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Fixed points theorems for enriched non-expansive mappings in geodesic spaces

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Abstract. The purpose of this paper is to extend a class of enriched non-expansive mappings from linear spaces to nonlinear spaces, namely, geodesic metric spaces of non-positive curvature. We prove that an enriched non-expansive mapping in complete CAT(0) space has fixed points. Moreover, we also propose simplified Mann iteration process to approximate fixed points of enriched non-expansive mappings by $\triangle$ and strong convergence in CAT(0) spaces.

1. Introduction and Preliminaries

Throughout this paper, $\mathbb{Z}_+$ denotes the set of all nonnegative integers. The study of fixed points in the setup of CAT(0) spaces was initiated by Kirk [1, 2]. He showed that every non-expansive mapping defined on a nonempty closed, convex and bounded subset of a complete CAT(0) space always has a fixed point. The notion of $\triangle$-convergence in general metric spaces was introduced by Lim [3] in 1976. Kirk and Panyanak [4] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [5] continued to work in this direction. Their results involved Mann and Ishikawa iteration processes involving one mapping.

A metric space (X, d) is a CAT(0) space (the term is due to M. Gromov, see [6]) if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. The precise definition is given below. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, one can see Bridson and Haefliger [6]. We note in particular that the complex Hilbert ball with a hyperbolic metric (see [7], also inequality (4.2) of [8] and subsequent comments) is a CAT(0) space.

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map γ from a closed interval $[a, b] \subset \mathbb{R}$ to X such that $\gamma(a) = x$, $\gamma(b) = y$, and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [a, b]$. The graph of γ is called a *geodesic* (or metric) segment joining x and y. We say that the geodesic γ joins x and y or that the geodesic segment $\gamma([a, b])$ joins x and y; x and y are also called the endpoints of γ. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be *geodesic space*

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if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. If $\gamma([a, b])$ is a geodesic segment joining x and y and $\lambda \in [0, 1]$, then $z := \gamma(\lambda a + (1 - \lambda)b)$ is the unique point in $\gamma([a, b])$ satisfying

$$d(z, x) = \lambda d(x, y) \quad \textit{and} \quad d(z, y) = (1 - \lambda)d(x, y). \tag{1}$$

In the sequel, we shall use the notation $[x, y]$ for the geodesic segment $\gamma([a, b])$ and we shall denote this z by $(1 - \lambda)x \oplus \lambda y$, provided that there is no possible ambiguity. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points, that is, $[x, y] \subset Y$ for all $x, y \in Y$.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits [9]. In fact (cf. [6], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

We now collect some basic facts about CAT(0) spaces.

Lemma 1.1. *[5] Let X be a CAT(0) space. Then*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z) \tag{2}$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 1.2. *[5] Let (X, d) be a CAT(0) space. Then*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2 \tag{3}$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Let $\{\tau_n\}$ be a bounded sequence in a complete CAT(0) space X. For $x \in X$, we set

$$r(x, \{\tau_n\}) = \lim_{n\to\infty} \sup d(x, \tau_n).$$

The asymptotic radius of $r(\{\tau_n\})$ of $\{\tau_n\}$ is given by

$$r(\{\tau_n\}) = \inf\{r(x, \{\tau_n\}) : x \in X\}.$$

The asymptotic center $A(\{\tau_n\})$ of $\{\tau_n\}$ is the set

$$A(\{\tau_n\}) = \{x \in X : r(x, \{\tau_n\}) = r(\{\tau_n\})\}.$$

It is well known that in a CAT(0) space, $A(\{\tau_n\})$ consists of exactly one point [10].

Definition 1.3. *[3] A sequence $\{\tau_n\}$ in a CAT(0) space X is called $\triangle$-convergence to $x \in X$, denoted by $\triangle - \lim_{n\to\infty}\{\tau_n\} = x$ if x is the unique asymptotic center of $\{u_n\}$, for every subsequence $\{u_n\}$ of $\{\tau_n\}$.*

Notice that for a given $\{\tau_n\} \subset X$ which $\triangle$-converges to x and for any $y \in X$ with $y \neq x$ (owing to uniquenes of asymptotic center), we have

$$\lim_{n\to\infty} \sup d(\tau_n, x) < \lim_{n\to\infty} \sup d(\tau_n, y).$$

Thus, every CAT(0) space satisfies the Opial's property.

