Filomat 37:12 (2023), 3681–3713
https://doi.org/10.2298/FIL2312681C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Stepanov multi-dimensional almost periodic functions and applications

Alan Chávez[a], Kamal Khalil[b], Marko Kostić[c], Manuel Pinto[d]

[a] *Departamento de Matemáticas, Facultad de Ciencias Físicas Y Matemáticas, Universidad Nacional de Trujillo, Trujillo, Perú*
[b] *LMAH, University of Le Havre Normandie, FR-CNRS-3335, ISCN, Le Havre 76600, France*
[c] *Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia*
[d] *Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Santiago de Chile, Chile*

Abstract. In this paper, we analyze several various classes of Stepanov multi-dimensional almost periodic functions in Lebesgue spaces with variable exponents. The introduced classes seem to be new and not considered elsewhere even in the case that the exponent $p(\cdot)$ has a constant value. We provide certain applications to the abstract (degenerate) Volterra integro-differential equations in Banach spaces.

1. Introduction and preliminaries

The class of almost periodic functions was introduced by the Danish mathematician H. Bohr around 1924-1926 and later reconsidered by many other authors. Suppose that I is either $\mathbb{R}$ or $[0,\infty)$ as well as that $f : I \rightarrow X$ is a given continuous function, where X is a complex Banach space equipped with the norm $\|\cdot\|$. Given $\varepsilon > 0$, we say that a positive real number $\tau > 0$ is a ε-period for $f(\cdot)$ if and only if $\|f(t+\tau) - f(t)\| \leq \varepsilon$, $t \in I$. The set consisting of all ε-periods for $f(\cdot)$ is denoted by $\vartheta(f,\varepsilon)$. The function $f(\cdot)$ is said to be almost periodic if and only if for each $\varepsilon > 0$ the set $\vartheta(f,\varepsilon)$ is relatively dense in $[0,\infty)$, which means that there exists $l > 0$ such that any subinterval of $[0,\infty)$ of length l intersects $\vartheta(f,\varepsilon)$.

The most important generalizations of the concept almost periodicity are those of Stepanov, Weyl and Besicovitch; in this paper, we consider Stepanov generalizations of almost periodic functions. Let $1 \leq p < \infty$ and let $f \in L^p_{loc}(I : X)$. Let us recall that the function $f(\cdot)$ is called Stepanov p-bounded (Stepanov bounded, if $p = 1$) if and only if

$$\|f\|_{S^p} := \sup_{t\in I}\left(\int_t^{t+1} \|f(s)\|^p\, ds\right)^{1/p} < \infty.$$

A function $f \in L^p_S(I : X)$ is said to be Stepanov p-almost periodic if and only if its Bochner transform $\hat{f} : I \rightarrow L^p([0,1] : X)$, defined by $\hat{f}(t)(s) := f(t+s)$, $t \in I$, $s \in [0,1]$, is almost periodic. If $f(\cdot)$ is an almost

2020 *Mathematics Subject Classification*. Primary 42A75; Secondary 43A60, 47D99
Keywords. Stepanov multi-dimensional almost periodic type functions, Lebesgue spaces with variable exponents, abstract Volterra integro-differential equations.
Received: 06 June 2022; Accepted: 09 August 2022
Communicated by Dragan S. Djordjević
Alan Chávez is partially supported by 038-2021-FONDECYT-Perú. Marko Kostić is partially supported by grant 451-03-68/2020/14/200156 of Ministry of Science and Technological Development, Republic of Serbia. Manuel Pinto is partially supported by Fondecyt 1170466.
Email addresses: ajchavez@unitru.edu.pe (Alan Chávez), kamal.khalil.00@gmail.com (Kamal Khalil), marco.s@verat.net (Marko Kostić), pintoj.uchile@gmail.com (Manuel Pinto)

periodic function, then $f(\cdot)$ is also Stepanov p-almost periodic for $1 \leq p < \infty$; the converse statement is false, however. For further information about almost periodic functions, various generalizations and applications, we refer the reader to the research monographs [4], [5], [12], [13], [16], [17], [23] and [28].

The notion of almost periodicity can be straightforwardly extended to the functions defined on $\mathbb{R}^n$. In our recent research study [3], we have investigated various notions of almost periodicity for continuous functions $F : I \times X \rightarrow Y$, where Y is a complex Banach space equipped with the norm $\| \cdot \|_Y$ and $\emptyset \neq I \subseteq \mathbb{R}^n$ (the notion of almost periodicity on (semi-)topological groups was analyzed by numerous authors; in [3], $\emptyset \neq I \subseteq \mathbb{R}^n$ generally does not satisfy the semigroup property $I + I \subseteq I$ or contain the zero vector).

In the existing literature concerning generalized almost periodic functions, we have not found any relevant reference which concerns multi-dimensional Stepanov almost periodic functions or (equi-)Weyl multi-dimensional almost periodic functions defined on some proper subsets of $\mathbb{R}^n$, even in the case that the exponent $p(\mathbf{u})$ has a constant value. With the exception of a recent paper [26] by T. Spindeler, in which the author has analyzed the Stepanov and equi-Weyl almost periodic functions in locally compact Abelian groups, the almost nothing has been said before about the Stepanov almost periodic functions defined on $\mathbb{R}^n$, where $n \geq 2$. Concerning the equi-Weyl almost periodic functions on $\mathbb{R}^n$ and general locally compact Abelian groups, we may refer the reader to the recent paper [14] by D. Lenz, T. Spindeler and N. Strungaru; the analysis of (equi-)Weyl multi-dimensional almost periodic functions defined on some proper subsets of $\mathbb{R}^n$ which does not satisfy the semigroup property or contain the zero vector has recently been conducted by V. E. Fedorov and M. Kostić. For the basic source of information about Besicovitch almost periodic functions on $\mathbb{R}^n$ and general topological groups, the reader may consult the important research monograph [25] by A. A. Pankov; the boundedness and almost periodicity in time for certain classes of evolution variational inequalities, positive boundary value problems for symmetric hyperbolic systems and nonlinear Schrödinger equations have been investigated in the third and fourth chapter of [25], while spatially Besicovitch almost periodic solutions for certain classes of nonlinear second-order elliptic equations, first-order hyperbolic systems, single higher-order hyperbolic equations and nonlinear Schrödinger equations have been investigated in the fifth chapter of this monograph. The main purpose of this paper is to continue our recent research study [3] by investigating various classes of Stepanov multi-dimensional almost periodic type functions $F : \Lambda \times X \rightarrow Y$, where Y is a complex Banach space and $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$. In [22], we have recently analyzed the Stepanov multi-dimensional almost automorphic functions in Lebesgue spaces with variable exponents (see also the recent research articles [20]-[21] by M. Kostić and W.-S. Du).

The organization and main ideas of this paper can be briefly described as follows. In Subsection 1.1, we collect the basic definitions and results from the theory of Lebesgue spaces with variable exponents; Subsection 1.2 provides a brief description of the necessary definitions and results about multi-dimensional almost periodic functions. Let Ω be a fixed compact subset of $\mathbb{R}^n$ with positive Lebesgue measure, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfy $\Lambda + \Omega \subseteq \Lambda$ and let $p : \Omega \rightarrow [1, \infty]$ belong to the space $\mathcal{P}(\Omega)$, introduced in Subsection 1.1. We assume henceforth that $(X, \| \cdot \|)$, $(Y, \| \cdot \|_Y)$ and $(Z, \| \cdot \|_Z)$ are complex Banach spaces, R is a non-empty collection of sequences in $\mathbb{R}^n$ and R_X is a non-empty collection of sequences in $\mathbb{R}^n \times X$. By $\mathcal{B}$ we denote a certain collection of non-empty subsets of X.

At the beginning of Section 2, we introduce the notions of multi-dimensional Bochner transform $\hat{F}_\Omega : \Lambda \times X \rightarrow Y^\Omega$. After that, in Subsection 2.1, we analyze the notions of Stepanov $(\Omega, p(\mathbf{u}))$-boundedness, Stepanov distance $D^{p(\cdot)}_{S_\Omega}(F, G)$ and Stepanov norm $\|F\|_{S^{p(\mathbf{u})}_\Omega}$ for functions $F : \Lambda \times X \rightarrow Y$ and $G : \Lambda \rightarrow Y$. We open Subsection 2.2 by introducing the notion of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodicity and the notion of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodicity (see Definition 2.4 and Definition 2.5, respectively). Our first structural result concerning the introduced notion is Proposition 2.6, in which we analyze the Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodicity for a given tuple $(F_1, \cdots, F_k)(\cdot; \cdot)$ of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic functions. After that, in Definition 2.7, we introduce the notions of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodicity and Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniform recurrence in a Bohr like manner. Remark 2.8 clarifies some sufficient conditions under which the multi-dimensional Bochner transform is continuous. It is well known that, for every almost periodic function $F : \mathbb{R} \rightarrow \mathbb{R}$ which can be analytically extended to the strip around the real axis, its composition with the signum function is always Stepanov p-almost periodic for any finite number $p \geq 1$; in Example 2.9, we transfer and extend this statement to multi-dimensional almost periodic functions (see also Proposition 2.10-Proposition 2.12 and Theorem 2.13).

Our first essential contributions are Theorem 2.14 and Theorem 2.15, in which we prove the uniqueness theorem for Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic functions and an extension type theorem for Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic functions. In Remark 2.16, we recondider the obtained results for convex polyhedrals in $\mathbb{R}^n$. The main aim of Proposition 2.17 is to reconsider the problematic analyzed in Proposition 2.6 for Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic functions.

The poinwise products of Stepanov multi-dimensional almost periodic functions with Stepanov multi-dimensional scalar-valued almost periodic functions are investigated in Propositon 2.18 and Proposition 2.19. Some other results concerning Stepanov multi-almost periodic type functions are given in Theorem 2.21, Proposition 2.22, Proposition 2.23 and Proposition 2.24.

Asymptotically Stepanov multi-dimensional almost periodic type functions are investigated in Section 3, composition theorems for Stepanov multi-dimensional almost periodic type functions in Lebesgue spaces with variable exponents are investigated in Section 4 and the invariance of Stepanov multi-dimensional almost periodicity under the actions of convolution products are investigated in Section 5. The final section of the paper is reserved for giving some applications of our abstract theoretical results to the abstract Volterra integro-differential equations in Banach spaces. Albeit we work with Lebesgue spaces with variable exponents, it is worthwhile to mention again that the introduced classes of Stepanov multi-dimensional almost periodic functions seem to be not analyzed elsewhere even in the case that the exponent $p(\cdot)$ has a constant value.

We use the standard notation throughout the paper. By $L(X, Y)$ we denote the Banach algebra of all bounded linear operators from X into Y with $L(X, X)$ being denoted $L(X)$. Assuming the function $F : \Lambda \to X$ is given, where $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, we define the function $\check{F} : -\Lambda \to X$ by $\check{F}(\mathbf{t}) := F(-\mathbf{t})$, $\mathbf{t} \in -\Lambda$. The Euler Gamma function is denoted by $\Gamma(\cdot)$. We also set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $\zeta > 0$. The convolution operator $*$ is defined by $f * g(t) := \int_0^t f(t-s)g(s)\, ds$. The Weyl-Liouville fractional derivative $D_{t,+}^{\gamma} u(t)$ of order $\gamma \in (0, 1)$ is defined for those continuous functions $u : \mathbb{R} \to X$ such that $t \mapsto \int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)\, ds$, $t \in \mathbb{R}$ is a well-defined continuously differentiable mapping, by

$$D_{t,+}^{\gamma} u(t) := \frac{d}{dt} \int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)\, ds, \quad t \in \mathbb{R}.$$

If $X,\ Y \neq \emptyset$ and $n \in \mathbb{N}$, then we set $Y^X := \{f \mid f : X \to Y\}$, $\mathbb{N}_n := \{1, \cdots, n\}$ and $\Delta_n := \{(t, t, \cdots, t) \in \mathbb{R}^n : t \in \mathbb{R}\}$.

The symbol $C(I : X)$, where $I = \mathbb{R}$ or $I = [0, \infty)$, stands for the space of all X-valued continuous functions on the interval I. By $C_b(I : X)$ (respectively, $BUC(I : X)$) we denote the subspaces of $C(I : X)$ consisting of all bounded (respectively, all bounded uniformly continuous functions). Both $C_b(I : X)$ and $BUC(I : X)$ are Banach spaces with the sup-norm. This also holds for the space $C_0(I : X)$ consisting of all continuous functions $f : I \to X$ such that $\lim_{|t| \to +\infty} f(t) = 0$.

1.1. *Lebesgue spaces with variable exponents* $L^{p(x)}$

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be a nonempty Lebesgue measurable subset and let $M(\Omega : X)$ denote the collection of all measurable functions $f : \Omega \to X$; $M(\Omega) := M(\Omega : \mathbb{R})$. Furthermore, $\mathcal{P}(\Omega)$ denotes the collection of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$. For any $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, we define

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, & t \geq 0,\ 1 \leq p(x) < \infty, \\ 0, & 0 \leq t \leq 1,\ p(x) = \infty, \\ \infty, & t > 1,\ p(x) = \infty \end{cases}$$

and

$$\rho(f) := \int_\Omega \varphi_{p(x)}(\|f(x)\|)\, dx.$$

We define the Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent as follows,

$$L^{p(x)}(\Omega : X) := \Big\{ f \in M(\Omega : X) : \lim_{\lambda \to 0+} \rho(\lambda f) = 0 \Big\}.$$

Equivalently

$$L^{p(x)}(\Omega : X) = \Big\{ f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty \Big\};$$

see, e.g., [9, p. 73]. For every $u \in L^{p(x)}(\Omega : X)$, we introduce the Luxemburg norm of $u(\cdot)$ by

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega:X)} := \inf\Big\{ \lambda > 0 : \rho(u/\lambda) \leq 1 \Big\}.$$

Equipped with the above norm, the space $L^{p(x)}(\Omega : X)$ becomes a Banach space (see e.g. [9, Theorem 3.2.7] for the scalar-valued case), coinciding with the usual Lebesgue space $L^p(\Omega : X)$ in the case that $p(x) = p \geq 1$ is a constant function. Further on, for any $p \in \mathcal{P}(\Omega)$, we define

$$p^- := \text{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \text{esssup}_{x\in\Omega} p(x).$$

Set

$$C_+(\Omega) := \Big\{ p \in \mathcal{P}(\Omega) : 1 < p^- \leq p(x) \leq p^+ < \infty \text{ for a.e. } x \in \Omega \Big\}$$

and

$$D_+(\Omega) := \Big\{ p \in \mathcal{P}(\Omega) : 1 \leq p^- \leq p(x) \leq p^+ < \infty \text{ for a.e. } x \in \Omega \Big\}.$$

For $p \in D_+([0, 1])$, the space $L^{p(x)}(\Omega : X)$ behaves nicely, with almost all fundamental properties of the Lesbesgue space with constant exponent $L^p(\Omega : X)$ being retained; in this case, we know that

$$L^{p(x)}(\Omega : X) = \Big\{ f \in M(\Omega : X) ; \text{ for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty \Big\}.$$

Set

$$E^{p(x)}(\Omega : X) := \Big\{ f \in L^{p(x)}(\Omega : X) ; \text{ for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty \Big\};$$

$E^{p(x)}(\Omega) \equiv E^{p(x)}(\Omega : \mathbb{C})$.

We will use the following lemma (cf. [9] for the scalar-valued case):

Lemma 1.1. *(i) (The Hölder inequality) Let $p,\ q,\ r \in \mathcal{P}(\Omega)$ such that*

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad x \in \Omega.$$

Then, for every $u \in L^{p(x)}(\Omega : X)$ and $v \in L^{r(x)}(\Omega)$, we have $uv \in L^{q(x)}(\Omega : X)$ and

$$\|uv\|_{q(x)} \leq 2\|u\|_{p(x)}\|v\|_{r(x)}.$$

(ii) Let Ω be of a finite Lebesgue's measure and let $p,\ q \in \mathcal{P}(\Omega)$ such $q \leq p$ a.e. on Ω. Then $L^{p(x)}(\Omega : X)$ is continuously embedded in $L^{q(x)}(\Omega : X)$, and the constant of embedding is less than or equal to $2(1 + m(\Omega))$.

(iii) Let $f \in L^{p(x)}(\Omega : X)$, $g \in M(\Omega : X)$ and $0 \leq \|g\| \leq \|f\|$ a.e. on Ω. Then $g \in L^{p(x)}(\Omega : X)$ and $\|g\|_{p(x)} \leq \|f\|_{p(x)}$.

(iv) (The dominated convergence theorem) Let $p \in \mathcal{P}(\Omega)$, and let $f_k,\ f \in M(\Omega : X)$ for all $k \in \mathbb{N}$. If $\lim_{k\to\infty} f_k(x) = f(x)$ for a.e. $x \in \Omega$ and there exists a real-valued function $g \in E^{p(x)}(\Omega)$ such that $\|f_k(x)\| \leq g(x)$ for a.e. $x \in \Omega$, then $\lim_{k\to\infty} \|f_k - f\|_{L^{p(x)}(\Omega:X)} = 0$.

We will use the following simple lemma, whose proof can be omitted:

Lemma 1.2. *Suppose that $f \in L^{p(x)}(\Omega : X)$ and $A \in L(X, Y)$. Then $Af \in L^{p(x)}(\Omega : Y)$ and $\|Af\|_{L^{p(x)}(\Omega:Y)} \leq \|A\| \cdot \|f\|_{L^{p(x)}(\Omega:X)}$.*

For further information concerning the Lebesgue spaces with variable exponents $L^{p(x)}$, we refer the reader to [9], [10] and [24].

1.2. $(\mathrm{R}_X, \mathcal{B})$*-Multi-almost periodic type functions and Bohr* $\mathcal{B}$*-almost periodic type functions*

Throughout this subsection, we assume that $n \in \mathbb{N}$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $\mathcal{B}$ is a non-empty collection of subsets of X, R is a non-empty collection of sequences in $\mathbb{R}^n$ and R_X is a non-empty collection of sequences in $\mathbb{R}^n \times X$; usually, $\mathcal{B}$ denotes the collection of all bounded subsets of X or all compact subsets of X. Henceforth we will always assume that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.

In this subsection, we recall the basic facts about $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic type functions and Bohr $\mathcal{B}$-almost periodic type functions; see [2] for more details.

Definition 1.3. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function, and the following condition holds:*

$$\text{If } \mathbf{t} \in I,\ \mathbf{b} \in \mathrm{R} \text{ and } l \in \mathbb{N}, \text{ then we have } \mathbf{t} + \mathbf{b}(l) \in I. \tag{1}$$

Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdot \cdot \cdot, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : I \times X \to Y$ such that

$$\lim_{l \to +\infty} F\Big(\mathbf{t} + (b_{k_l}^1, \cdot \cdot \cdot, b_{k_l}^n); x\Big) = F^*(\mathbf{t}; x)$$

uniformly for all $x \in B$ and $\mathbf{t} \in I$. By $AP_{(\mathrm{R},\mathcal{B})}(I \times X : Y)$ we denote the space consisting of all $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions.

