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Some spectral norms of RFPrLrR circulant matrices

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Abstract. In this article, based on combinatorial methods, the structure of RFPrLrR circulant matrices and the identities of matrix norms, we give three lower bounds and upper bounds for spectral norms of RFPrLrR circulant matrices involving exponential forms and trigonometric functions by using a different method. Moreover, we introduce new geometric RFPrLrR circulant matrix, and then we obtain three lower bounds for spectral norms and its upper bound.

1. Introduction and Preliminaries

Recently, studying the norms of circulant matrices has been a hot topic in matrix theory. Some scholars studied the norms of circulant matrices, r-circulant matrices, geometric circulant matrices, r-Hankel and r-Toeplitz matrices with some famous numbers and polynomials, and they examined various properties of these matrices $[3-7], [16-25], [28]$. These types of special matrices are used in many branches of science such as applied sciences, cryptology, encryption, and coding theory. A circulant matrix and r-circulant matrix with the first row $(c_0, c_1, c_2, \ldots, c_{n-1})$ is a square matrix as the following form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}, C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & c_0 \end{pmatrix},$$

where r is a nonzero complex number. If $r = 1$, r-circulant matrix become the circulant matrix. A Toeplitz matrix is defined by

$$\mathcal{T} = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-n+3} & t_{-n+2} \\ t_2 & t_1 & t_0 & \cdots & t_{-n+4} & t_{-n+3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 & t_0 \end{pmatrix}_{n\times n}. \tag{1}$$

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Especially, the $f(x)$-circulant matrix [2] are another natural extension of circulant matrices. The properties and structure of $x^n - rx - 1-$circulant matrices, which are called row-first-plus-$r$last right circulant matrices (or shortly, RFPrLR circulant matrices), $x^n + x - r-$circulant matrices are called RFMLrR circulant matrices. There are some interesting properties of $f(x)$-circulant matrices. For example, Shen [8] got the explicit determinants of RFMLR and RLMFL circulant matrices involving certain famous numbers. In 2012, Jiang [9] studied the norms of RFPLR circulant matrices with Fibonacci and Lucas numbers. Jiang [10] studied fast algorithms for solving RFPrLR circulant linear systems. In 2019, Shi obtained several results [11, 12] of the norms of matrices mentioned above with exponential forms $e\left(\frac{k}{n}\right)$ and trigonometric functions $\cos\left(\frac{k\pi}{n}\right)$, $\sin\left(\frac{k\pi}{n}\right)$. In this paper, we study another $f(x)$-circulant matrix called RFPrLrR circulant matrix, then we gave the spectral norms of $r-$circulant matrices and related results of the RFPrLrR circulant matrices. As far as we know, it seems that no one has studied the upper and lower estimate problems for the spectral norms of RFPrLrR circulant matrices involving exponential forms $e\left(\frac{k}{n}\right)$ and trigonometric functions $\cos\left(\frac{k\pi}{n}\right)$, $\sin\left(\frac{k\pi}{n}\right)$ yet. We shall give a different method to study the spectral norms of RFPrLrR circulant matrices. These results of this paper have potential applications in neural network for nonlinear system control based norms[13].

For exponential form $e(x)$, $e(x) = e^{2\pi i x}$, then $|e(x)| = 1$, by $e^{i\theta} = \cos\theta + i\sin\theta$, note that $e(0) = e(1) = e(-1) = e(n) = 1$, and by the trigonometric sums, we have

$$\sum_{k=1}^{n} e\left(\frac{km}{n}\right) = \begin{cases} n, & n|m \\ 0, & \text{otherwise.} \end{cases}$$

Particularly, $\sum_{k=0}^{n-1} e\left(\frac{k}{n}\right) = 0$, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. By the relationship between exponential forms $e\left(\frac{k}{n}\right)$ and trigonometric functions $\cos\left(\frac{k\pi}{n}\right)$, $\sin\left(\frac{k\pi}{n}\right)$, we can get some power sums of these functions. For more information, we can see reference [26 − 28].

A $n \times n$ row first-plus-$r$last r right circulant matrix with the first row $(a_0, a_1, \cdots, a_{n-1})$, denoted by $A = RFPrLrRcircfr_r(a_0, a_1, \cdots, a_{n-1})$, is defined by [14] Xu et al. as follows:

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ ra_{n-1} & a_0 + ra_{n-1} & a_1 & \cdots & a_{n-3} & a_{n-2} \\ ra_{n-2} & ra_{n-1} + ra_{n-2} & a_0 + ra_{n-1} & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ra_1 & ra_2 + ra_1 & ra_3 + ra_2 & \cdots & ra_{n-1} + ra_{n-2} & a_0 + ra_{n-1} \end{pmatrix}_{n\times n}. \quad (2)$$

Obviously, the RFPrLrR circulant matrix is determined by its first row, the authors defined $\Theta_{(r,r)}$ as the basic RFPrLrR circulant matrix with the first row$(0, 1, 0, \cdots, 0)$,

$$\Theta_{(r,r)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ r & r & 0 & \cdots & 0 \end{pmatrix}_{n\times n}.$$