Lemma 1.4. (*i*) *Every bounded sequence in X has a $\triangle$-convergent subsequence (cf. [4], p. 3690).*
(*ii*) *If $\mathcal{M}$ is a closed convex subset of X and if $\{\tau_n\}$ is a bounded sequence in $\mathcal{M}$, then the asymptotic center of $\{\tau_n\}$ is in $\mathcal{M}$ (cf. [11], Proposition 2.1).*
(*iii*) *If $\mathcal{M}$ is a closed convex subset of X and $\mathcal{G} : \mathcal{M} \to X$ is a non-expansive mapping, then the conditions, $\{\tau_n\}$ $\triangle$-converges to x and $d(\tau_n, \mathcal{G}\tau_n) \to 0$, imply $x \in \mathcal{M}$ and $\mathcal{G}(x) = x$ (cf. [4], Proposition 3.7).*

Lemma 1.5. *[5] If $\{\tau_n\}$ is a bounded sequence in X with $A(\{\tau_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{\tau_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(\tau_n, u)\}$ converges, then $x = u$.*

Definition 1.6. *[12] A mapping $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ is said to satisfy property (I), if there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ and $\psi(z) > 0$, $\forall z > 0$ such that $d(x, \mathcal{G}x) \geq \psi(d(x, F(\mathcal{G})))$, $\forall\, x \in \mathcal{M}$.*

Recall that a mapping $\mathcal{G} : \mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ a nonempty subset of a CAT(0) space X is said to be non-expansive if for all $x, y \in \mathcal{M}$

$$d(\mathcal{G}x, \mathcal{G}y) \leq d(x, y), \tag{4}$$

and if $\mathcal{G}$ has at least one fixed point then $\mathcal{G}$ is called quasi non-expansive mapping.

There are several generalizations of non-expansive mappings available in the literature, e.g. generalized non-expansive mappings due to Suzuki (2008) and due to Hardy and Rogers (1973). Most recently, Berinde [13] introduced enriched non-expansive mapping in normed space which is also a generalization of non-expansive mapping and is defined as follows:

Definition 1.7. *[13] Let X be a normed linear space. A mapping $\mathcal{G} : X \to X$ is said to be an enriched non-expansive mapping if there exists $b \in [0,\infty)$ such that*

$$\|b(x - y) + \mathcal{G}x - \mathcal{G}y\| \leq (b + 1)\|x - y\|, \qquad \forall\, x, y \in X. \tag{5}$$

Berinde proved existence and convergence results for such mappings. He also showed that every non-expansive mapping is enriched non-expansive, but the reverse is not true in general. Moreover, if $\mathcal{G}$ has at least one fixed point, then $\mathcal{G}$ need not be quasi non-expansive mapping. While generalized non-expansive mappings due to Suzuki (2008) and Hardy and Rogers (1973) are quasi non-expansive. In recent years authors also enriched other class of mappings, for example enriched contraction, enriched Kannan, enriched Chattarjea, enriched strictly pseudocontractive [cf. [14]].

In this paper, we define enriched non-expansive mapping in CAT(0) space and prove existence of fixed points for such mapping. We also define simplified Mann iterative process to approximate fixed points of enriched non-expansive mapping. Moreover, we discuss some relevant results for enriched non-expansive mappings.

2. Enriched non-expansive mapping in CAT(0) space and properties

Now, we define enriched non-expansive mapping in CAT(0) space and prove some basic properties and results for such mapping.

From now on, X is a complete CAT(0) space, $\mathcal{M}$ is a nonempty convex subset of X and $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ is a mapping. The mapping $\mathcal{G}$ is called enriched non-expansive if for each $x, y \in \mathcal{M}$ and $b \in [0, \infty)$,

$$d\Big(\frac{b}{(b+1)}x \oplus \frac{1}{(b+1)}\mathcal{G}x, \frac{b}{(b+1)}y \oplus \frac{1}{(b+1)}\mathcal{G}y\Big) \leq d(x, y). \tag{6}$$

A point $x \in \mathcal{M}$ is called a fixed point of $\mathcal{G}$ if $x = \mathcal{G}x$. We shall denote with $F(\mathcal{G})$ the set of fixed points of $\mathcal{G}$.