The notion introduced in Definition 1.3 is a special case of the notion introduced in the following definition:

Definition 1.4. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function, and the following condition holds:*

$$\text{If } \mathbf{t} \in I,\ (\mathbf{b}; \mathbf{x}) \in \mathrm{R}_X \text{ and } l \in \mathbb{N}, \text{ then we have } \mathbf{t} + \mathbf{b}(l) \in I. \tag{2}$$

Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n); x_k)) \in \mathrm{R}_X$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \cdot \cdot \cdot, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : I \times X \to Y$ such that

$$\lim_{l \to +\infty} F\Big(\mathbf{t} + (b_{k_l}^1, \cdot \cdot \cdot, b_{k_l}^n); x + x_{k_l}\Big) = F^*(\mathbf{t}; x)$$

uniformly for all $x \in B$ and $\mathbf{t} \in I$. By $AP_{(\mathrm{R}_X,\mathcal{B})}(I \times X : Y)$ we denote the space consisting of all $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic functions.

The domain I from the above two definitions is rather general. For example, if $n = 1$, $I = [0, \infty)$, $X = \{0\}$, $\mathcal{B} = \{X\}$ and R is the collection of all sequences in $[0, \infty)$, then the notion of $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity is equivalent with the usual notion of asymptotical almost periodicity.

The following notion is introduced in a Bohr like manner:

Definition 1.5. *Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I' \subseteq I$. Then we say that:*

(i) $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I')$-almost periodic (Bohr $\mathcal{B}$-almost periodic, if $I = I'$) if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that

$$\Big\|F(\mathbf{t} + \tau; x) - F(\mathbf{t}; x)\Big\|_Y \leq \epsilon, \quad \mathbf{t} \in I,\ x \in B.$$

(ii) $F(\cdot;\cdot)$ is $(\mathcal{B}, I')$-uniformly recurrent ($\mathcal{B}$-uniformly recurrent, if $I = I'$) if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_n) in I' such that $\lim_{n \to +\infty} |\tau_n| = +\infty$ and

$$\lim_{n \to +\infty} \sup_{\mathbf{t} \in I; x \in B} \Big\|F(\mathbf{t} + \tau_n; x) - F(\mathbf{t}; x)\Big\|_Y = 0.$$

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Bohr I'-almost periodic (I'-uniformly recurrent) [Bohr almost periodic (uniformly recurrent), if $I = I'$].

It is clear that, if $F(\cdot;\cdot)$ is $\mathcal{B}$-uniformly recurrent and $x \in X$, then we have the following supremum formula

$$\sup_{\mathbf{t}\in I}\|F(\mathbf{t};x)\|_Y = \sup_{\mathbf{t}\in I,|t|\geq a}\|F(\mathbf{t};x)\|_Y,$$

which in particular shows that for each $x \in X$ the function $F(\cdot;x)$ is identically equal to zero provided that the function $F(\cdot;\cdot)$ is $\mathcal{B}$-uniformly recurrent and $\lim_{\mathbf{t}\in I,|\mathbf{t}|\to+\infty} F(\mathbf{t};x) = 0$. The statement of [4, Theorem 7, p. 3] can be reformulated in this framework, as well.

We need the following lemma:

Lemma 1.6. *(i) Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $I + I \subseteq I$, I is closed, $F : I \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic and $\mathcal{B}$ is any family of compact subsets of X. If*

$$(\forall l > 0)\,(\exists \mathbf{t_0} \in I)\,(\exists k > 0)(\forall \mathbf{t} \in I)(\exists \mathbf{t}'_0 \in I)$$
$$(\forall \mathbf{t}''_0 \in B(\mathbf{t}'_0, l) \cap I)\,\mathbf{t} - \mathbf{t}''_0 \in B(\mathbf{t}_0, kl) \cap I,$$

then for each $B \in \mathcal{B}$ we have that the set $\{F(\mathbf{t};x) : \mathbf{t} \in I,\ x \in B\}$ is relatively compact in Y; in particular, $\sup_{\mathbf{t}\in I;x\in B} \|F(\mathbf{t};x)\|_Y < \infty$.

(ii) Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $I + I \subseteq I$, I is closed and $F : I \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic, where $\mathcal{B}$ is a family consisting of some compact subsets of X. If the following condition holds

$$(\exists \mathbf{t_0} \in I)\,(\forall \epsilon > 0)(\forall l > 0)\,(\exists l' > 0)\,(\forall \mathbf{t}',\ \mathbf{t}'' \in I)$$
$$B(\mathbf{t_0}, l) \cap I \subseteq B(\mathbf{t_0} - \mathbf{t}', l') \cap B(\mathbf{t_0} - \mathbf{t}'', l'),$$

then for each $B \in \mathcal{B}$ the function $F(\cdot;\cdot)$ is uniformly continuous on $I \times B$.

Lemma 1.7. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is continuous, $\mathcal{B}$ is any family of compact subsets of X and R is the collection of all sequences in $\mathbb{R}^n$. Then $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic if and only if $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic.*

We will also use the following lemmas:

Lemma 1.8. *Suppose that $h \in L^1(\mathbb{R}^n)$, the function $F(\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic and for each bounded subset D of X there exists a constant $c_D > 0$ such that $\|F(\mathbf{t};x)\|_Y \leq c_D$ for all $\mathbf{t} \in \mathbb{R}^n$, $x \in D$. Suppose, further, that for each sequence $((\mathbf{b};\mathbf{x})_k = ((b^1_k, b^2_k, \cdots, b^n_k); x_k)_k) \in \mathrm{R}_X$ and for each set $B \in \mathcal{B}$ we have that $B + \{x_k : k \in \mathbb{N}\}$ is a bounded set in X. Then the function*

$$(h * F)(\mathbf{t};x) := \int_{\mathbb{R}^n} h(\sigma)F(\mathbf{t} - \sigma;x)\,d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X$$

is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic and satisfies that for each bounded subset D of X there exists a constant $c'_D > 0$ such that $\|(h * F)(\mathbf{t};x)\|_Y \leq c'_D$ for all $\mathbf{t} \in \mathbb{R}^n$, $x \in D$.

Lemma 1.9. *Suppose that $I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$, the set I' is unbounded, $F : I \to Y$ is a uniformly continuous, Bohr I'-almost periodic function, resp. a uniformly continuous, I'-uniformly recurrent function, $S \subseteq \mathbb{R}^n$ is bounded and, for every $\mathbf{t}' \in \mathbb{R}^n$, there exists a finite real number $M > 0$ such that $\mathbf{t}' + I'_M \subseteq I$. Define $\Omega_S := [(I' \cup (-I')) + (I' \cup (-I'))] \cup S$. Then there exists a uniformly continuous, Bohr Ω_S-almost periodic, resp. a uniformly continuous, Ω-uniformly recurrent, function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$; furthermore, in almost periodic case, the uniqueness of such a function $\tilde{F}(\cdot)$ holds provided that $\mathbb{R}^n \setminus \Omega_S$ is a bounded set.*

Lemma 1.10. *Suppose that $I \subseteq \mathbb{R}^n$, $I + I \subseteq I$, $\mathbb{R}^n \setminus [(I \cup (-I)) + (I \cup (-I))]$ is a bounded set, and the following condition holds:*

(AP-E) For every $\mathbf{t}' \in \mathbb{R}^n$, there exists a finite real number $M > 0$ such that $\mathbf{t}' + I'_M \subseteq I$.

If $F : \mathbb{R}^n \to Y$ and $G : \mathbb{R}^n \to Y$ are two Bohr almost periodic functions and $F(\mathbf{t}) = G(\mathbf{t})$ for all $\mathbf{t} \in I$, then $F(\mathbf{t}) = G(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$.

The following definitions from [2] will be important in our further work:

Definition 1.11. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$ and $I + I \subseteq I$. Then we say that I is admissible with respect to the almost periodic extensions if and only if for any complex Banach space Y and for any uniformly continuous, Bohr almost periodic function $F : I \to Y$ there exists a unique Bohr almost periodic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$.*

Definition 1.12. *Suppose that $\mathbb{D} \subseteq \Omega \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded. By $C_{0,\mathbb{D},\mathcal{B}}(\Omega \times X : Y)$ we denote the vector space consisting of all continuous functions $Q : \Omega \times X \to Y$ such that, for every $B \in \mathcal{B}$, we have $\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\to+\infty} Q(\mathbf{t}; x) = 0$, uniformly for $x \in B$.*

Definition 1.13. *Suppose that the set $\mathbb{D} \subseteq \mathbb{R}^n$ is unbounded, and $F : I \times X \to Y$ is a continuous function. Then we say that $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. (strongly) $\mathbb{D}$-asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic, if and only if there exist an $(\mathrm{R}, \mathcal{B})$-multi-almost periodic function ($G : \mathbb{R}^n \times X \to Y$) $G : I \times X \to Y$, resp. an $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic function ($G : \mathbb{R}^n \times X \to Y$) $G : I \times X \to Y$, and a function $Q \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ such that $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in X$.*

Let $I = \mathbb{R}^n$. Then it is said that $F(\cdot;\cdot)$ is (strongly) asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. (strongly) asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic, if and only if $F(\cdot;\cdot)$ is (strongly) $\mathbb{R}^n$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. (strongly) $\mathbb{R}^n$-asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic.

We similarly introduce the notions of ((strong) $\mathbb{D}$-)asymptotical Bohr $\mathcal{B}$-almost periodicity and ((strong) $\mathbb{D}$-)asymptotical uniform recurrence.

2. Stepanov multi-dimensional almost periodic functions in Lebesgue spaces with variable exponents

This section investigates the generalized $(\mathrm{R}, \mathcal{B})$-multi-almost periodic type functions in Lebesgue spaces with variable exponents. In our analysis of Stepanov $p(\mathbf{u})$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions, we assume that Ω is a fixed compact subset of $\mathbb{R}^n$ with positive Lebesgue measure and $p \in \mathcal{P}(\Omega)$. Further on, Λ denotes a general non-empty subset of $\mathbb{R}^n$ satisfying $\Lambda + \Omega \subseteq \Lambda$ (in [2] and the previous subsection, this region has been denoted by I).

We introduce the multi-dimensional Bochner transform $\hat{F}_\Omega : \Lambda \times X \to Y^\Omega$ by

$$\Big[\hat{F}_\Omega(\mathbf{t}; x)\Big](u) := F(\mathbf{t} + \mathbf{u}; x), \quad \mathbf{t} \in \Lambda,\ \mathbf{u} \in \Omega,\ x \in X.$$

2.1. *Stepanov $(\Omega, p(\mathbf{u}))$-boundedness, Stepanov distance $D_{S_\Omega}^{p(\cdot)}(F, G)$ and Stepanov norm $\|F\|_{S_\Omega^{p(\mathbf{u})}}$*

The notion of Stepanov $(\Omega, p(\mathbf{u}))$-boundedness on $\mathcal{B}$ is introduced as follows:

Definition 2.1. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$ and $F : \Lambda \times X \to Y$ satisfies that for each $\mathbf{t} \in \Lambda$ and $x \in X$, the function $F(\mathbf{t} + \mathbf{u}; x)$ belongs to the space $L^{p(\mathbf{u})}(\Omega : Y)$. Then we say that $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded on $\mathcal{B}$ if and only if for each $B \in \mathcal{B}$ we have*

$$\sup_{\mathbf{t}\in\Lambda; x\in B} \Big\| \Big[\hat{F}_\Omega(\mathbf{t}; x)\Big](u) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = \sup_{\mathbf{t}\in\Lambda; x\in B} \Big\| F(\mathbf{t} + \mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} < \infty.$$

Denote by $L_{S,\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$ the set consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-bounded functions on $\mathcal{B}$.

If $n = 1$, $X = \{0\}$, $\Omega = [0,1]$ and $\Lambda = [0,\infty)$ or $\Lambda = \mathbb{R}$, then the notion introduced above reduces to the notion introduced recently in [6, Definition 4.1]. If $X = \{0\}$, then we abbreviate $L^{\Omega,p(\mathbf{u})}_{S,\mathcal{B}}(\Lambda \times X : Y)$ to $L^{\Omega,p(\mathbf{u})}_{S}(\Lambda : Y)$; in this case, we say that the function $F(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded and define $\|F\|_{S^{\Omega,p(\mathbf{u})}} := \sup_{\mathbf{t}\in\Lambda} \|F(\mathbf{t}+\mathbf{u})\|_{L^{p(\mathbf{u})}(\Omega:Y)}$.

Remark 2.2. *(i) The condition $\Lambda + \Omega \subseteq \Lambda$ used henceforth is clearly equivalent with the condition $\Lambda + \Omega = \Lambda$ if $0 \in \Omega$.*

(ii) Suppose that Ω_1 is also a compact subset of $\mathbb{R}^n$ with positive Lebesgue measure, $\Lambda + \Lambda \subseteq \Lambda$, $\Lambda + \Omega_1 \subseteq \Lambda$ and $1 \leq p < \infty$. It is clear that the existence of a finite subset $\{\mathbf{t}_1, \cdots, \mathbf{t}_k\}$ of Λ such that $\Omega \subseteq \bigcup_{i=1}^{k}(\mathbf{t}_i + \Omega_1)$ implies that for each $\mathbf{t} \in \Lambda$ we have $\mathbf{t} + \Omega \subseteq \bigcup_{i=1}^{k}(\mathbf{t} + \mathbf{t}_i + \Omega_1)$, so that the Stepanov $(\Omega_1, p(\mathbf{u}))$-boundedness on $\mathcal{B}$ implies the Stepanov $(\Omega, p(\mathbf{u}))$-boundedness on $\mathcal{B}$, for any function $F : \Lambda \times X \to Y$.

(iii) Let $1 \leq p < \infty$. In the one-dimensional case, the usual Stepanov p-boundedness of function $F : \Lambda \to Y$, where $\Lambda = [0,\infty)$ or $\Lambda = \mathbb{R}$, is equivalent with the Stepanov (Ω, p)-boundedness of function $F(\cdot)$, where $\Omega = [a, b]$ is any non-trivial segment in Λ.

In general case, it is very simple to show that:

1. $\alpha F + \beta G \in L^{\Omega,p(\mathbf{u})}_{S,\mathcal{B}}(\Lambda \times X : Y)$, provided $\alpha,\ \beta \in \mathbb{C}$ and $F,\ G \in L^{\Omega,p(\mathbf{u})}_{S,\mathcal{B}}(\Lambda \times X : Y)$.
2. Suppose that $\tau + \Lambda \subseteq \Lambda$, $x_0 \in X$ and for each $B \in \mathcal{B}$ there exists $B' \in \mathcal{B}$ such that $x_0 + B \subseteq B'$. Then we have $F(\cdot + \tau; \cdot + x_0) \in L^{\Omega,p(\mathbf{u})}_{S,\mathcal{B}}(\Lambda \times X : Y)$, provided that $F(\cdot;\cdot) \in L^{\Omega,p(\mathbf{u})}_{S,\mathcal{B}}(\Lambda \times X : Y)$.
3. If $1 \leq p_1(\mathbf{u}) \leq p(\mathbf{u})$ for a.e. $\mathbf{u} \in \Omega$ and $f \in L^{\Omega,p(\mathbf{u})}_{S,\mathcal{B}}(\Lambda \times X : Y)$, then we have $f \in L^{\Omega,p_1(\mathbf{u})}_{S,\mathcal{B}}(\Lambda \times X : Y)$.
4. $(L^{\Omega,p(\mathbf{u})}_{S}(\Lambda : Y), \|\cdot\|_{S^{\Omega,p(\mathbf{u})}})$ is a complex Banach space.

The translation invariance stated in the point [2.] does not generally hold if we follow the approach proposed by T. Diagana and M. Zitane in [8], as already mentioned in our previous investigations.

Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfy $\Lambda + \Omega \subseteq \Lambda$. Suppose first that $p(\mathbf{u}) \equiv p \in [1,\infty)$ and $F : \Lambda \to Y$ and $G : \Lambda \to Y$ are two functions for which $\|F(\mathbf{t}+\mathbf{u}) - G(\mathbf{t}+\mathbf{u})\|_Y \in L^p(\Omega)$ for all $\mathbf{t} \in \Lambda$. We define the Stepanov distance $D^p_{S_\Omega}(F, G)$ of the functions $F(\cdot)$ and $G(\cdot)$ by

$$D^p_{S_\Omega}(F,G) := \sup_{\mathbf{t}\in\Lambda}\Bigg[\Big(\frac{1}{m(\Omega)}\Big)^{1/p}\|F(\mathbf{t}+\mathbf{u}) - G(\mathbf{t}+\mathbf{u})\big\|_{L^p(\Omega:Y)}\Bigg].$$

Suppose now that $p,\ q \in \mathcal{P}(\Omega)$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) = 1$ for a.e. $\mathbf{u} \in \Omega$ and $q(\mathbf{u}) < +\infty$ for a.e. $\mathbf{u} \in \Omega$. In this case (the definition is consistent with the above given provided that $p(\mathbf{u}) \equiv p \in (1,\infty)$) , we define the Stepanov distance $D^{p(\cdot)}_{S_\Omega}(F, G)$ of the functions $F(\cdot)$ and $G(\cdot)$ by

$$D^{p(\cdot)}_{S_\Omega}(F,G) := \sup_{\mathbf{t}\in\Lambda}\Bigg[m(\Omega)^{-1}\|1\|_{L^{q(\mathbf{u})}(\Omega)}\|F(\mathbf{t}+\mathbf{u}) - G(\mathbf{t}+\mathbf{u})\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\Bigg].$$

The use of Hölder inequality (see Lemma 1.1(i)) enables one to see that the following proposition holds true:

Proposition 2.3. *Suppose that $1 \leq p_1(\mathbf{u}) \leq p_2(\mathbf{u})$ for a.e. $\mathbf{u} \in \Omega$, and $\|F(\mathbf{t}+\mathbf{u}) - G(\mathbf{t}+\mathbf{u})\|_Y \in L^{p_2(\mathbf{u})}(\Omega)$ for all $\mathbf{t} \in \Lambda$. Then*

$$D^{p_1(\cdot)}_{S_\Omega}(F,G) \leq 4D^{p_2(\cdot)}_{S_\Omega}(F,G).$$

Proof. It is clear that $\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\|_Y \in L^{p_1(\mathbf{u})}(\Omega)$ for all $\mathbf{t}\in\Lambda$. If $p_1(\mathbf{u})=1$ for a.e. $\mathbf{u}\in\Omega$, then we can apply the Hölder inequality once to conclude that $D^1_{S_\Omega}(F,G)\leq 2D^{p_2(\cdot)}_{S_\Omega}(F,G)$. Otherwise, if $1/p_i(\mathbf{u})+1/q_i(\mathbf{u})=1$ for a.e. $\mathbf{u}\in\Omega$ $(i=1,2)$, then $q_2(\mathbf{u})\leq q_1(\mathbf{u})<+\infty$ for a.e. $\mathbf{u}\in\Omega$. Applying the Hölder inequality twice, we get that for each $\mathbf{t}\in\Lambda$ we have

$$\begin{aligned}
&\|1\|_{L^{q_1(\mathbf{u})}(\Omega)}\big\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\big\|_{L^{p_1(\mathbf{u})}(\Omega:Y)}\\
&\quad\leq 2\|1\|_{L^{q_1(\mathbf{u})}(\Omega)}\|1\|_{L^{(q_1(\mathbf{u})-q_2(\mathbf{u}))^{-1}}(\Omega)}\big\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\big\|_{L^{p_2(\mathbf{u})}(\Omega:Y)}\\
&\quad\leq 4\|1\|_{L^{q_2(\mathbf{u})}(\Omega)}\big\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\big\|_{L^{p_2(\mathbf{u})}(\Omega:Y)}.
\end{aligned}$$

This simply completes the proof. □

Clearly, if $1\leq p_1(\mathbf{u})\equiv p_1\leq p_2\equiv p_2(\mathbf{u})$ for a.e. $\mathbf{u}\in\Omega$, then we have $D^{p_1}_{S_\Omega}(F,G)\leq D^{p_2}_{S_\Omega}(F,G)$. If $\Omega\equiv[0,l]^n$ for some $l>0$, then we also write $D^p_{S_l}(F,G)\equiv D^p_{S_\Omega}(F,G)$ and $D^{p(\cdot)}_{S_l}(F,G)\equiv D^{p(\cdot)}_{S_\Omega}(F,G)$.

