$ is n-power quasinormal, then T^n is quasinormal. Conversely, if T^n is quasinormal and $\ker T^{*n} \subset \ker T^n$, then T is n-power quasinormal.*

Proof. If T is n-power quasinormal, then $|T|^2$ commutes with T^n and $(T^n)^*$. Hence

$$[(T^n)^*T^n]T^n = T^{*n-1}|T|^2T^{n-1}T^n = T^{*n-1}T^{n-1}T^n|T|^2 = \cdots\cdots = T^n(|T|^2)^n.$$

Similarly, we obtain that

$$T^n[(T^n)^*T^n] = T^n[T^{*n-1}|T|^2T^{n-1}] = T^n[T^{*n-1}T^{n-1}]|T|^2 = \cdots\cdots = T^n(|T|^2)^n.$$

Hence $[(T^n)^*T^n]T^n = T^n[(T^n)^*T^n]$. Thus T^n is quasinormal.

Conversely, if T^n is quasinormal and $\ker T^{*n} \subset \ker T^n$, then it follows that $[(T^n)^*T^n - T^n(T^n)^*]T^n = 0$. Hence T^n is normal on $\overline{\operatorname{ran} T^n}$. Since $T^nT = TT^n$, Fuglede-Putnam theorem implies that $T^nT^* = T^*T^n$ on $\overline{\operatorname{ran} T^n}$. Since T^n is quasinormal and $\ker T^{*n} \subset \ker T^n$, we have $\ker T^{*n} = \ker T^n$. Moreover, since $T^{*n}T^n - T^nT^{*n} = 0$ on $\ker T^{*n} = \ker T^n$, T^n is normal on $\mathcal{H} = \overline{\operatorname{ran} T^n} \oplus \ker T^{*n}$. By the similar method above, $T^nT^* = T^*T^n$ on $\ker T^{*n} = \ker T^n$. Hence $T^nT^* = T^*T^n$ on $\mathcal{H} = \overline{\operatorname{ran} T^n} \oplus \ker T^{*n}$. That implies $(T^nT^* - T^*T^n)T = 0$. Thus T is n-power quasinormal. □

Theorem 3.2. *Every n-power quasinormal operator T in $\mathcal{L}(\mathcal{H})$ has the single-valued extension property.*

Proof. Let $f : D \to \mathcal{H}$ be an analytic function such that

$$(T-\lambda)f(\lambda) = 0 \tag{1}$$

where D is a disck. Since $T-\lambda$ is invertible on $D \setminus \sigma(T)$, it follows that $f(\lambda) = 0$. Hence we may assume that $D \subset \sigma(T)$. From (1),

$$0 = (T^n - \lambda^n)f(\lambda) = (T-\lambda)g(T,\lambda)$$

on D. Choose nonzero $\lambda_0 \in D$. Consider $D_0 = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\}$ with sufficiently small r in D such that $\frac{1}{\lambda^n}$ exists on D_0^n. Set $k(\mu) = f(\mu^{-n})$ on D_0^n. Then $(T^n - \mu)k(\mu) = 0$ on D_0^n. Since T^n is quasinormal by Lemma 3.1, T^n has the single-valued extension property. Hence $k(\mu) = 0$. Therefore, $f(\lambda) = 0$ on D_0. By the Identity Theorem, $f(\lambda) = 0$ on D. Thus T has the single-valued extension property. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *nilpotent* of order k if $T^k = 0$ for some positive integer k.

Corollary 3.3. *If $T \in \mathcal{L}(\mathcal{H})$ is n-power quasinormal, then the following statements hold.*
(i) $\sigma(T) = \cup_{x\in\mathcal{H}}\sigma_T(x)$ *and* $max\{|\lambda| : \lambda \in \sigma_T(x)\} = \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}$.
(ii) *If T is quasinilpotent* (*i.e.*, $\sigma(T) = \{0\}$), *then it is nilpotent of order n.*

Proof. (i) Since T has the single-valued extension property by Theorem 3.2, it follows from [16].

(ii) Since T^n is quasinormal from Lemma 3.1, T^n is normaloid, i.e., $r(T^n) = \|T^n\|$ where $r(T^n) = sup\{|\lambda| : \lambda \in \sigma(T^n)\}$. Since $\sigma(T^n) = \{0\}$, we have $\|T^n\| = 0$. Hence T is nilpotent of order n. □

The class of n-power quasinormal operators may not have the translation invariant property. For example, if $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ is defined as $T = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$, then T is 2-power quasinormal. However, $(T-\lambda)^2(T-\lambda)^*(T-\lambda) - (T-\lambda)^*(T-\lambda)^3 = -\lambda^2 T^2 T^* - 2\lambda T T^* T + 2\lambda^2 T T^* - 2\lambda^2 T^* T + 3\lambda T^* T^2 \neq 0$. Hence $T-\lambda$ is not 2-power quasinormal. In the following theorem, we consider the case when the traslation invariant property holds.

