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Measures of noncompactness in the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ and its application to infinite system of integral equation in two variables

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Abstract. The purpose of this paper is to study the existence of solutions to an infinite system of Volterra-Hammerstein type nonlinear integral equations in two variables in Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ using functions that are defined, continuous and bounded on $\mathbb{R}_+ \times \mathbb{R}_+$, taking values in a given Banach space E. The method used in our research is linked to the creation of a suitable measure of noncompactness in the space of functions defined, continuous and bounded on $\mathbb{R}_+ \times \mathbb{R}_+$ with values in the space ℓ_∞ consisting of real bounded sequences endowed with the standard supremum norm. An example exemplifies our investigations.

1. Introduction and Prelimaneries.

This section is for establishing the notation utilized in the paper. We also provide concepts that serve as the foundation for our research, as well as certain information about the theory of measures of noncompactness that are pertinent to our concerns.

Integral equations are well-known for their use in the description of a wide range of real-world occurrences, and they form a significant area of nonlinear functional analysis. Obviously, the theory of integral equations and the science of differential equations are intertwined (see[[1, 4, 7, 9, 10, 14, 17, 18]]). Recently, various effective attempts have been made to apply the idea of measure of noncompactness to the study of the existence and behaviour of nonlinear integral equation solutions (see[[5, 6, 12, 16]]).

The mentioned constraint is not addressed in this paper. To demonstrate the applicability of the constructed measures of noncompactness , we provide formulas that express the constructed measures in the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$, where ℓ_∞ denotes the classical Banach sequence space consisting of bounded real sequences and is equipped with the standard supremum norm. These measures of noncompactness are also used to prove the existence of solutions of an infinite system of quadratic integral equations of Volterra-Hammerstein type.

We will use the standard notation. Namely, by the symbol $\mathbb{R}$ we will denote the set of real numbers while $\mathbb{N}$ stands for the set of natural numbers.

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The Kuratowski measure of noncompactness for a bounded subset D of a metric space X is defined as

$$\alpha(D) = \inf\left\{\delta > 0 : D \subset \cup_{i=1}^{n} D_i, \mathrm{diam}(D_i) \leq \delta, \text{ for } 1 \leq i \leq m \leq \infty\right\},$$

where $\mathrm{diam}(D_i)$ denotes diameter of the set D_i.

Another important measure of non-compactness is the Hausdorff measure of non-compactness, which is defined as

$$\phi(D) = \inf\left\{\epsilon > 0 : D \text{ has a finite } \epsilon\text{-net in } E\right\}.$$

It can be shown that the Hausdorff measure of noncompactness ϕ is regular and it is equivalent to the Kuratowski measure $\alpha(X)$. More precisely, for an arbitrary set $X \in M_E$, the following inequalities hold (see[5]):

$$\phi(X) \leq \alpha(X) \leq 2\phi(X). \tag{1.1}$$

Let $(X, \|.\|)$ be a Banach space, $\mathbb{R}_+ = [0, \infty)$, the symbols $\bar{X}$ and $\mathrm{Conv}(X)$ denote closure of X and convex closure of X respectively. Let M_E denote the family of non-empty bounded subsets of E and N_E its subfamily consists of relatively compact subsets of E. We now define (MNC) axiomatically given by Banas and Goebel[5].

Definition 1.1 [5] Let X be a Banach space. A function $\phi : M_X \to [0, +\infty)$ is said to be measure of non-compactnes in X if it satisfies the following axioms:

1. The family $\ker \phi = \{E \in M_X : \phi(E) = 0\}$ is a nonempty and $\ker \phi \subset N_X$.
2. $E_1 \subset E_2 \Rightarrow \phi(E_1) \leq \phi(E_2)$.
3. $\phi(\overline{E}) = \phi(E)$.
4. $\phi(\mathrm{Conv}(E)) = \phi(E)$.
5. $\phi(\lambda E_1 + (1 - \lambda E_2) \leq \lambda\phi(E_1). + (1 - \lambda)\phi(E_2)$ for all $\lambda \in (0, 1)$.
6. If (E_m) is a sequence of closed sets from M_X such that $E_{m+1} \subset E_m$ and $\lim_{m\to\infty} \phi(E_m) = 0$, then the intersection set $E_\infty = \bigcap_{m=1}^{\infty} E_m$ is non-empty.

The family $\ker \phi$ appearing in axiom (i) will be called the kernel of the measure of noncompactness ϕ. Let us notice that the set X_∞ described in axiom (vi) is a member of the family $\ker \phi$. Indeed, it is a simple consequence of the inclusion $X_\infty \subset X_p$ for $p = 1, 2, ...$ and axiom (vi) which implies the inequality $\phi(X_\infty) \leq \phi(X_p)$ for $p = 1, 2, ...$ Hence we have $\phi(X_\infty) = 0$. Consequently, $\phi(X_\infty) \in ker\phi$. The above simple observation is quite important in applications.

Let $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ be the Banach space of all real bounded and continuous functions on $\mathbb{R}_+ \times \mathbb{R}_+$ equipped with the standard norm

$$\|x\| = \sup\{|x(w, s)| : w, s \geq 0\}$$

For any nonempty bounded subset X of $BC(\mathbb{R}_+ \times \mathbb{R}_+), x \in X, \zeta > 0$ and $\epsilon > 0$, let

$$\begin{aligned}
\Omega^{\zeta}(x, \epsilon) &= \sup\left\{|x(w, s) - x(u, v)| : w, s, u, v \in [0, \zeta], |w - u| \leq \epsilon, |s - v| \leq \epsilon\right\},\\
\Omega^{\zeta}(X, \epsilon) &= \sup\left\{\Omega^{\zeta}(x, \epsilon) : x \in X\right\},\\
\Omega_0^{\zeta}(X) &= \lim_{\epsilon\to 0}, \Omega^{\zeta}(X, \epsilon),\\
\Omega_0(X) &= \lim_{\zeta\to\infty} \Omega_0^{\zeta}(X),\\
\phi(X) &= \Omega_0(X) + \rho(X)
\end{aligned}$$

where

$$\rho(X) = \lim_{\zeta \to 0}\{\sup_{x\in X}\{\sup\{|x(w,s| : w,s \geq \zeta)\}\}\}.$$

Similar to [8], the function ϕ can be shown to be measure of noncompactness in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ (as defined in definition (1.1)).

The aim of this study is to create measures of noncompactnness in the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ using functions that are defined, continuous and bounded on $\mathbb{R}_+ \times \mathbb{R}_+$, taking values in a given Banach space E and its application to the solvability of infinite system of nonlinear integral equations of Volterra-Hammerstein type in two variables.

2. Measures of noncompactness in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$.

Assume that E is an infinite dimensional Banach space and that ϕ is a measure of noncompactness defined in E.

Consider the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ which consists of functions that are defined, continuous and bounded on $\mathbb{R}_+ \times \mathbb{R}_+$ and have values in the space E. We consider the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ with the supremum norm

$$\|x\|_\infty = \sup\left\{\|x(w,s)\|_E : w,s \in \mathbb{R}_+ \times \mathbb{R}_+\right\},$$

where the symbol $\|.\|_E$ denotes the norm of the space E. $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ is clearly a Banach space with the above mentioned norm.

Simultaneously, we consider the space $C_\zeta = C([0,\zeta]^2, E)$, where $\zeta > 0$ is arbitrarily fixed. Recall, that the C_ζ defines norm as

$$\|x\|_\zeta = \sup\left\{\|x(w,s)\|_E : w,s \in [0,\zeta]\right\}.$$

If we take a function $x \in BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$, we can consider the restriction $x_{|_{[0,\zeta]^2}}$ of x to the square $[0,\zeta]^2$ is an element of the space C_ζ.

Let us take an arbitrary and bounded set $X, X \subset BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ for the reminder of this section. Next, let us define the quantity $\Omega^\infty(x,\epsilon)$ for an arbitrarily fixed function $x \in X$ and for $\epsilon > 0$ as follows

$$\Omega^\infty(x,\epsilon) = \sup\left\{\|x(w,s) - x(u,v)\|_E : w,s \in \mathbb{R}_+ \times \mathbb{R}_+, |w-u| \leq \epsilon, |s-v| \leq \epsilon\right\}. \tag{2.1}$$

Observe that $\lim_{\epsilon\to 0}\Omega^\infty(x,\epsilon) = 0$ if and only if the function $x = x(w,s)$ is uniformly continuous on the interval $\mathbb{R}_+ \times \mathbb{R}_+$. On the other hand notice that for any $\zeta > 0$ we have

$$\Omega^\zeta(x,\epsilon) \leq \Omega^\infty(x,\epsilon), \tag{2.2}$$

where $\Omega^\zeta(x,\epsilon)$ denotes the modulus of continuity of restriction $x|_{[0,\zeta]}$ in the space C_ζ i.e.,

$$\Omega^\zeta(X,\epsilon) = \sup\left\{\|x(w,s) - x(u,v)\|_E : w,s \in [0,\zeta], |w-u| \leq \epsilon, |s-v| \leq \epsilon\right\}.$$

Next, we define

$$\Omega^\zeta(X,\epsilon) = \sup\left\{\Omega^\zeta(x,\epsilon) : x \in X\right\},$$
$$\Omega_0^\zeta(x) = \lim_{\epsilon\to 0}\Omega^\zeta(x,\epsilon).$$

Since the function $\epsilon \to \Omega^{\zeta}(X, \epsilon)$ is nondecreasing and nonnegative for $\epsilon > 0$, indicating that the above limit exists and is finite. Finally, we put

$$\Omega_0(X) = \lim_{\zeta\to\infty} \Omega_0^{\zeta}(X). \tag{2.2.1}$$

Further, assume that $\phi = \phi(X)$ is a given measure of noncompactness in the Banach space E. For an arbitrarily fixed number $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$ denoted by $X(w, s)$ the cross section of the set X at w, s; that is, $X(w, s) = \{x(w, s) : x \in X\}$. Obviously, $X(w, s)$ is a subset of the space E.

Next, for a fixed $\zeta > 0$, let us put

$$\overline{\phi}_{\zeta}(X) = \sup\{\phi(X(w, s)) : w, s \in [0, \zeta]\}. \tag{2.2.2}$$

Observe that the function $\zeta \to \overline{\chi}_{\zeta}(X)$ is nondecreasing and bounded from above since the set X is a bounded subset of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$. Indeed, we have

$$\|X(w, s)\|_E \leq \|X(w, s)\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)} < \infty$$

for any $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$. Consecutively, we define the following quantity

$$\overline{\phi}_{\infty}(X) = \lim_{\zeta\to\infty} \overline{\phi}_{\zeta}(X).$$

In addition, we have $\lim_{\epsilon\to 0} \Omega^{\zeta}(x, \epsilon) = 0$ for every arbitrary function $x \in BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$.

Take a look at the following example :

Example 2.1: Consider the space $BC(\mathbb{R}_+ \times \mathbb{R}_+) = BC(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R} \times \mathbb{R})$. Take the function $x = x(w, s)$ in the space defined on the interval $[0, 1] \times [0, 1]$ as the function with graph being the pyramid with base equal the interval $[0, 1] \times [0, 1]$ and with the height equal to 1. Analogously, we define consecutively the function x on the intervals $[1, 1 + \frac{1}{2}], [1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}]$ etc.

Then for any $\epsilon > 0$ we have that $\Omega^{\infty}(x, \epsilon) = 1$. Hence, we get that $\lim_{\epsilon\to 0} \Omega^{\infty}(x, \epsilon) = 1$. But on the other hand we have that $\lim_{\epsilon\to 0} \Omega^{\zeta}(x, \epsilon) = 0$ for any $\zeta > 0$.