They [14] obtained get $\Theta_{(r,r)}^n = rI_n + r\Theta_{(r,r)}$. According to the structure of the power of the basic RFPrLrR circulant matrix $\Theta_{(r,r)}$, it is clear that

$$A = RFPrLrRcircfr_r(a_0, a_1, \cdots, a_{n-1}) = \sum_{i=0}^{n-1} a_i \Theta_{(r,r)}^i.$$

In the light of the nice articles above, in this paper, we shall use identities of exponential forms $e\left(\frac{k}{n}\right)$ and trigonometric functions $\cos\left(\frac{k\pi}{n}\right)$, $\sin\left(\frac{k\pi}{n}\right)$ and power sums of $e\left(\frac{k}{n}\right)$, $\cos\left(\frac{k\pi}{n}\right)$, $\sin\left(\frac{k\pi}{n}\right)$ to study the norms of

RFPrLrR circulant matrices

$$\begin{aligned} A &= RFPrLrRcircfr_r\left(e\left(\frac{0}{n}\right), e\left(\frac{1}{n}\right), e\left(\frac{2}{n}\right), \cdots, e\left(\frac{n-1}{n}\right)\right), \\ B &= RFPrLrRcircfr_r\left(\cos\left(\frac{0\cdot\pi}{n}\right), \cos\left(\frac{1\cdot\pi}{n}\right), \cos\left(\frac{2\cdot\pi}{n}\right), \cdots, \cos\left(\frac{(n-1)\cdot\pi}{n}\right)\right), \\ C &= RFPrLrRcircfr_r\left(\sin\left(\frac{0\cdot\pi}{n}\right), \sin\left(\frac{1\cdot\pi}{n}\right), \sin\left(\frac{2\cdot\pi}{n}\right), \cdots, \sin\left(\frac{(n-1)\cdot\pi}{n}\right)\right). \end{aligned}$$

Let us now give the following definition and lemmas that we use throughout this article.

Definition 1[3] Let any matrix $A = (a_{ij}) \in M_{m\times n}(C)$, the spectral norm and the Euclidean norm of matrix A are defined by

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i(A^H A)}, \tag{3}$$

and

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}, \tag{4}$$

respectively, where, $\lambda_i(A^H A)$ is the eigenvalues of matrices $A^H A$ and A^H is the conjugate transpose of A. The following important inequalities hold between the Frobenius norm and spectral norm:

$$\frac{1}{\sqrt{n}}\|A\|_F \le \|A\|_2 \le \|A\|_F.$$

Lemma 1[12] For exponential forms $e\left(\frac{k}{n}\right)$,

$$\sum_{k=0}^{n-1} |e\left(\frac{k}{n}\right)| = n, \sum_{k=0}^{n-1} e\left(\frac{k}{n}\right) = \sum_{k=0}^{n-1} e\left(\frac{-k}{n}\right) = \sum_{k=0}^{n-1} e^2\left(\frac{k}{n}\right) = 0.$$

Lemma 2[12] For any positive integer $n \ge 2$, we have

$$\begin{aligned} &\sum_{k=0}^{n-1} \cos\left(\frac{k\pi}{n}\right) = 1, \sum_{k=0}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \cot\left(\frac{\pi}{2n}\right), \\ &\sum_{k=0}^{n-1} \cos^2\left(\frac{k\pi}{n}\right) = \sum_{k=0}^{n-1} \sin^2\left(\frac{k\pi}{n}\right) = \frac{n}{2}. \end{aligned}$$

Lemma 3[12] If n is even,

$$\sum_{j=1}^{n-1} |\cos\left(\frac{j\pi}{n}\right)| = \cot\left(\frac{\pi}{2n}\right) - 1.$$

If n is odd,

$$\sum_{j=1}^{n-1} |\cos\left(\frac{j\pi}{n}\right)| = \csc\left(\frac{\pi}{2n}\right) - 1.$$

$$\sum_{j=1}^{n-1} |\sin\left(\frac{j\pi}{n}\right)| = \sum_{j=1}^{n-1} \sin\left(\frac{j\pi}{n}\right) = \cot\left(\frac{\pi}{2n}\right).$$

By exponential sums, we can get easily

$$\sum_{j=1}^{n-1} |\sin\left(\frac{j\pi}{n}\right)| = \sum_{j=1}^{n-1} \sin\left(\frac{j\pi}{n}\right) = \cot\left(\frac{\pi}{2n}\right).$$

Lemma 4 The following identities hold:

$$\sum_{k=1}^{n-1} k\cos\left(\frac{k\pi}{n}\right) = \frac{1}{\cos\left(\frac{\pi}{n}\right)-1} + \frac{n}{2},$$

and

$$\sum_{k=1}^{n-1} k\sin\left(\frac{k\pi}{n}\right) = \frac{n}{2}\cot\left(\frac{\pi}{2n}\right).$$

