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Some results on higher order symmetric operators

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Abstract. For some operator $A \in \mathcal{B}(\mathcal{H})$, positive integers m and k, an operator $T \in \mathcal{B}(\mathcal{H})$ is called k-quasi-(A, m)-symmetric if $T^{*k}(\sum\limits_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}AT^j)T^k = 0$, which is a generalization of the m-symmetric operator. In this paper, some basic structural properties of k-quasi-(A, m)-symmetric operators are established with the help of operator matrix representation. We also show that if T and Q are commuting operators, T is k-quasi-(A, m)-symmetric and Q is n-nilpotent, then $T + Q$ is $(k + n - 1)$-quasi-$(A, m + 2n - 2)$-symmetric. In addition, we obtain that every power of k-quasi-(A, m)-symmetric is also k-quasi-(A, m)-symmetric. Finally, some spectral properties of k-quasi-(A, m)-symmetric are investigated.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on the complex separable Hilbert space $\mathcal{H}$. For $S, T \in \mathcal{B}(\mathcal{H})$, let L_S and $R_T \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ denote the operators $L_S(X) = SX$ and $R_T(X) = XT$ of left multiplication by S and right multiplication by T. Recall the definition of the usual derivation operator $\delta_{S,T}(X)$ given by $\delta_{S,T}(X) = SX - XT$ for $X \in \mathcal{B}(\mathcal{H})$. For every positive integer m, we have $\delta^m_{S,T}(X) = \delta_{S,T}(\delta^{m-1}_{S,T}(X))$ for $X \in \mathcal{B}(\mathcal{H})$. Given any positive integer m, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be m-symmetric (also called m-selfadjoint in the literature) if

$$\delta^m_{T^*,T}(I) = (L_{T^*} - R_T)^m(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j = 0,$$

where $\binom{m}{j}$ is the binomial coeffcient and T^* is the adjoint operator of T. The m-symmetric operators have applications in positive definite differential operators of odd order, conjugate point theory, and classical disconjugacy theory [1, 3, 12, 13]. In [11] Helton initiated the study of the m-symmetric operator, in a series of papers [11–13], he modelled these operators as multiplication t on a Sobolev space, established their connections to Sturm-Liouville operators. Note that T is 1-symmetric if and only if T is selfadjoint. It is clear that if T is m-symmetric, then T is n-symmetric for all $n \geq m$. In [17], McCullough and Rodman obtained some algebraic and spectral properties of m-symmetric operators. On the other hand, the perturbation of

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m-symmetric operators by nilpotent operators has been considered in [9, 16, 17], and products and sums of two commuting m-symmetric operators were discussed in [4, 5, 7–9]. In addition, m-symmetric weighted shift operators have been explored in [18]. Recently, in [14], Jeridi and Rabaoui extended the notion of m-symmetric operators to (A, m)-symmetric operators. For a positive $A \in \mathcal{B}(\mathcal{H})$ and positive integer m, an operator $T \in \mathcal{B}(\mathcal{H})$ is called (A, m)-symmetric if

$$\delta^m_{T^*,T}(A) = (L_{T^*} - R_T)^m(A) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}AT^j = 0.$$

(A, m)-symmetric operators inherit many interesting properties of m-symmetric operators, for example, if T and Q are commuting operators, T is an (A, m)-symmetric operator and Q is n-nilpotent, then $T + Q$ is an $(A, m + 2n - 2)$-symmetric operator; if T is an (A, m)-symmetric operator, then T is an (A, n)–symmetric operator for all $n \geq m$; the powers of an (A, m)-symmetric operator are also (A, m)-symmetric operators.

Now we consider an extension of the notion of the (A, m)-symmetric operator.

Definition 1.1. *For some operator $A \in \mathcal{B}(\mathcal{H})$, positive integers m and k, an operator $T \in \mathcal{B}(\mathcal{H})$ is called k-quasi-(A, m)-symmetric if*

$$T^{*k}\delta^m_{T^*,T}(A)T^k = T^{*k}(L_{T^*} - R_T)^m(A)T^k = T^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}AT^j)T^k = 0.$$

In particular, for $A = I$, the operator T is said to be k-quasi-m-symmetric if

$$T^{*k}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^j)T^k = 0.$$

In this paper, we study various properties of k-quasi-(A, m)-symmetric operators. The perturbation of k-quasi-(A, m)-symmetric operators by nilpotent operators is obtained. In addition, some spectral properties of k-quasi-(A, m)-symmetric are investigated.