Suppose now that $p(\mathbf{u})\equiv p\in[1,\infty)$ and $l_2>l_1>0$. Since, for every $\mathbf{t}\in\Lambda$, we have

$$\begin{aligned}
&\Big(\frac{1}{m([0,l_1]^n)}\Big)^{1/p}\big\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\big\|_{L^p(l_1\Omega:Y)}\\
&\leq\Big(\frac{m([0,l_2]^n)}{m([0,l_1]^n)}\Big)^{1/p}\Big(\frac{1}{m([0,l_2]^n)}\Big)^{1/p}\big\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\big\|_{L^p(l_2\Omega:Y)},
\end{aligned}$$

it follows that

$$D^p_{S_{l_1}}(F,G)\leq\Big[\frac{l_2}{l_1}\Big]^{n/p}\cdot D^p_{S_{l_2}}(F,G).$$

Suppose now that $l_2=kl_1+\theta l_1$ for some $k\in\mathbb{N}$ and $\theta\in[0,1)$. Since, for every $\mathbf{t}\in\Lambda$, we have

$$\begin{aligned}
&\Big(\frac{1}{m([0,l_2]^n)}\Big)^{1/p}\big\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\big\|_{L^p([0,l_2]^n:Y)}\\
&\leq\Big(\frac{1}{m([0,kl_1]^n)}\Big)^{1/p}\big\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\big\|_{L^p([0,(k+1)l_1]^n:Y)}\\
&\leq\Big(\frac{(k+1)^n m\big([0,l_1]^n\big)}{m\big([0,kl_1]^n\big)}\Big)^{1/p}\sup_{\mathbf{t}\in\Lambda}\Big(\frac{1}{m\big([0,l_1]^n\big)}\Big)^{1/p}\big\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\big\|_{L^p([0,l_1]^n:Y)}\\
&\leq\Big(\frac{k+1}{k}\Big)^{n/p}\cdot D^p_{S_{l_1}}(F,G), \qquad (3)
\end{aligned}$$

it follows that

$$D^p_{S_{l_2}}(F,G)\leq\Big(\frac{k+1}{k}\Big)^{n/p}\cdot D^p_{S_{l_1}}(F,G).$$

Therefore, if $p(\mathbf{t})\equiv p\in[1,\infty)$, the metrics $D^p_{S_{l_1}}(\cdot,\cdot)$ and $D^p_{S_{l_2}}(\cdot,\cdot)$ are topologically equivalent. Furthermore, the use of (3) enables one to see that in case $p(\mathbf{t})\equiv p\in[1,\infty)$ we have that

$$\limsup_{l\to\infty}D^p_{S_l}(F,G)\leq D^p_{S_{l_1}}p(F,G),\quad l_1>0.$$

Performing the limit inferior as $l_1\to\infty$, we get that

$$\limsup_{l\to\infty}D^p_{S_l}(F,G)\leq\liminf_{l\to\infty}D^p_{S_l}(F,G),$$

so that the limit

$$D^p_W(F,G):=\lim_{l\to\infty}D^p_{S_l}(F,G)$$

exists. Therefore, we can define the Weyl distance $D_W^p(F,G)$ of functions $F(\cdot)$ and $G(\cdot)$. This distance will play an important role in [11].

By $S_\Omega^p(\Lambda : Y)$ we denote the vector space of all functions $F : \Lambda \to Y$ for which $\|F(\mathbf{t}+\mathbf{u})\|_Y \in L^p(\Omega)$ for all $\mathbf{t} \in \Lambda$ and the Stepanov norm

$$\|F\|_{S_\Omega^p} := \sup_{\mathbf{t}\in\Lambda}\Biggl[\Bigl(\frac{1}{m(\Omega)}\Bigr)^{1/p}\bigl\|F(\mathbf{t}+\mathbf{u})\bigr\|_{L^p(\Omega:Y)}\Biggr]$$

is finite; if $\Omega \equiv [0,l]^n$, then we also write $S_l^p(\Lambda : Y) \equiv S_\Omega^p(\Lambda : Y)$ and $\|\cdot\|_{S_l^p} \equiv \|\cdot\|_{S_\Omega^p}$. If $p,\ q \in \mathcal{P}(\Omega)$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) = 1$ for a.e. $\mathbf{u} \in \Omega$ and $q(\mathbf{u}) < +\infty$ for a.e. $\mathbf{u} \in \Omega$, then (the definition is consistent with the above given provided that $p(\mathbf{u}) \equiv p \in (1,\infty)$), we define the Stepanov norm $\|F\|_{S_\Omega^{p(\mathbf{u})}}$ by

$$\|F\|_{S_\Omega^{p(\mathbf{u})}} := \sup_{\mathbf{t}\in\Lambda}\Biggl[m(\Omega)^{-1}\|1\|_{L^{q(\mathbf{u})}(\Omega)}\bigl\|F(\mathbf{t}+\mathbf{u})\bigr\|_{L^{p(\mathbf{u})}(\Omega:Y)}\Biggr];$$

again, $S_\Omega^{p(\mathbf{u})}(\Lambda : Y)$ denotes the vector space consisting of all functions $F : \Lambda \to Y$ satisfying that $\|F(\mathbf{t}+\mathbf{u})\|_Y \in L^{p(\mathbf{u})}(\Omega)$ for all $\mathbf{t} \in \Lambda$ and $\|F\|_{S_\Omega^{p(\mathbf{u})}} < \infty$. Since Fatou's lemma holds in our framework (see e.g., [9, p. 75]), using the arguments contained in the proof of [23, Theorem 5.2.1, p. 199] and Lemma 1.1(ii) we may conclude that $S_\Omega^{p(\mathbf{u})}(\Lambda : Y)$ is a Banach space equipped with the norm $\|\cdot\|_{S_\Omega^{p(\mathbf{u})}}$.

2.2. *Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic type functions and Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic type functions*

The notion of a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic function is introduced as follows:

Definition 2.4. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$, (1) holds with the set I replaced by the set Λ therein and the function $\hat{F} : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous. Then we say that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic if and only if the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, i.e., for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ such that*

$$\lim_{l\to+\infty}\Bigl\|F\Bigl(\mathbf{t}+\mathbf{u}+(b_{k_l}^1,\cdots,b_{k_l}^n);x\Bigr)-\Bigl[F^*(\mathbf{t};x)\Bigr](\mathbf{u})\Bigr\|_{L^{p(\mathbf{u})}(\Omega:Y)}=0,$$

uniformly for all $x \in B$ and $\mathbf{t} \in \Lambda$. By $APS_{(\mathrm{R},\mathcal{B})}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$ we denote the collection consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions $F : \Lambda \times X \to Y$. If $X = \{0\}$ and $\mathcal{B} = \{X\}$, then we also say that the function $F(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-R-multi-almost periodic and abbreviate $APS_{(\mathrm{R},\mathcal{B})}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$ to $APS_{\mathrm{R}}^{\Omega,p(\mathbf{u})}(\Lambda : Y)$.

In the following definition, we introduce the notion of a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic function:

Definition 2.5. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Omega \subseteq \Lambda$ and $F : \Lambda \times X \to Y$, (2) holds with the set I replaced by Λ therein and the function $\hat{F} : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous. Then we say that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic if and only if the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic, i.e., for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \cdots, b_k^n); x_k)) \in \mathrm{R}_X$ there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \cdots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^* : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ such that*

$$\lim_{l\to+\infty}\Bigl\|F\Bigl(\mathbf{t}+\mathbf{u}+(b_{k_l}^1,\cdots,b_{k_l}^n);x+x_{k_l}\Bigr)-\Bigl[F^*(\mathbf{t};x)\Bigr](\mathbf{u})\Bigr\|_{L^{p(\mathbf{u})}(\Omega:Y)}=0,$$

uniformly for all $x \in B$ and $\mathbf{t} \in \Lambda$. By $APS_{(\mathrm{R}_X,\mathcal{B})}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$ we denote the space consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic functions.

The following special cases should be emphasized (see also [3]):

L1. $\mathrm{R} = \{b : \mathbb{N} \to \mathbb{R}^n ;$ for all $j \in \mathbb{N}$ we have $b_j \in \{(a, a, a, \cdot \cdot \cdot, a) \in \mathbb{R}^n : a \in \mathbb{R}\}\}$. If $n = 2$ and $\mathcal{B}$ denotes the collection of all bounded subsets of X, then we also say that the function $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bi-almost periodic. The notion of Stepanov $(\Omega, p(\mathbf{u}))$-bi-almost periodicity seems to be new and not considered elsewhere even in the one-dimensional case $\Omega = [0, 1]$ with the constant exponent $p(\mathbf{u}) \equiv p \in [1, \infty)$.

L2. R is a collection of all sequences $b(\cdot)$ in $\mathbb{R}^n$. This is the limit case in our analysis because, in this case, any Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic function, is automatically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_1, \mathcal{B})$-multi-almost periodic, resp. Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{1\mathrm{X}}, \mathcal{B})$-multi-almost periodic, for any other collection R_1 of sequences $b(\cdot)$ in $\mathbb{R}^n$, resp. any other collection $\mathrm{R}_{1\mathrm{X}}$ of sequences in $\mathbb{R}^n \times X$.

Let $k \in \mathbb{N}$ and $F_i : \Lambda \times X \to Y_i$ $(1 \leq i \leq k)$. Then we define the function $(F_1, \cdot \cdot \cdot, F_k) : \Lambda \times X \to Y_1 \times \cdot \cdot \cdot \times Y_k$ by

$$(F_1, \cdot \cdot \cdot, F_k)(\mathbf{t}; x) := (F_1(\mathbf{t}; x), \cdot \cdot \cdot, F_k(\mathbf{t}; x)), \quad \mathbf{t} \in \Lambda,\ x \in X.$$

Almost immediately from definitions, we can clarify the following analogue of [2, Proposition 2.3]:

Proposition 2.6. *(i) Suppose that $k \in \mathbb{N}$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$,* (1) *holds with I replaced by Λ therein, and for any sequence which belongs to* R *we have that any its subsequence also belongs to* R*. If the function $F_i(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic for $1 \leq i \leq k$, then the function $(F_1, \cdot \cdot \cdot, F_k)(\cdot; \cdot)$ is also Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic.*

(ii) Suppose that $k \in \mathbb{N}$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, (1) *holds with I replaced by Λ therein, and for any sequence which belongs to R_{X} we have that any its subsequence also belongs to R_{X}. If the function $F_i(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic for $1 \leq i \leq k$, then the function $(F_1, \cdot \cdot \cdot, F_k)(\cdot; \cdot)$ is also Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic.*

The supremum formula for Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic functions, the conditions under which the range $\{\hat{F}_\Omega(\mathbf{t}; x) : \mathbf{t} \in \Lambda;\ x \in B\}$, for a given set $B \in \mathcal{B}$, is relatively compact in $L^{p(\mathbf{u})}(\Omega : Y)$ and the question when for a given Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic function $F : \Lambda \times X \to Y$ and a function $\phi : Y \to Z$ we have that $\phi \circ F : \Lambda \times X \to Z$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic can be deduced by appealing to [2, Proposition 2.5, Proposition 2.6, Proposition 2.9].

Now we will introduce the following notion in a Bohr like manner:

Definition 2.7. *Suppose that $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous.*

(i) Then we say that $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic, if $\Lambda' = \Lambda$) if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that

$$\left\| F(\mathbf{t} + \tau + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x) \right\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \epsilon, \quad \mathbf{t} \in \Lambda,\ x \in B.$$

By $APS^{\Omega, p(\mathbf{u})}_{\mathcal{B}, \Lambda'}(\Lambda \times X : Y)$ and $APS^{\Omega, p(\mathbf{u})}_{\mathcal{B}}(\Lambda \times X : Y)$ we denote the spaces consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodic functions and Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic functions, respectively.

(ii) Then we say that $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniformly recurrent (Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent, if $\Lambda' = \Lambda$) if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_n) in Λ' such that $\lim_{n \to +\infty} |\tau_n| = +\infty$ and

$$\lim_{n \to +\infty} \sup_{\mathbf{t} \in I; x \in B} \left\| F(\mathbf{t} + \tau_n + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x) \right\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0.$$

By $URS^{\Omega, p(\mathbf{u})}_{\mathcal{B}, \Lambda'}(\Lambda \times X : Y)$ and $URS^{\Omega, p(\mathbf{u})}_{\mathcal{B}}(\Lambda \times X : Y)$ we denote the spaces consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniformly recurrent functions and Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent functions, respectively.

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-uniformly recurrent) [Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-uniformly recurrent), if $\Lambda = \Lambda'$].

Remark 2.8. *(i) Suppose that $p \in D_+(\Omega)$ and there exists a finite constant $L \geq 1$ such that*

$$\|F(\mathbf{t}; x) - F(\mathbf{t}; y)\|_Y \leq L\|x - y\|, \quad \mathbf{t} \in \Lambda, \quad x, \ y \in X \tag{4}$$

and the mapping $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined. Then it is continuous. Towards this end, let $(\mathbf{t}_n; x_n) \to (\mathbf{t}; x)$ as $n \to +\infty$. Then (4) implies that

$$\begin{aligned}
&\|F(\mathbf{t}_n + \mathbf{u}; x_n) - F(\mathbf{t} + \mathbf{u}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\quad \leq \|F(\mathbf{t}_n + \mathbf{u}; x_n) - F(\mathbf{t}_n + \mathbf{u}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} + \|F(\mathbf{t}_n + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\quad \leq 2(1 + m(\Omega)) \cdot \Big[L\Big\|x_n - x\Big\|\Big]_{L^{p^+}(\Omega)} + \|F(\mathbf{t}_n + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)}.
\end{aligned}$$

The first addend clearly goes to zero since $\|x_n - x\| \to 0$ as $n \to +\infty$. For the second addend, we can apply the arguments used for proving the continuity of the translation mapping from the proof of [7, Proposition 5.1].

(ii) Suppose that $F : \Lambda \times X \to Y$ is continuous and $p \in D_+(\Omega)$. Then the continuity of mapping $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ follows directly by applying the dominated convergence theorem (see Lemma 1.1(iv)).

Example 2.9. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a Bohr Λ'-almost periodic function (Λ'-uniformly recurrent function). Define $sign(0) := 0$ and $H : \mathbb{R}^n \to \mathbb{R}$ by $H(\mathbf{t}) := sign(F(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$. Then, for every $p \in D_+(\Omega)$, the function $H(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-uniformly recurrent), provided that*

$$(\exists L \geq 1)\ (\forall \epsilon > 0)\ \Big(\forall y \in \mathbb{R}^n\Big)\ m\Big(\{x \in y + \Omega : |F(x)| \leq \epsilon\}\Big) \leq L\epsilon. \tag{5}$$

Let $\epsilon > 0$ be fixed. Then the required conclusion follows from the calculation

$$\begin{aligned}
&\Big\|H(\mathbf{t} + \tau + \mathbf{u}; x) - H(\mathbf{t} + \mathbf{u}; x)\Big\|_{L^{p(\mathbf{u})}(\Omega:\mathbb{R})} \\
&\quad \leq 2(1 + m(\Omega)) \cdot \Big\|H(\mathbf{t} + \tau + \mathbf{u}; x) - H(\mathbf{t} + \mathbf{u}; x)\Big\|_{L^{p^+}(\Omega:\mathbb{R})} \\
&\quad \leq 2(1 + m(\Omega)) \cdot \|1\|_{L^{p^+}\big((\mathbf{t}+\Omega)\cap E_\epsilon^c:\mathbb{R}\big)},
\end{aligned}$$

where E_ϵ denotes the set consisting of all tuples $y \in \mathbb{R}^n$ such that $|F(y)| \geq \epsilon$ and τ is a (Λ', ϵ)-period od $F(\cdot)$ (the inequality stated in the last line of computation follows from the fact that for any $y \in E_\epsilon$ and for any such a number τ we have $H(y + \tau) = H(y)$); see also [23, Theorem 5.3.1] for the first result in this direction. Suppose now that the function $F(\cdot)$ is Bohr almost periodic and there exist real numbers a and b such that $a < 0 < b$ and the function $F(\cdot)$ can be analytically extended to the region $\{(z_1, \cdot \cdot \cdot, z_n) \in \mathbb{C}^n : \Re z_i \in (a, b)$ for all $i \in \mathbb{N}_n\}$ (in particular, this holds for any trigonometric polynomial). Then we can repeat verbatim the argumentation contained in the proof of the last mentioned theorem (see also https://math.stackexchange.com/questions/3216833/holomorphic-function-on-mathbbcn-vanishing-on-a-positive-lebesgue-measure?rq=1) in order to see that $\lim_{\epsilon \to 0+} m(E_\epsilon^c \cap (\mathbf{t} + \Omega)) = 0$, uniformly for $\mathbf{t} \in \mathbb{R}^n$, which combined with the above calculation shows that the function $H(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic.

In connection with the above example, it should be noted that the function $H(\cdot)$ need not be Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-uniformly recurrent) for all $p \in \mathcal{P}(\Omega)$, even in the one-dimensional case. Strictly speaking, if $\Omega := [0, 1]$, $\Lambda' := \mathbb{R}$ and $p(u) := 1 - \ln u$, $u \in (0, 1]$, then we have proved, in [6, Example 4.11], that the function $x \mapsto sign(\sin x + \sin(\sqrt{2}x))$, $x \in \mathbb{R}$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded but not Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic. Suppose now that $\Omega = [0, 1]^n$ and $p(\mathbf{u}) := 1 - \ln(u_1 \cdot u_2 \cdot \cdot \cdot u_n)$, $\mathbf{u} = (u_1, u_2, \cdot \cdot \cdot, u_n) \in \Omega$ and $F(x_1, x_2, \cdot \cdot \cdot, x_n) := \sin(x_1 + x_2 + \cdot \cdot \cdot + x_n) + \sin(\sqrt{2}(x_1 + x_2 + \cdot \cdot \cdot + x_n))$, $(x_1, x_2, \cdot \cdot \cdot, x_n) \in \mathbb{R}^n$. Then $H(\cdot)$, defined as above, is essentially bounded and therefore Stepanov $(\Omega, p(\mathbf{u}))$-bounded. On the other hand, using the argumentation from the above-mentioned example, the Fubini theorem and the equality $\ln(u_1 \cdot u_2 \cdot \cdot \cdot u_n) = \ln u_1 + \ln u_2 + \cdot \cdot \cdot + \ln u_n$

for all $(u_1, u_2, \cdot \cdot \cdot, u_n) \in \Omega$, we get that, for every $\lambda \in (0, 2/e)$ and $l > 0$, we can find a ball $B(\mathbf{t_0}, l) \subseteq \mathbb{R}^n$ such that, for every $\tau \in B(\mathbf{t_0}, l)$, there exists $\mathbf{t} \in \mathbb{R}^n$ such that

$$\int_\Omega \Big(\frac{1}{\lambda}\Big)^{1-\ln(u_1 \cdot u_2 \cdots u_n)} \Big| sign\Big[\sin(\mathbf{u} + \mathbf{t} + \tau) + \sin(\sqrt{2}(\mathbf{u} + \mathbf{t} + \tau))\Big] \\ - sign\Big[\sin(\mathbf{u} + \mathbf{t}) + \sin(\sqrt{2}(\mathbf{u} + \mathbf{t}))\Big]\Big|^{1-\ln(u_1 \cdot u_2 \cdots u_n)} d\mathbf{u} = \infty.$$

This simply implies that the function $H(\cdot)$ is not Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic.