Proof. By using the identities $\cos\left(\frac{k\pi}{n}\right) = \frac{e\left(\frac{2k}{n}\right)+e\left(-\frac{2k}{n}\right)}{2}$, and $\sin\left(\frac{k\pi}{n}\right) = \frac{e\left(\frac{2k}{n}\right)-e\left(-\frac{2k}{n}\right)}{2i}$,

$$\sum_{k=1}^{n-1} ke\left(\frac{k}{2n}\right) = \frac{2e\left(\frac{1}{2n}\right)}{\left[1-e\left(\frac{1}{2n}\right)\right]^2} + \frac{n}{1-e\left(\frac{1}{2n}\right)},$$
$$\sum_{k=1}^{n-1} ke\left(\frac{-k}{2n}\right) = \frac{2e\left(\frac{-1}{2n}\right)}{\left[1-e\left(\frac{-1}{2n}\right)\right]^2} + \frac{n}{1-e\left(\frac{-1}{2n}\right)},$$

thus we have

$$\sum_{k=1}^{n-1} k\cos\left(\frac{k\pi}{n}\right) = \frac{1}{\cos\left(\frac{\pi}{n}\right)-1} + \frac{n}{2}, \quad \sum_{k=1}^{n-1} k\sin\left(\frac{k\pi}{n}\right) = \frac{n}{2}\cot\left(\frac{\pi}{2n}\right).$$

Lemma 5 For RFPrLrR circulant matrix A, we can get

$$\begin{aligned}
A &= \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\
ra_{n-1} & a_0 + ra_{n-1} & a_1 & \cdots & a_{n-3} & a_{n-2} \\
ra_{n-2} & ra_{n-1} + ra_{n-2} & a_0 + ra_{n-1} & \cdots & a_{n-4} & a_{n-3} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
ra_1 & ra_2 + ra_1 & ra_3 + ra_2 & \cdots & ra_{n-1} + ra_{n-2} & a_0 + ra_{n-1}
\end{pmatrix}_{n\times n} \\
&= \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\
ra_{n-1} & a_0 & c_1 & \cdots & a_{n-3} & a_{n-2} \\
ra_{n-2} & ra_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
ra_1 & ra_2 & ra_3 & \cdots & ra_{n-1} & a_0
\end{pmatrix}_{n\times n} \\
&+ r\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & a_{n-1} & 0 & \cdots & 0 & 0 \\
0 & a_{n-2} & a_{n-1} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1}
\end{pmatrix}_{n\times n} \\
&= C_r + rT,
\end{aligned}$$

where C_r is r-circulant matrix and T is lower triangular matrix. Then we have $\|A\|_2 \leq \|C_r\|_2 + |r|\|T\|_2$.
Lemma 6 For RFPrLrR circulant matrix A(even order), the matrix factorization of A is following form:

$$\begin{aligned} A &= \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ ra_{n-1} & a_0 + ra_{n-1} & a_1 & \cdots & a_{n-3} & a_{n-2} \\ ra_{n-2} & ra_{n-1} + ra_{n-2} & a_0 + ra_{n-1} & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ra_1 & ra_2 + ra_1 & ra_3 + ra_2 & \cdots & ra_{n-1} + ra_{n-2} & a_0 + ra_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} a_0 & a_1 - a_0 & a_2 - a_1 + a_0 & \cdots & a_{n-2} - a_{n-1} + \cdots + a_0 & a_{n-1} - a_{n-2} + \cdots - a_0 \\ ra_{n-1} & a_0 & a_1 - a_0 & \cdots & a_{n-3} - a_{n-2} + \cdots - a_0 & a_{n-2} - a_{n-1} + \cdots + a_0 \\ ra_{n-2} & ra_{n-1} & a_0 & \cdots & a_{n-4} - a_{n-3} + \cdots + a_0 & a_{n-3} - a_{n-2} + \cdots - a_0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ra_1 & ra_2 & ra_3 & \cdots & ra_{n-1} & a_0 \end{pmatrix} \\ &\quad \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \mathcal{T}\mathcal{P}. \end{aligned}$$

Proof. If we apply some elementary column operators to A, we can obtain

$$\begin{aligned} A &= \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ ra_{n-1} & a_0 + ra_{n-1} & a_1 & \cdots & a_{n-3} & a_{n-2} \\ ra_{n-2} & ra_{n-1} + ra_{n-2} & a_0 + ra_{n-1} & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ra_1 & ra_2 + ra_1 & ra_3 + ra_2 & \cdots & ra_{n-1} + ra_{n-2} & a_0 + ra_{n-1} \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} a_0 & a_1 - a_0 & a_2 - a_1 + a_0 & \cdots & a_{n-2} - a_{n-1} + \cdots + a_0 & a_{n-1} - a_{n-2} + \cdots - a_0 \\ ra_{n-1} & a_0 & a_1 - a_0 & \cdots & a_{n-3} - a_{n-2} + \cdots - a_0 & a_{n-2} - a_{n-1} + \cdots + a_0 \\ ra_{n-2} & ra_{n-1} & a_0 & \cdots & a_{n-4} - a_{n-3} + \cdots + a_0 & a_{n-3} - a_{n-2} + \cdots - a_0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ra_1 & ra_2 & ra_3 & \cdots & ra_{n-1} & a_0 \end{pmatrix} = \mathcal{T}. \end{aligned}$$

Namely, $AP = \mathcal{T}$, where

$$P = \begin{pmatrix} 1 & -1 & 1 & \cdots & -1 \\ & 1 & -1 & \cdots & 1 \\ & & \ddots & \ddots & \vdots \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix},$$

$$P^{-1} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ & 1 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} = \mathcal{P}.$$