Further, taking into account (2.1), for $X \in M_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$ we define

$$\Omega^{\infty}(X, \epsilon) = \sup\left\{\Omega^{\infty}(x, \epsilon) : x \in X\right\},$$
$$\Omega_0^{\infty}(X) = \lim_{\epsilon\to 0} \Omega^{\infty}(X, \epsilon). \tag{2.3}$$

It is self-evident that $\Omega^{\infty}(X) = 0$ if and only if functions from the set X are equicontinuous on $\mathbb{R}_+ \times \mathbb{R}_+$, or equivalently, functions from X are equiuniformly continuous on $\mathbb{R}_+ \times \mathbb{R}_+$.

let us have a look at the function $\overline{\phi}_{\infty}$ which is defined on the family $X \in M_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$ according to the formula

$$\overline{\phi}_{\infty}(X) = \lim_{\zeta\to\infty} \overline{\phi}_{\zeta}(X), \tag{2.4}$$

where

$$\overline{\phi}_{\zeta}(X) = \sup\left\{\phi(X(w, s)) : w, s \in [0, \zeta]\right\}. \tag{2.41}$$

It is worth noting that the existence of the limit in (2.4) is due to the fact that the function $\zeta \to \overline{\phi}_{\zeta}(X)$ is nondecreasing and bounded from above on $\mathbb{R}_+ \times \mathbb{R}_+$. Indeed, because the set X is a bounded subset in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$, a constant $c > 0$ exists such that

$$\sup\left\{\|x(w, s)|\_E : w, s \in \mathbb{R}_+ \times \mathbb{R}_+\right\} \leq c$$

for any $x \in X$. Thus fixing arbitrarily $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$ we conclude that
$\sup\left\{\|x(w,s)\|_E : x \in X\right\} \leq c$. This implies that the measures of noncompactness $\phi(X(w,s))$ are bounded from above for $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$.

Now, for $\zeta > 0$ let us put

$$\alpha_\zeta(X) = \sup_{x\in X}\left\{\sup\{\|x(w,s)\|_E : w, s \geq \zeta\right\}.$$

Let us note that the function $\zeta \to \alpha_\zeta(X)$ is nonincreasing and bounded on $\mathbb{R}_+ \times \mathbb{R}_+$. As a result, there exists a finite limit

$$\alpha_\infty(X) = \lim_{\zeta\to\infty} \alpha_\zeta(X). \tag{2.5}$$

Let us consider various values related to monitoring the behaviour of functions from the set X at infinity. Namely, for $\zeta > 0$ let us put:

$$\beta_\zeta(X) = \sup_{x\in X}\left\{\sup\{\|x(w,s) - x(u,v)\|_E : w, s, u, v \geq \zeta\right\},$$
$$\beta_\infty(X) = \lim_{\zeta\to\infty} \beta_\zeta(X). \tag{2.6}$$

Next, for $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$ let us define

$$\mathrm{diam}X(w,s) = \sup\left\{\|x(w,s) - y(w,s)\|_E : x, y \in X\right\}$$

and

$$e(X) = \lim_{w,s\to\infty} \mathrm{diam}X(w,s). \tag{2.7}$$

Finally, by linking (2.3)-(2.7), we can define the following quantities by linking (2.3)-(2.7):

$$\phi_\alpha(X) = \Omega_0^\infty(X) + \overline{\phi}_\infty(X) + \alpha_\infty(X), \tag{2.8}$$
$$\phi_\beta(X) = \Omega_0^\infty(X) + \overline{\phi}_\infty(X) + \beta_\infty(X), \tag{2.9}$$
$$\phi_\gamma(X) = \Omega_0^\infty(X) + \overline{\phi}_\infty(X) + \gamma_\infty(X). \tag{2.10}$$

We show that the function ϕ_α, ϕ_β and ϕ_γ defined by formulas (2.8)-(2.10) are measures of noncompctness in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ under some assumptions concerning the measure of noncompactness ϕ. Now we recall some results due to Nussbaum [20] which will be utilized in our reasoning process.

Lemma 2.2. *Let $\alpha_\zeta = \alpha_\zeta(X)$ denote the Kuratowski measure of noncompactness in the space $C_\zeta = C([0,\zeta], E)$. Then*

$$\max\left\{\frac{1}{2}\Omega_0^\zeta(X), \overline{\alpha}_\zeta(X)\right\} \leq \alpha_\zeta(X) \leq 2\Omega_0^\zeta(X) + \overline{\alpha}_\zeta(X), \tag{2.100}$$

where the quantity $\overline{\alpha}_\zeta$ was defined by (2.41)

In what follows let us notice that linking inequalities (2.100) and (1.1), we derive the estimates

$$\frac{1}{4}\left[\frac{1}{2}\Omega_0^\zeta(X) + \overline{\phi}_\zeta(X)\right] \leq \phi_\zeta(X) \leq 2[\Omega_0^\zeta(X) + \overline{\phi}_\zeta(X)] \tag{2.10.1}$$

for any $\zeta > 0$.

Then, we are ready to create our main result.

Theorem 2.2. Assume that ϕ is the Hausdorff measure of noncompactness in the Banach space E. Then the functions $\phi_\alpha(X), \phi_\beta(X)$ and $\phi_\gamma(X)$ defined by (2.8)-(2.10) are measures of noncompactness in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ such that

$$\phi(X) \leq 2\phi_\beta(X), \tag{2.11}$$

$$\phi(X) \leq 4\phi_\gamma(X), \tag{2.12}$$

$$\phi_\beta(X) \leq 2\phi_\alpha(X), \quad \phi_\gamma(X) \leq 2\phi_\alpha(X) \tag{2.13}$$

for an arbitrary set $X \in M_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$.

Proof: We first prove inequality (2.11). To this end, fix a set $X \in M_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$. From definition (2.2.1) and (2.2.2), we have

$$\Omega_0^\zeta(X) \leq \Omega_0(X), \tag{2.14}$$

$$\overline{\phi}_\zeta(X) \leq \overline{\phi}_\infty(X) \tag{2.15}$$

for a fixed $\zeta > 0$. On the other hand, taking an arbitrary fixed number $\epsilon > 0$ and using (2.6),we find a number $\zeta_0 > 0$ such that for any arbitrary $\zeta \geq \zeta_0$, we have

$$\beta_\zeta(X) \leq \beta_\infty(X) + \epsilon. \tag{2.16}$$

Using (2.16) and the definition of β_ζ, we infer that

$$\sup\left\{\|x(w,s) - x(u,v)\|_E : w,s,u,v \geq \zeta_0\right\} \leq \beta_\infty(X) + \epsilon \tag{2.17}$$

for an arbitrary function $x \in X$.
Let us fix an arbitrary number $\zeta, \zeta \geq \zeta_0$. Then keeping in mind estimate (2.10.1) and inequalities (2.14) and (2.15), we obtain the following innequality:

$$\phi_\zeta(X) \leq 2\Omega_0(X) + \overline{\phi}_\infty(X).$$

Hence we infer that, for an arbitrary fixed number $\delta > 0$, we can find $(2\Omega_0(X) + \overline{\phi}_\infty(X) + \delta)$-net $\overline{x_1}, \overline{x_2}, ..., \overline{x_m}$ of the set X in the space $C([0,\zeta], E)$. This means that for an arbitrary function $x \in X$ there exists $l \in \{1, 2, ..., m\}$ such that

$$\|x(w,s) - \overline{x}_l(w,s)\|_E \leq 2\Omega_0(X) + \overline{\phi}_\infty(X) + \delta \tag{2.18}$$

for $w, s \in [o, \zeta]$.

Now, consider the extension x_l of the function $\overline{x_l}(l = 1, 2, ..., m)$ on the interval $\mathbb{R}_+ \times \mathbb{R}_+$ defined in the following way:

$$x_l(w,s) = \begin{cases} \overline{x_l}(w,s) & \text{for } w,s \in [0,\zeta], \\ \\ \overline{x_l}(\zeta) & \text{for } w,s > \zeta. \end{cases} \tag{2.19}$$

Obviously, we have $x_l \in BC(\mathbb{R}_+ \times \mathbb{R}_+, E)(l = 1, 2, .., m)$. Further, using (2.17) and (2.18), for an arbitrary $w, s \geq \zeta$ we get

$$\begin{aligned} \|x(w,s) - x_l(w,s)\|_E &\leq \|x(w,s) - x(\zeta)\|_E + \|x(\zeta) - x_l(w,s)\|_E \\ &\leq \beta_\infty(X) + \epsilon + \|x(\zeta) - \overline{x_l}(\zeta)\|_E \leq \beta_\infty(X) + \epsilon + 2\Omega_0(X) + \overline{\phi}_\infty(X) + \delta \\ &\leq 2\Omega_0(X) + 2\overline{\phi}_\infty(X) + 2\beta_\infty(X) + \epsilon + \delta. \end{aligned}$$

From the above estimate, it follows that the functions $x_1, x_2, ..., x_m$ form a finite $\Big(2\Omega_0(X) + 2\overline{\phi}_\infty(X) + 2\beta_\infty(X) + \epsilon + \delta\Big)$-net of the set X in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$. Consiquently, we have

$$\phi(X) \leq 2\Omega_0(X) + 2\overline{\phi}_\infty(X) + 2\beta_\infty(X) + \epsilon + \delta.$$

Since , ϵ and δ were choosen arbitrary, we obtain

$$\phi(X) \leq 2\phi_\beta(X).$$

This proves inequality (2.11).

In order to prove (2.12), take an arbitrary $\epsilon > 0$. Then, we can find a number $\zeta_0 > 0$ such that for $w, s \geq \zeta_0$ the following inequality is satisfied:

$$\operatorname{diam} X(w, s) \leq e(X) + \epsilon. \tag{2.20}$$

Furthermore, arguing in the same way as previously, we deduce that, for an arbitrary fixed number $\zeta > \zeta_0$, the set X considered in the space $C([0, \zeta], E)$, that is, the set

$$\overline{X}_\zeta = \Big\{x|_{[0,\zeta]} : x \in X\Big\},$$

has, for an arbitrary $\delta > 0$, a finite $(2\Omega_0(X) + \overline{\phi}_\infty(X) + \delta)$-net composed by functions $\overline{x_1}, \overline{x_2}, ..., \overline{x_m}$ belonging to the space $C([0, \zeta], E)$.

Now, let us choose arbitrary functions $z_1, z_2, ..., z_m \in X$ such that, for any $i \in \{1, 2, ..., m\}$, the inequality

$$||z_i(w, s) - \overline{x_i}(w, s)||_E \leq 2\Omega_0(X) + \overline{\phi}_\infty(X) + \delta \tag{2.21}$$

is satisfied for $w, s \in [0, \zeta]$.
Further, taking an arbitrary function $x \in X$, we can find $i \in \{1, 2, .., m\}$ such that

$$||x(w, s) - \overline{x_i}(w, s)||_E \leq 2\Omega_0(X) + \overline{\phi}_\infty(X) + \delta \tag{2.22}$$

for an arbitrary $w, s \in [0.\zeta]$. Next taking (2.21) and (2.22), we get

$$\begin{aligned} ||x(w, s) - z_i(w, s)||_E &\leq ||x(w, s) - \overline{x_i}(w, s)||_E + ||\overline{x_i}(w, s) + z_i(w, s)||_E \\ &\leq 2(\Omega_0(X) + \overline{\phi}_\infty(X)) + 2\delta \end{aligned} \tag{2.222}$$

for an arbitrary $w, s \in [0.\zeta]$.

Now, combining (2.20) and (2.222) for an arbitrary number $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$, we obtain

$$\begin{aligned} ||x(w, s) - z_i(w, s)||_E &\leq \max\Big\{2(2\Omega_0(X) + \overline{\phi}_\infty(X)) + 2\delta, e(X) + \epsilon\Big\} \\ &\leq 4\Omega_0(X) + 2\overline{\phi}_\infty(X) + e(X) + \epsilon + 2\delta. \end{aligned}$$

From the above estimate, we deduce that the functions $z_1, z_2, ..., z_m$ form a finite $(4\Omega_0(X) + 2\overline{\phi}_\infty(X) + e(X) + \epsilon + 2\delta)$-net of the set X in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$. Thus, we have

$$\phi(X) \leq 4\phi_\gamma(X) + \epsilon + 2\delta.$$

Hence, taking into account the arbitrariness of the numbers ϵ and δ, we derive the inequality (2.12).

It is simple to verify that $\beta_\infty(X) \leq 2\alpha_\infty(X)$ and $e(X) \leq 2\alpha_\infty(X)$ for an arbitrary set $X \in M_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$.

Next, consider the kernals of the functions ker ϕ_α, ker ϕ_β and ker ϕ_γ which are represented by the families ϕ_α, ϕ_β and ϕ_γ, respectively. It is worth noting that the family ker ϕ_α is nonempty because it contains the set consisting of the function equivalent to θ on $\mathbb{R}_+ \times \mathbb{R}_+$. We can infer the inclusions ker $\phi_\alpha \subset$ ker ϕ_β and ker $\phi_\alpha \subset$ ker ϕ_γ from the inequalities mentioned before. This demonstrates that the families ker ϕ_β and ker ϕ_γ are both nonempty.