Theorem 3.4. *Let $T \in \mathcal{L}(\mathcal{H})$. Then $T - \lambda I$ is n-power quasinormal for all $\lambda \in \mathbb{C}$ if and only if T is normal.*

Proof. If $T - \lambda I$ is n-power quasinormal for all $\lambda \in \mathbb{C}$, then

$$(T-\lambda I)^n (T-\lambda I)^* (T-\lambda I) = (T-\lambda I)^* (T-\lambda I)^{n+1}.$$

Since $(T-\lambda I)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \lambda^j T^{n-j}$, we get that

$$\begin{aligned} &(\sum_{j=0}^{n}(-1)^j\binom{n}{j}\lambda^j T^{n-j})(T^*T - \overline{\lambda}T - \lambda T^* + |\lambda|^2) \\ = \; &(T^*T - \overline{\lambda}T - \lambda T^* + |\lambda|^2)(\sum_{j=0}^{n}(-1)^j\binom{n}{j}\lambda^j T^{n-j}). \end{aligned}$$

Calculating the above equation, we obtain that

$$\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\lambda^j[T^{n-j}T^*T - T^*T^{n-j+1}] - \sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\lambda^{j+1}[T^{n-j}T^* - T^*T^{n-j}] = 0.$$

Set $\lambda = re^{i\theta}$ for every $0 \le \theta < 2\pi$ and $r > 0$. Dividing both sides by λ^n, for each positive r

$$\begin{aligned} 0 &= \sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\frac{1}{r^{n-j}e^{i(n-j)}}(T^{n-j}T^*T - T^*T^{n-j+1}) \\ &\quad - \sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\frac{1}{r^{n-j-1}e^{i(n-j-1)}}(T^{n-j}T^* - T^*T^{n-j}) \\ &= \frac{1}{r}[\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\frac{1}{r^{n-j-1}e^{i(n-j)}}(T^{n-j}T^*T - T^*T^{n-j+1}) \\ &\quad - \sum_{j=0}^{n-2}(-1)^j\binom{n}{j}\frac{1}{r^{n-j-1}e^{i(n-j-1)}}(T^{n-j}T^* - T^*T^{n-j})] - (-1)^j\binom{n}{j}(TT^* - T^*T). \end{aligned}$$

Letting $r \to \infty$ in above equation, we have $TT^* = T^*T$. Thus T is normal.

The converse implication is trivial. □

Proposition 3.5. *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following statements hold.*
(i) *Let $\{T_k\}$ be a sequence of n-power qusinormal operators in $\mathcal{L}(H)$. If $T_k \to T$ in norm, then T is n-power quasinormal.*
(ii) *T is n-power quasinormal if and only if $|T|$ commutes with $Re\ T^n$ and $Im\ T^n$ where $Re\ A = \frac{1}{2}\{A + A^*\}$ and $Im\ A = \frac{1}{2i}\{A - A^*\}$.*
(iii) *If T is n-power quasinormal and compact, then T is n-power normal.*

Proof. (i) Since $T_k \to T$ in norm, we get that

$$\begin{aligned} ||T^nT^*T - T^*T^{n+1}|| &\le ||T^n - T_k^n||||T^*T|| + ||T_k||^n||T^* - T_k^*||||T|| \\ &\quad + ||T_k||^n||T_k^*||||T - T_k|| + ||T_k^* - T^*||||T_k^{n+1}|| \\ &\quad + ||T^*||||T_k^{n+1} - T^{n+1}|| \to 0 \end{aligned}$$

as $k \to \infty$. Hence $T^nT^*T = T^*T^{n+1}$. Thus T is n-power quasinormal.

(ii) If T is n-power quasinormal, then $T^n|T|^2 = |T|^2T^n$. Since $T^np(|T|^2) = p(|T|^2)T^n$ for any polynomial $p(t)$ with $p(0) = 0$, take $p_k(t) \to t^{\frac{1}{2}}$. Then $T^n|T| = |T|T^n$ since the square root $|T|$ of a positive operator $|T|^2$ is approximated uniformly by polynomials of $|T|^2$. Since $|T|T^{*n} = T^{*n}|T|$, $|T|(Re\ T^n) = (Re\ T^n)|T|$ and $|T|(Im\ T^n) = (Im\ T^n)|T|$ hold. Conversely, if $|T|$ commutes with $Re\ T^n$ and $Im\ T^n$, then $|T|$ commutes with T^n. Thus $T^n|T|^2 = |T|^2T^n$. So T is n-power quasinormal.