Concerning the convolution invariance of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic functions, we will clarify the following result:

Proposition 2.10. *Suppose that $h \in L^1(\mathbb{R}^n)$, $p \in D_+(\Omega)$, the function $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic and for each bounded subset D of X there exists a constant $c_D > 0$ such that $\|F(\mathbf{t}; x)\|_Y \leq c_D$ for a.e. $\mathbf{t} \in \mathbb{R}^n$ and all $x \in D$. Suppose, further, that for each sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n); x_k)) \in \mathrm{R}_X$ and for each set $B \in \mathcal{B}$ we have that $B + \{x_k : k \in \mathbb{N}\}$ is a bounded set in X. Then the function*

$$(h * F)(\mathbf{t}; x) := \int_{\mathbb{R}^n} h(\sigma) F(\mathbf{t} - \sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X$$

is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic and satisfies that for each bounded subset D of X there exists a constant $c_D' > 0$ such that $\|(h * F)(\mathbf{t}; x)\|_Y \leq c_D'$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in D$.

Proof. The prescribed assumptions imply that for each bounded subset D of X there exists a constant $c_D' > 0$ such that $\|\hat{F}_\Omega(\mathbf{t}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq c_D$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in D$, as well as that $\|(h * F)(\mathbf{t}; x)\|_Y \leq c_D'$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in D$. Applying Lemma 1.8, we get that the function $[h * \hat{F}_\Omega](\cdot; \cdot)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic. The result now simply follows from the equality

$$h * \hat{F}_\Omega = h \hat{*} F_\Omega \tag{6}$$

and a corresponding definition of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodicity. □

Using [2, Proposition 2.8] and the corresponding definition, we can immediately deduce the following result which can be also formulated for the (asymptotical) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodicity and (asymptotical) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodicity (see [2] for more details):

Proposition 2.11. *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic. If for each $B \in \mathcal{B}$ there exists $\epsilon_B > 0$ such that*

$$\lim_{j \to +\infty} \sup_{\mathbf{t} \in \Lambda; x \in B'} \Big\| F_j(\mathbf{t} + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0,$$

where $B' \equiv B^\circ \cup \bigcup_{x \in \partial B} B(x, \epsilon_B)$, then the function $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic.

The subsequent result is trivial and follows almost immediately from the above definitions:

Proposition 2.12. *Suppose that $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous. Then the function $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniformly recurrent) if and only if the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is Bohr $(\mathcal{B}, \Lambda')$-almost periodic ($(\mathcal{B}, \Lambda')$-uniformly recurrent).*

Since every Bohr almost periodic function $F : \mathbb{R}^n \to Y$ is immediately Bohr Δ_n-almost periodic ([2]), we may deduce from the previous proposition that a Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic function $F : \mathbb{R}^n \to Y$ is immediately Stepanov $(\Omega, p(\mathbf{u}))$-Δ_n-almost periodic. Using Lemma 1.7 we can simply deduce the following:

Theorem 2.13. *Suppose that $\hat{F}_\Omega : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous, $\mathcal{B}$ is any family of compact subsets of X and R is the collection of all sequences in $\mathbb{R}^n$. Then $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic if and only if $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic.*

The notion of strong $\mathcal{B}$-almost periodicity was also introduced and analyzed in [2]. Keeping in mind Proposition 2.12, the notion of a strong Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodicity can be introduced in the following way: a function $F : \Lambda \times X \to Y$ is said to be strongly Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic if and only if the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is strongly almost periodic. We will skip all related details concerning this theme for brevity.

Using Lemma 1.10 and Proposition 2.12, we can deduce the following result:

Theorem 2.14. *(The uniqueness theorem for Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic functions) Suppose that $\Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda \subseteq \Lambda$, condition* (AP-E) *holds with the sets I and I' replaced therein with the sets Λ and Λ', as well as $\mathbb{R}^n \setminus [(\Lambda \cup (-\Lambda)) + (\Lambda \cup (-\Lambda))]$ is a bounded set. If $F : \mathbb{R}^n \to Y$ and $G : \mathbb{R}^n \to Y$ are two Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic functions and $F(\mathbf{t}) = G(\mathbf{t})$ for a.e. $\mathbf{t} \in \Lambda$, then $F(\mathbf{t}) = G(\mathbf{t})$ for a.e. $\mathbf{t} \in \mathbb{R}^n$.*

Proof. By Proposition 2.12, $\hat{F} : \mathbb{R}^n \to L^{p(\mathbf{u})}(\Omega : Y)$ and $\hat{G} : \mathbb{R}^n \to L^{p(\mathbf{u})}(\Omega : Y)$ are Bohr almost periodic functions. Let $\mathbf{t} \in \Lambda$ be fixed. Then our assumption implies $F(\mathbf{t} + \mathbf{u}) = G(\mathbf{t} + \mathbf{u})$ for a.e. $\mathbf{u} \in \Omega$ so that $\hat{F}(\mathbf{t}) = \hat{G}(\mathbf{t})$. Applying Lemma 1.10, we get $\hat{F}(\mathbf{t}) = \hat{G}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$, which simply implies that $F(\mathbf{t}) = G(\mathbf{t})$ for a.e. $\mathbf{t} \in \mathbb{R}^n$. $\square$

Now we will state and prove the following important result about extensions of Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic functions:

Theorem 2.15. *Suppose that $\Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, the set Λ' is unbounded, $m(\partial\Lambda) = 0$, $\Omega^\circ \neq \emptyset$, $F : \Lambda \to Y$ satisfies that $\hat{F}_\Omega : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is a uniformly continuous, Bohr Λ'-almost periodic function, resp. a uniformly continuous, Λ'-uniformly recurrent function, $S \subseteq \mathbb{R}^n$ is bounded and, for every $\mathbf{t}' \in \mathbb{R}^n$, there exists a finite real number $M > 0$ such that $\mathbf{t}' + \Lambda'_M \subseteq \Lambda$. Define $\Lambda_S := [(\Lambda' \cup (-\Lambda')) + (\Lambda' \cup (-\Lambda'))] \cup S$. Then there exists a Stepanov $(\Omega, p(\mathbf{u}))$-Λ_S-almost periodic, resp. a Stepanov $(\Omega, p(\mathbf{u}))$-Λ_S-uniformly recurrent, function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for a.e. $\mathbf{t} \in \Lambda$; furthermore, in Stepanov almost periodic case, if $\mathbb{R}^n \setminus \Lambda_S$ is a bounded set and the function $\tilde{G}(\cdot)$ satisfies the same requirements as the function $\tilde{F}(\cdot)$, then there exists a set $N \subseteq \mathbb{R}^n$ such that $m(N) = 0$ and $\tilde{F}(\mathbf{t}) = \tilde{G}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n \setminus N$.*

Proof. We will consider only Stepanov almost periodicity. By Proposition 2.12, we have that the function $\hat{F}_\Omega : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is Bohr Λ'-almost periodic. Due to the prescribed assumptions, we can apply Lemma 1.9 in order to see that there exists a uniformly continuous Bohr Λ_S-almost periodic function $H : \mathbb{R}^n \to L^{p(\mathbf{u})}(\Omega : Y)$ such that $\hat{F}_\Omega(\mathbf{t}) = H(\mathbf{t})$ for all $\mathbf{t} \in \Lambda$. Furthermore, by the corresponding proof of Lemma 1.9, given in [2], there exists a sequence (τ_k) in Λ' such that $H(\mathbf{t}) = \lim_{k\to+\infty} \hat{F}_\Omega(\mathbf{t} + \tau_k)$, where the limit is uniform in $\mathbf{t} \in \mathbb{R}^n$, and $\lim_{k\to+\infty} |\tau_k| = +\infty$. Now we will prove the following:

- ($\diamond$) Let $\mathbf{t}_1,\ \mathbf{t}_2 \in \mathbb{R}^n$ be fixed. Then there exists a set $N \subseteq \Omega$ such that $m(N) = 0$ and, for every $\mathbf{u}_1,\ \mathbf{u}_2 \in \Omega \setminus N$, the assumption $\mathbf{t}_1 + \mathbf{u}_1 = \mathbf{t}_2 + \mathbf{u}_2$ implies $[H(\mathbf{t}_1)](\mathbf{u}_1) = [H(\mathbf{t}_2)](\mathbf{u}_2)$.

In actual fact, we have that there exists a set $N_i \subseteq \Omega$ such that $m(N_i) = 0$ and $[H(\mathbf{t}_i)](\mathbf{u}_i) = \lim_{k\to+\infty} F(\mathbf{t}_i + \tau_k + \mathbf{u}_i)$ for $i = 1, 2$, so that ($\diamond$) follows immediately by plugging $N \equiv N_1 \cup N_2$. Define now $\tilde{F} : \mathbb{R}^n \to Y$ by $\tilde{F}(\mathbf{t}) := [H(\mathbf{x_t})](\mathbf{t} - \mathbf{x_t})$, if $\mathbf{x_t} \in \mathbb{Q}^n$ and $\mathbf{t} \in \mathbf{x_t} + \Omega^\circ$. Using ($\diamond$) and our assumption $\Omega^\circ \neq \emptyset$, it is very simple to prove that the function $\tilde{F}(\cdot)$ is well defined as well as that the Bochner transform of $\tilde{F}(\cdot)$ is $H(\cdot)$, i.e., that for each $\mathbf{t} \in \mathbb{R}^n$ there exists a set $N_\mathbf{t} \subseteq \Omega$ such that $\tilde{F}(\mathbf{t} + \mathbf{u}) = [H(\mathbf{t})](\mathbf{u})$ for all $\mathbf{u} \in \Omega \setminus N_\mathbf{t}$. Applying again Proposition 2.12, we get that the function $\tilde{F}(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-Λ_S-almost periodic. Now we will prove that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for a.e. $\mathbf{t} \in \Lambda$. By the foregoing, for every $\mathbf{t} \in \Lambda$, there exists a set $N_\mathbf{t} \subseteq \Omega$ such that $m(N_\mathbf{t}) = 0$ and

$$F(\mathbf{t} + \mathbf{u}) = [H(\mathbf{t})](\mathbf{u}) = \tilde{F}(\mathbf{t} + \mathbf{u}), \quad \mathbf{u} \in \Omega \setminus N_\mathbf{t}. \tag{7}$$

Let $\mathbf{x} \in \mathbb{Q}^n$ be fixed. Denote $\Lambda_k := \{\mathbf{t} \in (\mathbf{x}+\Omega)\cap\Lambda : \operatorname{dist}(\mathbf{t}, \partial\Omega) \geq 1/k\}$ $(k \in \mathbb{N})$. Then $[(\mathbf{x}+\Omega)\cap\Lambda]\setminus\partial\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ so that the required statement easily follows from our assumption $m(\partial\Lambda) = 0$ and the fact that for each $k \in \mathbb{N}$ and $\mathbf{t} \in \Lambda_k$ we have $\mathbf{t} \in \Lambda^\circ$ and therefore $\Lambda_k \subseteq \bigcup_{\mathbf{t}\in(x+\Omega)\cap\Lambda}(\mathbf{t}+\Omega^\circ)$; by the Heine-Borel theorem, for every $k \in \mathbb{N}$, this implies the existence of a finite sequence of numbers $\mathbf{t}_1, \cdot\cdot\cdot, \mathbf{t}_{a_k} \in \Omega^\circ$ such that $\Lambda_k \subseteq \bigcup_{k=1}^{a_k}(\mathbf{t} + \Omega^\circ)$ and we can apply (7) to achieve our aims. Finally, if $\mathbb{R}^n \setminus \Lambda_S$ is a bounded set and the function $\tilde{G}(\cdot)$ satisfies the same requirements as the function $\tilde{F}(\cdot)$, then the foregoing arguments simply imply that $\hat{\tilde{F}}(\mathbf{t}) = \hat{\tilde{G}}(\mathbf{t})$ for all $\mathbf{t} \in \Lambda$. Moreover, the both functions $\hat{\tilde{F}}(\cdot)$ and $\hat{\tilde{G}}(\cdot)$ must be Bohr almost periodic on $\mathbb{R}^n$ and therefore compactly almost automorphic so that the arguments used in [2] yield that these functions are equal identically on $\mathbb{R}^n$, which completes the proof in a routine manner. □

Remark 2.16. *(i) It is clear that Theorem 2.15 is applicable provided that $(\mathbf{v}_1, \cdot\cdot\cdot, \mathbf{v}_n)$ is a basis of $\mathbb{R}^n$,*

$$\Lambda' = \Lambda = \Big\{\alpha_1\mathbf{v}_1 + \cdot\cdot\cdot + \alpha_n\mathbf{v}_n : \alpha_i \geq 0 \text{ for all } i \in \mathbb{N}_n\Big\}$$

is a convex polyhedral in $\mathbb{R}^n$ and Ω is any compact subset of Λ with non-empty interior; in this case, we have that there exists a unique Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic extension of the function $F : \Lambda \to Y$ to the whole Euclidean space. This enables to see that Proposition 2.17 and the statement (ii) preceding directly the third section of paper continue to hold with the set $\mathbb{R}^n$ replaced therein with any convex polyhedral in $\mathbb{R}^n$. It is also worth noting that Theorem 2.15 is applicable in the following special case: $\Lambda = [r_1, \infty) \times [r_2, \infty) \times \cdot\cdot\cdot \times [r_n, \infty)$ for some real numbers $r_i \in \mathbb{R}$ $(1 \leq i \leq n)$, $\Lambda' = [r_1', \infty) \times [r_2', \infty) \times \cdot\cdot\cdot \times [r_n', \infty)$ for some non-negative real numbers r_i, $r_i' \geq 0$ $(1 \leq i \leq n)$ and Ω is any compact subset of $[0, \infty)^n$ with non-empty interior, when the function $\tilde{F}(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic.

(ii) It is well known that a compact set with positive Lebesgue measure in $\mathbb{R}^n$, like the famous Smith–Volterra–Cantor set in the one-dimesional case, can have the empty interior.

Keeping in mind Proposition 2.12, we may conclude that the notion introduced in Definition 2.7 generalizes the notion introduced in [6, Definition 4.2(i), Definition 5.2(i)]. Furthermore, combining Proposition 2.6 and Lemma 1.7, we immediately get:

Proposition 2.17. *Suppose that $k \in \mathbb{N}$ and $\mathcal{B}$ is any family of compact subsets of X. If the function $F_i : \mathbb{R}^n \times X \to Y_i$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic for $1 \leq i \leq k$, then the function $(F_1, \cdot\cdot\cdot, F_k)(\cdot; \cdot)$ is also Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic.*

It is clear that Lemma 1.6(i) can be particularly used to profile when, for a given Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic function $F : \Lambda\times X \to Y$, we have that for each $B \in \mathcal{B}$ we have $\sup_{\mathbf{t}\in\Lambda; x\in B} \|F(\mathbf{t}+\mathbf{u}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} < \infty$; if for each $x \in X$ we define the function $F_x : \Lambda \to Y$ by $F_x(\mathbf{t}) := F(\mathbf{t}; x)$, $\mathbf{t} \in \Lambda$, then the above means that $\sup_{x\in B} \|F_x\|_{S_\Omega^{p(\cdot)}} < \infty$ for each fixed set $B \in \mathcal{B}$. Furthermore, Lemma 1.6(ii) can be used to profile when, for a given Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic function $F : \Lambda\times X \to Y$, we have that for each $B \in \mathcal{B}$ the function $\hat{F}_\Omega(\cdot; \cdot)$ is uniformly continuous on $\Lambda \times B$.

Now we will prove the following extension of [23, Theorem 5.2.5] concerning pointwise products of multi-dimensional Stepanov $p(\mathbf{u})$-almost periodic type functions with scalar-valued Stepanov $r(\mathbf{u})$-almost periodic functions (for simplicity, we consider here case $\Lambda = \mathbb{R}^n$, only, albeit we can formulate a corresponding result in case that Λ is admissible with respect to the almost periodic extensions):

Proposition 2.18. *Suppose that p, q, $r \in \mathcal{P}(\Omega)$, $1/p(\mathbf{u}) + 1/r(\mathbf{u}) = 1/q(\mathbf{u})$, $f : \mathbb{R}^n \to \mathbb{C}$ is a Stepanov-$(\Omega, r(\mathbf{u}))$-almost periodic function and $F : \mathbb{R}^n \times X \to Y$ is a Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic function, where $\mathcal{B}$ denotes any family of compact subsets of X. Define $F_1(\mathbf{t}; x) := f(\mathbf{t})F(\mathbf{t}; x)$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$. Then the function $F_1(\cdot; \cdot)$ is Stepanov-$(\Omega, q(\mathbf{u}))$-$\mathcal{B}$-almost periodic.*

Proof. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be given. We have

$$\begin{aligned}&\hat{F}_{1\Omega}(\mathbf{t}'; x') - \hat{F}_{1\Omega}(\mathbf{t}; x)\\ &\quad= \hat{f}_\Omega(\mathbf{t}') \cdot \Big[\hat{F}_\Omega(\mathbf{t}'; x') - \hat{F}_\Omega(\mathbf{t}; x)\Big] + \Big[\hat{f}_\Omega(\mathbf{t}') - \hat{f}_\Omega(\mathbf{t})\Big] \cdot \hat{F}_\Omega(\mathbf{t}; x)\end{aligned}$$

for every $\mathbf{t}$, $\mathbf{t}' \in \mathbb{R}^n$ and x, $x' \in X$. Since the mapping $\hat{f}_\Omega(\cdot)$ is uniformly continuous and bounded on $\mathbb{R}^n$ as well as the mapping $\hat{F}_\Omega(\cdot;\cdot)$ is continuous, we can apply the above equality and the Hölder inequality (see Lemma 1.1(i)) in order to see that the mapping $\hat{F}_{1\Omega}(\cdot;\cdot)$ is continuous, as well. Due to Proposition 2.17, there exists $l > 0$ such that for every $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l)$ such that $\|F(\mathbf{t}+\tau+\mathbf{u};x) - F(\mathbf{t}+\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \epsilon$, $\mathbf{t} \in \mathbb{R}^n$, $x \in B$ and $\|f(\mathbf{t}+\tau+\mathbf{u}) - f(\mathbf{t}+\mathbf{u})\|_{L^{r(\mathbf{u})}(\Omega:Y)} \leq \epsilon$, $\mathbf{t} \in \mathbb{R}^n$. Since

$$\begin{aligned} &F_1(\mathbf{t}+\tau+\mathbf{u};x) - F_1(\mathbf{t}+\tau;x) \\ &\quad = \hat{f}_\Omega(\mathbf{t}+\tau+\mathbf{u}) \cdot \Big[F(\mathbf{t}+\tau+\mathbf{u};x) - F(\mathbf{t}+\tau;x)\Big] \\ &\quad + \Big[f(\mathbf{t}+\tau+\mathbf{u}) - f(\mathbf{t}+\tau)\Big] \cdot F(\mathbf{t}+\mathbf{u};x) \end{aligned}$$

for every $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{u} \in \Omega$ and $x \in B$, we can apply the Hölder inequality again, along with the estimates $\sup_{\mathbf{t}\in\mathbb{R}^n} \|\hat{f}_\Omega(\mathbf{t})\|_{L^{r(\mathbf{u})}(\Omega)} < \infty$ and

$$\sup_{\mathbf{t}\in\mathbb{R}^n; x\in B} \|\hat{F}_\Omega(\mathbf{t};x)\|_{L^{p(\mathbf{u})}(\Omega)} < \infty,$$

to complete the whole proof. □

We can similarly prove the following:

Proposition 2.19. *Suppose that* $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $f : \Lambda \to \mathbb{C}$ *is Stepanov-*$(\Omega, r(\mathbf{u}))$*-bounded and Stepanov* $(\Omega, r(\mathbf{u}))$*-*R*-multi-almost periodic and* $F : \Lambda \times X \to Y$ *is a* $(\Omega, p(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B})$*-multi-almost periodic function satisfying that* $\sup_{\mathbf{t}\in\Lambda; x\in B} \|\hat{F}_\Omega(\mathbf{t};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} < \infty$. *Define* $F_1(\mathbf{t};x) := f(\mathbf{t})F(\mathbf{t};x)$, $\mathbf{t} \in \Lambda$, $x \in X$. *Then* $F_1(\cdot;\cdot)$ *is Stepanov-*$(\Omega, q(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, provided that for each sequence* $(\mathbf{b}_k)$ *in* R *we have that any its subsequence also belongs to* R.