Thus $A = \mathcal{T}P^{-1} = \mathcal{T}\mathcal{P}$, where $\mathcal{T}$ is Toeplitz matrix.
Lemma 7 For Toeplitz matrix $\mathcal{T}$ as (1.1), its decomposition is as following form:

$$\mathcal{T} = \sum_{k=1}^{n-1} t_{-k}\mathcal{B}^k + \sum_{k=0}^{n-1} t_k C^k.$$

$$\|\mathcal{T}\|_2 \leq \sum_{k=1}^{n-1} |t_{-k}| + \sum_{k=0}^{n-1} |t_k|.$$

where

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & & & & 0 \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}.$$

Proof. We can verify $\mathcal{T} = \sum_{k=1}^{n-1} t_{-k}\mathcal{B}^k + \sum_{k=0}^{n-1} t_k C^k$, and $\|\mathcal{B}\|_2 = \|C\|_2 = 1$, Therefore, $\|\mathcal{T}\|_2 = \|\sum_{k=1}^{n-1} t_{-k}B^k + \sum_{k=0}^{n-1} t_k C^k\|_2 \leq \sum_{k=1}^{n-1} |t_{-k}| + \sum_{k=0}^{n-1} |t_k|$.

Lemma 8[28] Let $A \in C^{m\times n}$, then

$$\|A\|_2 \geq \frac{1}{\sqrt{q}}\|A\|_E, q = \min(m,n), \quad \|A\|_2 \geq \frac{2|r_1 + r_2 + \cdots + r_m|}{m+n}.$$

In particular, for $A \in C^{n\times n}$,

$$\|A\|_2 \geq \frac{|r_1 + r_2 + \cdots + r_n|}{n}, \quad \|A\|_2 \geq \sqrt{\frac{|r_1|^2 + |r_2|^2 + \cdots + |r_n|^2}{n}},$$

where r_i denote the row sums of A.

Our aim in this paper is to examine RFPrLrR circulant matrices whose entries are exponential forms $e\left(\frac{k}{n}\right)$ and trigonometric functions $\cos\left(\frac{k\pi}{n}\right)$, $\sin\left(\frac{k\pi}{n}\right)$, respectively and obtain their three lower bounds and upper bounds for spectral norm. Moreover we define a new geometric RFPrLrR circulant matrix and mentioned some norm calculations for further research.

2. Main results

We give Frobenius norms, the three lower and upper bounds for the spectral norms of these matrices as following theorems.

Theorem 1 Let $A = RFPrLrRcircfr_r\left(e\left(\frac{0}{n}\right), e\left(\frac{1}{n}\right), e\left(\frac{2}{n}\right), \cdots, e\left(\frac{n-1}{n}\right)\right)$ be a $n \times n$ RFPrLrR circulant matrix, then we have three lower bounds

$$\|A\|_2 \geq \begin{cases} \sqrt{\frac{\alpha}{n}}, \\ \frac{\left||2r-1|\sum_{k=0}^{n-2}\sum_{j=0}^{k} e\left(\frac{-k+j-1}{n}\right)\right|}{n}, \\ \sqrt{\frac{\left||2r-1|^2\sum_{k=0}^{n-2}|\sum_{j=0}^{k} e\left(\frac{-k+j-1}{n}\right)|^2\right|}{n}} \end{cases}$$

Where

$$\alpha = \frac{n^2+n}{2} + (n^2-n)r^2 + (n-1)\cos\left(\frac{2\pi}{n}\right)(nr^2 - 2r^2 + 2r).$$

Two upper bounds for spectral norm,

- $$\|A\|_2 \leq \begin{cases} |r|(2n-1), & |r|>1; \\ n+|r|(n-1), & |r|\leq 1. \end{cases}$$

- $\|A\|_2 \leq \frac{(n-1)(n+2)}{\sqrt{2}} + \sqrt{2}|r|n.$

Proof. The matrix $A = RFPrLrRcircfr_r\left(e\left(\frac{0}{n}\right), e\left(\frac{1}{n}\right), e\left(\frac{2}{n}\right), \cdots, e\left(\frac{n-1}{n}\right)\right)$ is of the following form:

$$A = \begin{pmatrix} e(\frac{0}{n}) & e(\frac{1}{n}) & e(\frac{2}{n}) & \cdots & e(\frac{n-2}{n}) & e(\frac{n-1}{n}) \\ re(\frac{n-1}{n}) & e_1 & e(\frac{1}{n}) & \cdots & e(\frac{n-3}{n}) & e(\frac{n-2}{n}) \\ re(\frac{n-2}{n}) & e_2 & e_1 & \cdots & e(\frac{n-4}{n}) & e(\frac{n-3}{n}) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ re(\frac{1}{n}) & re(\frac{2}{n})+re(\frac{1}{n}) & re(\frac{3}{n})+re(\frac{2}{n}) & \cdots & e_2 & e_1 \end{pmatrix}_{n\times n},$$

where $e_1 = e\left(\frac{0}{n}\right) + re\left(\frac{n-1}{n}\right), e_2 = re\left(\frac{n-1}{n}\right) + re\left(\frac{n-2}{n}\right)$.
Using the definition of Euclidean norm and Lemma 1, $a_k = e\left(\frac{k}{n}\right)$, we have