Definition 2.1. *[15] For a self map $\mathcal{G}$ on a convex subset $\mathcal{M}$ of a complete CAT(0) space X and for any $\alpha \in (0, 1]$, the averaged (or α-Krasnoselskii) mapping $\mathcal{G}_\alpha$ given by*

$$\mathcal{G}_\alpha(x) = (1 - \alpha)x \oplus \alpha\mathcal{G}x, \qquad \forall x \in \mathcal{M}. \tag{7}$$

Remark 2.2. *For a self mapping $\mathcal{G}$ on a convex subset $\mathcal{M}$ of a CAT(0) space X and for any $\alpha \in (0, 1]$, we have*

$$F(\mathcal{G}_\alpha) = F(\mathcal{G}).$$

Now, we state and prove first result of this section as follows.

Theorem 2.3. *Let X be a complete CAT(0) space and $\mathcal{G} : X \to X$ be enriched non-expansive mapping. Then, α-Krasnoselskii map $\mathcal{G}_\alpha : X \to X$ is non-expansive mapping.*

Proof. Since $\mathcal{G}$ is an enriched non-expansive mapping, we have for all $x, y \in X$,

$$d\Big(\frac{b}{(b+1)}x \oplus \frac{1}{(b+1)}\mathcal{G}x, \frac{b}{(b+1)}y \oplus \frac{1}{(b+1)}\mathcal{G}y\Big) \leq d(x, y).$$

Set $\alpha = \frac{1}{b+1}$, we have

$$d\Big(\alpha(\frac{1}{\alpha} - 1)x \oplus \alpha\mathcal{G}x, \alpha(\frac{1}{\alpha} - 1)y \oplus \alpha\mathcal{G}y\Big) \leq d(x, y)$$
$$d\Big((1 - \alpha)x \oplus \alpha\mathcal{G}x, (1 - \alpha)y \oplus \alpha\mathcal{G}y\Big) \leq d(x, y).$$

This gives

$$d\Big(\mathcal{G}_\alpha x, \mathcal{G}_\alpha y\Big) \leq d(x, y).$$

Hence $\mathcal{G}_\alpha$ is a non-expansive mapping. □

Lemma 2.4. *Let $\mathcal{M}$ be a nonempty closed convex subset of a complete CAT(0) space X satisfying Opial's condition. Let $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ be an enriched non-expansive map. Then, $\mathcal{G}x = x$.*

Proof. From Theorem 2.3, we know that $\mathcal{G}_\alpha$ is a non-expansive map for $\alpha = \frac{1}{b+1}$. Now, let $\{\tau_n\}$ be a sequence that $\triangle$-converges to $x \in \mathcal{M}$ and $\lim_{n\to\infty} d(\tau_n, \mathcal{G}\tau_n) = 0$. However

$$d(\tau_n, \mathcal{G}_\alpha\tau_n) \leq \alpha d(\tau_n, \mathcal{G}\tau_n),$$

so that

$$\lim_{n\to\infty} d(\tau_n, \mathcal{G}_\alpha\tau_n) \leq \alpha \lim_{n\to\infty} d(\tau_n, \mathcal{G}\tau_n) = 0.$$

By Lemma 1.4(iii), we have

$$\mathcal{G}_\alpha(x) = x.$$

It can be easily seen from Remark 2.2, $F(\mathcal{G}_\alpha)$=$F(\mathcal{G})$. Hence, $\mathcal{G}(x) = x$. □

To estimate fixed points of enriched non-expansive mapping, we define simplified Mann iterative process as follows. Let $\mathcal{M}$ be a convex subset of a CAT(0) space X, x_0 be an arbitrary point in $\mathcal{M}$ and $b \in [0, \infty)$, the modified/simplified Mann iteration process is defined as follows:

$$\tau_{n+1} = \Big[\frac{b}{b+1}\tau_n \oplus \frac{1}{b+1}\mathcal{G}\tau_n\Big], \qquad n \in \mathbb{Z}_+, \tag{8}$$

$\{\tau_{n+1}\}$ is a point on the geodesic segment $[\tau_n, \mathcal{G}\tau_n]$.