2. Main Results

Henceforth, let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ be the set of natural numbers, real numbers and complex numbers, respectively. A will denote a bounded linear operator unless explicitly stated otherwise, $\overline{\mathcal{M}}$ will denote the closure of a set $\mathcal{M}$. If $T \in \mathcal{B}(\mathcal{H})$, we shall write $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\sigma(T)$ for the null space, the range space and the spectrum of T, respectively.

Theorem 2.1. *Let $A = A_1 \oplus A_2$ be an operator on $\mathcal{H}$ where $A_1 = A|_{\overline{\mathcal{R}(T^k)}}$ and $A_2 = A|_{\mathcal{N}(T^{*k})}$. Suppose that $\mathcal{R}(T^k)$ is not dense. Then the following statements are equivalent:*
(1) *T is a k-quasi-(A, m)-symmetric operator;*
(2) $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ *on $\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$, where T_1 is an (A_1, m)-symmetric operator and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. (1) ⇒ (2) Consider the matrix representation of T with respect to the decomposition $\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

Let P be the projection onto $\overline{\mathcal{R}(T^k)}$. Since T is a k-quasi-(A, m)-symmetric operator, we have

$$P(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}AT^j)P = 0.$$

Therefore

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}A_1T_1^j = 0.$$

On the other hand, for any $x = (x_1, x_2)^T \in \mathcal{H}$, we have

$$(T_3^k x_2, x_2) = (T^k(I-P)x, (I-P)x) = ((I-P)x, T^{*k}(I-P)x) = 0,$$

which implies $T_3^k = 0$. Since $\sigma(T_1) \cap \{0\}$ has no interior point, by [10, Corollary 7] $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) ⇒ (1) Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$, where T_1 is an (A_1, m)-symmetric operator and $T_3^k = 0$. We have

$$T^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix}.$$

Let $F = \sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}A_1T_1^j$. Then $F = 0$. Since

$$\begin{aligned}
&T^{*k}\Big(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}AT^j\Big)T^k \\
=&\begin{pmatrix} T_1^{*k} & 0 \\ \big(\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}\big)^* & 0 \end{pmatrix}\begin{pmatrix} \sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{*m-j}A_1T_1^j & * \\ * & * \end{pmatrix}\begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix} \\
=&\begin{pmatrix} T_1^{*k}FT_1^k & T_1^{*k}F\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ \big(\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}\big)^*FT_1^k & \big(\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}\big)^*F\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \end{pmatrix} \\
=&0
\end{aligned}$$

for some non specified entries $*$. Hence T is a k-quasi-(A, m)-symmetric operator. □

Corollary 2.2. ([19]) *Suppose that $\mathcal{R}(T^k)$ is not dense. Then the following statements are equivalent:*
(1) *T is a k-quasi-m-symmetric operator;*
(2) *$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$, where T_1 is an m-symmetric operator and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. This is a result of Theorem 2.1. □

Corollary 2.3. *Suppose that T is a k-quasi-(A, m)-symmetric operator and $\mathcal{R}(T^k)$ is dense. Then T is an (A, m)-symmetric operator.*

Proof. This is a result of Definition 1.1. □

Proposition 2.4. *Suppose that T is a k-quasi-(A, m)-symmetric operator. Then T^n is also a k-quasi-(A, m)-symmetric operator for any $n \in \mathbb{N}$.*

Proof. Since T is a k-quasi-(A, m)-symmetric operator, we have

$$T^{*k}\delta^m_{T^*,T}(A)T^k = T^{*k}(L_{T^*} - R_T)^m(A)T^k = T^{*k}\Big(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}AT^j\Big)T^k = 0.$$

Therefore

$$\begin{aligned}T^{*nk}\delta^m_{T^{*n},T^n}(A)T^{nk} =&T^{*nk}(L_{T^{*n}}-R_{T^n})^m(A)T^{nk}\\ =&T^{*nk}(L^n_{T^*}-R^n_T)^m(A)T^{nk}\\ =&T^{*nk}\{L^{n-1}_{T^*}\delta_{T^*,T}+L^{n-2}_{T^*}\delta_{T^*,T}R_T+L^{n-3}_{T^*}\delta_{T^*,T}R^2_T\\ &+\cdots+L_{T^*}\delta_{T^*,T}R^{n-2}_T+\delta_{T^*,T}R^{n-1}_T\}^m(A)T^{nk}\\ =&T^{*(n-1)k}\{L^{n-1}_{T^*}+L^{n-2}_{T^*}R_T+L^{n-3}_{T^*}R^2_T+\cdots\\ &+L_{T^*}R^{n-2}_T+R^{n-1}_T\}^m\{T^{*k}\delta^m_{T^*,T}(A)T^k\}T^{(n-1)k}\\ =&0,\end{aligned}$$

i.e., T^n is a k-quasi-(A,m)-symmetric operator for any $n\in\mathbb{N}$. □

Remark The converse of Proposition 2.4 is not true in general as shown in the following example.