Now we would like to present the following illustrative example:

Example 2.20. *Suppose that* α, $\beta \in \mathbb{R} \setminus \{0\}$ *and* $\alpha\beta^{-1}$ *is an irrational number. As is well known, the functions*

$$f_{\alpha,\beta}(t) = \sin\left(\frac{1}{2 + \cos\alpha t + \cos\beta t}\right), \quad t \in \mathbb{R}$$

and

$$g_{\alpha,\beta}(t) = \cos\left(\frac{1}{2 + \cos\alpha t + \cos\beta t}\right), \quad t \in \mathbb{R}$$

are Stepanov p-almost periodic but not almost periodic ($1 \leq p < \infty$)*. Suppose now that*

$$F\Big(t_1, t_2, \cdots, t_n\Big) = f_1(t_1)f_2(t_2)\cdots f_n(t_n), \quad \mathbf{t} = \Big(t_1, t_2, \cdots, t_n\Big) \in \mathbb{R}^n$$

and for each $i \in \mathbb{N}_n$ *there exist real numbers* α_i, $\beta_i \in \mathbb{R} \setminus \{0\}$ *such that* $\alpha_i\beta_i^{-1}$ *is an irrational number and* $f_i = f_{\alpha_i,\beta_i}$ *or* $f_i = g_{\alpha_i,\beta_i}$. *Applying Proposition 2.18, we inductively may conclude that the function* $\mathbf{t} \mapsto F(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$ *is Stepanov-*$(\Omega, p(\mathbf{u}))$*-almost periodic with* $\Omega = [0,1]^n$ *and* $p \in D_+(\Omega)$.

Using Lemma 1.6(ii) and Theorem 1.7, we can repeat verbatim the argumentation used in the one-dimensional case in order to see that the following result holds:

Theorem 2.21. *Suppose that* $\mathcal{B}$ *is any family of compact subsets of* X *and* $p \in D_+(\Omega)$. *If* $F : \mathbb{R}^n \times X \to Y$ *is uniformly continuous and Stepanov* $(\Omega, p(\mathbf{u}))$*-*$\mathcal{B}$*-almost periodic, then* $F(\cdot;\cdot)$ *is Bohr* $\mathcal{B}$*-multi-almost periodic.*

A sufficient condition for a function $F : \Lambda \times X \to Y$ to be Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic is given as follows:

Proposition 2.22. *Let* $\Lambda + \Lambda \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $\mathcal{B}$ *is any family of compact subsets of* X *and* $F : \Lambda \times X \to Y$ *satisfy the following conditions:*

(i) For each $x \in X$, $F(\cdot; x) \in APS^{\Omega,p(\mathbf{u})}(\Lambda : Y)$.

(ii) $F(\cdot;\cdot)$ is $S^{p(\mathbf{u})}$-uniformly continuous with respect to the second argument on each compact subset B in $\mathcal{B}$ in the following sense: for all $\varepsilon > 0$ there exists $\delta_{B,\varepsilon} > 0$ such that for all $x_1, x_2 \in B$ one has

$$\|x_1 - x_2\| \leq \delta_{B,\varepsilon} \Longrightarrow \Big\| F(\mathbf{t} + \cdot; x_1) - F(\mathbf{t} + \cdot; x_2) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \varepsilon \quad \text{for all } \mathbf{t} \in \Lambda. \tag{8}$$

Then $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic.

Proof. We may assume that $p(\mathbf{u}) \equiv p \in [1, \infty)$ since the proof in general case can be deduced along the same lines. Let $\varepsilon > 0$ and $B \subseteq X$ be a compact set. It follows that there exists a finite subset $\{x_1, ..., x_n\} \subseteq B$ $(n \in \mathbb{N})$ such that $B \subseteq \bigcup_{i=1}^{n} B(x_i, \delta_{B,\varepsilon})$. Therefore, for every $x \in B$, there exists $i \in \mathbb{N}_n$ satisfying $\|x - x_i\| \leq \delta_{B,\varepsilon}$. Let $\tau \in \Lambda$. Then we have

$$\begin{aligned}
&\left(\int_\Omega \|F(\mathbf{t} + \mathbf{s} + \tau; x) - F(\mathbf{t} + \mathbf{s}; x)\|_Y^p \, d\mathbf{s} \right)^{\frac{1}{p}} \\
&\leq \left(\int_\Omega \|F(\mathbf{t} + \mathbf{s} + \tau; x) - F(\mathbf{t} + \mathbf{s} + \tau; x_i)\|_Y^p \, d\mathbf{s} \right)^{\frac{1}{p}} \\
&\quad + \left(\int_\Omega \|F(\mathbf{t} + \mathbf{s} + \tau; x_i) - F(\mathbf{t} + \mathbf{s}; x_i)\|_Y^p \, d\mathbf{s} \right)^{\frac{1}{p}} \\
&\quad + \left(\int_\Omega \|F(\mathbf{t} + \mathbf{s}; x_i) - F(\mathbf{t} + \mathbf{s}; x)\|_Y^p \, d\mathbf{s} \right)^{\frac{1}{p}}, \ \mathbf{t} \in \Lambda.
\end{aligned} \tag{9}$$

Using (i), we have that for each $i = 1, \ldots, n$ there exists $l_{B,\varepsilon} > 0$ such that for all $\mathbf{t}_0 \in \Lambda$ there exists $\tau \in B(\mathbf{t}_0, l_{B,\varepsilon})$ satisfying

$$\left(\int_\Omega \|F(\mathbf{t} + \mathbf{s} + \tau; x_i) - F(\mathbf{t} + \mathbf{s}; x_i)\|_Y^p \, d\mathbf{s} \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda. \tag{10}$$

Since $\|x - x_i\| \leq \delta_{K,\delta}$, by (ii) we claim that

$$\left(\int_\Omega \|F(\mathbf{t} + \mathbf{s} + \tau; x) - F(\mathbf{t} + \mathbf{s} + \tau; x_i)\|_Y^p \, d\mathbf{s} \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda, \tag{11}$$

and

$$\left(\int_\Omega \|F(\mathbf{t} + \mathbf{s}; x) - F(\mathbf{t} + \mathbf{s}; x_i)\|_Y^p \, d\mathbf{s} \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda. \tag{12}$$

Inserting (10), (11) and (12) in (9), we obtain

$$\sup_{x \in B} \left(\int_\Omega \|F(\mathbf{t} + \mathbf{s} + \tau; x) - F(\mathbf{t} + \mathbf{s}; x)\|_Y^p \, d\mathbf{s} \right)^{\frac{1}{p}} \leq \varepsilon \quad \text{for all } \mathbf{t} \in \Lambda.$$

Hence, $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic. □

Almost directly from Definition 2.4, we may conclude the following; the similar statements can be formulated for the notion introduced in Definition 2.5-Definition 2.7 (cf. Lemma 1.1):

Proposition 2.23. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous.*

(i) For every $p \in \mathcal{P}(\Omega)$, we have that $APS^{\Omega,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\Lambda \times X : Y)$ is a subset of $APS^{\Omega,1}_{(\mathrm{R},\mathcal{B})}(\Lambda \times X : Y)$.

(ii) For every $p,\ q \in \mathcal{P}(\Omega)$, we have that the assumption $q(\mathbf{u}) \leq p(\mathbf{u})$ for a.e. $\mathbf{u} \in \Omega$ implies that $APS^{\Omega,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\Lambda\times X : Y)$ is a subset of $APS^{\Omega,q(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\Lambda \times X : Y)$.

(iii) If $p \in D_+(\Omega)$ and $1 \leq p^- \leq p(\mathbf{u}) \leq p^+ < +\infty$ for a.e. $\mathbf{u} \in \Omega$, then

$$APS^{\Omega,p^+}_{(\mathrm{R},\mathcal{B})}(\Lambda \times X : Y) \subseteq APS^{\Omega,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\Lambda \times X : Y) \subseteq APS^{\Omega,p^-}_{(\mathrm{R},\mathcal{B})}(\Lambda \times X : Y).$$

Keeping in mind Remark 2.8(ii) and the proof of [6, Proposition 4.5], we may deduce the following:

Proposition 2.24. *Suppose that $p \in D_+(\Omega)$ and the function $F : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R},\mathcal{B})$-multi-almost periodic [Bohr $\mathcal{B}$-almost periodic/$\mathcal{B}$-uniformly recurrent]. Then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $p(\mathbf{u})$-$\mathcal{B}$-uniformly recurrent].*

Furthermore, we have the following simple result which can be shown with the help of Lemma 1.2:

Proposition 2.25. *Suppose that $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent] and $A \in L(X, Z)$. Then $AF(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent].*

The main structural properties of $(\mathrm{R},\mathcal{B})$-multi-almost periodic type functions clarified in [2, Proposition 2.16] can be simply reformulated for the corresponding Stepanov classes. For example, we have the following:

(i) Suppose that $c \in \mathbb{C}$ and $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X,\mathcal{B})$-multi-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent]. Then $cF(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X,\mathcal{B})$-multi-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent].

(ii) Suppose that $\alpha,\ \beta \in \mathbb{C}$ and, for every sequence which belongs to R (R_X), we have that any its subsequence belongs to R (R_X). If $F(\cdot;\cdot)$ and $G(\cdot;\cdot)$ are Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X,\mathcal{B})$-multi-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent]. Then $(\alpha F + \beta G)(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X,\mathcal{B})$-multi-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/ Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent].

3. Asymptotically Stepanov multi-dimensional almost periodic functions in Lebesgue spaces with variable exponents

In this section, we will generalize the notion introduced in Definition 1.12 by investigating several various classes of multi-dimensional ergodic components in the Lebesgue spaces with variable exponent; the introduced notion is new even for the multi-dimensional ergodic components with constant coefficients.

We start by introducing the following notion:

Definition 3.1. *Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Omega \subseteq \Lambda$ and the set $\mathbb{D}$ is unbounded. By $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ we denote the vector space consisting of all functions $Q : \Lambda \times X \to Y$ such that, for every $\mathbf{t} \in \Lambda$ and $x \in X$, we have $[\hat{Q}_\Omega(\mathbf{t}; x)](\mathbf{u}) \in L^{p(\mathbf{u})}(\Omega : Y)$ as well as that, for every $B \in \mathcal{B}$, we have $\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\to+\infty}[\hat{Q}_\Omega(\mathbf{t}; x)](\mathbf{u}) = 0$ in $L^{p(\mathbf{u})}(\Omega : Y)$, uniformly for $x \in B$. In the case that $X = \{0\}$ and $\mathcal{B} = \{X\}$, then we abbreviate $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ to $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D}}(\Lambda : Y)$.*

Using the dominated convergence theorem, it immediately follows that $C_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X : Y) \subseteq S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X : Y)$.

We continue by providing two illustrative examples:

Example 3.2. *(i) Let $1 \leq p < \infty$. Consider the function $f : [0,\infty) \to \mathbb{R}$ defined by*

$$f(s) := \begin{cases} k, & \text{if } k \leq s \leq k + k^{-p} \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the function $f(\cdot)$ is neither continuous nor bounded but for each $t \in \mathbb{R}$ we have

$$\begin{aligned} \int_{[t,t+1]} |f(s)|^p\, ds &\leq \int_{[\lfloor t\rfloor,\lfloor t\rfloor+2]} |f(s)|^p\, ds \\ &= \sum_{k=\lfloor t\rfloor}^{\lfloor t\rfloor+1} \int_{[k,k+k^{-p}]\cap[k,k+1]} |f(s)|^p\, ds \\ &= \sum_{k=\lfloor t\rfloor}^{\lfloor t\rfloor+1} \int_{[k,k+k^{-p}]} k^p\, ds = 2. \end{aligned}$$

Hence, $f(\cdot)$ is Stepanov p-bounded in the ususal sense. Fix now a number $t \geq 0$. Then there exists a unique integer $k \in \mathbb{N}_0$ such that $k \leq t < k+1$. There exists two possibilities: $k \leq t < k + k^{-p}$ or $k + k^{-p} \leq t < k+1$. In the first case, we have

$$\begin{aligned} \int_t^{t+1} |f(s)|^p\, ds &= \int_t^{k+k^{-p}} k^p\, ds + \int_{k+1}^{t+1} (k+1)^p\, ds \\ &= (t-k)\Big[(k+1)^p - k^p\Big] + 1 \geq 1. \end{aligned}$$

In the second case, we have

$$\int_t^{t+1} |f(s)|^p\, ds = \int_{k+1}^{t+1} (k+1)^p\, ds = (t-k)(k+1)^p \geq k^{-p}(k+1)^p \geq 1.$$

Summa summarum, $\inf_{t\geq 0} \int_t^{t+1} |f(s)|^p\, ds \geq 1$ so that there does not exist an unbounded set $\mathbb{D} \subseteq [0,\infty)$ such that $\lim_{t\to+\infty, t\in\mathbb{D}} \int_t^{t+1} |f(s)|^p\, ds = 0$.

(ii) Let (Ω_n) be a sequence of pairwise disjoint Lebesgue measurable subsets of $\mathbb{R}^n$, let $\Omega = [0,1]^n$ and let $f_n : \Omega_n \to Y$ ($n \in \mathbb{N}$) satisfy

$$\sup_{n\in\mathbb{N}} \Big\| f_n(\cdot)\Big\|_{L^\infty(\Omega_n:Y)} < \infty. \tag{13}$$

Define the function $f : \mathbb{R}^n \to Y$ by $f(\mathbf{t}) := 0$ if $\mathbf{t} \notin \cup_{n\in\mathbb{N}}\Omega_n$ and $f(\mathbf{t}) := f_n(\mathbf{t})$ if $\mathbf{t} \in \Omega_n$ for some $n \in \mathbb{N}$. Then it can be easily seen that the function $f(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded for any $p \in \mathcal{P}(\Omega)$, provided that there exists an integer $l \in \mathbb{N}$ such that for each $\mathbf{t} \in \mathbb{R}^n$ there exist at most l distinct positive integers s such that $(\mathbf{t} + \Omega) \cap \Omega_s \neq 0$. In actual fact, we have

$$\Big\| F(\mathbf{t}+\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:X)} \leq 4 \Big\| F(\mathbf{t}+\mathbf{u})\Big\|_{L^\infty(\Omega:X)} \leq 4l \sup_{n\in\mathbb{N}} \Big\| f_n(\cdot)\Big\|_{L^\infty(\Omega_n:X)}, \quad \mathbf{t} \in \mathbb{R}^n$$

and we can apply (13)*. Furthermore, if $\mathbb{D}$ is any unbounded subset of $\mathbb{R}^n$ such that dist$(\mathbb{D}, \cup_{n\in\mathbb{N}}\Omega_n) \geq$ diam(Ω), we have $f \in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D}}(\mathbb{R}^n : Y)$ for any $p \in \mathcal{P}(\Omega)$.*

Using the idea proposed in [17, Example 2.5.39], we can extend the notion of space $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ in the following three ways: Let $G:\mathbb{R}^n\rightarrow(0,\infty)$ and $\phi:[0,\infty)\rightarrow[0,\infty)$. Then we say that:

(i) a function $Q:\Lambda\times X\rightarrow Y$ belongs to the space $S^{\Omega,p(\mathbf{u}),\phi,G}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ if and only if for every $\mathbf{t}\in\Lambda$ and $x\in X$, we have $\phi(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y)\in L^{p(\mathbf{u})}(\Omega)$ as well as that, for every $B\in\mathcal{B}$, we have

$$\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\rightarrow+\infty}G(\mathbf{t})\Big[\phi\Big(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y\Big)\Big]_{L^{p(\mathbf{u})}(\Omega)}=0,$$

uniformly for $x\in B$;

(ii) a function $Q:\Lambda\times X\rightarrow Y$ belongs to the space $S^{\Omega,p(\mathbf{u}),\phi,G,1}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ if and only if for every $\mathbf{t}\in\Lambda$ and $x\in X$, we have $[\hat{Q}_\Omega(\mathbf{t};x)](\mathbf{u})\in L^{p(\mathbf{u})}(\Omega:Y)$ as well as that, for every $B\in\mathcal{B}$, we have

$$\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\rightarrow+\infty}G(\mathbf{t})\phi\Big(\big\|Q(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\Big)=0,$$

uniformly for $x\in B$;

(iii) a function $Q:\Lambda\times X\rightarrow Y$ belongs to the space $S^{\Omega,p(\mathbf{u}),\phi,G,2}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ if and only if for every $\mathbf{t}\in\Lambda$ and $x\in X$, we have $[\hat{Q}_\Omega(\mathbf{t};x)](\mathbf{u})\in L^{p(\mathbf{u})}(\Omega:Y)$ as well as that, for every $B\in\mathcal{B}$, we have

$$\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\rightarrow+\infty}\phi\Big(G(\mathbf{t})\big\|Q(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\Big)=0,$$

uniformly for $x\in B$.