$$\|A\|_F^2 = \sum_{j=0}^{n-1}|a_j|^2 + r^2\sum_{j=1}^{n-1}|a_j|^2 + \left(n-1\right)|a_0 + ra_{n-1}|^2 + r^2\sum_{j=1}^{n-2} j|a_{j+1}+a_j|^2 + \sum_{j=1}^{n-2}(n-j-1)|a_j|^2,$$

then $|a_k| = |e\left(\frac{k}{n}\right)| = 1$,

$$\begin{aligned} |a_0 + ra_{n-1}|^2 &= |e\left(\frac{0}{n}\right) + re\left(\frac{n-1}{n}\right)|^2 = \left|e\left(\frac{0}{n}\right) + re\left(\frac{-1}{n}\right)\right|^2 = \left|e\left(\frac{1}{n}\right) + r\right|^2 \\ &= \left|\cos\left(\frac{2\pi}{n}\right) + r + i\sin\left(\frac{2\pi}{n}\right)\right|^2 = 2r\cos\left(\frac{2\pi}{n}\right) + r^2 + 1, \\ |a_{j+1} + a_j|^2 &= \left|e\left(\frac{j+1}{n}\right) + e\left(\frac{j}{n}\right)\right|^2 = \left|e\left(\frac{j}{n}\right)(e\left(\frac{1}{n}\right) + 1)\right|^2 = \left|e\left(\frac{1}{n}\right) + 1\right|^2 \\ &= 2\cos\left(\frac{2\pi}{n}\right) + 2 \end{aligned}$$

therefore,

$$\begin{aligned} \|A\|_F^2 &= \frac{n^2-n+2}{2} + (n-1)r^2 + (n-1)\left|e\left(\frac{1}{n}\right) + r\right|^2 + r^2\left|e\left(\frac{1}{n}\right) + 1\right|^2 \cdot \frac{(n-1)(n-2)}{2} \\ &= \frac{n^2+n}{2} + (n^2-n)r^2 + (n-1)\cos\left(\frac{2\pi}{n}\right)(nr^2 - 2r^2 + 2r) = \alpha. \end{aligned}$$

Using $\|A\|_2 \geq \frac{1}{\sqrt{n}}\|A\|_F$, we can get the lower bound $\|A\|_2 \geq \sqrt{\frac{\alpha}{n}}$,
where $\alpha = \frac{n^2+n}{2} + (n^2-n)r^2 + (n-1)\cos\left(\frac{2\pi}{n}\right)(nr^2 - 2r^2 + 2r)$.

By Lemma 8, we can get the other lower bounds of $\|A\|_2$,

$$\begin{aligned} r_1 &= 0, \\ r_2 &= (2r-1)e\left(\frac{-1}{n}\right), \\ r_3 &= (2r-1)e\left(\frac{-2}{n}\right)\left(1+e\left(\frac{1}{n}\right)\right), \\ r_4 &= (2r-1)e\left(\frac{-3}{n}\right)\left(1+e\left(\frac{1}{n}\right)+e\left(\frac{2}{n}\right)\right), \\ &\vdots \\ r_n &= (2r-1)e\left(\frac{-n+1}{n}\right)\left(1+e\left(\frac{1}{n}\right)+\cdots+e\left(\frac{n-2}{n}\right)\right), \end{aligned}$$

and then we have

$$\begin{aligned} \|A\|_2 &\geq \frac{|r_1+r_2+\cdots+r_n|}{n} = \frac{\left| |2r-1| \sum_{k=0}^{n-2}\sum_{j=0}^{k} e\left(\frac{-k+j-1}{n}\right)\right|}{n}, \\ \|A\|_2 &\geq \sqrt{\frac{|r_1|^2+|r_2|^2+\cdots+|r_n|^2}{n}} = \sqrt{\frac{\left| |2r-1|^2 \sum_{k=0}^{n-2} |\sum_{j=0}^{k} e\left(\frac{-k+j-1}{n}\right)|^2\right|}{n}} \end{aligned}$$

On the other hand,

$$\begin{aligned} A &= \begin{pmatrix} e(\frac{0}{n}) & e(\frac{1}{n}) & \cdots & e(\frac{n-2}{n}) & e(\frac{n-1}{n}) \\ re(\frac{n-1}{n}) & e\left(\frac{0}{n}\right)+re\left(\frac{n-1}{n}\right) & \cdots & e(\frac{n-3}{n}) & e(\frac{n-2}{n}) \\ re(\frac{n-2}{n}) & re\left(\frac{n-1}{n}\right)+re\left(\frac{n-2}{n}\right) & \cdots & e(\frac{n-4}{n}) & e(\frac{n-3}{n}) \\ \vdots & \vdots & & \vdots & \vdots \\ re(\frac{1}{n}) & re(\frac{2}{n})+re(\frac{1}{n}) & \cdots & re\left(\frac{n-1}{n}\right)+re\left(\frac{n-2}{n}\right) & e\left(\frac{0}{n}\right)+re\left(\frac{n-1}{n}\right) \end{pmatrix}_{n\times n} \\ &= \begin{pmatrix} e(\frac{0}{n}) & e(\frac{1}{n}) & \cdots & e(\frac{n-2}{n}) & e(\frac{n-1}{n}) \\ re(\frac{n-1}{n}) & e\left(\frac{0}{n}\right) & \cdots & e(\frac{n-3}{n}) & e(\frac{n-2}{n}) \\ re(\frac{n-2}{n}) & re\left(\frac{n-1}{n}\right) & \cdots & e(\frac{n-4}{n}) & e(\frac{n-3}{n}) \\ \vdots & \vdots & & \vdots & \vdots \\ re(\frac{1}{n}) & re(\frac{2}{n}) & \cdots & re(\frac{n-1}{n}) & e(\frac{0}{n}) \end{pmatrix} \\ &+ r\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & e(\frac{n-1}{n}) & \cdots & 0 & 0 \\ 0 & e(\frac{n-2}{n}) & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & e(\frac{1}{n}) & \cdots & e(\frac{n-2}{n}) & e(\frac{n-1}{n}) \end{pmatrix}. \\ &= C_r + rT. \end{aligned}$$