Further, fix arbitrary $\zeta > 0$ and consider the quantity $\phi_{\alpha,\zeta}$ on the space $C_\zeta = C([0,\zeta], E)$ defined for M_{C_ζ} by the formula

$$\phi_{\alpha,\zeta}(X) = \Omega_0^\zeta(X) + \overline{\phi}_\zeta(X).$$

Obviously in the space C_ζ, $\phi_{\alpha,\zeta}$ is a measure of noncompactness. This means that ϕ_α meets the requirements (1)-(6) of definition (1.1) on the family $M_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$.
Similarly we can show that the quantities ϕ_β and ϕ_γ also satisfy the conditions (2)-(6) of definition (1.1).

Now, we prove that ϕ_α satisfies the condition (1) of definition (2.1). To this end assume that $\phi_\alpha(X) = 0$ i.e., assume that $X \in \ker\phi_\alpha$. then in view of (2.8) we have that $\Omega_0^\infty(X) = 0$ and $\overline{\phi}_\infty(X) = 0$ and $\alpha_\infty(X) = 0$. Therefore, for each $\epsilon > 0$ there is $\zeta > 0$ such that $\alpha_\zeta(X) < \epsilon$.

In view of compactness of the set $X|_{[0,\zeta]}$ in the space C_ζ we deduce that there is a finite set $T \subset X$ such that the restriction $T|_{[0,\zeta]}$ is an ϵ-net of the set $X|_{[0,\zeta]}$. Hence we conclude that T is a 2ϵ-net of X in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$. But this implies that X is relatively compact and we have that ker $\phi_\alpha \subset N_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$.
This shows that (1) is true . In the same way, we can show ϕ_β and ϕ_γ satisfy condition (1).

In what follows we show that ϕ_α, ϕ_β and ϕ_γ satisfy axiom (6) of definition (1.1).
Note that in view of Example (2.1), axiom cannot be expressed in the same way as axioms (2)-(5). This is due to the fact that the equality

$$\Omega^\infty(X,\epsilon) = \sup\{\Omega^\zeta(X,\epsilon) : \zeta \geq 0\}$$

is not true, in general, for $\epsilon > 0$.
Obviously, the equality

$$\Omega^\infty(X,\epsilon) = \sup\{\Omega^\zeta(X,\epsilon) : \zeta \geq 0\}$$

is also not true.

As an example, consider a sequence closed sets (X_n) from the family $M_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$ ssuch that $X_{n+1} \subset X_n$ for $n = 1, 2, ...$ and $\lim_{n\to\infty} \phi_\alpha(X_n) = 0$. As a result, in view of (2.8) we have

$$\lim_{n\to\infty} \Omega_0^\infty(X_n) = 0, \tag{2.23}$$

$$\lim_{n\to\infty} \overline{\phi}_\infty(X_n) = 0, \tag{2.24}$$

$$\lim_{n\to\infty} \alpha_\infty(X_n) = 0. \tag{2.25}$$

Using (2.3), we can also derive that the following inequality holds for any $k > 0$.

$$\Omega^\infty(X_{n+1}, k) \leq \Omega^\infty(X_n, k).$$

Now, let us pretend that (w_i, s_i) is a sequence of nonnegative real numbers dense in the interval $\mathbb{R}_+\times\mathbb{R}_+$. Next, consider the sequence of functions $x_n = x_n(w,s)$ for $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $x_n \in X_n$ for $n = 1, 2, ...$ Using the diagonal procedure, without loss of generality we may assume that the sequence (x_n) is pointwise convergent on the set of points of the sequence (w_i, s_i). Finally, let us define the function x_∞ on the set of points of the sequence (w_i, s_i) by putting

$$x_\infty = \lim_{n\to\infty} x_n(w_i, s_i)$$

for each $i = 1, 2, ...$ We show that the function x_∞ is uniformly continuous on the set of points of the sequence (w_i, s_i).

To this end let us observe that for arbitrary fixed indices i, j and for arbitrary natural number n we obtain

$$\begin{aligned}\|x_\infty(w_i, s_i) - x_\infty(w_j, s_j)\|_E &\leq \|x_\infty(w_i, s_i) - x_n(w_i, s_i)\|_E + \|x_n(w_i, s_i) - x_n(w_j, s_j)\|_E \\ &+ \|x_n(w_j, s_j) - x_\infty(w_j, s_j)\|_E \\ &\leq \|x_\infty(w_i, s_i) - x_n(w_i, s_i)\|_E + \Omega^\infty(X_n, |w_i - w_j|, |s_i - sj|) \\ &+ \|x_n(w_j, s_j) - x_\infty(w_j, s_j)\|_E.\end{aligned}$$

Hence, letting $n \to \infty$ we get

$$\|x_\infty(w_i, s_i) - x_\infty(w_j, s_j)\|_E \leq \lim_{n\to\infty} \Omega^\infty(X_n, |w_i - w_j|, |s_i - sj|). \tag{2.26}$$

From the above estimate and (2.23) it follows that the function x_∞ is uniformly continuous on the points of sequence (w_i, s_i) .

Now, applying a theorem on the extension of functions , we deduce that the function x_∞ can be extended uniquely to a function being uniformly continuous on $\mathbb{R}_+ \times \mathbb{R}_+$. Obviously, from (2.26), we get

$$\|x_\infty(w, s) - x_\infty(u, v)\|_E \leq \lim_{n\to\infty} \Omega^\infty(X_n, |w - u|, |s - v|) \tag{2.27}$$

for arbitrary $w, s, u, v \in \mathbb{R}_+ \times \mathbb{R}_+$.

The function x_∞ is then shown to be the uniform limit of the function sequence (x_n). let us fix arbitrarily a number $\epsilon > 0$ and choosing $\delta > 0$ such that

$$\lim_{n\to\infty} \Omega^\infty(X_n, k) \leq \frac{\epsilon}{2} \tag{2.28}$$

for any number k such that $0 < k \leq \delta$.

Indeed, to demonstrate the above mentioned fact, we can deduce fromm equality (2.23) that for a fixed $\epsilon > 0$ we can find a natural number n_0 such that

$$\Omega^\infty(X_n) \leq \frac{\epsilon}{4}$$

for $n \geq n_0$. As a result of of (2.3), we may deduce that there exists a number $\delta > 0$ such that

$$\Omega^\infty(X_n, k) \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$$

for each k such that $0 < k \leq \delta$ and for $n \geq n_0$.

From this fact we infer inequality (2.28) for k such that $0 < k \leq \delta$.

Now, let us choose (w_j, s_j) such that $\|(w,s)-(w_j,s_j)\| < k$. Then, we have

$$\begin{aligned}\|x_\infty(w,s)-x_n(w,s)\|_E &\le \|x_\infty(w,s)-x_\infty(w_i,s_i)\|_E + \|x_\infty(w_j,s_j)-x_n(w_j,s_j)\|_E \\ &\quad + \|x_n(w_j,s_j)-x_n(w,s)\|_E.\end{aligned}$$

Hence, in view of (2.27) and (2.28) we obtain

$$\|x_\infty(w,s)-x_n(w,s)\|_E \le \lim_{n\to\infty}\Omega^\infty(X_n,k) + \|x_\infty(w_j,s_j)-x_n(w_j,s_j)\|_E + \Omega^\infty(X_n,k).$$

From the above estimate, we get

$$\lim_{n\to\infty}\|x_n(w,s)-x_\infty(w,s)\|_E \le \epsilon$$

for all $w,s \in \mathbb{R}_+\times\mathbb{R}_+$. Hence we derive that

$$\lim_{n\to\infty}\|x_n - x_\infty\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)} = 0. \tag{2.29}$$

As indicated by the above equality, the function x_∞ is uniform limit of the function sequence (x_n) on the interval $\mathbb{R}_+\times\mathbb{R}_+$. Particularly, from (2.29) we conclude that x_∞ is a cluster point of all sets $X_n (n = 1,2,...)$. As a result, we infer that $x_\infty \in X_n$ for $n = 1,2,....$ Thus $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty intersection.

Finally, we deduce that the function ϕ_α satisfies axiom (6) of Definition 1.1 by linking the obtained conclusion with equalities (2.24) and (2.25).

Similarly, we can show that functions ϕ_β and ϕ_γ satisfy axiom (6) of Definition 1.1

Thus, functions ϕ_α, ϕ_β and ϕ_γ are measures of noncompactness in the space $BC(\mathbb{R}_+\times\mathbb{R}_+,E)$. This completes the proof.

We will now look at the kernels of the measures of noncompactness ϕ_α, ϕ_β and ϕ_γ which are defined by formulas (2.8), (2.9) and (2.10), respectively.

It is worth mentioning that the kernel ker ϕ_α of the measure ϕ_α is made up of all bounded subsets X of the space $BC(\mathbb{R}_+\times\mathbb{R}_+,E)$ such that the functions from X are uniformly continuous and equicontinuous on $\mathbb{R}_+\times\mathbb{R}_+$ and with the same rate, it tends to zero at infinity. Furthermore, all cross sections $X(w,s)$ of the set X are relatively compact in Banach space E. Similarly, the kernel ker ϕ_β of measure ϕ_β defined by (2.9) consists of all X of the space $BC(\mathbb{R}_+\times\mathbb{R}_+,E)$ such that the functions from X are uniformly continuous and equicontinuous on $\mathbb{R}_+\times\mathbb{R}_+$ and in Banach space E, all cross sections $X(w,s)$ of the set X are relatively compact. Furthermore, all functions from X tend to limits uniformly with respect to the set X.

Finally, to describe the kernel ker ϕ_γ of measure of noncompactness ϕ_γ defined by (2.10), note that it contains all bounded subsets X of $BC(\mathbb{R}_+\times\mathbb{R}_+,E)$ which are locally continuous on $\mathbb{R}_+\times\mathbb{R}_+$ and such that the cross section $X(w,s)$ of X are relatively compact in E for any $w,s \in \mathbb{R}_+\times\mathbb{R}_+$. Apart from this, at infinity the thickness of the bundle formed by graphs of functions from X tends to zero.

Also, note that measures of noncompactess ϕ_α, ϕ_β and ϕ_γ defined by formulas (2.8)-(2.10) are not complete. That is to say, the kernels ker ϕ_α, ker ϕ_β and ker ϕ_γ are proper subfamilies of the family $N_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$. let us fix a nonzero vector $x_0 \in E$. In the space $BC(\mathbb{R}_+\times\mathbb{R}_+,E)$ consider the functions $x = x(w,s), y = y(w,s)$ defined as follows:

$$x(w,s) = x_0\sin(w,s), \quad y(w,s) = y_0\cos(w,s)$$

for $w,s \in \mathbb{R}_+\times\mathbb{R}_+$. Take the set $X = \{x,y\}$. Obviously X is a compact subset of the space $BC(\mathbb{R}_+\times\mathbb{R}_+,E)$ since it is finite. Moreover, it is easy to check that $\Omega_0^\infty(X) = 0$ and $\overline{\phi}_\infty(X) = 0$, where the quantities Ω^∞ and $\overline{\phi}_\infty$ are defined by (2.3) and (2.4), respectively.

On the other hand, when the values $\alpha_\infty, \beta_\infty$ and e defined consecutively by formulas(2.5), (2.6) and (2.7), it clear that

$$\alpha_\infty(X) = \|x_0\|_E, \ \ \beta_\infty(X) = 2\|x_0\|_E, \ \ e(X) = \sqrt{2}\|x_0\|_E.$$

Thus the set X does not belong to the families ker ϕ_α, ker ϕ_β and ker ϕ_γ.

Taking into mind our subsequent applications of the measures of noncompactness ϕ_α, ϕ_β and ϕ_γ to the theory of infinite system of integral equations, we shall consider as the Banach space E the sequence space ℓ_∞ containing of all sequences (x_p) being bounded. We limit ourselves to the study of real sequences. Obviously, the space ℓ_∞ will be endowed with the classical supremum norm

$$\|x\| = \|(x_p)\| = \sup\left\{|x_p| : p = 1, 2, ...\right\},$$

where $x = (x_p) \in \ell_\infty$.