(iii) If T is compact, then T^n is compact and quasinormal by Lemma 3.1. Hence T^n is normal by [7, Corollary 4.10]. Since $T^nT = TT^n$, by Fuglede-Putnam $T^nT^* = T^*T^n$. Thus T is n-power normal. □

The following propositions provide several examples for n-power quasinormal operators.

Proposition 3.6. *Every nilpotent operator $T \in \mathcal{L}(\mathcal{H})$ of order $n - 1$ is n-power quasinormal.*

Proof. Since $T \in \mathcal{L}(\mathcal{H})$ is nilpotent of order $n - 1$, by Halmos characterization T is unitarily equivalent an operator matrix S, where $S = \begin{pmatrix} 0 & S_{12} & \cdots & S_{1n} \\ 0 & 0 & \cdots & S_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & S_{(n-1)n} \\ 0 & & \cdots & 0 \end{pmatrix}$. Thus $[S^n, S^*]S = 0$. Hence S is n-power quasinormal.
Since T is unitarily equivalent to S, T is n-power quasinormal. □

Proposition 3.7. *Let W be a unilateral weighted shift defined by $We_k = \alpha_k e_{k+1}$ for $k = 1, 2, \cdots$ where $\{e_k\}$ is an orthonormal basis for $\mathcal{H}$. Then the following statements hold.*
(i) *W is n-power quasinormal if and only if $|\alpha_k| = |\alpha_{k+n}|$ for $k = 1, 2, \cdots$. In this case, if W is hyponormal, then $|\alpha_1| = |\alpha_k|$ for all $k = 1, 2, \cdots$.*
(ii) *W^n is quasinormal if and only if $|\alpha_k| \cdots |\alpha_{k+n-1}| = |\alpha_{k+n}| \cdots |\alpha_{k+2n-1}|$ for $k = 1, 2, \cdots$.*

Proof. (i) Since $W^n W^* W e_k = |\alpha_k|^2 \alpha_k \cdots \alpha_{k+n-1} e_{k+n}$ and $W^* W^{n+1} e_k = \alpha_k \cdots \alpha_{k+n-1} |\alpha_{k+n}|^2 e_{k+n}$ for $k = 1, 2, \cdots$, $|\alpha_k| = |\alpha_{k+n}|$ for $k = 1, 2, \cdots$. The converse implication is similar. In this case, if W is hyponormal, then $\{|\alpha_k|\}$ is increasing. Hence

$$|\alpha_k| \le |\alpha_{k+1}| \le \cdots \le |\alpha_{k+n}| = |\alpha_k|$$

for $k = 1, 2, \cdots$. Thus $|\alpha_1| = |\alpha_k|$ for all $k = 1, 2, \cdots$.

(ii) Since $[(W^n)^* W^n] W^n e_k = \alpha_k \cdots \alpha_{k+n-1} |\alpha_{k+n}|^2 \cdots |\alpha_{k+2n-1}|^2 e_{k+n}$ and $W^n [(W^n)^* W^n] e_k = |\alpha_k|^2 \cdots |\alpha_{k+n-1}|^2 \cdots \alpha_k \cdots \alpha_{k+n-1} e_{k+n}$, W^n is quasinormal if and only if $|\alpha_k| \cdots |\alpha_{k+n-1}| = |\alpha_{k+n}| \cdots |\alpha_{k+2n-1}|$ for $k = 1, 2, \cdots$. □

We observe from Proposition 3.7 that the following implications hold. However, the converse implications do not hold, in general.

$$\{\text{quasinormality of } T\} \Rightarrow \{n\text{-power quasinormality of } T\} \Rightarrow \{\text{quasinormality of } T^n\}$$

Moreover, there exist n-power quasinormal operators which are neither hyponormal nor p-hyponormal, in general (see Example 3.9).

Proposition 3.8. *Let T be any 2×2 matrix in $\mathcal{L}(\mathbb{C}^2)$. Then T is n-power quasinormal if and only if T is unitarily equivalent to one of the following matrices;*

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}, \text{ and } \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \text{ where } \sum_{j=0}^{n-1} a^{n-1-j} c^j = 0.$$

Proof. Since T is unitarily equivalent to $S = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, it suffices to consider the n-power quasinormality of S. It is easy to show that S is n-power quasinormal if and only if the following identities hold.
(i) $[a^n, \overline{a}]a + (\sum_{j=0}^{n-1} a^{n-1-j} b c^j) \overline{b} a = 0$.
(ii) $[a^n, \overline{a}]b + (\sum_{j=0}^{n-1} a^{n-1-j} b c^j)(|b|^2 + |c|^2) - \overline{a}(\sum_{j=0}^{n-1} a^{n-1-j} b c^j) c = 0$.
(iii) $c^n \overline{b} a - \overline{b} a^{n+1} = 0$.
(iv) $c^n |b|^2 - a^n |b|^2 + [c^n, \overline{c}]c - \overline{b}(\sum_{j=0}^{n-1} a^{n-1-j} b c^j) c = 0$.