Example 2.5. *Let* $A=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}\in B(\mathbb{C}^4)$ *and* $T=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}\in B(\mathbb{C}^4)$. *A simple calculation shows that* $T^{*2}(T^{*6}A-3T^{*4}AT^2+3T^{*2}AT^4-AT^6)T^2=0$ *and* $T^*(T^{*3}A-3T^{*2}AT+3T^*AT^2-AT^3)T\neq 0$. *So, we obtain that* T^2 *is a quasi-*$(A,3)$*-symmetric operator, but* T *is not a quasi-*$(A,3)$*-symmetric operator.*

Corollary 2.6. *Suppose that* T *is an invertible* k*-quasi-*(A,m)*-symmetric operator. Then* T^{-1} *is a* k*-quasi-*(A,m)*-symmetric operator.*

Proof. Suppose that T is an invertible k-quasi-(A,m)-symmetric operator. Then T is an (A,m)-symmetric operator, and so is T^{-1}. Hence T^{-1} is a k-quasi-(A,m)-symmetric operator. □

Proposition 2.7. *Suppose that* $\{T_n\}$ *is a sequence of* k*-quasi-*(A,m)*-symmetric operators such that* $\lim_{n\to\infty}\|T_n-T\|=0$. *Then* T *is a* k*-quasi-*(A,m)*-symmetric operator.*

Proof. Suppose that $\{T_n\}$ is a sequence of k-quasi-(A,m)-symmetric operators such that $\lim_{n\to\infty}\|T_n-T\|=0$. Then

$$\begin{aligned}&\|T^{*k}_n(\sum_{j=0}^m(-1)^j\tbinom{m}{j}T^{*m-j}_nAT^j_n)T^k_n-T^{*k}(\sum_{j=0}^m(-1)^j\tbinom{m}{j}T^{*m-j}AT^j)T^k\|\\ \le&\|T^{*k}_n(\sum_{j=0}^m(-1)^j\tbinom{m}{j}T^{*m-j}_nAT^j_n)T^k_n-T^{*k}_n(\sum_{j=0}^m(-1)^j\tbinom{m}{j}T^{*m-j}AT^j)T^k\|\\ +&\|T^{*k}_n(\sum_{j=0}^m(-1)^j\tbinom{m}{j}T^{*m-j}AT^j)T^k-T^{*k}(\sum_{j=0}^m(-1)^j\tbinom{m}{j}T^{*m-j}AT^j)T^k\|\\ \le&\|T^{*k}_n\|\|\sum_{j=0}^m(-1)^j\tbinom{m}{j}T^{*m-j}_nAT^{j+k}_n-\sum_{j=0}^m(-1)^j\tbinom{m}{j}T^{*m-j}AT^{j+k}\|\\ +&\|T^{*k}_n-T^{*k}\|\|\sum_{j=0}^m(-1)^j\tbinom{m}{j}T^{*m-j}AT^{j+k}\|\to 0.\end{aligned}$$

Since $\{T_n\}$ is a k-quasi-(A,m)-symmetric operator,

$$T^{*k}_n(\sum_{j=0}^m(-1)^j\tbinom{m}{j}T^{*m-j}_nAT^j_n)T^k_n=0,$$

we have

$$T^{*k}\Big(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}AT^j\Big)T^k=0,$$

i.e., T is a k-quasi-(A,m)-symmetric operator. □

Lemma 2.8. ([6, Proposition 2.2]) *Suppose that T is an (A,m)-symmetric operator and Q is an n-nilpotent operator such that $TQ=QT$. Then $T+Q$ is an $(A,m+2n-2)$-symmetric operator.*

Theorem 2.9. *Let $A=A_1\oplus A_2$ be an operator on $\mathcal{H}$ where $A_1=A|_{\overline{\mathcal{R}(T^k)}}$ and $A_2=A|_{\overline{\mathcal{N}(T^{*k})}}$. Suppose that T is a k-quasi-(A,m)-symmetric operator and Q is an n-nilpotent operator such that $TQ=QT$. Then $T+Q$ is a $(k+n-1)$-quasi-$(A,m+2n-2)$-symmetric operator.*