Consider the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ consisting of functions $x : \mathbb{R}_+ \times \mathbb{R}_+ \to \ell_\infty$ which are continuous and bounded on $\mathbb{R}_+ \times \mathbb{R}_+$. Obviously, such a function can be written in the form

$$x(w, s) = (x_p(w, s)) = (x_1(w, s), x_2(w, s), ...)$$

for any $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$, where the sequence $(x_p(w, s))$ is an element of the space ℓ_∞ for fixed (w, s). The norm of the function $x = x(w, s) = (x_n(w, s))$ is defined by the equality

$$\|x\| = \sup\left\{\|x(w, s)\|_{\ell_\infty} : w, s \in \mathbb{R}_+ \times \mathbb{R}_+\right\} = \sup_{w,s\in\mathbb{R}_+\times\mathbb{R}_+}\left\{\sup\{|x_p(w, s)| : p = 1, 2, ...\}\right\}.$$

We then provide formulas that expresses measures of noncompactness ϕ_α, ϕ_β and ϕ_γ in connection with measures of noncompactness in the space ℓ_∞.

At the beginning let us fix a set $X \in M_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$. For $\epsilon > 0$ and for an arbitrary function $x(w, s) = (x_n(w, s))$ belonging to the set X consider the modulus $\Omega^\infty(x, \epsilon)$ defined before, which is now stated in the following form

$$\Omega^\infty(x, \epsilon) = \sup\{\|x(w, s) - x(u, v)\|\|_{\ell_\infty} : w, s, u, v \in \mathbb{R}_+ \times \mathbb{R}_+, |w - u| \le \epsilon, |s - v| \le \epsilon\}$$

$$= \sup\Big\{\sup\{|x_p(w) - x_p(u)|, |x_p(s) - x_p(v)| : p = 1, 2, ...\} : w, s, u, v \in \mathbb{R}_+ \times \mathbb{R}_+, |w - u| \le \epsilon, |s - v| \le \epsilon\Big\}.$$

Then, using the aforementioned formula and (2.3), we get

$$\Omega^\infty(X, \epsilon) = \sup_{x\in X}\Big\{\sup\Big\{\sup_{p\in\mathbb{N}}\{|x_p(w) - x_p(u)|, |x_p(s) - x_p(v)| : p = 1, 2, ...\} : w, s, u, v \in \mathbb{R}_+ \times \mathbb{R}_+, |w - u| \le \epsilon, |s - v| \le \epsilon\}\Big\}.$$

In the end, we put

$$\begin{aligned}\Omega_0^\infty(X) &= \lim_{\epsilon\to 0}\Omega^\infty(X, \epsilon)\\ &= \lim_{\epsilon\to 0}\Big\{\sup_{x\in X}\Big\{\sup\Big\{\sup_{p\in\mathbb{N}}\{|x_p(w) - x_p(u)|, |x_p(s) - x_p(v)| : p = 1, 2, ...\} : w, s, u, v \in \mathbb{R}_+ \times \mathbb{R}_+,\\ &\qquad |w - u| \le \epsilon, |s - v| \le \epsilon\}\Big\}\Big\}. \qquad (2.30)\end{aligned}$$

To define the second term $\overline{\phi}_\infty$ of the measures ϕ_α, ϕ_β and ϕ_γ given by formulas (2.8)-(2.10), we will assume that in the space ℓ_∞, we take into account the measures of noncompactness ϕ^1, ϕ^2 and ϕ^3 defined on the family M_{ℓ_∞} as follows:

$$\phi^1(X) = \lim_{p\to\infty}\left\{\sup_{x=x_i\in X}\left\{\sup\{|x_l| : l \geq p\}\right\}\right\},$$

$$\phi^2(X) = \lim_{n\to\infty}\left\{\sup_{x=x_i\in X}\left\{\sup\{|x_p - x_q| : p, q \geq n\}\right\}\right\},$$

$$\phi^3(X) = \lim_{p\to\infty} \sup \operatorname{diam} X_p$$

where

$$X_p = \{x_p : x = (x_i) \in X\}$$

and

$$\operatorname{diam} X_p = \sup\{|x_p - y_p| : x = (x_i), y = (y_i) \in X\}.$$

We can now define the terms $\phi_\infty^{-i} (i = 1, 2, 3)$ related with these formulas based on the above mentioned formulas. Namely $X \in M_{BC(\mathbb{R}_+\times\mathbb{R}_+,E)}$ and for a fixed $\zeta > 0$ we put :

$$\phi_\zeta^{-1}(X) = \sup\{\phi^1(X(w, s)) : w, s \in [0.\zeta]\}$$

$$= \sup_{w,s\in[0,\zeta]}\left\{\lim_{p\to\infty}\left\{\sup_{x=x_i\in X}\left\{\sup\{|x_l(w, s)| : l \geq p\}\right\}\right\}\right\}, \tag{2.31}$$

$$\phi_\zeta^{-2}(X) = \sup\{\phi^2(X(w, s)) : w, s \in [0.\zeta]\}$$

$$= \sup_{w,s\in[0,\zeta]}\left\{\lim_{n\to\infty}\left\{\sup_{x=x_i\in X}\left\{\sup\{|x_p(w, s) - x_q(w, s)| : p, q \geq n\}\right\}\right\}\right\}, \tag{2.32}$$

$$\phi_\zeta^{-3}(X) = \sup\{\phi^3(X(w, s)) : w, s \in [0.\zeta]\}$$

$$= \sup_{w,s\in[0,\zeta]}\left\{\lim_{n\to\infty} \sup\left\{\sup\{|x_p - y_p| : x = x(w, s), y = y(w, s) \in X\}\right\}\right\}. \tag{2.33}$$

As a result, we arrive at the following formulas:

$$\phi_\infty^{-1}(X) = \lim_{\zeta\to\infty} \phi_\zeta^{-1}(X)$$

$$= \lim_{\zeta\to\infty}\left\{\sup_{w,s\in[0,\zeta]}\left\{\lim_{p\to\infty}\left\{\sup_{x=x_i\in X}\left\{\sup\{|x_l(w, s)| : l \geq p\}\right\}\right\}\right\}\right\}, \tag{2.34}$$

$$\phi_\infty^{-2}(X) = \lim_{\zeta\to\infty} \phi_\zeta^{-2}(X)$$

$$= \lim_{\zeta\to\infty}\left\{ \sup_{w,s\in[0,\zeta]} \left\{ \lim_{n\to\infty}\left\{ \sup_{x=x_i\in X} \left\{ \sup\{|x_p(w,s) - x_q(w,s)| : p,q \geq n\}\right\}\right\}\right\}\right\}, \tag{2.35}$$

$$\phi_\infty^{-3}(X) = \lim_{\zeta\to\infty} \phi_\zeta^{-3}(X)$$

$$= \lim_{\zeta\to\infty}\left\{ \sup_{w,s\in[0,\zeta]} \left\{ \lim_{n\to\infty} \sup\left\{ \sup\{|x_p - y_p| : x = x(w,s), y = y(w,s) \in X\}\right\}\right\}\right\}. \tag{2.36}$$

Now, we define Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ as the third term of the constructed measures of noncompactness. let us observe that based on formulas (2.5), (2.6) and (2.7), we get:

$$\alpha_\infty(X) = \lim_{\zeta\to\infty} \alpha_\zeta(X)$$

$$= \lim_{\zeta\to\infty}\left\{ \sup_{x=x(w,s)\in X} \left\{ \sup\{\sup_{p\in\mathbb{N}} |x_p(w,s)| : w,s \geq \zeta\}\right\}\right\}, \tag{2.37}$$

$$\beta_\infty(X) = \lim_{\zeta\to\infty} \beta_\zeta(X)$$

$$= \lim_{\zeta\to\infty}\left\{ \sup_{x=x_i\in X} \left\{ \sup\{\sup_{p\in\mathbb{N}} |x_p(w,s) - x_p(u,v)| : w,s,u,v \geq \zeta\}\right\}\right\}, \tag{2.38}$$

$$e(X) = \lim_{w,s\to\infty} \sup \operatorname{diam} X(w,s)$$

$$= \lim_{w,s\to\infty}\left\{ \sup\left\{ \sup\{\sup_{p\in\mathbb{N}} |x_p(w,s) - y_p(w,s)| : x = x(w,s), y = y(w,s) \in X\}\right\}\right\}. \tag{2.39}$$

Finally,we can present nine formulas expressing suitable measures of noncompactness in the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ by remembering formulas (2.8)-(2.10) expressing measures of noncompactness in the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ and taking into account the above obtainned formulas (2.30)-(2.39). As a result, we have:

$$\phi_\alpha^i(X) = \Omega_0^\infty(X) + \phi_\infty^{-i}(X) + \alpha_\infty(X) \tag{2.40}$$

for $i = 1,2,3$. Similarly, we obtain

$$\phi_\beta^i(X) = \Omega_0^\infty(X) + \phi_\infty^{-i}(X) + \beta_\infty(X) \tag{2.41}$$

for $i = 1,2,3$. Finally, we can define the measures of noncompactness related to the term $e = e(X)$, by putting

$$\phi_e^i(X) = \Omega_0^\infty(X) + \phi_\infty^{-i}(X) + e(X) \tag{2.42}$$

for $i = 1,2,3$.

In order to accomplish this, we prove the following lemma.

Lemma 2.3. The following equality is satisfied

$$\overline{\phi}_\infty(X) = \sup\{\phi(X(w,s)) : w, s \in \mathbb{R}_+ \times \mathbb{R}_+\},$$

where $\overline{\phi}_\infty$ is defined by formula (2.4).

Proof. Obviously, for any $\zeta > 0$ we have

$$\sup\{\phi(X(w,s)) : w, s \in [0,\zeta]\} \leq \sup\{\phi(X(w,s)) : w, s \in \mathbb{R}_+ \times \mathbb{R}_+\}.$$

Hence, we get

$$\overline{\phi}_\infty(X) = \lim_{\zeta\to\infty}\Big\{\sup\{\phi(X(w,s)) : w, s \in [0,\zeta]\}\Big\} \leq \sup\{\phi(X(w,s)) : w, s \in \mathbb{R}_+ \times \mathbb{R}_+\}. \quad (2.43)$$

To prove the converse inequality, let us denote

$$\delta = \sup\{\phi(X(w,s)) : w, s \in \mathbb{R}_+ \times \mathbb{R}_+\}.$$

Further, fix an arbitrary number $\epsilon > 0$. Then we can find $w_0, s_0 \in \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$\delta - \epsilon \leq \phi(X(w_0, s_0)).$$

Hence, for $\zeta \geq w_0, s_0$ we obtain

$$\delta - \epsilon \leq \sup\{\phi(X(w,s)) : w, s \in [0,\zeta]\}. \quad (2.44)$$

Since the function $\zeta \to \sup\{\phi(X(w,s)) : w, s \in [0,\zeta]\}$ is nondecreasing, we get

$$\sup\{\phi(X(w,s)) : w, s \in [0,\zeta]\} \leq \lim_{\zeta\to\infty}\Big\{\sup\{\phi(X(w,s)) : w, s \in [0,\zeta]\}\Big\}. \quad (2.45)$$

Combining (2.44) and (2.45), we have

$$\delta - \epsilon \leq \lim_{\zeta\to\infty}\Big\{\sup\{\phi(X(w,s)) : w, s \in [0,\zeta]\}\Big\}. \quad (2.46)$$

Consequently, in view of the arbitratiness of the number ϵ, we derive the following inequality

$$\delta \leq \lim_{\zeta\to\infty}\Big\{\sup\{\phi(X(w,s)) : w, s \in [0,\zeta]\}\Big\} = \overline{\phi}_\infty(X). \quad (2.47)$$

Finally, linking (2.43) and (2.47) we obtain the desired equality.

Now, let us notice that taking into account Lemma (2.3) and formula (2.34) expressing the quantity $\overline{\phi}_\infty$ in the case of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$, we obtain the following corollary.

Corollary 2.4. The quantity (2.34) can be expressed by the formula

$$\phi_\infty^{-1}(X) = \sup_{w,s\geq 0}\Bigg\{\lim_{p\to\infty}\Bigg\{\sup_{x=x_i\in X}\Big\{\sup\{|x_l(w,s)| : l \geq p\}\Big\}\Bigg\}\Bigg\}.$$

We recall a useful fixed point theorem of Darbo type [8, 13] at the end of this section.
Let us assume that E is a Banach space and ϕ is a measure of noncompactness (as defined in Definition 2.1) in the space E.