If $a = c = 0$, $a = b = 0$, or $b = 0$, then (i), (ii), (iii), and (iv) are satisfied. Hence S is n-power quasinormal. If $\sum_{j=0}^{n-1} a^{n-1-j} c^j = 0$, then $a^n - c^n = (a - c) \sum_{j=0}^{n-1} a^{n-1-j} c^j = 0$. Since (i), (ii), (iii), and (iv) hold, S is also n-power quasinormal.

Conversely, if S is n-power quasinormal, then from (i), $(\sum_{j=0}^{n-1} a^{n-1-j} c^j)|b|^2 a = 0$. Hence $a = 0$, $b = 0$, or $\sum_{j=0}^{n-1} a^{n-1-j} c^j = 0$. If $\sum_{j=0}^{n-1} a^{n-1-j} c^j = 0$, then it is clear. If $a = 0$, from (ii) and (iv) $b = 0$ or $c = 0$. If $b = 0$, (i), (ii), (iii), and (iv) hold. Hence we complete the proof. □

We observe from Proposition 3.8 that every n-power quasinormal operator is not necessary to be normal on a finite dimensional space. Hence it is neither hyponormal nor p-hyponormal, in general.

Example 3.9. Let w be a root of $z^n - 1 = 0$. Then $S = \begin{bmatrix} 1 & b \\ 0 & w \end{bmatrix}$ is n-power quasinormal. Indeed, since $a = 1$, $c = w$, and $\sum_{j=0}^{n-1} w^j = 0$ in Proposition 3.8, S is n-power quasinormal. Moreover, if $b \neq 0$, S is not normal. Thus if $b \neq 0$, S is neither hyponormal nor p-hyponormal, in general.

Recall that an antilinear map $C : \mathcal{H} \to \mathcal{H}$ is called a conjugation on $\mathcal{H}$ if $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. We say that $T \in \mathcal{L}(H)$ is *complex symmetric* if there exists a conjugation C such that $CTC = T^*$. An operator $T \in \mathcal{L}(\mathcal{H})$) is a *quasiaffinity* if T has trivial kernel and dense range. We next consider complex symmetric operators which are n-power quasinormal.

Theorem 3.10. *If $T \in \mathcal{L}(H)$ is n-power quasinormal and complex symmetric, then there exists a nilpotent operator R of order n and an n-power normal operator S such that $T = R \oplus S$.*

Proof. Assume that $CTC = T^*$ for some conjugation C and $T^nT^*T = T^*T^{n+1}$. Then

$$\begin{aligned} CT^{*n}TCT &= T^nCTCT = T^nT^*T = T^*T^{n+1} \\ &= CTCT^{n+1} = CTT^{*n}CT. \end{aligned}$$

Hence

$$T^{*n}TT^*C = T^{*n}TCT = TT^{*n}CT = TT^{*n+1}C.$$

Thus $T^{*n}TT^* = TT^{*n+1}$, i.e., T^* is n-power quasinormal. Since both T and T^* are n-power quasinormal, both T^n and $(T^n)^*$ are quasinormal from Lemma 3.1. Since $[(T^n)^*T^n - T^n(T^n)^*]T^n = 0$, $(T^n)^*T^n - T^n(T^n)^* = 0$ on $\overline{ran\ T^n}$. Since both T^n and $(T^n)^*$ are quasinormal, it is clear that $ker\ T^n = ker\ (T^n)^*$. Hence $(T^n)^*T^n - T^n(T^n)^* = 0$ on $ker\ (T^n)^*$. Thus T^n is normal. By [11, Theorem 3.1], there exists a nilpotent operatos R of order n and an operator S which is quasisimilar to a normal operator N with $\sigma(S) = \sigma(N)$ such that $T = R \oplus S$. Let X be a quasiaffinity such that $S^nX = XN^n$. By [8, Theorem 7], S^n is normal. Hence $T = R \oplus S$ where R is nilpotent operator of order n and S is n-power normal. □

Recall that an operator $T \in \mathcal{L}(H)$ has finite ascent if there exists an $n \in \mathbb{N}$ such that $ker\ T^n = ker\ T^{n+1}$.