Proof. Assume that $\mathcal{R}(T^k)$ is dense. Then T is an (A,m)-symmetric operator, $T+Q$ is an $(A,m+2n-2)$-symmetric operator by Lemma 2.8, hence $T+Q$ is a $(k+n-1)$-quasi-$(A,m+2n-2)$-symmetric operator. Now we may assume that T^k does not have dense range. Then by Theorem 2.1 the k-quasi-(A,m)-symmetric T can be decomposed as follows:

$$T=\begin{pmatrix} T_1 & T_2\\ 0 & T_3\end{pmatrix} \text{ on } \mathcal{H}=\overline{\mathcal{R}(T^k)}\oplus\mathcal{N}(T^{*k}),$$

where T_1 is an (A_1,m)-symmetric operator and $T_3^k=0$. Since $TQ=QT$, it follows that Q has the upper triangular representation

$$Q=\begin{pmatrix} Q_1 & Q_2\\ 0 & Q_3\end{pmatrix} \text{ on } \mathcal{H}=\overline{\mathcal{R}(T^k)}\oplus\mathcal{N}(T^{*k}),$$

hence $T_iQ_i=Q_iT_i$ and $Q_i^n=0 (i=1,3)$. Since T_1 is an (A_1,m)-symmetric operator, by Lemma 2.8, T_1+Q_1 is an $(A_1,m+2n-2)$-symmetric operator. We have

$$\begin{aligned}
\delta^{m+2n-2}_{T^*+Q^*,T+Q}(A) &= \sum_{j=0}^{m+2n-2}(-1)^j\binom{m+2n-2}{j}(T+Q)^{*(m+2n-2-j)}A(T+Q)^j\\
&= \sum_{j=0}^{m+2n-2}(-1)^j\binom{m+2n-2}{j}\begin{pmatrix} T_1+Q_1 & T_2+Q_2\\ 0 & T_3+Q_3\end{pmatrix}^{*(m+2n-2-j)}\\
&\quad \begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}\begin{pmatrix} T_1+Q_1 & T_2+Q_2\\ 0 & T_3+Q_3\end{pmatrix}^{j}\\
&= \begin{pmatrix} \delta^{m+2n-2}_{T_1^*+Q_1^*,T_1+Q_1}(A_1) & F_1\\ F_2 & F_3\end{pmatrix}\\
&= \begin{pmatrix} 0 & F_1\\ F_2 & F_3\end{pmatrix}
\end{aligned}$$

for some operators $F_i (i=1,2,3)$ and

$$\begin{aligned}
(T+Q)^{k+n-1} &= \begin{pmatrix} T_1+Q_1 & T_2+Q_2\\ 0 & T_3+Q_3\end{pmatrix}^{k+n-1}\\
&= \begin{pmatrix} (T_1+Q_1)^{k+n-1} & F\\ 0 & (T_3+Q_3)^{k+n-1}\end{pmatrix}\\
&= \begin{pmatrix} (T_1+Q_1)^{k+n-1} & F\\ 0 & 0\end{pmatrix},
\end{aligned}$$

for some operator F. Hence

$$\begin{aligned}&(T^* + Q^*)^{k+n-1}\delta^{m+2n-2}_{T^*+Q^*,T+Q}(A)(T + Q)^{k+n-1}\\ &= \begin{pmatrix}(T_1^* + Q_1^*)^{k+n-1} & 0\\ F^* & 0\end{pmatrix}\begin{pmatrix}0 & F_1\\ F_2 & F_3\end{pmatrix}\begin{pmatrix}(T_1 + Q_1)^{k+n-1} & F\\ 0 & 0\end{pmatrix}\\ &= 0,\end{aligned}$$

i.e., $T + Q$ is a $(k + n - 1)$-quasi-$(A, m + 2n - 2)$-symmetric operator. □

In the sequel, let $\sigma_{ap}(T)$, $\sigma_p(T)$, $\sigma_{su}(T)$, $\sigma_w(T)$, $\sigma_b(T)$ and $\sigma_T(x)$ for the approximate point spectrum of T, the point spectrum of T, the surjective spectrum of T, the Weyl spectrum of T, the Browder spectrum of T and the local spectrum of T at x, respectively.