Now we are ready to introduce the following notion:

Definition 3.3. *(i) Suppose that $\emptyset\neq\Lambda\subseteq\mathbb{R}^n$ satisfies $\Lambda+\Omega\subseteq\Lambda$, $\mathbb{D}\subseteq\Lambda\subseteq\mathbb{R}^n$, the set $\mathbb{D}$ is unbounded, $F:\Lambda\times X\rightarrow Y$ and* (1)*, resp.* (2)*, holds with the set I replaced by the set Λ therein. Then we say that the function $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic, resp. (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R}_X,\mathcal{B})$-multi-almost periodic, if and only if there exist a Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic function ($H:\mathbb{R}^n\times X\rightarrow Y$) $H:\Lambda\times X\rightarrow Y$, resp. a Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R}_X,\mathcal{B})$-multi-almost periodic function ($H:\mathbb{R}^n\times X\rightarrow Y$) $H:\Lambda\times X\rightarrow Y$, and a function $Q\in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ such that $F(\mathbf{t};x)=H(\mathbf{t};x)+Q(\mathbf{t};x)$ for a.e. $\mathbf{t}\in\Lambda$ and all $x\in X$. If $X=\{0\}$ and $\mathcal{B}=\{X\}$, then we also say that the function $F(\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega,p(\mathbf{u}))$-R-multi-almost periodic.*

(ii) Suppose that $\emptyset\neq\Lambda\subseteq\mathbb{R}^n$ satisfies $\Lambda+\Lambda\subseteq\Lambda$, $\Lambda+\Omega\subseteq\Lambda$, $\mathbb{D}\subseteq\Lambda\subseteq\mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded.

(ii.1) Then we say that $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-almost periodic if and only if there exist a Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-almost periodic function ($H:\mathbb{R}^n\times X\rightarrow Y$) $H:\Lambda\times X\rightarrow Y$ and a function $Q\in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ such that $F(\mathbf{t};x)=H(\mathbf{t};x)+Q(\mathbf{t};x)$ for a.e. $\mathbf{t}\in\Lambda$ and all $x\in X$.

(ii.2) Then we say that $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent if and only if there exist a Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent function ($H:\mathbb{R}^n\times X\rightarrow Y$) $H:\Lambda\times X\rightarrow Y$ and a function $Q\in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ such that $F(\mathbf{t};x)=H(\mathbf{t};x)+Q(\mathbf{t};x)$ for a.e. $\mathbf{t}\in\Lambda$ and all $x\in X$.

If $X\in\mathcal{B}$, then we also say that the function $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega,p(\mathbf{u}))$-almost periodic ((strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega,p(\mathbf{u}))$-uniformly recurrent). If $\mathbb{D}=\Lambda$, then we omit the "prefix $\mathbb{D}$-" and say that the function $F(\cdot;\cdot)$ is (strongy) asymptotically Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic, for example.

We can use [2, Proposition 2.27] to simply deduce when the decompositions in Definition 3.3 are unique; [2, Proposition 2.25(ii), Proposition 2.29] can be reformulated in our context, as well.

Suppose that $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$ and $\Lambda + \Omega \subseteq \Lambda$. The notion of $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodicity and the notion of $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniform recurrence can be also introduced and analyzed. We will skip all related details for brevity. For applications, we need the following definition:

Definition 3.4. *Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ is a continuous function and $\Lambda + \Lambda' \subseteq \Lambda$. Then we say that:*

(i) $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodic of type 1 if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exist $l > 0$ and $M > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that

$$\left\| F(\mathbf{t} + \tau + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x) \right\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \epsilon, \text{ provided } \mathbf{t},\ \mathbf{t} + \tau \in \mathbb{D}_M,\ x \in B. \tag{14}$$

(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniformly recurrent of type 1 if and only if for every $B \in \mathcal{B}$ there exist a sequence (τ_n) in Λ' and a sequence (M_n) in $(0, \infty)$ such that $\lim_{n\to+\infty} |\tau_n| = \lim_{n\to+\infty} M_n = +\infty$ and

$$\lim_{n\to+\infty} \sup_{\mathbf{t}, \mathbf{t}+\tau_n \in \mathbb{D}_{M_n}; x \in B} \left\| F(\mathbf{t} + \tau_n + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x) \right\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0.$$

If $\Lambda' = \Lambda$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic of type 1 ($\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent of type 1); furthermore, if $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-almost periodic of type 1 ($\mathbb{D}$-asymptotically Stepanov Λ'-uniformly recurrent of type 1). If $\Lambda' = \Lambda$ and $X \in \mathcal{B}$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov almost periodic of type 1 ($\mathbb{D}$-asymptotically Stepanov uniformly recurrent of type 1). As before, we remove the prefix "$\mathbb{D}$-" in the case that $\mathbb{D} = \Lambda$ and remove the prefix "$(\mathcal{B},)$" in the case that $X \in \mathcal{B}$.

4. Composition theorems for Stepanov multi-dimensional almost periodic functions in Lebesgue spaces with variable exponents

In this section, we will analyze the $(\mathrm{R}, \mathcal{B})$-multi-almost periodic properties of the following multi-dimensional Nemytskii operator $W : \Lambda \times X \to Z$, given by

$$W(\mathbf{t}; x) := G\Big(\mathbf{t}; F(\mathbf{t}; x)\Big), \quad \mathbf{t} \in \Lambda,\ x \in X.$$

First of all, we will state and prove the following composition result for Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic functions:

Theorem 4.1. *Suppose that Λ is admissible with respect to the almost periodic extensions, $x : \Lambda \to X$ is a uniformly continuous, Bohr almost periodic function, $\mathcal{B}$ is any family consisting of compact subsets of X containing $\overline{R(x(\cdot))}$, and $F : \Lambda \times X \to Y$ satisfies the item* (ii) *of Proposition 2.22 as well as that, for every $z \in \overline{R(x(\cdot))}$, the function $\hat{F}_\Omega(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is uniformly continuous, Bohr almost periodic. Then the function $F(\cdot; x(\cdot))$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic.*

Proof. Without loss of generality, we may assume that $p(\mathbf{u}) \equiv p \in [1, \infty)$ and $\Lambda = \mathbb{R}^n$ (the assumptions prescribed imply that the function $x(\cdot)$ can be extended to a Bohr almost periodic function defined on $\mathbb{R}^n$ as well as that, for every $z \in \overline{R(x(\cdot))}$, the function $\hat{F}_\Omega(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ can be extended to a Bohr almost periodic function defined on $\mathbb{R}^n$ so that the functions $x(\cdot)$ and the finite collection of functions of the form

$\hat{F}_{\Omega}(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ can share the same ϵ-periods for each positive real number $\epsilon > 0$; we only need this fact and the relative compactness of range of range of function $x(\cdot)$ below). Let $\mathbf{t},\ \tau \in \mathbb{R}^n$. Then we have

$$\begin{aligned}&\left(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau; x(\mathbf{t}+\mathbf{s}+\tau)) - F(\mathbf{t}+\mathbf{s}; x(\mathbf{t}+\mathbf{s}))\|^p \, d\mathbf{s}\right)^{\frac{1}{p}} \\ \leq\ & \left(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau; x(\mathbf{t}+\mathbf{s}+\tau)) - F(\mathbf{t}+\mathbf{s}+\tau; x(\mathbf{t}+\mathbf{s}))\|^p d\mathbf{s}\right)^{\frac{1}{p}} \\ &+ \left(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau; x(\mathbf{t}+\mathbf{s})) - F(\mathbf{t}+\mathbf{s}; x(\mathbf{t}+\mathbf{s}))\|^p d\mathbf{s}\right)^{\frac{1}{p}}.\end{aligned}$$

Let $\varepsilon > 0$ be fixed. Due to our assumption, $K := \overline{\{x(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^n\}}$ is a compact subset of X. We know that there exists $\delta_{\varepsilon,K} > 0$ such that (8) holds. Moreover, there exists $l_\varepsilon > 0$ such that every ball of center l_ε contains an element τ such that $\|x(\mathbf{s}+\tau) - x(\mathbf{s})\| \leq \delta_{\varepsilon,K}$ for all $\mathbf{s} \in \mathbb{R}^n$. Moreover, for each $\mathbf{s} \in \mathbb{R}^n$, we have $x(\mathbf{s}) \in K$. Hence,

$$\left(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau; x(\mathbf{t}+\mathbf{s}+\tau)) - F(\mathbf{t}+\mathbf{s}+\tau; x(\mathbf{t}+\mathbf{s}))\|^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{4}. \tag{15}$$

Since K is compact, it follows that there exists a finite subset $\{x_1, ..., x_n\} \subseteq K$ ($n \in \mathbb{N}$) such that $K \subseteq \bigcup_{i=1}^{n} B(x_i, \delta_{K,\varepsilon})$. Then, for all $\mathbf{t} \in \mathbb{R}^n$ there exists $i(\mathbf{t}) \in \mathbb{N}_n$ such that $\|x(\mathbf{t}) - x_{i(\mathbf{t})}\| \leq \delta_{K,\varepsilon}$. Thus,

$$\left(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau; x(\mathbf{t}+\mathbf{s})) - F(\mathbf{t}+\mathbf{s}+\tau; x_{i(\mathbf{t})})\|^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{4}, \tag{16}$$

and

$$\left(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}; x(\mathbf{t}+\mathbf{s})) - F(\mathbf{t}+\mathbf{s}; x_{i(\mathbf{t})})\|^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{4}. \tag{17}$$

By Proposition 2.22, we have

$$\left(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau; x_{i(\mathbf{t})}) - F(\mathbf{t}+\mathbf{s}; x_{i(\mathbf{t})})\|^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{4}. \tag{18}$$

Consequently, by (15), (16), (17) and (18), we obtain that

$$\left(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau; x(\mathbf{t}+\mathbf{s}+\tau)) - F(\mathbf{t}+\mathbf{s}; x(\mathbf{t}+\mathbf{s}))\|^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,$$

for any $\mathbf{t} \in \mathbb{R}^n$. This proves the result. □

Now we will state the following simple consequence of Theorem 4.1 in which $F(\cdot; \cdot)$ is Lipschitzian with respect to the second argument; more precisely, we assume that there exists a non-negative scalar-valued function $L_F(\cdot)$ such that $\sup_{\mathbf{t}\in\Lambda} \|L_F(\mathbf{t}+\mathbf{u})\|_{L^{p(\mathbf{u})}(\Omega)} < +\infty$ and

$$\|F(\mathbf{t}; x) - F(\mathbf{t}; y)\| \leq L_F(\mathbf{t})\|x - y\|, \quad x,\ y \in X,\ \mathbf{t} \in \Lambda. \tag{19}$$

Corollary 4.2. *Suppose that Λ is admissible with respect to the almost periodic extensions, $x : \Lambda \to X$ is a uniformly continuous, Bohr almost periodic function, $\mathcal{B}$ is any family consisting of compact subsets of X containing $\overline{R(x(\cdot))}$, and $F : \Lambda \times X \to Y$ satisfies that, for every $z \in \overline{R(x(\cdot))}$, the function $\hat{F}_{\Omega}(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is uniformly continuous, Bohr almost periodic. Then the function $F(\cdot; x(\cdot))$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic, provided that there exists a non-negative scalar-valued function $L_F(\cdot)$ such that $\sup_{t\in\Lambda} \|L_F(\mathbf{t}+\mathbf{u})\|_{L^{p(\mathbf{u})}(\Omega)} < +\infty$ and (19) holds.*

The following composition principle generalizes [6, Theorem 5.4] and can be proved by using the argumentation contained in the proofs of [15, Lemma 2.1, Theorem 2.2] (the assumptions prescribed imply that we can pass to the case in which $\Lambda = \mathbb{R}^n$, as in the proof of Theorem 4.1):

Theorem 4.3. *Suppose that Λ is admissible with respect to the almost periodic extensions, $\hat{x} : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is a uniformly continuous, Bohr almost periodic function, $\mathcal{B}$ is any family consisting of compact subsets of X containing $\overline{R(x(\cdot))}$, $p \in \mathcal{P}(\Omega)$, and $F : \Lambda \times X \to Y$ satisfies that, for every $z \in \overline{R(x(\cdot))}$, the function $\hat{F}_\Omega(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is uniformly continuous, Bohr almost periodic. Let the following conditions hold:*

(i) There exist a function $r \in \mathcal{P}(\Omega)$ such that $r(\cdot) \geq \max(p(\cdot), p(\cdot)/p(\cdot) - 1)$ and a function $L_F \in L_S^{\Omega, r(\mathbf{u})}(\Lambda)$ such that:

$$\|F(\mathbf{t}; x) - F(\mathbf{t}; y)\| \leq L_F(\mathbf{t})\|x - y\|_Y, \quad \mathbf{t} \in \Lambda,\ x,\ y \in Y; \tag{20}$$

(ii) There exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{x(\mathbf{t}) : \mathbf{t} \in \Lambda \setminus \mathrm{E}\}$ is relatively compact in X.

Define $q \in \mathcal{P}(\Omega)$ by $q(\mathbf{u}) := p(\mathbf{u})r(\mathbf{u})/[p(\mathbf{u}) + r(\mathbf{u})]$, if $\mathbf{u} \in \Omega$ and $r(\mathbf{u}) < \infty$, $q(\mathbf{u}) := p(\mathbf{u})$, if $\mathbf{u} \in \Omega$ and $r(\mathbf{u}) = \infty$. Then $q(\mathbf{u}) \in [1, p(\mathbf{u}))$ for $\mathbf{u} \in \Omega$, $r(\mathbf{u}) < \infty$ and $F(\cdot, x(\cdot)) \in APS_{\mathcal{B}}^{\Omega, q(\mathbf{u})}(\Lambda : Y)$.

The following composition principle generalizes [18, Theorem 2.1] and it is not comparable with Theorem 4.3 in general (see [18] for more details):

Theorem 4.4. *Suppose that Λ is admissible with respect to the almost periodic extensions, $\hat{x} : \Lambda \to L^{q(\mathbf{u})}(\Omega : Y)$ is a uniformly continuous, Bohr almost periodic function, $\mathcal{B}$ is any family consisting of compact subsets of X containing $\overline{R(x(\cdot))}$, $p \in \mathcal{P}(\Omega)$, and $F : \Lambda \times X \to Y$ satisfies that, for every $z \in \overline{R(x(\cdot))}$, the function $\hat{F}_\Omega(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is uniformly continuous, Bohr almost periodic. Suppose, further, that $p,\ q,\ r \in \mathcal{P}(\Omega)$, $1/p = 1/q + 1/r$ and the following conditions hold:*

(i) There exists a function $L_F \in L_S^{\Omega, r(\mathbf{u})}(\Lambda)$ such that (20) holds.

(ii) There exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{x(\mathbf{t}) : \mathbf{t} \in \Lambda \setminus \mathrm{E}\}$ is relatively compact in X.

Then $F(\cdot, x(\cdot)) \in APS_{\mathcal{B}}^{\Omega, p(\mathbf{u})}(\Lambda : Y)$.

Keeping in mind the above two results, we can simply extend the statements of [6, Proposition 5.5] and [18, Proposition 2.2] for $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic functions; the proofs are completely similar to the proofs of these statements given in the one-dimensional case. For simplicity, in the formulations of the following two theorems, we will assume that $\Lambda = \mathbb{R}^n$, albeit we can also assume that Λ is admissible with respect to the almost periodic extensions:

Theorem 4.5. *Let $\mathcal{B}$ be any family consisting of compact subsets of X, $p \in \mathcal{P}(\Omega)$ and the following conditions hold:*

(i) $G \in APS_{\mathcal{B}}^{\Omega, p(\mathbf{u})}(\mathbb{R}^n \times X : Y)$ and there exist a function $r \in \mathcal{P}(\Omega)$ such that $r(\cdot) \geq \max(p(\cdot), p(\cdot)/p(\cdot) - 1)$ and a function $L_G \in L_S^{\Omega, r(\mathbf{u})}(\mathbb{R}^n)$ such that (20) holds with the function $F(\cdot; \cdot)$ replaced therein with the function $G(\cdot; \cdot)$;

(ii) $u \in APS^{\Omega, p(\mathbf{u})}(\mathbb{R}^n : X)$, and there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{u(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^n \setminus \mathrm{E}\}$ is relatively compact in X;

(iii) $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$, where $Q \in S_{0, \mathbb{D}, \mathcal{B}}^{\Omega, q(\mathbf{u})}(\mathbb{R}^n \times X : Y)$ and $q(\cdot)$ being defined as in the formulation of Theorem 4.3;

(iv) $x(\mathbf{t}) = u(\mathbf{t}) + \omega(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$, where $\omega \in S_{0, \mathbb{D}}^{\Omega, p(\mathbf{u})}(\mathbb{R}^n : X)$;

(v) There exists a set $E' \subseteq I$ with $m(E') = 0$ such that $K' = \{x(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^n \setminus E'\}$ is relatively compact in X.

Then $F(\cdot, x(\cdot)) \in AAPS^{\Omega,q(\mathbf{u})}_{\mathcal{B}}(\mathbb{R}^n : Y)$.

Theorem 4.6. *Let* $\mathcal{B}$ *be any family consisting of compact subsets of* X. *Suppose that* $p,\ q,\ r \in \mathcal{P}(\Omega)$, $1/p = 1/q + 1/r$ *and the following conditions hold:*

- *(i)* $G \in APS^{\Omega,p(\mathbf{u})}_{\mathcal{B}}(\mathbb{R}^n \times X : Y)$ *and there exists a function* $L_G \in L^{\Omega,r(\mathbf{u})}_{S}(\mathbb{R}^n)$ *such that* (20) *holds with the function* $F(\cdot;\cdot)$ *replaced therein with the function* $G(\cdot;\cdot)$;
- *(ii)* $u \in APS^{\Omega,q(\mathbf{u})}(\mathbb{R}^n : X)$, *and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{u(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^n \setminus \mathrm{E}\}$ *is relatively compact in* X;
- *(iii)* $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ *for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$, *where* $Q \in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$;
- *(iv)* $x(\mathbf{t}) = u(\mathbf{t}) + \omega(\mathbf{t})$ *for all* $\mathbf{t} \in \mathbb{R}^n$, *where* $\omega \in S^{\Omega,q(\mathbf{u})}_{0,\mathbb{D}}(\mathbb{R}^n : X)$;
- *(v) There exists a set* $E' \subseteq I$ *with* $m(E') = 0$ *such that* $K' = \{x(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^n \setminus E'\}$ *is relatively compact in* X.

Then $F(\cdot, x(\cdot)) \in APS^{\Omega,p(\mathbf{u})}_{\mathcal{B}}(\mathbb{R}^n : Y)$.

The interested reader may try to formulate composition principles for Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent functions following the approach obeyed in [19].

5. Invariance of Stepanov multi-dimensional almost periodicity under the actions of convolution products

Let $\Omega_{\mathbf{k}} := \Omega + \mathbf{k}$, $\mathbf{k} \in \mathbb{N}_0^n$. If any component of $\mathbf{t} = (t_1, t_2, \cdots, t_n)$ is strictly positive, then we simply write $\mathbf{t} > \mathbf{0}$.