Theorem 2.5. Assume that Q is a nonempty, bounded, closed and convex subset of a Banach space E and $T : Q \to Q$ is a continuous operator such that there exists a constant $k \in [0.1)$ for which $\phi(T(X)) \leq k\phi(X)$ for an arbitrary nonempty subset X of Q. Then there exists atleast one fixed point of the operator T in the set Q.

Remark 2.6. It can be shown that the set Fix T of all fixed points of the operator T belongs to the family ker ϕ.

3. Existence of solutions of infinite systems of integral equations on the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$.

We will look at the infinite system of Volterra-Hammerstein type nonlinear quadratic integral equations with the form

$$\begin{aligned} x_p(w,s) &= \alpha_p(w,s) + f_p(w,s,x_p(w,s),x_{p+1}(w,s),...) \\ &\times \int_0^w \int_0^s k_p(w,s,u,v) g_p(u,v,x_1(u,v),x_2(u,v),...)\,dudv \end{aligned} \tag{3.1}$$

for $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ and for $p = 1,2,....$

Our considerations concerning the solvability of the infinite system of integral equations (3.1) will proceed by a lemma which will be used in our later arguments.

Lemma 3.1. Let the function $x(w,s) = (x_p(w,s))$ be an element of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$. Then the space (x_p) is equibounded and locally conxvex on $\mathbb{R}_+ \times \mathbb{R}_+$.

Proof. First, let us note that the function $x = x(w,s)$ acts continuously from $\mathbb{R}_+ \times \mathbb{R}_+$ into ℓ_∞. Hence, we deduce that, for each $\zeta > 0$, the function $x(w,s)$ is uniformy continuous on the interval $[0,\zeta]$. Thus for a given $\epsilon > 0$, we choose a $\delta > 0$ such that $\|(w_2,s_2) - (w_1,s_1)\| \le \delta$ for $w_1,w_2,s_1,s_2 \in [0,\zeta]$ implies that

$$\|x(w_2,s_2) - (w_1.s_1)\|_{\ell_\infty} = \sup\{|x_p(w_2) - x_p(w_1)|, |x_p(s_2) - x_p(s_1)| : p = 1,2,...\} \le \epsilon.$$

This means that $|x_p(w_2) - x_p(w_1)| \le \epsilon, |x_p(s_2) - x_p(s_1)| \le \epsilon$ for $p = 1,2,....$

Summing up, we conclude that for any $\epsilon > 0$ there exists $\delta > 0$ such that, for arbitrary $w_1,w_2,s_1,s_2 \in [0,\zeta]$ such that $\|(w_2,s_2)-(w_1,s_1)\| \le \delta$ and for each $p = 1,2,...,$ we have $|x_p(w_2)-x_p(w_1)| \le \epsilon, |x_p(s_2)-x_p(s_1)| \le \epsilon$. Thus, the function sequence (x_p) is equicontinuous om the interval $[0,\zeta]$. Hence it follows that the mentioned function sequence (x_p) is locally equicontinuous on $\mathbb{R}_+ \times \mathbb{R}_+$.

On the other hand the function $x = x(w,s)$ is bounded on $\mathbb{R}_+ \times \mathbb{R}_+$ implies that there exists a constant $M > 0$ such that $\|x(w,s)\|_{\ell_\infty} \le M$ for $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$. Thus, we obtain the desired equiboundedness of the sequence (x_p) on the interval $\mathbb{R}_+ \times \mathbb{R}_+$.

Now we will look at the assumptions that will be used to study the infinite system of integral equations (3.1).

(i) The sequence $(\alpha_p(w,s))$ is an element of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$. Moreover, the functions $\alpha_p = a_p(w,s)$ are equicontinuous on $\mathbb{R}_+ \times \mathbb{R}_+$.

(ii) The functions $k_p(w,s,u,v) = k_p : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are continuous on the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ (p = 1,2,...)$. Apart from this the functions $w,s \to k_p(w,s,u,v)$ are equicontinuous on the set $\mathbb{R}_+ \times \mathbb{R}_+$ uniformly with respect to $u,v \in \mathbb{R}_+ \times \mathbb{R}_+$ i.e, the following condition is satisfied

$$\begin{aligned} \forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{p\in\mathbb{N}}\ \forall_{u,v\in\mathbb{R}_+\times\mathbb{R}_+}\ \forall_{w_1,w_2,s_1,s_2\in\mathbb{R}_+\times\mathbb{R}_+} \Big[&|w_2 - w_1| \le \delta, |s_2 - s_1| \le \delta \\ &\Longrightarrow |k_p(w_2,s_2,u,v) - k_p(w_1,s_1,u,v)| \le \epsilon \Big]. \end{aligned}$$

(iii) There exists a constant $G_1 > 0$ such that

$$\int_o^w \int_o^s |k_p(w,s,u,v)|\,dudv \le G_1$$

for any $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ and $p = 1,2,....$

(iv) The sequence $(k_p(w,s,u,v))$ is equibounded on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ i.e, there exists a constant $G_2 > 0$ such that $|k_p(w,s,u,v)| \le G_2$ for $w,s,u,v \in \mathbb{R}_+ \times \mathbb{R}_+$ and $p = 1,2,....$

(v) The functions f_p are defined on the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\infty$ and take real values for $p = 1,2,....$ Moreover, the functions $w,s \to f_p(w,s,x_1,x_2,...)$ are equicontinuous on $\mathbb{R}_+ \times \mathbb{R}_+$ uniformly with respect to $x = (x_p) \in \ell_\infty$ i.e., the following condition is satisfied

$$\forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{p\in\mathbb{N}}\ \forall_{u,v\in\mathbb{R}_+\times\mathbb{R}_+}\ \forall_{w_1,w_2,s_1,s_2\in\mathbb{R}_+\times\mathbb{R}_+}\Big[|w_2 - w_1| \le \delta, |s_2 - s_1| \le \delta \implies |f_p(w_2,s_2,x_1,x_2,...) - f_p(w_1,s_1,x_1,x_2,...)| \le \epsilon\Big].$$

(vi) There exists a function $l : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}_+$ such that l is nondecreasing on $\mathbb{R}_+ \times \mathbb{R}_+$, $l(0) = 0, l$ is continuous at 0 and the following is satisfied

$$|f_p(w,s,x_1,x_2,...) - f_p(w,s,y_1,y_2,...)| \le l(r)\sup\Big\{|x_i - y_i| : i \ge p\Big\}$$

for any $r > 0$, for $x = (x_i), y = (y_i) \in \ell_\infty$ such that $\|x\|_{\ell_\infty} \le r, \|y\|_{\ell_\infty} \le r$ and for all $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ and $p = 1,2,....$

(vii) The sequence of functions $(\overline{f}_p)$ where $\overline{f}_p(w,s) = |f_p(w,s,0,0,0,...)|$ is an element of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$.

Assume that we can define the finite constant based on assumption (vii).

$$\overline{F} = \sup\{\overline{f}_p)(w,s) : w,s \in \mathbb{R}_+ \times \mathbb{R}_+, p = 1,2,...\}.$$

Now we formulate the final assumption about the infinite system (3.1).

(viii) The functions g_p are defined on the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\infty$ and take real values for $p = 1,2,....$ Moreover, there exists a function $m : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}_+$ on $\mathbb{R}_+ \times \mathbb{R}_+$, continuous at $r = 0, m(0) = 0$ and such that the following condition is satisfied

$$|g_p(w,s,x_1,x_2,...) - g_p(w,s,y_1,y_2,...)| \le m(r)\sup\{|x_i - y_i| : i \ge p\}$$

for any $r > 0$, for $x = (x_i), y = (y_i) \in \ell_\infty$ such that $\|x\|_{\ell_\infty} \le r, \|y\|_{\ell_\infty} \le r$ and for all $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ and $p = 1,2,....$

(ix) The operator g defined on the space $\mathbb{R}_+ \times \mathbb{R}_+ \times \ell_\infty$ by the formula

$$(gx)(w,s) = (g_p(w,s,x)) = (g_1(w,s,x), g_2(w,s,x), ...)$$

is bounded i.e., there exists a positive constant $\overline{g}$ such that $\|(gx)(w,s)\|_{\ell_\infty} \le \overline{g}$ for any $x \in \ell_\infty$ and for each $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$.

(x) There exists a positive constant $\overline{M}$ suc that for any $w,s \in \mathbb{R}_+ \times \mathbb{R}_+, n \in \mathbb{N}$ and for each $x = x(w,s) = (x_n(w,s)) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ the following inequality holds

$$\int_o^w \int_o^s |g_p(u,v,x(u,v)|\,dudv = \int_o^w \int_o^s |g_p(u,v,x_1(u,v),x_1(u,v),...|\,dudv \le \overline{M}.$$

(xi) There exists a positive solution r_0 of the inequality

$$A + \overline{F}\overline{g}G_1 + \overline{g}G_1 r l(r) \le r$$

such that

$$\overline{g}G_1 l(r_0) + (r_0 l(r_0) + \overline{F})G_1 m(r_0) < 1$$

where the constants $\overline{F}, \overline{g}, G_1$ were defined above and the constant A was defined in the following way

$$A = \sup\{|\alpha_p(w,s) : w,s \in \mathbb{R}_+ \times \mathbb{R}_+, p = 1,2,...\}.$$

Remark 3.2. Observe that from assumption (*vi*) we deduce that for any $r > 0$ and for $x = x_i, y = y_i \in \ell_\infty$ such that $\|x\|_{\ell_\infty} \leq r, \|y\|_{\ell_\infty} \leq r$ and for $w, s \in \mathbb{R}_+ \times \mathbb{R}_+, p \in \mathbb{N}$, the following inequality is satisfied

$$|f_p(w,s,x_1,x_2,...) - f_p(w,s,x_1,x_2,...)| \leq l(r)\|x-y\|_{\ell_\infty},$$

where $l = l(r)$ is the function is the function from assumption (*vi*).
Similarly, from assumption (*viii*) we infer that

$$|g_p(w,s,x_1,x_2,...) - g_p(w,s,y_1,y_2,...)| \leq m(r)\|x-y\|_{\ell_\infty}$$

for $w, s \in \mathbb{R}_+ \times \mathbb{R}_+, p \in \mathbb{N}$ and for $r > 0$, provided $x = x_i, y = y_i \in \ell_\infty$ such that $\|x\|_{\ell_\infty} \leq r, \|y\|_{\ell_\infty} \leq r$. The function $m = m(r)$ appears in assumption (*viii*).

Now we can express our existence result in terms of an infinite system (3.1).

Theorem 3.3. Under assumptions (*i*) − (*xi*) the infinite system of integral equations (3.1) has atleast one solution $x(w,s) = (x_p(w,s))$ in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$. Moreover, the function $x = x(w,s)$ is uniformly continuous on the interval $\mathbb{R}_+ \times \mathbb{R}_+$.

Proof. We start with defining three operators F, V, Q on the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ in the following way:

$$(Fx)(w,s) = ((F_px)(w,s)) = (f_p(w,s,x(w,s))) = (f_p(w,s,x_1(w,s),x_2(w,s),...)),$$

$$(Vx)(w,s) = ((V_px)(w,s)) = \Big(\int_o^w \int_o^s k_p(w,s,u,v)g_p(u,v,x_1(u,v),x_2(u,v),...)\,dudv\Big),$$

$$(Qx)(w,s) = ((Q_px)(w,s)) = (\alpha_p(w,s) + (F_px)(w,s)(V_px)(w,s)).$$

At the begining we show that the operator F transforms the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ into itself.