Corollary 3.11. *If $T \in \mathcal{L}(H)$ is n-power quasinormal and complex symmetric, the following statements hold.*
(i) *$ker\ T^n = ker\ T^{n+k}$ for all positive integer k. Hence T has finite ascent.*
(ii) *Both T and T^* have the single-valued extension property.*

Proof. (i) If $T \in \mathcal{L}(H)$ is n-power quasinormal and complex symmetric, then $T = R \oplus S$ where $R^n = 0$ and S^n is normal from Theorem 3.10. Now it suffices to show that $ker\ T^{n+1} \subset \ker\ T^n$. If $T^{n+1}x = 0$, then

$$0 = T^{n+1}x = \begin{pmatrix} 0 & 0 \\ 0 & S^{n+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ S^{n+1}x_2 \end{pmatrix}.$$

Hence $S^{n+1}x_2 = 0$, i.e., $Sx_2 \in \ker S^n = \ker S^{*n}$. Therefore, $S^{*n}Sx_2 = 0$. Since $S^nS = SS^n$, by Fuglede-Putnam Theorem $S^{*n}S = SS^{*n}$. Moreover, since $S^*SS^{*n}x_2 = S^*S^{*n}Sx_2 = 0$, it follows that $\|SS^{*n}x_2\|^2 = 0$. Hence $S^nS^{*n}x_2 = 0$, and so $\|S^{*n}x_2\|^2 = 0$. Then $x_2 \in ker\ S^{*n} = ker\ S^n$. Thus $x \in ker\ T^n$.

(ii) If $T \in \mathcal{L}(H)$ is n-power quasinormal and complex symmetric, then both T^n and $(T^n)^*$ are quasinormal by Lemma 3.1. Hence both T^n and $(T^n)^*$ have the single-valued extension property by Theorem 3.2. □

Theorem 3.12. *Let $T \in \mathcal{L}(\mathcal{H})$ be n-power quasinormal. If $ran\ T = ran\ T^{n+1}$, then T has the following matrix representation,*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_3 \end{bmatrix} : \overline{ran\ T} \oplus ker\ T^* \to \overline{ran\ T} \oplus ker\ T^*$$

where $T_1 = T|_{\overline{ranT}}$ is n-power normal and T_3 is nilpotent of order n, and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Since $[T^*, T^n]T = 0$, we get that $[T^*, T^n] = 0$ on $\overline{ran\ T}$. If T has dense range in $\mathcal{H}$, then T is n-power normal. Otherwise, $\overline{ran\ T} \neq \mathcal{H}$ and $\overline{ran\ T} \in Lat\ T$. Hence T has the matrix representation, $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$ on $\overline{ran\ T} \oplus ker\ T^*$. If $y \in \overline{ranT}$, then there is a sequence $\{y_k\}$ in $ranT$ such that $y_k \to y$. Since $ran\ T = ran\ T^{n+1}$, we get $y_k \in ran\ T = ran\ T^{n+1}$. Then there is a sequence $\{x_k\} \in \mathcal{H}$ such that $y_k = T^{n+1}x_k$. $T^*y_k = T^*T^{n+1}x_k = T^nT^*Tx_k \in ran\ T^n = ran\ T$. Therefore $T^*y_k \in ran\ T$, and so $T^*y \in \overline{ran\ T}$. Thus $T^*(\overline{ran\ T}) \subset \overline{ran\ T}$ and $\overline{ran\ T}$ reduces T. Since $T^*T^{n+1} = T^nT^*T$ and $\overline{ran\ T}$ is a reducing subspace for T, $T_1 = T|_{\overline{ran\ T}}$ is n-power normal. Let P be the orthogonal projection onto $\overline{ran\ T}$. For any $z = \binom{z_1}{z_2} \in \mathcal{H} = \overline{ran\ T} \oplus ker\ T^*$, $(I - P)z \in ker\ T^*$ and

$$\begin{aligned} \langle T_3^n z_2, z_2 \rangle &= \langle T^n(I-P)z, (I-P)z \rangle \\ &= \langle (I-P)z, T^{*n}(I-P)z \rangle \\ &= 0. \end{aligned}$$

Then T_3 is nilpotent of order n and $\sigma(T) = \sigma(T_1) \cup \{0\}$. □

Corollary 3.13. Let $T \in \mathcal{L}(\mathcal{H})$ be n-power quasinormal. If $\overline{ran\ T}$ is a reducing subspace of T, then $T|_{\overline{ran\ T}}$ is n-power normal and $T|_{ker\ T^*}$ is nilpotent.

Proof. As in the proof of Theorem 3.12, we get the results. □

Corollary 3.14. Let $T \in \mathcal{L}(\mathcal{H})$ be n-power quasinormal. If $ran\ T = ran\ T^{n+1}$, then $\sigma(T) = \sigma_{ap}(T)$.