The following result is very similar to [16, Proposition 2.6.11] (see also [6, Proposition 6.1]):

Theorem 5.1. *Let* $\Omega = [0,1]^n$, $p \in D_+(\Omega)$, $q \in \mathcal{P}(\Omega)$, $1/p(x) + 1/q(x) = 1$ *for all* $x \in \Omega$, *and* $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X, Y)$ *is a strongly continuous operator family satisfying that* $M := \sum_{\mathbf{k}\in\mathbb{N}_0^n} \|R(\cdot + \mathbf{k})\|_{L^{q(\mathbf{u})}(\Omega)} < \infty$. *If* $\check{f} : \mathbb{R}^n \to X$ *is Stepanov* $(\Omega, p(\mathbf{u}))$*-almost periodic, then the function* $F : \mathbb{R}^n \to Y$, *given by*

$$F(\mathbf{t}) := \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_n} R(\mathbf{t} - \mathbf{s}) f(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{21}$$

is well defined and almost periodic.

Proof. The proof of theorem can be deduced by using the argumentation given in the proof of the above-mentioned propositions and we will only present the main details. Since

$$F(\mathbf{t}) := \int_{0}^{+\infty} \int_{0}^{+\infty} \cdots \int_{0}^{+\infty} R(\mathbf{s}) f(\mathbf{t} - \mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{22}$$

the Hölder inequality holds in our framework (see Lemma 1.1(ii)) and the function $f(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded, the above integral converges absolutely. The proof of fact that for each $\epsilon > 0$ the set of all ϵ-periods of $F(\cdot)$ is relatively dense in $\mathbb{R}^n$ can be repeated verbatim. Since any element of $L^{p(\mathbf{u})}(\Omega : X)$ is absolutely continuous with respect to the norm $\|\cdot\|_{L^{p(\mathbf{u})}}$ (see [10, Definition 1.12, Theorem 1.13]) and the function $\hat{\check{f}}(\cdot)$ is uniformly continuous, the proof of continuity of function $F(\cdot)$ can be deduced along the same lines as in the one-dimensional case. □

Using the argumentation contained in the proof of [7, Proposition 5.1] with regards to the continuity of mapping $F(\cdot)$, we can similarly deduce the following result:

Theorem 5.2. *Let* $\Omega = [0,1]^n$, $p \in D_+(\Omega)$, $q \in \mathcal{P}(\Omega)$, $1/p(x) + 1/q(x) = 1$ *for all* $x \in \Omega$, *and* $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that* $M := \sum_{\mathbf{k}\in\mathbb{N}_0^n} \|R(\cdot + \mathbf{k})\|_{L^{q(\mathbf{u})}(\Omega)} < \infty$. *If* $\check{f} : \mathbb{R}^n \to X$ *is Stepanov* $(\Omega, p(\mathbf{u}))$*-bounded and Stepanov* $(\Omega, p(\mathbf{u}))$-R*-multi-almost periodic, then the function* $F : \mathbb{R}^n \to Y$, *given by* (21), *is well defined and* R*-multi-almost periodic.*

Set, for brevity, $I_{\mathbf{t}} := (-\infty, t_1] \times (-\infty, t_2] \times \cdots \times (-\infty, t_n]$ and $\mathbb{D}_{\mathbf{t}} := I_{\mathbf{t}} \cap \mathbb{D}$ for any $\mathbf{t} = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n$. Now we are able to state and prove the following analogue of [2, Proposition 2.44] for strong $\mathbb{D}$-asymptotical Stepanov $(\Omega, p(\mathbf{u}))$-almost periodicity (see also [16, Proposition 2.6.13, Remark 2.6.14]):

Proposition 5.3. *Suppose that* $\Omega = [0,1]^n$, $p \in D_+(\Omega)$, $q \in \mathcal{P}(\Omega)$, $1/p(x) + 1/q(x) = 1$ *for all* $x \in \Omega$, *and* $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that* $M := \sum_{\mathbf{k}\in\mathbb{N}_0^n} \|R(\cdot + \mathbf{k})\|_{L^{q(\mathbf{u})}(\Omega)} < \infty$. *Suppose, further, that* $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ *satisfies* $\Lambda + \Omega \subseteq \Lambda$, $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ *and the set* $\mathbb{D}$ *is unbounded. Let* $\check{g} : \mathbb{R}^n \to X$ *be Stepanov* $(\Omega, p(\mathbf{u}))$*-almost periodic (Stepanov* $(\Omega, p(\mathbf{u}))$*-bounded and Stepanov* $(\Omega, p(\mathbf{u}))$-R*-multi-almost periodic), let* $q : \Lambda \to X$, *and let* $f(\mathbf{t}) := g(\mathbf{t}) + q(\mathbf{t})$ *for all* $\mathbf{t} \in \Lambda$. *Then the function* $F : \Lambda \to Y$, *defined by*

$$F(\mathbf{t}) := \int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s}) f(\mathbf{s})\, ds, \quad \mathbf{t} \in \Lambda, \tag{23}$$

is strongly $\mathbb{D}$*-asymptotically Stepanov* $(\Omega, p(\mathbf{u}))$*-almost periodic (strongly* $\mathbb{D}$*-asymptotically Stepanov* $(\Omega, p(\mathbf{u}))$-R*-multi-almost periodic), provided that*

$$\lim_{|\mathbf{t}|\to\infty, \mathbf{t}\in\mathbb{D}} \sum_{\mathbf{k}\in\mathbb{N}_0^n} \left\| R(\mathbf{s}+\mathbf{k}) \right\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[I_{\mathbf{t}}\cap\mathbb{D}^c])\cap\Omega)} = 0, \tag{24}$$

and for each $\epsilon > 0$ *there exists* $r > 0$ *such that for each* $\mathbf{t} \in \mathbb{D}$ *with* $|\mathbf{t}| \geq r$ *there exists a finite real number* $r_{\mathbf{t}} > 0$ *such that*

$$\sum_{\mathbf{k}\in\mathbb{N}_0^n} \Bigg\{ \left\| R(\mathbf{s}+\mathbf{k}) \right\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[I_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega)} \times \left\| \check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t}) \right\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega)} \Bigg\} < \epsilon/2 \tag{25}$$

and

$$\sum_{\mathbf{k}\in\mathbb{N}_0^n} \Bigg\{ \left\| R(\mathbf{s}+\mathbf{k}) \right\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega)} \times \left\| \check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t}) \right\|_{L^{p(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega)} \Bigg\} < \epsilon/2. \tag{26}$$

Proof. We will consider only strong $\mathbb{D}$-asymptotical Stepanov $(\Omega, p(\mathbf{u}))$-almost periodicity. Clearly, we have the decomposition

$$F(\mathbf{t}) = \int_{I_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s}) g(\mathbf{s})\, d\mathbf{s} + \left[\int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s}) q(\mathbf{s})\, d\mathbf{s} - \int_{I_{\mathbf{t}}\cap\mathbb{D}^c} R(\mathbf{t}-\mathbf{s}) g(\mathbf{s})\, d\mathbf{s} \right], \quad \mathbf{t} \in \Lambda.$$

Keeping in mind Theorem 5.1, it suffices to show that the function

$$\mathbf{t} \mapsto \int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s}) q(\mathbf{s})\, d\mathbf{s} - \int_{I_{\mathbf{t}}\cap\mathbb{D}^c} R(\mathbf{t}-\mathbf{s}) g(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \Lambda$$

belongs to the class $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D}}(\Lambda : X)$. For the second addend, this immediately follows from the following calculus and condition (24):

$$\int_{I_{\mathbf{t}}\cap\mathbb{D}^c} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\,d\mathbf{s} = \int_{\mathbf{t}-[I_{\mathbf{t}}\cap\mathbb{D}^c]} R(\mathbf{s})\check{g}(\mathbf{s}-\mathbf{t})\,d\mathbf{s}$$
$$= \sum_{\mathbf{k}\in\mathbb{N}_0^n} \int_{(\mathbf{t}-\mathbf{k}-[I_{\mathbf{t}}\cap\mathbb{D}^c])\cap\Omega} R(\mathbf{s}+\mathbf{k})\check{g}(\mathbf{s}+\mathbf{k}-\mathbf{t})\,d\mathbf{s}$$
$$\leq 2\sum_{\mathbf{k}\in\mathbb{N}_0^n} \big\|R(\mathbf{s}+\mathbf{k})\big\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[I_{\mathbf{t}}\cap\mathbb{D}^c])\cap\Omega)} \cdot \sup_{\mathbf{t}\in\mathbb{R}^n}\big\|\hat{\check{g}}(\mathbf{t})\big\|_{L^{p(\mathbf{u})}(\Omega)}.$$

Let $\epsilon > 0$ be given. Then there exists $r > 0$ such that for each $\mathbf{t} \in \mathbb{D}$ with $|\mathbf{t}| \geq r$ there exists a finite real number $r_{\mathbf{t}} > 0$ such that (25)-(26) hold. If $\mathbf{t} \in \mathbb{D}$ and $|\mathbf{t}| \geq r$, then we have

$$\int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} = \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} + \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s}.$$

For the first addend in the above sum, we can use the following calculation and condition (25):

$$\int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} = \int_{\mathbf{t}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})]} R(\mathbf{s})\check{q}(\mathbf{s}-\mathbf{t})\,d\mathbf{s}$$
$$= \sum_{\mathbf{k}\in\mathbb{N}_0^n} \int_{(\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega} R(\mathbf{s}+\mathbf{k})\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\,d\mathbf{s}$$
$$\leq 2\sum_{\mathbf{k}\in\mathbb{N}_0^n} \big\|R(\mathbf{s}+\mathbf{k})\big\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega)}$$
$$\cdot \big\|\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\big\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega)}.$$

For the second addend in the above sum, we can use the following calculation and condition (26):

$$\int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} = \int_{\mathbf{t}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c]} R(\mathbf{s})\check{q}(\mathbf{s}-\mathbf{t})\,d\mathbf{s}$$
$$= \sum_{\mathbf{k}\in\mathbb{N}_0^n} \int_{(\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega} R(\mathbf{s}+\mathbf{k})\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\,d\mathbf{s}$$
$$\leq 2\sum_{\mathbf{k}\in\mathbb{N}_0^n} \big\|R(\mathbf{s}+\mathbf{k})\big\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega)}$$
$$\cdot \big\|\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\big\|_{L^{p(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega)}.$$

The proof of the proposition is thereby completed. □

Before we move ourselves to the final section of paper, it should be recalled that any Stepanov p-almost periodic function $F : \mathbb{R} \to Y$ is equi-Weyl-p-almost periodic ($1 \leq p < \infty$), so that the Bohr-Fourier coefficients $P_r(F)$ of $F(\cdot)$, defined by

$$P_r(F) := \lim_{t\to+\infty} \frac{1}{t}\int_{\alpha}^{\alpha+t} e^{-irs}F(s)\,ds, \quad r \in \mathbb{R},$$

exist and do not depend on the choice of a real number α; see e.g. [23, Chapter 5]. On the other hand, it is well known that, for every Bohr almost periodic function $F : \mathbb{R}^n \to Y$, the Bohr-Fourier coefficients $P_\lambda(F)$ of $F(\cdot)$, defined by

$$P_\lambda(F) := \lim_{T\to+\infty} \frac{1}{(2T)^n}\int_{\mathbf{s}+[-T,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle}F(\mathbf{t})\,d\mathbf{t}, \quad \lambda \in \mathbb{R}^n,$$

exist and do not depend on the choice of a tuple $\mathbf{s} \in \mathbb{R}^n$. Similar statements hold for multi-dimensional Stepanov p-almost periodic functions and multi-dimensional equi-Weyl-p-almost periodic functions, which will be considered in our forthcoming paper [11] in more detail.

6. Examples and applications to the abstract Volterra integro-differential equations

In this section, we apply our results established so far in the analysis of existence and uniqueness of the Stepanov multi-almost periodic type solutions for various classes of abstract Volterra integro-differential equations.

We start with two examples concerning Stepanov almost periodic type solutions (with respect to the space variable) of the multi-dimensional heat equations:

1. Let Y be one of the spaces $L^p(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$, where $1 \leq p < \infty$. It is well known that the Gaussian semigroup

$$(G(t)F)(x) := \Big(4\pi t\Big)^{-(n/2)} \int_{\mathbb{R}^n} F(x-y)e^{-\frac{|y|^2}{4t}}\, dy, \quad t > 0,\ f \in Y,\ x \in \mathbb{R}^n,$$

can be extended to a bounded analytic C_0-semigroup of angle $\pi/2$, generated by the Laplacian Δ_Y acting with its maximal distributional domain in Y; see [1, Example 3.7.6] for more details (recall that the semigroup $(G(t))_{t>0}$ is not strongly continuous at zero on $L^\infty(\mathbb{R}^n)$ and $C_b(\mathbb{R}^n)$). Suppose now that $\emptyset \neq \Lambda' \subseteq \Lambda = \mathbb{R}^n$ and $F(\cdot)$ is bounded Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic function, where $p \in D_+(\Omega)$. Then an application of Proposition 2.10 shows that for each $t_0 > 0$ the function $\mathbb{R}^n \ni x \mapsto u(x,t_0) \equiv (G(t_0)F)(x)$ is likewise bounded and Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic; further on, if $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, then we can use Proposition 2.12, Lemma 1.8 and the equation (6) in order to conclude that for each $t_0 > 0$ the function $x \mapsto u(x,t_0)$, $x \in \mathbb{R}^n$ is bounded and Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-almost periodic provided that the function $F(\cdot)$ has the same properties. Similar statements hold in the case of consideration of the Poisson semigroup (see e.g., [1, Example 3.7.9]).

2. Suppose that $0 < T < \infty$. Set $\Lambda := \{(x,t) : x > 0,\ t > 0\}$,

$$E_1(x,t) := \Big(\pi t\Big)^{-1/2} \int_0^x e^{-y^2/4t}\, dy, \quad x \in \mathbb{R},\ t > 0$$

and suppose that $\mathbb{D}$ is any unbounded subset of Λ satisfying that

$$\lim_{|(x,t)|\to+\infty,(x,t)\in\mathbb{D}} \min\left(\frac{x^2}{4(t+T)}, t\right) = +\infty.$$

Following the formula proposed by F. Trèves [27, p. 433]:

$$u(x,t) = \frac{1}{2}\int_{-x}^{x} \frac{\partial E_1}{\partial y}(y,t)u_0(x-y)\, dy - \int_0^t \frac{\partial E_1}{\partial t}(x,t-s)g(s)\, ds, \quad x > 0,\ t > 0, \tag{27}$$

for the solution of the following mixed initial value problem

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \ \ x > 0,\ t > 0;\\ &u(x,0) = u_0(x),\ x > 0,\ \ u(0,t) = g(t),\ t > 0, \end{aligned} \tag{28}$$

in the final section of [2] we have recently analyzed $\mathbb{D}$-asymptotically I'-almost periodic solutions of type 1 to (28) ($\mathbb{D}$-asymptotically I'-uniformly recurrent solutions of type 1 to (28)). We have assumed there that the function $u_0 : [0,\infty) \to \mathbb{C}$ is bounded Bohr I_0-almost periodic, resp. bounded I_0-uniformly recurrent, for a certain non-empty subset I_0 of $[0,\infty)$.

Suppose now that $g(t) \equiv 0$ as well as that the function $u_0 : [0,\infty) \to \mathbb{C}$ is both Stepanov bounded and Stepanov $([0,1],1)$-Λ_0-almost periodic, resp. Stepanov bounded and Stepanov $([0,1],1)$-Λ_0-uniformly recurrent, for a certain non-empty subset Λ_0 of $[0,\infty)$. Set $\Lambda' := \Lambda_0 \times (0,T)$. We will prove that the solution $u(x,t)$ of (28) is $\mathbb{D}$-asymptotically Stepanov $([0,1]^2,1)$-Λ'-almost periodic of type 1, resp. $\mathbb{D}$-asymptotically Stepanov $([0,1]^2,1)$-Λ'-uniformly recurrent of type 1 (see Definition 3.4). In our concrete situation, the formula (27) takes the following form:

$$u(x,t) = \frac{1}{2}\int_{-x}^{x}\Bigl(\pi t\Bigr)^{-1/2}e^{-y^2/4t}u_0(x-y)\,dy, \quad x>0,\ t>0.$$

For any $(x,t) \in \Lambda$ and $(\tau_1,\tau_2) \in \Lambda$, we have:

$$\begin{aligned}
&\int_0^1\int_0^1\Bigl|u\Bigl(x+\tau_1+u_1,t+\tau_2+u_2\Bigr)-u\Bigl(x+u_1,t+u_2\Bigr)\Bigr|\,du_1\,du_2\\
&\leq \frac{1}{2}\int_0^1\int_0^1\int_{x+u_1}^{x+\tau_1+u_1}\Bigl(\pi\Bigl(t+\tau_2+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&\times\Bigl|u_0(x+\tau_1+u_1-y)\Bigr|\,dy\,du_1\,du_2\\
&+\frac{1}{2}\int_0^1\int_0^1\int_{-(x+\tau_1+u_1)}^{-(x+u_1)}\Bigl(\pi\Bigl(t+\tau_2+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&\times\Bigl|u_0(x+\tau_1+u_1-y)\Bigr|\,dy\,du_1\,du_2\\
&+\frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\Biggl|\Bigl(\pi\Bigl(t+\tau_2+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}u_0\Bigl(x+\tau_1+u_1-y\Bigr)\\
&-\Bigl(\pi\Bigl(t+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+u_2)}u_0\Bigl(x+u_1-y\Bigr)\Biggr|\,dy\,du_1\,du_2.
\end{aligned}\tag{29}$$

Let $\epsilon>0$ be given. Then we know that there exists $l>0$ such that for each $x_0 \in \Lambda_0$ there exists $\tau_1 \in (x_0-l,x_0+l)\cap\Lambda_0$ such that

$$\int_x^{x+1}\Bigl|u_0(t+\tau_1)-u_0(t)\Bigr|\,dt \leq \epsilon, \quad x\geq 0.\tag{30}$$

Furthermore, there exists a finite real number $M_0>0$ such that $\int_v^{+\infty}e^{-x^2}\,dx<\epsilon$ for all $v\geq M_0$. Let $M>0$ be such that

$$\min\Biggl(\frac{x^2}{4(t+T)},t\Biggr) > M_0^2+\frac{1}{\epsilon},\ \text{provided } (x,t)\in\mathbb{D} \text{ and } |(x,t)|>M.\tag{31}$$