To this end let us choose a function $x = (x_n(w,s)) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$. Fix a number $n \in \mathbb{N}$ and take $w, s \in \mathbb{R}^\infty \times \mathbb{R}_+$. Then, in view of the imposed asssumptions and Remark (3.2), we obtain

$$\begin{aligned}|(F_px)(w,s)| &\leq |f_p(w,s,x_1(w,s),x_2(w,s),...) - f_p(w,s,0,0,...)| + |f_p(w,s,0,0,...)| \\ &\leq l(\|x(w,s)\|_{\ell_\infty}) \sup\{|xI(w,s) : i \geq p\} + |\overline{f}_n(w,s)| \\ &\leq l(\|x\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)})\|x\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} + \overline{F}. \end{aligned} \tag{3.2}$$

Next, we show that the function Fx is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$. In order to show this fact we will utilize the continuity of an arbitrary function

$$x = x(w,s) = (x_p(w,s)) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$$

on the interval $\mathbb{R}_+ \times \mathbb{R}_+$. This means that the following condition holds

$$\begin{aligned}\forall_{\epsilon>0}\, \exists_{\delta>0}\, \forall_{w_0,s_0 \in \mathbb{R}_+\times\mathbb{R}_+}\, \forall_{w,s,\in\mathbb{R}_+\times\mathbb{R}_+}\Big[&|w-w_0| \leq \delta, |s-s_0| \leq \delta \\ &\implies \|x(w,s) - x(w_0,s_0)\|\{\ell_\infty\} \leq \epsilon\Big].\end{aligned} \tag{3.3}$$

Further, fix $\epsilon > 0$ and $w_0, s_0 \in \mathbb{R}_+ \times \mathbb{R}_+$. Next, choose $\delta > 0$ according to condition (3.3). Then, for $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $\|(w,s) - (w_0,s_0)\| \leq \delta$, in view of remark (3.2), we obtain

$$\begin{aligned}|(F_px)(w,s) - (F_px)(w_0,s_0)| &\leq |f_q(w,s,x_1(w,s),x_2(w,s),...) - f_n(w_0,s_o,x_1(w,s),x_2(w,s),...)| \\ &\quad + l(\|x(w,s)\|_{\ell_\infty})\|x(w,s) - x(w_0,s_0)\|_{\ell_\infty} \\ &\leq |f_p(w,s,x_1(w,s),x_2(w,s),...) - f_p(w_0,s_0,x_1(w,s),x_2(w,s),...)| \\ &\quad + l(\|x\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)})\epsilon.\end{aligned} \tag{3.4}$$

Now, keeping assumption (v) in mind, we can select a number $\delta > 0$ in such that

$$|f_p(w,s,x_1(w,s),x_2(w,s),...) - f_p(w_0,s_0,x_1(w,s),x_2(w,s),...)| \leq \epsilon$$

for $\|(w,s)-(w_0,s_0)\| \leq \delta$ and for $n = 1,2,....$ We can get the following estimate by combining this fact with (3.4).

$$|(F_px)(w,s) - (F_px)(w_0,s_0)| \leq (1 + l(\|x\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)}))\epsilon$$

for $p = 1,2,...$ and for any $w,s \in \mathbb{R}_+\times\mathbb{R}_+$ such that $\|(w,s)-(w_0,s_0)\| \leq \delta$. This shows that the function Fx is continuous at point $w_0,s_0 \in \mathbb{R}_+\times\mathbb{R}_+$. Since w_0,s_0 was choosen arbitrary we conclude that the function Fx is continuous on $\mathbb{R}_+\times\mathbb{R}_+$. Joining the above deduced property of Fx with the earlier established boundedness of Fx we infer that the operator F transforms the space $BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)$ into itself.

We now are going to show that the above mentioned operator V transforms the space $BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)$ into itself. To this end, similarly as above, take a function $x = x(w,s) = (x_n(w,s)) \in BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)$. Then, for arbitrarily fixed numbers $w,s \in \mathbb{R}_+\times\mathbb{R}_+$ and $p \in \mathbb{N}$, based on assumptions (iii) and (ix), we get

$$|(V_px)(w,s)| \leq \int_o^w\int_o^s |k_p(w,s,u,v)||g_p(u,v,x_1(u,v),x_2(u,v),...)|\,dudv$$

$$\leq \int_o^w\int_o^s |k_p(w,s,u,v)|\overline{g}\,dudv \leq \overline{g}\int_o^w\int_o^s |k_p(w,s,u,v)|\,dudv \leq \overline{g}G_1. \tag{3.5}$$

The derived estimate, in particular, shows that the function Vx is bounded on the interval $\mathbb{R}_+\times\mathbb{R}_+$. Next, fix $\epsilon > 0$ and determine a number $\delta > 0$ according to assumption (ii). Then, for arbitrary $w_1,w_2,s_1,s_2 \in \mathbb{R}_+\times\mathbb{R}_+$ such that $\|(w_2,s_2)-(w_1,s_1)\| \leq \delta$, on the basis of assumptions (ii) and (ix)(assuming, for example, that $(w_1,s_1) < (w_2,s_2)$), we have

$$\begin{aligned}
&|(V_px)(w_2,s_2) - (V_px)(w_1,s_1)| \\
&\leq \left|\int_o^{w_2}\int_o^{s_2} k_p(w_2,s_2,u,v)g_p(u,v,x_1(u,v),x_2(u,v),...)\,dudv\right. \\
&\left.- \int_o^{w_2}\int_o^{s_2} k_p(w_1,s_1,u,v)g_p(u,v,x_1(u,v),x_2(u,v),...)\,dudv\right| \\
&+ \left|\int_o^{w_2}\int_o^{s_2} k_p(w_1,s_1,u,v)g_p(u,v,x_1(u,v),x_2(u,v),...)\,dudv\right. \\
&\left.- \int_o^{w_1}\int_o^{s_1} k_p(w_1,s_1,u,v)g_p(u,v,x_1(u,v),x_2(u,v),...)\,dudv\right| \\
&\leq \int_o^{w_2}\int_o^{s_2} |k_p(w_2,s_2,u,v) - k_p(w_1,s_1,u,v)||g_p(u,v,x_1(u,v),...)|\,dudv \\
&+ \int_{w_1}^{w_2}\int_{s_1}^{s_2} |k_p(w_1,s_1,u,v)||g_p(u,v,x_1(u,v),...)|\,dudv
\end{aligned}$$

$$\leq \Omega_k(\delta)|g_p(u,v,x_1(u,v),x_2(u,v)...)|\,dudv$$
$$+\int_{w_1}^{w_2}\int_{s_1}^{s_2} G_2|g_p(u,v,x_1(u,v),...)|\,dudv,$$

where G_2 is a constant from assumption (iv) and $\Omega_k(\delta)$ denotes a common modulus of equicontnuity of the sequence of functions $w,s \to k_p(w,s,u,v)$ (according to the assumption (iii)). Obviously we have $\Omega_k(\delta) \to 0$ as $\delta \to 0$.

Let us now notice that, using assumptions (ix) and (x), we can obtain the following estimate from the previous one.

$$|(V_px)(w_2,s_2) - (V_px)(w_1,s_1)| \leq \overline{M}\Omega_k(\delta) + \overline{g}G_2\delta. \tag{3.6}$$

Hence, we get

$$\|(Vx)(w_2,s_2) - (Vx)(w_1,s_1)\|_{\ell_\infty} \leq \overline{M}\Omega_k(\delta) + \overline{g}G_2\delta.$$

This shows that the function Vx is continuous on the interval $\mathbb{R}_+ \times \mathbb{R}_+$. We conclude that the operator V transforms the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ into itself by linking the boundedness of the function Vx with its continuity on $\mathbb{R}_+ \times \mathbb{R}_+$.

Taking into account the fact the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ is a Banach algebra in terms of coordinatewise multiplication of function sequences and keeping in mind the definition of the operator Q and assumption (i), we deduce that for an arbitrarily fixed function $x = x(w,s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ the function $(Qx)(w,s) = ((Q_px)(w,s)) = (\alpha_p(w,s) + (F_px)(w,s)(V_px)(w,s))$ transforms the interval $\mathbb{R}_+ \times \mathbb{R}_+$ into the space ℓ_∞.
Indeed, in virtue of the fact that $((F_px)(w,s)) \in \ell_\infty$ for any $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ and in the light of estimate (3.5), we get

$$|(Q_px)(w,s)| \leq |\alpha_p(w,s)| + \overline{g}G_1|(F_px)(w,s)|$$

for any $p \in \mathbb{N}$. In view of (3.2) this yields that $(Qx)(w,s) = ((Q_px)(w,s)) \in \ell_\infty$ for every $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$.

Next, let us notice that the continuity of the function Qx on $\mathbb{R}_+ \times \mathbb{R}_+$ follows easily from the continuity of the functions Fx and Vxv= on the interval $\mathbb{R}_+ \times \mathbb{R}_+$. Similarly, if we use assumption (i), we may infer the boundedness of the function Qx on $\mathbb{R}_+ \times \mathbb{R}_+$.
Finally, by combining all the above established properties of the function Qx we infer that the operator Q transforms the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ into itself.

Now, let us observe that in view of estimates (3.2) and (3.5), for an arbitrarily fixed $p \in \mathbb{N}$ and $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$, we have

$$\begin{aligned}|(Q_px)(w,s)| &\leq |\alpha_p(w,s)| + |(F_px)(w,s)||(V_px)(w,s)|\\ &\leq A + \Big[l(\|x(w,s\|_{\ell_\infty})\|x(w,s\|_{\ell_\infty}) + \overline{F}\Big]\overline{g}G_1\end{aligned}$$

As a result, we arrive at the following estimate:

$$\|Qx\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leq A + \overline{F}\overline{g}G_1 + \overline{g}G_1 l(\|x\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)})\|x\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)}.'$$

Based on the aforementioned estimate and assumption (xi) we conclude that there exists a number $r_0 > 0$ such that the operator Q transforms the ball B_{r_0} (in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$) into itself.

In what follows we show that the operator Q is continuous on the ball B_{r_0}. To achive this, it is sufficient to show the continuity of the operator F and V seperately, taking into account the representation of the operator Q.
So, let us fix an arbitrary $\epsilon > 0$ and choose $x \in B_{r_0}$. Next, take an arbitrary point $y \in B_{r_0}$ such that

$\|x-y\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leq \epsilon$. Then, for a fixed $p \in \mathbb{N}$ and for $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$, in view of assumption (vi) and Remark (3.5), we have

$$
\begin{aligned}
|(F_n x)(w,s) - (F_n y)(w,s)| &= |f_n(w,s,x_1(w,s),x_2(w,s),...) - f_n(w,s,y_1(w,s),y_2(w,s),...)| \\
&\leq l(r_0)\|x-y\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leq l(r_0)\epsilon.
\end{aligned}
$$

Hence, we obtain

$$
\|Fx - Fy\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leq l(r_0)\epsilon.
$$

We may deduce the intended continuity of the operator F on the ball B_{r_0} based on this approximation.

In what follows, let us choose arbitrary points $x = (x_i), y = (y_i) \in B_{r_0}$. Thus in view of assumption (viii), for fixed $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$ and $p \in \mathbb{N}$, we obtain

$$
\begin{aligned}
&|(V_p x)(w,s) - (V_p y)(w,s)| \\
&\leq \int_o^w \int_o^s |k_p(w,s,u,v)||g_p(u,v,x_1(u,v),x_2(u,v),...) - g_p(u,v,y_1(u,v),y_2(u,v),...)\, dudv \\
&\leq \int_o^w \int_o^s |k_p(w,s,u,v)| m(r_0) \sup\{|x_i(u,v) - y_p(u,v)| : i \geq p\}\, dudv \\
&\leq m(r_0) \int_o^w \int_o^s |k_p(w,s,u,v)|(\|x(u,v) - y(u,v)\|_{\ell_\infty})\, dudv \\
&\leq m(r_0) \sup\{\|x(u,v) - y(u,v)\|_{\ell_\infty} : u,v \in \mathbb{R}_+ \times \mathbb{R}_+\} \int_o^w \int_o^s |k_p(w,s,u,v)|\, dudv.
\end{aligned}
$$

Keeping assumption (iii) in mind, we arrive at the following inequality

$$
|(V_p x)(w,s) - (V_p y(w,s))| \leq G_1 m(r_0)\|x-y\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)}.
$$

We deduce that the operator V is continuous on the ball B_{r_0} based on the above-mentioned approximation.