By the identities of matrix norms, we have

$$\|A\|_2 = \|C_r + rT\|_2 \leq \|C_r\|_2 + |r|\|T\|_2.$$

Where C_r is r-circulant matrix, T is lower Triangular matrix. For the spectral norm of C_r, we defined the basic r-circulant matrix $\Theta_r = Circ_r(0, 1, \cdots, 0)$,

$C_r = \sum_{i=0}^{n-1} c_i\Theta_r^i$, we can get $\|C_r\|_2 = \|\sum_{i=0}^{n-1} c_i\Theta_r^i\|_2 \leq \sum_{i=0}^{n-1} |c_i|\|\Theta_r\|_2^i$.
Since

$$\Theta_r^H\Theta_r = \begin{pmatrix} |r|^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n},$$

when $|r| > 1, \|\Theta_r\|_2 = |r|$,

$$\|C_r\|_2 \leq |r| \sum_{i=0}^{n-1} \left| e\left(\frac{i}{n}\right) \right| = |r|n.$$

$|r| \leq 1, \|\Theta_r\|_2 = 1$.

$$\|C_r\|_2 \leq \sum_{i=0}^{n-1} \left| e\left(\frac{i}{n}\right) \right| = n.$$

For matrix T, let matrices $Q_1, Q_2, \cdots, Q_{n-1}$ be as follows:

$$Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix}_{n\times n}, Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_{n-2} & 0 \end{pmatrix}_{n\times n}, \cdots,$$

$$Q_{n-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}.$$

Then we can get

$$T = a_{n-1}Q_1 + a_{n-2}Q_2 + \cdots + a_1Q_{n-1},$$

and the spectral norms of $Q_1, Q_2, \cdots, Q_{n-1}$,

$$\|Q_1\|_2 = \|Q_2\|_2 = \cdots = \|Q_{n-1}\|_2 = 1,$$

hence,

$$\begin{aligned} \|T\|_2 &= \|a_{n-1}Q_1 + a_{n-2}Q_2 + \cdots + a_1Q_{n-1}\|_2 \\ &\leq |a_{n-1}|\|Q_1\|_2 + |a_{n-2}|\|Q_2\|_2 + \cdots + |a_1|\|Q_{n-1}\|_2 = \sum_{j=1}^{n-1} |a_j| \\ &= \sum_{j=1}^{n-1} \left| e\left(\frac{j}{n}\right) \right| = n-1. \end{aligned}$$

thus we have

$$\|A\|_2 \leq \begin{cases} |r|(2n-1), & |r| > 1; \\ n + |r|(n-1), & |r| \leq 1. \end{cases}$$

On the other hand, we give another upper bound for spectral norm, by using Lemma 6 and Lemma 7, $\|\mathcal{P}\|_2 = \sqrt{2}$, $\|A\|_2 = \|\mathcal{T}\mathcal{P}\| \leq \|\mathcal{T}\|_2\|\mathcal{P}\|_2 = \sqrt{2}\|\mathcal{T}\|_2$, now we discuss $\|\mathcal{T}\|_2$,
since

$$\mathcal{T} = \sum_{k=1}^{n-1} (\sum_{i=0}^{k} (-1)^{n-i+1} a_i)\mathcal{B}^k + \sum_{k=0}^{n-1} r a_{n-k} C^k,$$

so

$$\|\mathcal{T}\|_2 = \|\sum_{k=1}^{n-1}(\sum_{i=0}^{k}(-1)^{n-i+1}a_i)\mathcal{B}^k + \sum_{k=0}^{n-1} ra_{n-k}C^k\|_2 \le \sum_{k=1}^{n-1}\sum_{i=0}^{k}|a_i| + |r|\sum_{k=0}^{n-1}|a_{n-k}|,$$

by $|a_k| = |e(\frac{k}{n})| = 1$, thus $\|\mathcal{T}\|_2 \le \frac{(n-1)(n+2)}{2} + |r|n$,
and then we get

$$\|A\|_2 \le \frac{(n-1)(n+2)}{\sqrt{2}} + \sqrt{2}|r|n,$$

we can find the upper bound is terrible. Hence, we always use Lemma 5 to compute the upper bounds of spectral norm.
So the proof is completed.