So, let $(x,t)\in\mathbb{D}$ and $|(x,t)|>M$. For the first addend in (29), we use the Fubini theorem and the following estimates (see (31)):

$$\begin{aligned}
&\frac{1}{2}\int_0^1\int_0^1\int_{x+u_1}^{x+\tau_1+u_1}\Big(\pi\Big(t+\tau_2+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&\times\Big|u_0(x+\tau_1+u_1-y)\Big|\,dy\,du_1\,du_2\\
&\leq\frac{1}{2}\int_0^1\int_{x+1}^{x+\tau_1}\int_0^1\Big(\pi\Big(t+\tau_2+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&\times\Big|u_0(x+\tau_1+u_1-y)\Big|\,du_1\,dy\,du_2\\
&+\frac{1}{2}\int_0^1\int_{x}^{x+1}\int_0^1\Big(\pi\Big(t+\tau_2+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&\times\Big|u_0(x+\tau_1+u_1-y)\Big|\,du_1\,dy\,du_2\\
&+\frac{1}{2}\int_0^1\int_{x+\tau_1}^{x+\tau_1+1}\int_0^1\Big(\pi\Big(t+\tau_2+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&\times\Big|u_0(x+\tau_1+u_1-y)\Big|\,du_1\,dy\,du_2\\
&\leq\frac{\|u_0\|_{S^1}}{2}\int_0^1\int_x^{x+\tau_1+1}\Big(\pi\Big(t+\tau_2+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\,dy\,du_2\\
&\leq\frac{\|u_0\|_{S^1}}{2}\int_0^1\int_x^{\infty}\Big(\pi\Big(t+\tau_2+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\,dy\,du_2\\
&\leq\pi^{-1/2}\|u_0\|_{S^1}\int_{x/2\sqrt{t+T}}^{+\infty}e^{-v^2}\,dv\leq\pi^{-1/2}\|u_0\|_{S^1}\epsilon.
\end{aligned}\tag{32}$$

The second addend in (29) can be estimated in the same manner. For the third addend in (29), we use the following decomposition:

$$\begin{aligned}
&\frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\Big|\Big(\pi\Big(t+\tau_2+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}u_0\Big(x+\tau_1+u_1-y\Big)\\
&-\Big(\pi\Big(t+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+u_2)}u_0\Big(x+u_1-y\Big)\Big|\,dy\,du_1\,du_2\\
&\leq\frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\Big(\pi\Big(t+\tau_2+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&\times\Big|u_0\Big(x+\tau_1+u_1-y\Big)-u_0\Big(x+u_1-y\Big)\Big|\,dy\,du_1\,du_2\\
&+\frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\Big|\Big(\pi\Big(t+\tau_2+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}u_0\Big(x+u_1-y\Big)\\
&-\Big(\pi\Big(t+u_2\Big)\Big)^{-1/2}e^{-y^2/4(t+u_2)}u_0\Big(x+u_1-y\Big)\Big|\,dy\,du_1\,du_2.
\end{aligned}\tag{33}$$

The second addend in (33) can be estimated similarly as the first addend in (29) and the corresponding term from the computation given in [2]. We get:

$$\begin{aligned}
&\frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\Bigl|\Bigl(\pi\Bigl(t+\tau_2+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}u_0\Bigl(x+u_1-y\Bigr)\\
&-\Bigl(\pi\Bigl(t+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+u_2)}u_0\Bigl(x+u_1-y\Bigr)\Bigr|\,dy\,du_1\,du_2\\
&\leq\frac{\|u_0\|_{S^1}}{2}\int_0^1\int_{-(x+1)}^{x+1}\Bigl|\Bigl(\pi\Bigl(t+\tau_2+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&-\Bigl(\pi\Bigl(t+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\Bigr|\,dy\,du_2\\
&\leq\frac{\|u_0\|_{S^1}}{2}\int_0^1 2\pi^{-1/2}\int_{-\infty}^{+\infty}\Biggl|\sqrt{\frac{t+u_2}{t+\tau_2+u_2}}e^{-v^2\cdot\frac{t+u_2}{t+\tau_2+u_2}}-e^{-v^2}\Biggr|\,dv\,du_2\\
&\leq\|u_0\|_{S^1}\pi^{-1/2}\int_0^1\Biggl|\sqrt{\frac{t+u_2}{t+\tau_2+u_2}}-1\Biggr|\,du_2\times\int_{-\infty}^{+\infty}e^{-\frac{M_0^2}{M_0^2+T}v^2}\Bigl(1+2v^2\Bigr)\,dv\\
&\|u_0\|_{S^1}\pi^{-1/2}\int_0^1\frac{\tau_2}{t+u_2+\sqrt{(t+u_2)^2+(t+u_2)\tau_2}}du_2\times\int_{-\infty}^{+\infty}e^{-\frac{M_0^2}{M_0^2+T}v^2}\Bigl(1+2v^2\Bigr)\,dv\\
&\leq\|u_0\|_{S^1}\pi^{-1/2}\frac{T}{t}\times\int_{-\infty}^{+\infty}e^{-\frac{M_0^2}{M_0^2+T}v^2}\Bigl(1+2v^2\Bigr)\,dv.
\end{aligned}\tag{34}$$

The first addend in (33) can be estimated similarly; we have:

$$\begin{aligned}
&\frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\Bigl(\pi\Bigl(t+\tau_2+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&\times\Bigl|u_0\Bigl(x+\tau_1+u_1-y\Bigr)-u_0\Bigl(x+u_1-y\Bigr)\Bigr|\,dy\,du_1\,du_2\\
&\leq\frac{1}{2}\int_0^1\int_{-(x+1)}^{x+1}\Bigl(\pi\Bigl(t+\tau_2+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&\times\Bigl[\sup_{\xi\geq 0}\int_0^1\Bigl|u_0\Bigl(\xi+\tau_1\Bigr)-u_0\Bigl(\xi\Bigr)\Bigr|\,du_1\Bigr]\,dy\,du_2\\
&\leq\frac{\epsilon}{2}\int_0^1\int_{-\infty}^{+\infty}\Bigl(\pi\Bigl(t+\tau_2+u_2\Bigr)\Bigr)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\,dy\,du_2\leq\epsilon\pi^{-1/2}\int_{-\infty}^{+\infty}e^{-v^2}\,dv.
\end{aligned}\tag{35}$$

This finally implies the required conclusion.

3. As explained in [2], Theorem 5.1 and Theorem 5.2 are applicable in the analysis of existence of almost periodic solutions for a wide class of the abstract partial differential equations, which can be constructed in a little bit artificial way. For example, let A be the infinitesimal generator of an exponentially decaying, strongly continuous semigroup $(T(t))_{t\geq 0}$ on X $(i=1,2)$, let $\gamma\in(0,1)$ and let $(T_\gamma(t))_{\geq 0}$ be the subordinated γ-times resolvent family generated by A (see [16] for more details). Suppose that $1<p<\infty$, $F:\mathbb{R}^2\rightarrow X$ is a Stepanov $([0,1]^2,p)$-almost periodic function satisfying that the improper integral in (36) is absolutely convergent. Define

$$u\Bigl(t_1,t_2\Bigr):=\int_{[0,\infty)^2}\Bigl[-T_\gamma\Bigl(s_1\Bigr)+T\Bigl(s_2\Bigr)\Bigr]F\Bigl(t_1-s_1,t_2-s_2\Bigr)\,ds_1\,ds_2,\quad t_1,\ t_2\in\mathbb{R}.\tag{36}$$

Due to Theorem 5.1 (see also the equation (22)), we have that $u:\mathbb{R}^2\rightarrow X$ is almost periodic; furthermore, under certain conditions, we have (see also [16]):

$$\begin{aligned}
u_{t_2}\big(t_1,t_2\big) &= -\int_{[0,\infty)} T_\gamma\big(s_1\big)\Bigg(\int_0^\infty \frac{\partial}{\partial t_2}F\big(t_1-s_1,t_2-s_2\big)\,ds_2\Bigg)\,ds_1 \\
&+ \int_0^\infty \Bigg(\frac{\partial}{\partial t_2}\int_0^\infty T\big(s_2\big)F\big(t_1-s_1,t_2-s_2\big)\,ds_2\Bigg)\,ds_1 \\
&= \int_{[0,\infty)} T_\gamma\big(s_1\big)F_{t_2}\big(t_1-s_1,t_2-s_2\big)\,ds_2\,ds_1 \\
&+ \int_0^\infty \Bigg(A\int_0^\infty T\big(s_2\big)F\big(t_1-s_1,t_2-s_2\big)\,ds_2 + F\big(t_1-s_1,t_2\big)\Bigg)\,ds_1,
\end{aligned}$$

for any $t_1,\ t_2 \in \mathbb{R}$. Since the unique solution of the abstract fractional differential equation

$$D_{t,+}^{\gamma}u(t) = (-A)u(t) + f(t),\ t \in \mathbb{R}$$

is given by $t \mapsto \int_0^\infty T_\gamma(s)f(t-s)\,ds$, $t \in \mathbb{R}$, we similarly obtain

$$\begin{aligned}
-D_{t_1,+}^{\gamma}u\big(t_1,t_2\big) &= -\int_0^\infty T\big(s_2\big)\Bigg(\int_0^\infty D_{t_1,+}^{\gamma}F\big(t_1-s_1,t_2-s_2\big)\,ds_1\Bigg)\,ds_2 \\
&+ \int_0^\infty \Bigg((-A)\int_0^\infty T_\gamma\big(s_1\big)F\big(t_1-s_1,t_2-s_2\big)\,ds_1 + F\big(t_1,t_2-s_2\big)\Bigg)\,ds_2,
\end{aligned}$$

so that

$$\begin{aligned}
u_{t_2}\big(t_1,t_2\big) - D_{t_1,+}^{\gamma}u\big(t_1,t_2\big) &= Au\big(t_1,t_2\big) + \int_0^\infty F\big(t_1-s_1,t_2\big)\,ds_1 \\
&+ \int_0^\infty F\big(t_1,t_2-s_2\big)\,ds_2 + \int_{[0,\infty)} T_\gamma\big(s_1\big)F_{t_2}\big(t_1-s_1,t_2-s_2\big)\,ds_2\,ds_1 \\
&- \int_0^\infty T\big(s_2\big)\Bigg(\int_0^\infty D_{t_1,+}^{\gamma}F\big(t_1-s_1,t_2-s_2\big)\,ds_1\Bigg)\,ds_2, \quad t_1,\ t_2 \in \mathbb{R}.
\end{aligned}$$

Unfortunately, it is very difficult to find some applications or interpretations of these types of abstract fractional PDEs in the world of real phenomena.

4. The existence and uniqueness of almost periodic solutions for a wide class of abstract semilinear integral equations of the form

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_{-\infty}^{\mathbf{t}} R(\mathbf{t}-\mathbf{s})F(\mathbf{s},u(\mathbf{s}))\,d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n$$

can be shown by using the Banach contraction principle and our results about the convolution invariance of almost periodicity under the actions of infinite convolution products and established composition principles; here, we assume that $f(\cdot)$ is almost periodic, $(R(\mathbf{t}))_{\mathbf{t}>0}$ has a similar growth rate as in Theorem 5.1 and $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic for a certain function $p \in D_+(\Omega)$; $\Omega \equiv [0,1]^n$. The consideration is quite similar to the corresponding considerations given in the proofs of [16, Theorem 2.7.6, Theorem 2.7.7] and therefore omitted. Observe, however, that we can similarly analyze the existence and uniqueness of asymptotically almost periodic solutions for a wide class of abstract semilinear integral equations of the form

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_0^{\mathbf{t}} R(\mathbf{t}-\mathbf{s})F(\mathbf{s},u(\mathbf{s}))\,d\mathbf{s}, \quad \mathbf{t} \in [0,\infty)^n$$

by using a similar argumentation containing our results about the convolution invariance of asymptotical almost periodicity under the actions of finite convolution products and established composition principles (see e.g., [16, Theorem 2.9.10, Theorem 2.9.11], which must be slightly reformulated for our new purposes).

5. Let A generate a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X whose elements are certain complex-valued functions defined on $\mathbb{R}^n$. Under some assumptions, we have that the function

$$u(t,x)=\Big(T(t)u_0\Big)(x)+\int_0^t [T(t-s)f(s)](x)\,ds,\quad t\geq 0,\ x\in\mathbb{R}^n$$

is a unique classical solution of the abstract Cauchy problem

$$u_t(t,x)=Au(t,x)+F(t,x),\ t\geq 0,\ x\in\mathbb{R}^n;\ u(0,x)=u_0(x),$$

where $F(t,x):=[f(t)](x)$, $t\geq 0$, $x\in\mathbb{R}^n$. In many concrete situations (for example, this holds for the Gaussian semigroup on $\mathbb{R}^n$), there exists a kernel $(t,y)\mapsto E(t,y)$, $t>0$, $y\in\mathbb{R}^n$ which is integrable on any set $[0,T]\times\mathbb{R}^n$ $(T>0)$ and satisfies that

$$[T(t)f(s)](x)=\int_{\mathbb{R}^n} F(s,x-y)E(t,y)\,dy,\quad t>0,\ s\geq 0,\ x\in\mathbb{R}^n.$$

Suppose that this is the case and fix a positive real number $t_0>0$. In [2, Example 0.1], we have observed that the almost periodic behaviour of function $x\mapsto u_{t_0}(x)\equiv\int_0^{t_0}[T(t_0-s)f(s)](x)\,ds$, $x\in\mathbb{R}^n$ depends on the almost periodic behaviour of function $F(t,x)$ in the space variable x. Suppose, for example, that the function $F(t,x)$ is Stepanov $(\Omega,1)$-almost periodic with respect to the variable $x\in\mathbb{R}^n$, uniformly in the variable t on compact subsets of $[0,\infty)$. Then we have $(x,\ \tau\in\mathbb{R}^n;\ \mathbf{u}\in\Omega)$:

$$\Big|u_{t_0}(x+\tau+\mathbf{u})-u_{t_0}(x+\mathbf{u})\Big|$$
$$\leq\int_0^{t_0}\int_{\mathbb{R}^n}|F(s,x+\tau-y+\mathbf{u})-F(s,x-y+\mathbf{u})|\cdot\Big|E\Big(t_0,y\Big)\Big|\,dy\,ds.$$

Integrating this estimate over Ω and using the Fubini theorem, we get that $(x,\ \tau\in\mathbb{R}^n)$:

$$\int_\Omega\Big|u_{t_0}(x+\tau+\mathbf{u})-u_{t_0}(x+\mathbf{u})\Big|\,du$$
$$\leq\int_0^{t_0}\int_{\mathbb{R}^n}\Bigg[\int_\Omega|F(s,x+\tau-y+\mathbf{u})-F(s,x-y+\mathbf{u})|\,d\mathbf{u}\Bigg]\cdot\Big|E\Big(t_0,y\Big)\Big|\,dy\,ds$$
$$\leq\epsilon\int_0^{t_0}\int_{\mathbb{R}^n}\Big|E\Big(t_0,y\Big)\Big|\,dy\,ds;$$

see the corresponding definitions. It follows that the function $u_{t_0}(\cdot)$ is Stepanov $(\Omega,1)$-almost periodic, as well.

Finally, we would like to mention that the research article [2] has been divided into two separate parts as well as that the first part, entitled "Almost periodic type functions of several variables and applications", has recently been published in Journal of Mathematical Analysis and Applications.

References

[1] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace Transforms and Cauchy Problems, Birkhäuser/Springer Basel AG, Basel, 2001.

[2] A. Chávez, K. Khalil, M. Kostić, M. Pinto, Multi-dimensional almost periodic type functions and applications, preprint. 2021. arXiv:2012.00543.

[3] A. Chávez, K. Khalil, M. Kostić, M. Pinto, Multi-dimensional almost automorphic type functions and applications, Bull. Braz. Math. Soc., New Series (2022). https://doi.org/10.1007/s00574-022-00284-x.

[4] A. S. Besicovitch, Almost Periodic Functions, Dover Publ, New York, 1954.
[5] T. Diagana, Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces, Springer-Verlag, New York, 2013.
[6] T. Diagana, M. Kostić, Generalized almost periodic and generalized asymptotically almost periodic type functions in Lebesgue spaces with variable exponents $L^{p(x)}$, Filomat 34(5) (2020) 1629–1644.
[7] T. Diagana, M. Kostić, Generalized almost automorphic and generalized asymptotically almost automorphic type functions in Lebesgue spaces with variable exponents $L^{p(x)}$, Book chapter in: Recent Studies in Differential Equations, H. Forster (ed.), pp. 1–28, Nova Science Publishers, New York, 2020.
[8] T. Diagana, M. Zitane, Weighted Stepanov-like pseudo-almost periodic functions in Lebesgue space with variable exponents $L^{p(x)}$, Afr. Diaspora J. Math. 15 (2013) 56–75.
[9] L. Diening, P. Harjulehto, P. Hästüso, M. Ruzicka, Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, 2011. Springer, Heidelberg, 2011.
[10] X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(O)$ and $W^{m,p(x)}(O)$, J. Math. Anal. Appl. 263 (2001) 424–446.
[11] V. E. Fedorov, M. Kostić, Multi-dimensional Weyl almost periodic type functions and applications, preprint. 2021. arXiv:2101.11754.pdf.
[12] A. M. Fink, Almost Periodic Differential Equations, Springer-Verlag, Berlin, 1974.
[13] G. M. N'Guérékata, Almost Automorphic and Almost Periodic Functions in Abstract Spaces, Kluwer Acad. Publ, Dordrecht, 2001.
[14] D. Lenz, T. Spindeler, N. Strungaru, Pure point diffraction and mean, Besicovitch and Weyl almost periodicity, preprint, arXiv:2006.10821.
[15] W. Long, H.-S. Ding, Composition theorems of Stepanov almost periodic functions and Stepanov-like pseudo-almost periodic functions, Adv. Difference Equ., Vol. 2011, Article ID 654695, 12 pages, doi:10.1155/2011/654695.
[16] M. Kostić, Almost Periodic and Almost Automorphic Type Solutions to Integro-Differential Equations, W. de Gruyter, Berlin, 2019.
[17] M. Kostić, Selected Topics in Almost Periodicity, W. de Gruyter, Berlin, 2022.
[18] M. Kostić, Composition principles for generalized almost periodic functions, Bull. Cl. Sci. Math. Nat. Sci. Math. 43 (2018) 65–80.
[19] M. Kostić, Composition principles for almost periodic type functions and applications, J. Fract. Calc. Appl. 12 (2021) 1–15.
[20] M. Kostić, W.-S. Du, Generalized almost periodicity in Lebesgue spaces with variable exponents, in: Fixed Point Theory and Dynamical Systems with Applications, special issue of Mathematics, Mathematics 8 (2020), 928; doi:10.3390/math8060928.
[21] M. Kostić, W.-S. Du, Generalized almost periodicity in Lebesgue spaces with variable exponents, Part II, in: Fixed Point Theory and Dynamical Systems with Applications, special issue of Mathematics, Mathematics 8(7) (2020), 1052; https://doi.org/10.3390/math8071052.
[22] M. Kostić, V. Kumar, M. Pinto, Stepanov multi-dimensional almost automorphic type functions and applications, J. Nonl. Evol. Equ. Appl. 1 (2022) 1–24.
[23] M. Levitan, Almost Periodic Functions, G.I.T.T.L., Moscow, 1959 (in Russian).
[24] P. Q. H. Nguyen, On variable Lebesgue spaces, Thesis Ph.D., Kansas State University. Pro- Quest LLC, Ann Arbor, MI, 2011. 63 pp.
[25] A. A. Pankov, Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations, Kluwer Acad. Publ., Dordrecht, 1990.
[26] T. Spindeler, Stepanov and Weyl almost periodicity functions in locally compact Abelian groups, preprint, arXiv:2006.07266v1.
[27] F. Trèves, Basic Linear Partial Differential Equations, Dover Publications, New York, 2006 (republication of work published by Academic Press, New York, 1975).
[28] S. Zaidman, Almost-Periodic Functions in Abstract Spaces, Pitman Research Notes in Math, Vol. **126**, Pitman, Boston, 1985.