In the sequel, let us fix an arbitrary number $\epsilon > 0$. Next, choose $w, s, u, v \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $\|(w,s) - (u,v)\| \leq \epsilon$ and take a nonempty set X of the ball B_{r_0}. Then, for a function $x = x(w,s) = (x_p(w,s)) \in X$ and for an arbitrarily fixed natural number p, estimating similarly as in (3.4), we get

$$
\begin{aligned}
|(F_p x)(w,s) - (F_p x)(w,s)| \leq\; & l(r_0) \sup\{|x_i(w,s) - x_i(u,v)| : i \geq p\} \\
&+ \sup\{|f_p(w,s,x_1,x_2,...) - f_p(u,v,x_1,x_2,...)| : |w-u| \leq \epsilon, \\
&|s-v| \leq \epsilon, \|x\|_{\ell_\infty} = \|(x_p)\|_{\ell_\infty} \leq r_0\} \\
&\leq l(r_0)\Omega^\infty(x,\epsilon) + \Omega^1_\infty(f,\epsilon),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
\Omega^1_\infty(f,\epsilon) = \sup_{p\in\mathbb{N}}\{\sup |f_p(w,s,x_1,x_2,...) - f_p(u,v,x_1,x_2,...)| : |w-u| \leq \epsilon, |s-v| \leq \epsilon, \\
\|x\|_{\ell_\infty} = \|(x_p)\|_{\ell_\infty} \leq r_0\}.
\end{aligned}
$$

Obviously, in view of assumption (v) we have $\Omega^1_\infty(f,\epsilon) \to 0$ as $\epsilon \to 0$.

Now, from estimate (3.7) we deduce that

$$\Omega^\infty(Fx,\epsilon) \le l(r_0)\Omega^\infty(x,\epsilon) + \Omega^1_\infty(f,\epsilon). \tag{3.8}$$

Further, let us observe that the same assumptions as above, asssuming additionally that $(w,s) > (u,v)$, similarly as in (3.6) we can obtain the following estimate

$$\|(V_p x)(w,s) - (V_p x(u,v))| \le \overline{M}\Omega_k(\epsilon) + \overline{g}G_2\epsilon,$$

where the symbol $\Omega_k(\epsilon)$ denotes the modulus of equicontinuity of the sequence of functions $w,s \to k_p(w,s,\tau_1,\tau_2)$ i.e.,

$$\Omega_k(\epsilon) = \sup_{p\in\mathbb{N}}\Big\{\sup\{|k_p(w,s,\tau_1,\tau_2) - k_p(u,v,\tau_1,\tau_2)| : w,s,,u,v,\tau_1,\tau_2 \in \mathbb{R}_+\times\mathbb{R}_+,$$
$$\tau_1,\tau_2 \le w,s,\tau_1,\tau_2 \le u,v,|w-u| \le \epsilon, |S-v| \le \epsilon\}\Big\}.$$

Obviously $\Omega_k(\epsilon) \to 0$ as $\epsilon \to 0$.

Let us now take note of the fact that, based on the preceding calculation, we have

$$\Omega^\infty(Vx,\epsilon) \le \overline{M}\Omega_k(\epsilon) + \overline{g}G_2\epsilon. \tag{3.9}$$

Now, for a fixed function $x \in X$ and for arbitrary numbers $w,s,u,v \in \mathbb{R}_+\times\mathbb{R}_+$, taking into account the representation of the operator Q, we have

$$\|(Qx)(w,s) - (Qx)(u,v)\|_{\ell_\infty}$$
$$\le \|\alpha(w,s) - \alpha(u,v)\|_{\ell_\infty} + \|(Vx)(w,s)\|_{\ell_\infty}\|(Fx)(w,s) - (Fx)(u,v)\|_{\ell_\infty}$$
$$\|(Fx)(u,v)\|_{\ell_\infty}\|(Vx)(w,s) - (Vx)(u,v)\|_{\ell_\infty},$$

where we denoted $\alpha(w,s) = (\alpha_p(w,s))$.

Further, fix $\epsilon > 0$ and assume that $\|(w,s) - (u,v)\| \le \epsilon$. Utilizing (3.3), (3.5), (3.8) and (3.9), from the above inequality we get

$$\begin{aligned}\Omega^\infty(Qx,\epsilon) &\le \Omega^\infty(\alpha,\epsilon) + \overline{g}G_1\Omega^\infty(Fx,\epsilon) + (r_0 l(r_0) + \overline{F})(\overline{M}\Omega_k(\epsilon) + \overline{g}G_2\epsilon)\\ &\le \Omega^\infty(\alpha,\epsilon) + \overline{g}G_1\Big[l(r_0)\Omega^\infty(x,\epsilon) + \Omega^1_\infty(f,\epsilon)\Big]\\ &\quad + (r_0 l(r_0) + \overline{F})(\overline{M}\Omega_k(\epsilon) + \overline{g}G_2\epsilon).\end{aligned}$$

As a result, keeping in mind the above established properties of functions $\epsilon \to \Omega^1_\infty(f,\epsilon), \epsilon \to \Omega_k(\epsilon)$ and assumption (i), we obtain

$$\Omega^\infty_0(QX) \le \overline{g}G_1 l(r_0)\Omega^\infty(X). \tag{3.10}$$

In what follows we will consider the second term of the measure of noncompactness ϕ^3_e defined by the formula (2.42) for $i = 3$. That term is denoted by ϕ^{-3}_∞ and is expressed by formula (2.33). To this end fix a set $X \subset B_{r_0}$ and take arbitrary $x = x(w,s), y = y(w,s) \in X$. Then, for arbitrarily $w,s \in \mathbb{R}_+\times\mathbb{R}_+$ and $k \in \mathbb{N}$, we have :

$$\begin{aligned}|(Q_k x)(w,s) - (Q_k y)(w,s)| &= |(F_k x)(w,s)(V_k x)(w,s) - (F_k y)(w,s)(V_k y)(w,s)|\\ &\le |(V_k)(w,s)||(F_k x)(w,s) - (F_k y)(w,s)|\\ &\quad + |(F_k y)(w,s)||(V_k x)(w,s) - (V_k y)(w,s)|.\end{aligned} \tag{3.11}$$

Further on, we are going to estimate the terms appearing on the right hand side of inequality(3.11). To this end, fix a natural number n and a number $\zeta > 0$. Then, for $t \in [0,\zeta]$ and for $p \in \mathbb{N}, q \geq p$, based on assumptions (viii) and (iii), for arbitrary functions $x, y \in X$, we obtain

$$
\begin{aligned}
&|(V_q x)(w,s) - (V_q y)(w,s)| \\
&\quad \leq \int_o^w \int_o^s |k_q(w,s,u,v)||g_q(u,v,x_1(u,v),x_2(u,v),...) \\
&\qquad - g_q(u,v,y_1(u,v),y_2(u,v),...)|\, dudv \\
&\quad \leq m(r_0) \int_o^w \int_o^s |k_q(w,s,u,v)|\Big(\sup\{|x_i(u,v) - y_i(u.v)| : i \geq q\}\Big) dudv \\
&\quad \leq m(r_0) \int_o^w \int_o^s |k_q(w,s,u,v)|\Big\{\sup_{w,s\in[0,\zeta]} \Big\{\sup_{i\geq p} |x_i(w,s) - y_i(w,s)|\Big\}\Big\} dudv \\
&\quad \leq G_1 m(r_0)\Big\{\sup_{w,s\in[0,\zeta]} \Big\{\Big\{\sup_{i\geq p}\{\sup\{|x_i(w,s) - y_i(w,s)| : x = x(w,s), y = y(w,s) \in X\}\}\Big\}\Big\}.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&\sup_{w,s\in[0,\zeta]} \Big\{\Big\{\sup_{i\geq p}\{\sup\{|x_i(w,s) - y_i(w,s)| : x = x(w,s), y = y(w,s) \in X\}\}\Big\} \\
&\leq G_1 m(r_0)\Big\{\sup_{w,s\in[0,\zeta]} \Big\{\Big\{\sup_{i\geq p}\{\sup\{|x_i(w,s) - y_i(w,s)| : x = x(w,s), y = y(w,s) \in X\}\}\Big\}\Big\}.
\end{aligned}
$$

The above estimate yields (cf.formula 2.33):

$$\phi_\infty^{-3}(VX) \leq G_1 m(r_0)\phi_\infty^{-3}(X). \tag{3.12}$$

Similarly as above, for an arbitrarily fixed $p \in \mathbb{N}, w, s \in \mathbb{R}_+ \times \mathbb{R}_+$ and for $x = x(w,s), y = y(w,s) \in X$, utilizing assumption (vi), we obtain

$$|(F_p x)(w,s) - (F_p y)(w,s)| \leq l(r_0)\sup\{|x_i(w,s) - y_i(w,s)| : i \geq p\}.$$

As a result, we arrive to the following estimate:

$$
\begin{aligned}
&\sup_{w,s\in[0,\zeta]} \Big\{\sup_{i\geq p} \Big\{\sup\{|(F_i x)(w,s) - (F_i y)(w,s)| : x = x(w,s), y = y(w,s) \in X\}\Big\}\Big\} \\
&\leq l(r_0)\Big\{\sup_{w,s\in[0,\zeta]} \Big\{\Big\{\sup_{i\geq p}\{\sup\{|x_i(w,s) - y_i(w,s)| : x = x(w,s), y = y(w,s) \in X\}\}\Big\}\Big\}.
\end{aligned}
$$

We can now conclude the following inequality using the above estimate and formula (2.33).

$$\phi_\infty^{-3}(FX) \leq l(r_0)\phi_\infty^{-3}(X). \tag{3.13}$$

Finally, joining estimates (3.2), (3.5), (3.11), (3.12) and (3.13), we obtain

$$\phi_\infty^{-3}(QX) \leq \overline{g} G_1 l(r_0)\phi_\infty^{-3}(X) + \Big(l(r_0)r_0 + \overline{F}\Big) G_1 m(r_0)\phi_\infty^{-3}(X). \tag{3.14}$$

In the sequel we will consider the third term of the measure of noncompactness ϕ_e^3 defined by (2.42) i.e., the term $e(X)$ expressed by formula (2.39).

Thus, let us fix a nonempty subset X of the ball B_{r_0} and the functions $x = x(w,s), y = y(w,s) \in X$. Next, fix $\zeta > 0$ and take $w, s \geq \zeta$. Then, for an arbitrary natural number p, on the basis of calculations performed before estimate (3.12), we obtain

$$|(V_p x)(w,s) - (V_p y)(w,s)| \leq G_1 m(r_0)\Big\{\sup_{w,s\geq\zeta}\Big\{\sup_{i\geq p}|x_i(w,s) - y_i(w,s)|\Big\}\Big\}.$$

The above estimate yields

$$\sup_{w,s\geq\zeta}\Big\{\sup\Big\{\sup_{p\in\mathbb{N}}|(V_p x)(w,s) - (V_p y)(w,s)| : x = x(w,s), y = y(w,s) \in X\Big\}\Big\}$$
$$\leq G_1 m(r_0)\Big\{\sup_{w,s\geq\zeta}\Big\{\sup\Big\{\sup_{p\in\mathbb{N}}|(x_p)(w,s) - (y_p)(w,s)| : x = x(w,s), y = y(w,s) \in X\Big\}\Big\}\Big\}.$$

Consiquently, we get

$$e(VX) \leq G_1 m(r_0) e(X). \tag{3.15}$$

Following that, we derive the following inequality using the same reasoning as in the calculations that preceding estimate (3.13).

$$e(FX) \leq l(r_0) e(X). \tag{3.16}$$

Finally, linking estimates (3.2), (3.5), (3.11), (3.15) and (3.16), we obtain

$$e(QX) \leq \overline{g} G_1 l(r_0) e(X) + \Big(l(r_0) r_0 + \overline{F}\Big) G_1 m(r_0) e(X). \tag{3.17}$$

Now, combining estimates (3.10), (3.14), (3.17) and keeping in md formula (2.42) expressing the measure of noncompactness ϕ_e^3, we get

$$\begin{aligned}\phi_e^3(QX) \leq\ & \overline{g} G_1 l(r_0) \Omega_0^\infty(X) \\ & + \Big[\overline{g} G_1 l(r_0) + (r_0 l(r_0) + \overline{F}) G_1 m(r_0)\Big]\phi_e^{-3}(X) \\ & + \Big[\overline{g} G_1 l(r_0) + (r_0 l(r_0) + \overline{F}) G_1 m(r_0)\Big] e(X).\end{aligned}$$

Hence, we derive the following estimate

$$\phi_e^3(QX) \leq \Big[\overline{g} G_1 l(r_0) + (r_0 l(r_0) + \overline{F}) G_1 m(r_0)\Big]\phi_e^3(X) \tag{3.18}$$

Further, taking into account the above obtained estimate, in view of the facts established in the conducted proof, assumption (xi) and Theorem 2.5 we deduce that there exists atleast one element $x \in B_{r_0}$which is the fixed point of the operator Q in the ball B_{r_0}. Obviously the function $x = x(w,s)$ is a solution of infinite system of integral equations (3.1) in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$.