3. Existence and approximation results

Theorem 3.1. *Let $\mathcal{M}$ be a nonempty bounded closed convex subset of a complete CAT(0) space X and $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ be enriched non-expansive mapping. Then the set $F(\mathcal{G})$ is nonempty.*

Proof. Since $\mathcal{G}$ is enriched non-expansive mapping, by definition, it follows that there exists a constant $b \in [0, \infty)$ such that

$$d\Big(\frac{b}{(b+1)}x \oplus \frac{1}{(b+1)}\mathcal{G}x, \frac{b}{(b+1)}y \oplus \frac{1}{(b+1)}\mathcal{G}y\Big) \leq d(x, y), \qquad \forall x, y \in \mathcal{M}.$$

By putting $b = \frac{1}{\alpha} - 1$ for $b > 0$, it follows that $\alpha \in (0, 1)$ and previous inequality is equivalent to

$$d\Big((1-\alpha)x \oplus \alpha\mathcal{G}x, (1-\alpha)y \oplus \alpha\mathcal{G}y\Big) \leq d(x, y). \tag{9}$$

Denote $\mathcal{G}_\alpha(x) = (1-\alpha)x \oplus \alpha\mathcal{G}x$. Then inequality (9) expresses the fact that

$$d(\mathcal{G}_\alpha x, \mathcal{G}_\alpha y) \leq d(x, y) \qquad \forall x, y \in \mathcal{M}$$

i.e. the averaged operator $\mathcal{G}_\alpha$ is non-expansive. So, by Kirk [1], it follows that $\mathcal{G}_\alpha$ has at least one fixed point. Next in view of Remark 2.2, $F(\mathcal{G})=F(\mathcal{G}_\alpha)\neq\emptyset$. □

Now, we prove the following useful lemmas which are used to prove the next results of this section.

Lemma 3.2. *Let $\{\tau_n\}$ be a sequence developed by the iteration process (8), then $\lim_{n\to\infty} d(\tau_n, t)$ exists for all $t \in F(\mathcal{G})$.*

Proof. Let $t \in F(\mathcal{G})$. From Theorem 2.3, we know that for $\alpha = \frac{1}{b+1}$, $\mathcal{G}_\alpha$ is a non-expansive map. Therefore, we have

$$\begin{aligned} d(\tau_{n+1}, t) &= d\Big([\frac{b}{b+1}\tau_n \oplus \frac{1}{b+1}\mathcal{G}\tau_n], t\Big) \\ &\leq d(\tau_n, t). \end{aligned} \tag{10}$$

Thus the sequence $\{d(\tau_n, t)\}$ is bounded below and decreasing for all $t \in F(\mathcal{G})$. Hence $\lim_{n\to\infty} d(\tau_n, t)$ exists. □

Lemma 3.3. *Let $\{\tau_n\}$ be a sequence developed by the iteration process* (8) *and $F(\mathcal{G}) \neq \emptyset$. Then* $\lim_{n\to\infty} d(\tau_n, \mathcal{G}\tau_n) = 0$.

Proof. Consider $\mathcal{G}_\alpha : \mathcal{M} \to \mathcal{M}$, for $\alpha = \frac{1}{b+1}$. Then from Theorem 2.3, we know that $\mathcal{G}_\alpha$ is non-expansive. Also from Remark 2.2, we know that $F(\mathcal{G}) = F(\mathcal{G}_\alpha) \neq \emptyset$. Moreover, for the same initial guess $\tau_0 \in \mathcal{M}$, the sequence generated by Mann type iteration process (8) using $\mathcal{G}$ is the same as that generated by the Mann iteration process using $\mathcal{G}_\alpha$. Hence by Lemma 2.12 in [5], we have $\lim_{n\to\infty} d(\tau_n, \mathcal{G}_\alpha(\tau_n)) = 0$. By using the definition of $\mathcal{G}_\alpha$, we get

$$\alpha \lim_{n\to\infty} d(\tau_n, \mathcal{G}(\tau_n)) = 0.$$

Since $\alpha \neq 0$, then $\lim_{n\to\infty} d(\tau_n, \mathcal{G}(\tau_n)) = 0$. □

Now, we prove the following $\triangle$-convergence theorem for enriched non-expansive mapping via the iteration process (8).