Theorem 2 Let $B = RFPrLrRcircfr_r\left(\cos\left(\frac{0\cdot\pi}{n}\right), \cos\left(\frac{1\cdot\pi}{n}\right), \cos\left(\frac{2\cdot\pi}{n}\right), \cdots, \cos\left(\frac{(n-1)\cdot\pi}{n}\right)\right)$ be a $n\times n$ RFPrLrR circulant matrix, then we have three lower bounds

$$\|B\|_2 \ge \begin{cases} \sqrt{\frac{\beta}{n}}, \\ \left|1 + \frac{2r-1}{n}\sum_{k=1}^{n-1} k\cos\left(\frac{k\pi}{n}\right)\right|, \\ \sqrt{\frac{1+\sum_{k=2}^{n}\left[(2r-1)\sum_{j=1}^{k-1}\cos\left(\frac{(n-j)\pi}{n}\right)+1\right]^2}{n}} \end{cases}$$

where

$$\beta = \frac{n^2}{2} + (2r^2-1)\frac{n^2-2n}{4} + 2r^2\left(\frac{n^2-5n+4}{4}\right)\cos\left(\frac{\pi}{n}\right) - 2(n-1)r\cos\left(\frac{\pi}{n}\right).$$

When n is even,
if $|r| \ge 1$,

$$\|B\|_2 \le 2|r|\sum_{j=0}^{n-1}\left|\cos\left(\frac{j\pi}{n}\right)\right| - |r| = |r|\left(2\cot\left(\frac{\pi}{2n}\right) - 1\right),$$

if $|r| < 1$,

$$\|B\|_2 \le \sum_{j=0}^{n-1}\left|\cos\left(\frac{j\pi}{n}\right)\right| + |r|\sum_{j=1}^{n-1}\left|\cos\left(\frac{j\pi}{n}\right)\right| = (|r|+1)\cot\left(\frac{\pi}{2n}\right) - |r|.$$

When n is odd, if $|r| \ge 1$,

$$\|B\|_2 \le 2|r|\sum_{j=0}^{n-1}\left|\cos\left(\frac{j\pi}{n}\right)\right| - |r| = |r|\left(2\csc\left(\frac{\pi}{2n}\right) - 1\right),$$

if $|r| < 1$,

$$\|B\|_2 \le \sum_{j=0}^{n-1}\left|\cos\left(\frac{j\pi}{n}\right)\right| + |r|\sum_{j=1}^{n-1}\left|\cos\left(\frac{j\pi}{n}\right)\right| = (|r|+1)\csc\left(\frac{\pi}{2n}\right) - |r|.$$

Proof.

$$B = \begin{pmatrix} \cos(\frac{0\pi}{n}) & \cos(\frac{1\pi}{n}) & \cdots & \cos(\frac{(n-2)\pi}{n}) & \cos(\frac{(n-1)\pi}{n}) \\ r\cos(\frac{(n-1)\pi}{n}) & m_1 & \cdots & \cos(\frac{(n-3)\pi}{n}) & \cos(\frac{(n-2)\pi}{n}) \\ r\cos(\frac{(n-2)\pi}{n}) & m_2 & \cdots & \cos(\frac{(n-4)\pi}{n}) & \cos(\frac{(n-3)\pi}{n}) \\ \vdots & \vdots & & \vdots & \vdots \\ r\cos(\frac{1\pi}{n}) & r\cos(\frac{2\pi}{n}) + r\cos(\frac{1\pi}{n}) & \cdots & m_2 & m_1 \end{pmatrix}_{n\times n},$$

where $m_1 = \cos\left(\frac{0\pi}{n}\right) + r\cos\left(\frac{(n-1)\pi}{n}\right)$, $m_2 = r\cos\left(\frac{(n-1)\pi}{n}\right) + r\cos\left(\frac{(n-2)\pi}{n}\right)$.
Using the Frobenius norms and Theorem 1, $b_j = \cos\left(\frac{j\pi}{n}\right)$, we have

$$\begin{aligned}
||B||_F^2 &= \sum_{j=0}^{n-1} b_j^2 + r^2\sum_{j=1}^{n-1} b_j^2 + \left(n-1\right)(b_0 + rb_{n-1})^2 + r^2\sum_{j=1}^{n-2} j(b_{j+1}+b_j)^2 + \sum_{j=1}^{n-2}(n-j-1)b_j^2 \\
&= n\sum_{j=0}^{n-1} b_j^2 + (r^2-1)\sum_{k=1}^{n-1}\sum_{j=n-k}^{n-1} b_j^2 + 2r^2\sum_{k=1}^{n-2}\sum_{j=n-k-1}^{n-2} b_jb_{j+1} + 2(n-1)rb_0b_{n-1} \\
&= n\sum_{j=0}^{n-1} b_j^2 + (r^2-1)\sum_{k=1}^{n-1}\left(\sum_{j=0}^{n-1} b_j^2 - \sum_{j=0}^{n-k-1} b_j^2\right) + 2r^2\sum_{k=1}^{n-2}\left(\sum_{j=0}^{n-2} b_jb_{j+1} - \sum_{j=0}^{n-k-2} b_jb_{j+1}\right) \\
&+ 2(n-1)rb_0b_{n-1},
\end{aligned}$$

where $b_j = \cos\left(\frac{j\pi}{n}\right)$, by Lemma 1, $\sum_{j=0}^{n-1}\cos^2\left(\frac{j\pi}{n}\right) = \frac{n}{2}$.