Theorem 2.10. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a k-quasi-(A, m)-symmetric operator for some positive $A \in \mathcal{B}(\mathcal{H})$ and $0 \notin \sigma_p(A)$. The following statements hold:*
(1) $\sigma_p(T) \subset \mathbb{R}$;
(2) *For distinct non-zero real numbers a, b and non-zero vecters $x, y \in \mathcal{H}$, if $Tx = ax$ and $Ty = by$, then $(Ax, y) = 0$;*
(3) *For distinct non-zero real numbers a, b and sequences of unit vectors $\{x_n\}, \{y_n\} \subset \mathcal{H}$, if $\lim_{n\to\infty}(T - a)x_n = 0$ and $\lim_{n\to\infty}(T - b)y_n = 0$, then $\lim_{n\to\infty}(Ax_n, y_n) = 0$.*

Proof. (1) We argue by contradiction. Assume that $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If $\lambda \in \sigma_p(T)$, then there exists a non-zero vecter $x \in \mathcal{H}$ such that $(T - \lambda)x = 0$. Thus, for each integer l, $(T^l - \lambda^l)x = 0$. Moreover,

$$\begin{aligned}0 =&(T^{*k}(\sum_{j=0}^{m}(-1)^j\tbinom{m}{j}T^{*m-j}AT^j)T^kx, x)\\ =&|\lambda|^{2k}(\overline{\lambda} - \lambda)^m(Ax, x)\\ =&|\lambda|^{2k}(-2Im(\lambda))^m\|A^{\frac{1}{2}}x\|,\end{aligned}$$

which implies that $Im(\lambda) = 0$ since $\|A^{\frac{1}{2}}x\| \neq 0$, this is a contradiction. Hence, $\sigma_p(T) \subset \mathbb{R}$.
(2) Since a, b are two non-zero eigenvalues of T and $Tx = ax$ and $Ty = by$, we have

$$0 = (T^{*k}(\sum_{j=0}^{m}(-1)^j\tbinom{m}{j}T^{*m-j}AT^j)T^kx, y) = a^kb^k(a - b)^m(Ax, y).$$

Hence $(Ax, y) = 0$.
(3) By similar arguments of the proof of (2), we have

$$0 = \lim_{n\to\infty}(T^{*k}(\sum_{j=0}^{m}(-1)^j\tbinom{m}{j}T^{*m-j}AT^j)T^kx_n, y_n) = a^kb^k(a - b)^m\lim_{n\to\infty}(Ax_n, y_n).$$

Hence $\lim_{n\to\infty}(Ax_n, y_n) = 0$. □

Definition 2.11. [15] *An operator $T \in \mathcal{B}(\mathcal{H})$ has the single-valued extension property, abbreviated SVEP, if, for every open set $\mathcal{G} \subseteq \mathbb{C}$, the only analytic solution $f : \mathcal{G} \to \mathcal{H}$ of the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{G}$ is the zero function on $\mathcal{G}$.*

Theorem 2.12. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a k-quasi-(A, m)-symmetric operator for some positive $A \in \mathcal{B}(\mathcal{H})$ and $0 \notin \sigma_p(A)$. Then T has SVEP.*

Proof. Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a k-quasi-(A, m)-symmetric operator for some positive $A \in \mathcal{B}(\mathcal{H})$ and $0 \notin \sigma_p(A)$. Then by Theorem 2.10 $\sigma_p(T) \subset \mathbb{R}$. An operator such that its point spectrum has empty interior has SVEP [2, Remark 2.4(d)], hence T has SVEP. □

Corollary 2.13. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a k-quasi-(A, m)-symmetric operator for some positive $A \in \mathcal{B}(\mathcal{H})$ and $0 \notin \sigma_p(A)$. The following statements hold:*
(1) $\sigma(T) = \sigma_{su}(T) = \cup\{\sigma_T(x) : x \in \mathcal{H}\}$;
(2) $\sigma_w(T) = \sigma_b(T)$.

Proof. Note that T has SVEP. For (1) we can apply [15, Proposition 1.3.2]. For (2) we can apply [2, Corollary 3.53]. □

Corollary 2.14. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a k-quasi-m-symmetric operator. The following statements hold:*
(1) $\sigma(T) = \sigma_{su}(T) = \cup\{\sigma_T(x) : x \in \mathcal{H}\}$;
(2) $\sigma_w(T) = \sigma_b(T)$.

Proof. This is a result of Corollary 2.13. □

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