Proof. From Theorem 3.12, $T = T_1 \oplus T_3$ where T_1 is n-power normal and T_3 is nilpotent of order n. Since T_1 is n-power normal, T_1^n is normal as in the proof of Corollary 3.11. Hence T_1 has the single-valued extension property by Theorem 3.2. Since T_3 has also the single-valued extension property, T has the single-valued extension property. Since $T^* = T_1^* \oplus T_3^*$ where T_1^* is n-power normal and T_3^* is nilpotent of order n, T^* has the single-valued extension property. Hence the proof follows from [2, Corollary 2.45]. □

We next consider the operator transforms of an n-power quasinormal operator.

Theorem 3.15. *Let $T = U|T|$ be the polar decomposition of an n-power qusinormal operator $T \in \mathcal{L}(H)$. Then the following statements hold.*
(i) *If $T^nU^* = U^*T^n$, then the Aluthge transform $\widetilde{T}$ of T is n-power quasinormal.*
(ii) *The Duggal transform $\widetilde{T}^D$ of T is also n-power quasinormal.*

Proof. (i) If T is n-power quasinormal, then $T^n|T|^2 = |T|^2T^n$. Since $T^np(|T|^2) = p(|T|^2)T^n$ for any polynormial $p(t)$ with $p(0) = 0$, take $p_k(t) \to t^{\frac{1}{2}}$. Then $T^n|T| = |T|T^n$ since the square root $|T|$ of a positive operator $|T|^2$ is approximated uniformly by polynomials of $|T|^2$. Since $T^nU^* = U^*T^n$ and $T^n|T| = |T|T^n$,

$$\widetilde{T}^n(\widetilde{T}^*\widetilde{T}) - (\widetilde{T}^*\widetilde{T})\widetilde{T}^n = |T|^{\frac{1}{2}}[T^nU^*|T|U - U^*|T|T^nU]|T|^{\frac{1}{2}} = 0.$$

Hence $\widetilde{T}$ is n-power quasinormal.

(ii) Since $T^n|T|^2 = |T|^2T^n$, we get that

$$(\widetilde{T}^D)^n((\widetilde{T}^D)^*\widetilde{T}^D) - ((\widetilde{T}^D)^*\widetilde{T}^D)(\widetilde{T}^D)^n = U^*[T^n|T|^2 - |T|^2T^n]U = 0. \tag{2}$$

Hence $\widetilde{T}^D$ is also n-power quasinormal. □

Corollary 3.16. *Let $T = U|T|$ be the polar decomposition of an n-power quasinormal operator $T \in \mathcal{L}(H)$. If U is unitary, then T is n-power quasinormal if and only if $\widetilde{T}^D$ is.*

Proof. Since U is unitary, the proof follows from (2). □

Recall that given $x, y \in \mathcal{H}$, we define $x \otimes y$ mapping $\mathcal{H}$ into itself by $(x \otimes y)h = \langle h, y\rangle x$. We next consider the case of rank one operators.

Theorem 3.17. *Let T be a rank one operator defined by $T = x \otimes y$. Then the following statements are equivalent.*
(i) *T is n-power quasinormal.*
(ii) *T^n is normal.*
(iii) *T^n is quasinormal.*
(iv) *$x = \frac{\langle x,y\rangle}{||y||^2} y$ holds.*

Proof. If $\langle x, y\rangle = 0$, it is trivial. So we may assume that $\langle x, y\rangle \neq 0$.

(i)⇔(ii) If (i) holds, then T^n is normal by Lemma 3.1. Conversely, if (ii) holds, $T^nT^* = T^*T^n$ by Fuglede-Putnam theorem since $T^nT = TT^n$. Thus $T^nT^*T = T^*T^{n+1}$.

(i)⇔(iv) Since $T^n = \langle x, y\rangle^{n-1} x \otimes y$ and $T^*T = ||x||^2 y \otimes y$,

$$T^n(T^*T) = \langle x, y\rangle^{n-1}||x||^2||y||^2 x \otimes y \text{ and } (T^*T)T^n = \langle x, y\rangle^n ||x||^2 y \otimes y.$$

Then T is n-power quasinormal if and only if

$$\langle x, y\rangle^{n-1}||x||^2||y||^2 x \otimes y = \langle x, y\rangle^n||x||^2 y \otimes y.$$

Hence T is n-power quasinormal if and only if $||y||^2 x \otimes y = \langle x, y\rangle y \otimes y$ if and only if $||y||^2 x = \gamma\langle x, y\rangle y$ and $y = \overline{\gamma} y$ for some nonzero $\gamma \in \mathbb{C}$. Since $\gamma = 1$, T is n-power quasinormal if and only if $x = \frac{\langle x,y\rangle}{||y||^2} y$ holds.