Theorem 3.4. *Presume that X satisfies Opial's property, then the sequence $\{\tau_n\}$ developed by modified Mann iteration process* (8) *$\triangle$-converges to a fixed point of the mapping $\mathcal{G}$.*

Proof. By Lemmas 3.2 and 3.3, we observe that $\{\tau_n\}$ is a bounded sequence with

$$\lim_{n\to\infty} d(\tau_n, \mathcal{G}\tau_n) = 0.$$

Let $W_w(\{\tau_n\}) := \cup A(\{u_n\})$, where union is taken over all subsequence $\{u_n\}$ over $\{\tau_n\}$. In order to prove that $\triangle$-convergence of $\{\tau_n\}$ to a fixed point of $\mathcal{G}$, firstly we will prove $W_w(\{\tau_n\}) \subset F(\mathcal{G})$ and thereafter argue that $W_w(\{\tau_n\})$ is singleton set. To show $W_w(\{\tau_n\}) \subset F(\mathcal{G})$, let $y \in W_w(\{\tau_n\})$. Then, there exists a subsequence $\{y_n\}$ of $\{\tau_n\}$ such that $A(\{y_n\}) = \{y\}$. By Lemma 1.4(i) and (ii) there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\triangle\text{-}\lim_{n\to\infty} z_n = z \in \mathcal{M}$. By Lemma 1.4(iii), $z \in F(\mathcal{G})$. By Lemma 1.5, $z = y$. With a view to prove that $W_w(\{\tau_n\})$ is singleton, let $\{y_n\}$ be a subsequence of $\{\tau_n\}$. In view of Lemma 1.4(i) and (ii), there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\triangle\text{-}\lim_{n\to\infty} z_n = z$. Let $A(\{y_n\}) = \{y\}$ and $A(\{\tau_n\}) = \{x\}$. Earlier, we have shown that $y = z$, therefore it is enough to show $z = x$. If $z \neq x$, then by Lemma 3.2 $\{d(\tau_n, z)\}$ is convergent. By uniqueness of asymptotic centers

$$\begin{aligned}\limsup_{n\to\infty} d(z_n, z) &= \limsup_{n\to\infty} d(z_n, x) \le \limsup_{n\to\infty} d(\tau_n, x) < \limsup_{n\to\infty} d(\tau_n, z) \\ &= \limsup_{n\to\infty} d(z_n, z),\end{aligned}$$

which is a contradiction. So that the conclusion follows. □

Now, we prove two strong convergence results.

Theorem 3.5. *The sequence $\{\tau_n\}$ developed by the iteration process* (8) *converges strongly to a fixed point of $\mathcal{G}$ if and only if $\liminf_{n\to\infty} d(\tau_n, F(\mathcal{G})) = 0$.*

Proof. First part is trivial. Now, we prove the converse part. Presume that $\liminf_{n\to\infty} d(\tau_n, F(\mathcal{G})) = 0$. From Lemma 3.2, $\lim_{n\to\infty} d(\tau_n, t)$ exists, for all $t \in F(\mathcal{G})$ and by hypothesis $\lim_{n\to\infty} d(\tau_n, F(\mathcal{G})) = 0$.
Now our assertion is that $\{\tau_n\}$ a Cauchy sequence in $\mathcal{M}$. As $\lim_{n\to\infty} d(\tau_n, F(\mathcal{G})) = 0$, for a given $\lambda > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$,

$$\begin{aligned} d(\tau_n, F(\mathcal{G})) &< \frac{\lambda}{2} \\ \implies \inf\{d(\tau_n, t) : t \in F(\mathcal{G})\} &< \frac{\lambda}{2}. \end{aligned}$$

Specifically, $\inf\{d(\tau_N, t) : t \in F(\mathcal{G})\} < \frac{\lambda}{2}$. So, there exists $t \in F(\mathcal{G})$ such that

$$d(\tau_N, t) < \frac{\lambda}{2}.$$

Now, for $m, n \ge N$,

$$\begin{aligned} d(\tau_{n+m}, \tau_n) &\le d(\tau_{n+m}, t) + d(\tau_n, t) \\ &\le d(\tau_N, t) + d(\tau_N, t) \\ &= 2d(\tau_N, t) < \lambda. \end{aligned}$$

Thus, $\{\tau_n\}$ is a Cauchy sequence in $\mathcal{M}$, so there exists an element $\ell \in \mathcal{M}$ such that $\lim_{n\to\infty} \tau_n = \ell$. Now, $\lim_{n\to\infty} d(\tau_n, F(\mathcal{G})) = 0$ implies $d(\ell, F(\mathcal{G})) = 0$, hence we get $\ell \in F(\mathcal{G})$. □

By applying condition (I), we prove another strong convergence result.