Moreover, in view of remark and the description of the kernel of measure of noncompactness ϕ_α, ϕ_β and ϕ_γ located after the proof of Theorem 2.2, we conclude that the function $= x(w,s)$ is uniformly continuous on the interval $\mathbb{R}_+ \times \mathbb{R}_+$. This completes the proof.

The following example exemplifies the above result:

Example 3.4. Let us consider the following infinite system 0f nonlinear quadratic integral eqyuations of the Volterra-Hammerstein type

$$X_p(w,s) = \frac{a(w+s)}{1+p^2+(ws)^2} + \Big(\frac{b}{p^2+(ws)^2} + \frac{zx_p(w,s)}{1+x_1^2(w,s)} + \frac{zx_{p+1}}{p+x_2^2(w,s)}\Big)$$

$$\times \int_o^w \int_o^s \frac{uv}{1+p\Big((uv)^2+(ws)^2\Big)} \arctan\Big(\frac{x_1(u,v)+x_p(u,v)}{p+(uv)^2}\Big) dudv \tag{3.19}$$

for $p = 1, 2, ...$ and $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$. Also, we assume a, b, z appearing in the above are positive constants. Observe that infinite system (3.19) is a particular case of system (3.1) if we put

$$\alpha_p(w,s) = \frac{a(w+s)}{1+p^2+(ws)^2}, \tag{3.20}$$

$$f_p(w,s,x_1,x_2,...) = \frac{b}{p^2+(ws)^2} + \frac{zx_p(w,s)}{1+x_1^2(w,s)} + \frac{zx_{p+1}}{p+x_2^2(w,s)}, \tag{3.21}$$

$$k_p(w,s,u,v) = \frac{uv}{1+p\Big((uv)^2+(ws)^2\Big)}, \tag{3.22}$$

$$g_p(w,s,x_1,x_2,...) = \arctan\Big(\frac{x_1(u,v)+x_p(u,v)}{p+(uv)^2}\Big) \tag{3.23}$$

for $p = 1, 2, ...$ and $w, s \in \mathbb{R}_+ \times \mathbb{R}_+$.

In order to show that the infinite system of integral equations (3.19) has a solution in the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ it is sufficient to apply Theorem (3.3). To this end, we have to show that the functions defined by formulas (3.20)-(3.23) satisfy assumptions (i)-(xi) of Theorem (3.3).

At the begining let us observe that the functions $\alpha_p(w,s)$ defined by (3.20) satisfy the Lipschitz condition with the constant $l = 1$ for $p = 1, 2, ...$ Thus, these functions are equicontinuous on $\mathbb{R}_+ \times \mathbb{R}_+$. Moreover, we have

$$A = \sup\{|\alpha_p(w,s)| : p = 1, 2, ..., w, s \in \mathbb{R}_+ \times \mathbb{R}_+\} = 1.$$

This shows that the assumption (i) is satisfied.

Further, let us notice that the function $k_p(w,s,u,v)$ defined by (3.22) $(p = 1, 2, ...)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$. additionally, uusing standard tools of differential calculus it is easy seen that

$$|k_p(w_2,s_2,u,v) - k_p(w_1,s_1,u,v)| \leq \frac{1}{p}|(w_2,s_2) - (w_1,s_1)|$$

for $p = 1, 2, 3, ...$ and for $w_1, w_2, s_1, s_2 \in \mathbb{R}_+ \times \mathbb{R}_+$. This means that the sequence of functions $(k_p(.,u,v))$ is equicontinuous on $\mathbb{R}_+ \times \mathbb{R}_+$ uniformly with respect to $u, v \in \mathbb{R}_+ \times \mathbb{R}_+$.
Summing up, we see that there is satisfied assumption (ii).

Next, let us observe that for each $p \in \mathbb{N}$ and for arbitrary $w, s, u, v \in \mathbb{R}_+ \times \mathbb{R}_+$ we have the following estimate

$$|k_p(w,s,u,v) \leq \frac{uv}{1+p(uv)^2} \leq \frac{uv}{1+(uv)^2} \leq \frac{1}{2}.$$

Hence it follows that the sequence $k_p(w,s,u,v)$ is equibounded on $\mathbb{R}_+ \times \mathbb{R}_+$ with the constant $K_2 = \dfrac{1}{2}$. This shows that there is satisfied assumption (iv).

On the other hand we obtain

$$\int_o^w\int_o^s |k_p(w,s,u,v)|\,dudv = \int_o^w\int_o^s \frac{uv}{1+p((uv)^2+(ws)^2)}\,dudv = \frac{1}{2}\Big(\frac{1+2p(ws)^2}{1+p(ws)^2}\Big)$$
$$\leq \frac{1}{2p}\ln 2 \leq \frac{1}{2}\ln 2.$$

Next, let us notice that the functions $f_p = f_p(w,s,x_1,x_2,...)$ given by (3.21) act from $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^\infty \to \mathbb{R}(p = 1,2,...)$. Additionally, taking into account that the functions f_p do not depend explicity on (w,s), we conclude that there is satisfied assumption (v).

In order to verify assumption (*vi*) let us fix a number $r > 0$ and take $x = (x_i)$ such that $\|x\|_{\ell_\infty} \leq r$. Then, keeping in mind formula (3.21), for an arbitrary natural number p and $w,s \in \mathbb{R}_+\times\mathbb{R}_+$, we have

$$\begin{aligned}|f_p(w,s,x_1,x_2,...)| &\leq \frac{b}{p^2+(ws)^2} + z\Big[\frac{|x_p|}{1+x_1^2} + \frac{|x_{p+1}|}{p+x_2^2}\Big]\\ &\leq \frac{b}{p^2+(ws)^2} + z(|x_p)| + |x_{p+1}|)\\ &\leq \frac{b}{p^2+(ws)^2} + z(|x_p) + 2z\sup\{|x_i| : i \geq p\}.\end{aligned}$$

This shows that the inequality from assumption (vi) is satisfied with the following functions

$$\begin{aligned}\overline{f}_p(w,s) &= \frac{b}{p^2+(ws)^2},\\ l(r) &= 2z\end{aligned}$$

for $p = 1,2,....$ Since $\overline{f}_p(w,s) = \dfrac{b}{p^2+(ws)^2}$ we infer that $\lim_{w,s\to\infty}\overline{f}_p(w,s) = 0$ uniformly with respect to $p \in \mathbb{N}$.
Apart from this we have that $\lim_{p\to\infty}\overline{f}_p(w,s) = 0$ for any $w,s \in \mathbb{R}_+\times\mathbb{R}_+$.
Summing up we see that assumption (*vi*) is satisfied. Moreover, let us notice that

$$\overline{F} = \sup\{\overline{f}_p(w,s) : w,s \in \mathbb{R}_+\times\mathbb{R}_+, p = 1,2,...\} = b.$$

Next, let us fix a number $r > 0$ and take $x = (x_i), y = (y_i)$ such that $\|x\|_{\ell_\infty} \leq r, \|y\|_{\ell_\infty} \leq r$. Then, keeping in mind formula (3.21), for an arbitrary natural number p and $w,s \in \mathbb{R}_+\times\mathbb{R}_+$, we have

$$
\begin{aligned}
&|f_p(w,s,x_1,x_2,...) - f_p(w,s,y_1,y_2,...)| \\
&\leq z\left|\frac{x_p}{1+x_1^2} - \frac{y_p}{1+y_1^2}\right| + z\left|\frac{x_{p+1}}{p+x_2^2} - \frac{y_{p+1}}{p+y_2^2}\right| \\
&\leq z\frac{|x_p + x_p y_1^2 - y_p - y_p x_1^2|}{(1+x_1^2)(1+y_1^2)} + z\frac{|px_{p+1} + x_{p+1}y_2^2 - py_{p+1} - y_{p+1}x_2^2|}{(p+x_2^2)(p+y_2^2)} \\
&\leq z|x_p - y_p| + z\frac{(x_p y_1^2 - y_p x_1^2) + (y_p y_1^2 - y_p x_1^2)}{(1+x_1^2)(1+y_1^2)} + y_p\frac{|x_{p+1} - y_{p+1}}{(p+x_2^2)(p+y_2^2)} \\
&+ z\frac{(x_p y_1^2 - y_p x_1^2) + (y_{p+1}y_2^2 - y_{p+1}x_2^2)}{(p+x_2^2)(p+y_2^2)} \\
&\leq 2z|x_p - y_p| + yr\left(\frac{|y_1|}{(1+x_1^2)(1+y_1^2} + \frac{|x_1|}{(1+x_1^2)(1+y_1^2}\right)|x_1 - y_1| \\
&+ 2z|x_{p+1} - y_{p+1}| + yr\left(\frac{|y_2|}{(1+x_2^2)(1+y_1^2} + \frac{|x_2|}{(1+x_2^2)(1+y_2^2)}\right)|x_2 - z_2| \\
&\leq 2z|x_p - y_p| + yr|x_1 - y_1| + 2z|x_{p+1} - z_{p+1}| + zr|x_2 - z_2| \\
&\leq (4z + 2rz)\|x - z\|_{\ell_\infty} = 2z(2+r)\|x - z\|_{\ell_\infty}.
\end{aligned}
$$

Thus see that assumption (*vii*) is satisfied with the function $m(r) = 2z(2 + r)$.

In the next step of our proof we are going to verify assumptuion (*viii*). To this end fix arbitrarily $\in \mathbb{N}$ and consider the function $g_p(w,s,x) = g_p(w,s,x_1,x_2,...)$ defined by formula (3.23) i.e,.

$$g_p(w,s,x_1,x_2,...) = \arctan\left(\frac{x_1 + x_p}{p + (ws)^2}\right).$$

Then, from the estimate

$$g_p(w,s,x_1,x_2,...) \leq \frac{|x_1| + |x_p|}{p + (ws)^2} \leq \frac{|x_1| + |x_p|}{p}.$$

We deduce that the operator g defined in assumption (*viii*) by the equality

$$(gx)(w,s) = (g_p(w,s,x)) = (g_1(w,s,x), g_2(w,s,x), ...)$$

transforms the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \ell_\infty$ into ℓ_∞.
Further on, fix $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ and take $x = (x_i), y = (y_i) \in \ell_\infty$. Then we have

$$|g_p(w,s,x) - g_p(w,s,z)| \leq \left|\frac{x_1 + x_p}{p + (ws)^2} + \frac{z_1 + z_p}{p + (ws)^2}\right| \leq \frac{|x_1 - z_1}{p} + \frac{|x_p - z_p|}{p}.$$

. This allows us to derive te following estimate:

$$
\begin{aligned}
\|(gx)(w,s) - (gz)(w,s)\|_{\ell_\infty} &= \sup\{|g_p(w,s,x) - g_p(w,s,z)| : p \in \mathbb{N}\} \\
&\leq \sup\left\{\frac{|x_1 - z_1}{p} + \frac{|x_p - z_p|}{p} : p \in \mathbb{N}\right\} \\
&\leq 2\sup\left\{\frac{|x_p - z_p|}{p} : p \in \mathbb{N}\right\} \leq 2\|x - z\|_{\ell_\infty}.
\end{aligned}
$$

From the above estimate we infer that the operator g satisfies assumption (*viii*).
Moreover, it is easily seen that for an arbitrary $x \in \ell_\infty$ and $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ we get

$$\|(gx)(w,s)\|_{\ell_\infty} = \sup\{g_p(w,s,x) : p \in \mathbb{N}\} \leq \frac{\pi}{2}.$$

This means that the operator g satisfies the assumption (ix) with constant $\overline{G} = \frac{\pi}{2}$.

Finally, let us consider the first inequality from assumption (x). Obviously, in our case that inequality has the form

$$\frac{a}{2\sqrt{2}} + \frac{\pi}{4}\ln 2(b + 2zr) < r. \tag{3.24}$$

On the other hand, taking the second inequality required in assumption (x), we get

$$z\frac{\pi}{2}\ln 2(2 + r_0) < 1. \tag{3.25}$$

It is easy to check that choosing $z < \frac{1}{\pi \ln 2}$ and taking $r_0 > \frac{a}{\sqrt{2}} + \frac{b}{2y}$, we can easily verify that both inequalities (3.24) and (3.25) are satisfied.
Thus, in the light of Theorem (3.3), we infer that infinite system of nonlinear integral equations (3.19) has atleast one solution belonging to the ball B_{r_0} in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$.

Declarations

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