(iii)⇔(iv) Since $T^n = \langle x, y\rangle^{n-1} x \otimes y$ and $(T^n)^* = \langle y, x\rangle^{n-1} y \otimes x$,

$$(T^n)^*T^n = |\langle x, y\rangle|^{2(n-1)}||x||^2 y \otimes y.$$

Then

$$[(T^n)^*T^n]T^n = |\langle x, y\rangle|^{2(n-1)}\langle x, y\rangle^n||x||^2 y \otimes y$$

and

$$T^n[(T^n)^*T^n] = |\langle x, y\rangle|^{2(n-1)}\langle x, y\rangle^{n-1}||x||^2||y||^2 x \otimes y.$$

Hence T^n is quasinormal if and only if

$$|\langle x, y\rangle|^{2(n-1)}\langle x, y\rangle^n||x||^2 y \otimes y = |\langle x, y\rangle|^{2(n-1)}\langle x, y\rangle^{n-1}||x||^2||y||^2 x \otimes y.$$

Hence T^n is quasinormal if and only if $||y||^2 x \otimes y = \langle x, y\rangle y \otimes y$ if and only if $||y||^2 x = \gamma\langle x, y\rangle y$ and $y = \overline{\gamma} y$ for some nonzero $\gamma \in \mathbb{C}$. Since $\gamma = 1$, T^n is quasinormal if and only if $x = \frac{\langle x,y\rangle}{||y||^2} y$ holds. □

We next consider the n-power quasinormality of operator matrices.

Lemma 3.18. *Let $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ be defined as $T = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$. Then T is n-power quasinormal if and only if the following identities hold.*

(i) $[A^n, A^*]A + ZB^*A = 0$.
(ii) $[A^n, A^*]B + Z(B^*B + C^*C) - A^*ZC = 0$.
(iii) $C^nB^*A - B^*A^{n+1} = 0$.
(iv) $C^nB^*B - B^*A^nB + [C^n, C^*]C - B^*ZC = 0$
where $Z = \sum_{j=0}^{n-1} A^{n-1-j}BC^j$.

Proof. Set $Z = \sum_{j=0}^{n-1} A^{n-1-j}BC^j$. Then

$$T^n = \begin{bmatrix} A^n & \sum_{j=0}^{n-1} A^{n-1-j}BC^j \\ 0 & C^n \end{bmatrix} = \begin{bmatrix} A^n & Z \\ 0 & C^n \end{bmatrix}.$$

Since T is n-power quasinormal, an easy calculation shows that

$$T^nT^* = \begin{bmatrix} A^nA^* + ZB^* & ZC^* \\ C^nB^* & C^nC^* \end{bmatrix} \quad \text{and} \quad T^*T^n = \begin{bmatrix} A^*A^n & A^*Z \\ B^*A^n & B^*Z + C^*C^n \end{bmatrix}.$$

Hence we get that

$$\begin{aligned} 0 &= [T^n, T^*]T \\ &= \begin{bmatrix} [A^n, A^*] + ZB^* & ZC^* - A^*Z \\ C^nB^* - B^*A^n & [C^n, C^*] - B^*Z \end{bmatrix} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} \end{aligned}$$

where R_1, R_2, R_3, and R_4 satisfy the following identities;
$R_1 = [A^n, A^*]A + ZB^*A$,
$R_2 = [A^n, A^*]B + Z(B^*B + C^*C) - A^*ZC$,
$R_3 = C^nB^*A - B^*A^{n+1}$, and
$R_4 = C^nB^*B - B^*A^nB + [C^n, C^*]C - B^*ZC$
where $Z = \sum_{j=0}^{n-1} A^{n-1-j}BC^j$. So we complete the proof. □

Proposition 3.19. *Let $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ be defined as $T = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$. Then the following statements hold.*
(i) *If Q is unitarily equivalent to $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$, then Q is n-power quasinormal.*
(ii) *When $B = 0$, T is n-power quasinormal if and only if both A and C are n-power quasinormal.*
(iii) *If $\sum_{j=0}^{n-1} A^{n-1-j}BC^j = 0$ and $\ker (A^n)^* \subset \ker A^n$, then A^n is normal. In addition, if T is hyponormal and $n = 2$, then A is normal.*
(iv) *If $\sum_{j=0}^{n-1} A^{n-1-j}BC^j = 0$ and $A = B$, then A and C have the single valued extension property.*

Proof. (i) If Q is unitarily equivalent to $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$, then there exists a unitary operator U such that $U^*QU = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$. Since $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$ is n-power quasinormal by Lemma 3.18 with $A = B = 0$, we get that

$$\begin{aligned} Q^nQ^*Q &= (UTU^*)^n(UTU^*)^*(UTU^*) \\ &= (UT^nU^*)(UT^*U^*)(UTU^*) \\ &= UT^nT^*TU^* \\ &= U(T^*T^{n+1})U^* \\ &= U(U^*QU)^*(U^*QU)^{n+1}U^* \\ &= Q^*Q^{n+1}. \end{aligned}$$

Hence Q is n-power quasinormal.