Theorem 3.6. *Presume that the mapping $\mathcal{G}$ satisfies condition* (I). *Then the sequence $\{\tau_n\}$ developed by the iteration process* (8) *converges strongly to a fixed point of $\mathcal{G}$.*

Proof. We already proved in Lemma 3.3 that

$$\lim_{n\to\infty} d(\tau_n, \mathcal{G}(\tau_n)) = 0. \tag{11}$$

Applying condition (I) and equation (11), we obtain

$$\begin{aligned} 0 \le \lim_{n\to\infty} \psi(d(\tau_n, F(\mathcal{G}))) &\le \lim_{n\to\infty} d(\tau_n, \mathcal{G}(\tau_n)) = 0 \\ \implies \lim_{n\to\infty} \psi(d(\tau_n, F(\mathcal{G}))) &= 0. \end{aligned}$$

And hence,

$$\lim_{n\to\infty} d(\tau_n, F(\mathcal{G})) = 0.$$

Now, in view of Theorem 3.5, we are through. □

4. Conclusions

In this paper, we extend enriched non-expansive mappings from linear spaces to nonlinear spaces and prove existence result for such mappings. Further, we introduce simplified Mann iteration process to approximate the fixed points of enriched non-expansive mappings in CAT(0) spaces.

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References

[1] W. A. Kirk, Geodesic geometry and fixed point theory, Seminar of Mathematical Analysis, Vol. 64 of Colecciń Abierta, Univ. Sevilla Secr. Publ., Seville, 2003.
[2] W. A. Kirk, Geodesic geometry and fixed point theory II. In International Conference on Fixed Point Theory and Applications, Yokohama Publication, Yokohama, 2004.
[3] T. C. Lim, Remarks on some fixed point theorems, Proceedings of the American Mathematical Society 60 (1976) 179–182.
[4] W. A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Analysis 68 (2008) 3689–3696.
[5] S. Dhompongsa, B. Panyanak, On △-convergence theorems in CAT(0) spaces, Computers and Mathematics with Applications 56 (2008) 2572–2579.
[6] M. Bridson, A. Haefliger, Metric spaces of non-positive curvature, Springer-Verlag, Berlin, Heidelberg, 1999.
[7] K. Goebel, S. Reich, Uniform convexity, hyperbolic geometry and non-expansive mappings, Marcel Dekker, Inc., New York, 1984.
[8] S. Reich, I. Shafrir, Nonexpansive iterations in hyperbolic space, Nonlinear Analysis 15(6) (1990) 537–558.
[9] F. Bruhat, J. Tits, Groupes reductifs sur un corps local.I. Donnees radicielles valuees, Inst. Hautes Etudes Sci. Publ. Math., 1972.
[10] S. Dhompongsa, W. A. Kirk, B. Panyanak, Fixed points of uniformly lipschitzian mappings, Nonlinear Analysis 65 (2006) 762–772.
[11] S. Dhompongsa, W. A. Kirk, B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, Journal of Nonlinear and Convex Analysis 8 (2007) 35–45.
[12] H. F. Senter, W. G. Dotson, Approximating fixed points of nonexpansive mappings, Proceedings of the American Mathematical Society 44(2) (1974) 375–380.
[13] V. Berinde, Approximating fixed points of enriched nonexpansive mappings by Krasnoselskii iteration in Hilbert spaces, Carpathian Journal of Mathematics 35(3) (2019) 293–304.
[14] A. Marchiş, Common fixed point theorems for enriched Jungck contractions in Banach spaces, Journal of Fixed Point Theory Applications (2021) 23:76.
[15] J. B. Baillon, R. E. Bruck, S. Reich, On the asymptotic behaviour of nonexpansive mappings and semigroups in Banach spaces, Houston Journal of Mathematics 4(1) (1978) 1–9.