(ii) If $B = 0$, then $Z = \sum_{j=0}^{n-1} A^{n-1-j}BC^j = 0$ in Lemma 3.18. Hence the proof follows from Lemma 3.18.

(iii) If $\sum_{j=0}^{n-1} A^{n-1-j}BC^j = 0$, then A is n-power quasinormal from Lemma 3.18 and hence A^n is quasinormal from Lemma 3.1. Then $[A^n, (A^n)^*]A^n = 0$, i.e., $[A^n, (A^n)^*] = 0$ on $\overline{ran\, A^n}$. Since A^n is quasinormal, it is clear that $ker\ A^n \subset ker\ (A^n)^*$. Thus $ker\ A^n = ker\ (A^n)^*$. Then $[A^n, (A^n)^*] = 0$ on $ker\ (A^n)^*$. Hence A^n is normal. In addition, if T is hyponormal and $n = 2$, then

$$0 \leq T^*T - TT^* = \begin{bmatrix} A^*A - AA^* - BB^* & A^*B - BC^* \\ B^*A - CB^* & B^*B + C^*C + CC^* \end{bmatrix}.$$

Hence $A^*A - AA^* - BB^* \geq 0$ from [19]. Thus A is hyponormal. Since A^2 is normal and A is hyponormal,

$$A(A^*A)A^* \geq A(AA^*)A^* = A^*(A^*A)A \geq A^*(AA^*)A.$$

Hence

$$(AA^*)^2 \geq (A^*A)^2.$$

By Löwner's theorem (see [17]), $AA^* \geq A^*A$. Hence A is normal.

(iv) If $\sum_{j=0}^{n-1} A^{n-1-j}BC^j = 0$ and $A = B$, then A and C are n-power quasinormal from Lemma 3.18. Hence A and C are nth roots of quasinormal operators from Lemma 3.1. Since A^n and C^n have the single valued extension property from Theorem 3.2. □

Recall that $T \in \mathcal{L}(H)$ is said to be binormal if T^*T and TT^* commute. In the following examples, we observe that there are no inclusion relationships between the binormality and the n-power quasinormality.

Proposition 3.20. *Let T be any 2×2 matrix in $\mathcal{L}(\mathbb{C}^2)$. Assume that T is n-power quasinormal. Then T is binormal if and only if it is unitarily equivalent to $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ for $a \neq 0$ and $b \neq 0$ where $\sum_{j=0}^{n-1} a^{n-1-j}c^j = 0$, $(|a|^2 - |c|^2)(\overline{a} - \overline{c}) + |b|^2(|a|^2 + |c|^2) = 0$, and $a\overline{c} \in \mathbb{R}$.*

Proof. Since T is n-power quasinormal, we get from Proposition 3.8 that T is unitarily equivalent to one of the following matrices;

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}, \text{ and } \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \text{ where } \sum_{j=0}^{n-1} a^{n-1-j}c^j = 0.$$

Since the first, second, third cases are binormal, it suffice to check the fourth case with $a \neq 0$, $b \neq 0$, and $c \neq 0$. Moreover, it is an elementary calculation that T^*T and TT^* commute if and only if $(|a|^2 - |c|^2)(\overline{a} - \overline{c}) + |b|^2(|a|^2 + |c|^2) = 0$, and $a\overline{c} \in \mathbb{R}$. Thus we complete the proof. □

Example 3.21. Let T be a 2×2 matrix in $\mathcal{L}(\mathbb{C}^2)$ defined as $T = \begin{bmatrix} 2 & 1 \\ 0 & -1 + \sqrt{3}i \end{bmatrix}$. Then T is 3-power quasinormal, but is not binormal. Indeed, since $\sum_{j=0}^{2} 2^{2-j}(-1 + \sqrt{3}i)^j = 0$, T is 3-power quasinormal from Proposition 3.8. However, T is not binormal from Proposition 3.20. On the other hand, if T is a 2×2 matrix in $\mathcal{L}(\mathbb{C}^2)$ defined as $T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, then T is binormal, but is not n-power quasinormal for any odd number n from Propositions 3.8 and 3.20.

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