Using the identities $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $e(x) = e^{2\pi i x}$, $\cos\frac{(2j+1)\pi}{n} = \frac{e\left(\frac{2j+1}{2n}\right)+e\left(-\frac{2j+1}{2n}\right)}{2}$,
we can get

$$\begin{aligned}
\sum_{k=1}^{n-1}\sum_{j=0}^{n-k-1}\cos^2\left(\frac{j\pi}{n}\right) &= \frac{n^2}{4}, \\
\sum_{j=0}^{n-1}\cos\left(\frac{(2j+1)\pi}{n}\right) &= \frac{1}{2}\sum_{j=0}^{n-1}\left(e\left(\frac{2j+1}{2n}\right) + e\left(-\frac{2j+1}{2n}\right)\right) \\
&= \frac{1}{2}\left(\frac{e\left(\frac{1}{2n}\right)(1-e(1))}{1-e(\frac{1}{n})} + \frac{e\left(\frac{-1}{2n}\right)(1-e(-1))}{1-e\left(\frac{-1}{n}\right)}\right) \\
&= 0, \\
\sum_{j=0}^{n-1}\cos\left(\frac{j\pi}{n}\right)\cos\left(\frac{(j+1)\pi}{n}\right) &= \frac{1}{2}\sum_{j=0}^{n-1}\cos\left(\frac{(2j+1)\pi}{n}\right) + \frac{n}{2}\cos\left(\frac{\pi}{n}\right) \\
&= \frac{n}{2}\cos\left(\frac{\pi}{n}\right).
\end{aligned}$$

Hence, $\sum_{j=0}^{n-2}\cos\left(\frac{j\pi}{n}\right)\cos\left(\frac{(j+1)\pi}{n}\right) = \frac{n-2}{2}\cos\left(\frac{\pi}{n}\right)$, and then we can get

$$\begin{aligned}
&\sum_{k=1}^{n-2}\sum_{j=0}^{n-k-2}\cos\left(\frac{j\pi}{n}\right)\cos\left(\frac{(j+1)\pi}{n}\right)\\
=\ &\frac{1}{2}\sum_{k=1}^{n-2}\sum_{j=0}^{n-k-2}\cos\left(\frac{(2j+1)\pi}{n}\right)+\cos\left(\frac{j\pi}{n}\right)\sum_{k=1}^{n-2}\frac{n-k-1}{2}\\
=\ &\frac{1}{2}\sum_{k=1}^{n-2}\sum_{j=0}^{n-k-2}\left(e\left(\frac{2j+1}{2n}\right)+e\left(-\frac{2j+1}{2n}\right)\right)+\frac{n^2-3n+2}{4}\cos\left(\frac{j\pi}{n}\right)\\
=\ &\frac{n^2-3n+4}{4}\cos\left(\frac{\pi}{n}\right).
\end{aligned}$$

Therefore,

$$\|B\|_F^2=\frac{n^2}{2}+(2r^2-1)\frac{n^2-2n}{4}+2r^2\left(\frac{n^2-5n+4}{4}\right)\cos\left(\frac{\pi}{n}\right)-2(n-1)r\cos\left(\frac{\pi}{n}\right)=\beta.$$

We can get $\|B\|_2\geq\frac{1}{\sqrt{n}}\|B\|_F=\sqrt{\frac{\beta}{n}}$,
where

$$\beta=\frac{n^2}{2}+(2r^2-1)\frac{n^2-2n}{4}+2r^2\left(\frac{n^2-5n+4}{4}\right)\cos\left(\frac{\pi}{n}\right)-2(n-1)r\cos\left(\frac{\pi}{n}\right).$$

Next, we give the row sums of B, denote as $r_1,r_2,r_3,\cdots,r_n$,

$$\begin{aligned}
r_1&=\sum_{k=0}^{n-1}\cos\left(\frac{k\pi}{n}\right)=1,\\
r_2&=(2r-1)\cos\left(\frac{(n-1)\pi}{n}\right)+1,\\
r_3&=(2r-1)\left[\cos\left(\frac{(n-2)\pi}{n}\right)+\cos\left(\frac{(n-1)\pi}{n}\right)\right]+1,\\
r_4&=(2r-1)\left[\cos\left(\frac{(n-3)\pi}{n}\right)+\cos\left(\frac{(n-2)\pi}{n}\right)+\cos\left(\frac{(n-1)\pi}{n}\right)\right]+1,\\
&\vdots\\
r_n&=(2r-1)\left[\cos\left(\frac{1\pi}{n}\right)+\cos\left(\frac{2\pi}{n}\right)+\cdots+\cos\left(\frac{(n-1)\pi}{n}\right)\right]+1=1,
\end{aligned}$$

so we have $r_1=1$, and $r_k=(2r-1)\sum_{j=1}^{k-1}\cos\left(\frac{(n-j)\pi}{n}\right)+1$.

By Lemma 5, we can get the other lower bounds of $\|B\|_2$,

$$\|B\|_2\geq\frac{|r_1+r_2+\cdots+r_n|}{n}=\left|1+\frac{2r-1}{n}\sum_{k=1}^{n-1}k\cos\left(\frac{k\pi}{n}\right)\right|,$$

$$\|B\|_2\geq\sqrt{\frac{|r_1|^2+|r_2|^2+\cdots+|r_n|^2}{n}}=\sqrt{\frac{1+\sum_{k=2}^{n}\left[(2r-1)\sum_{j=1}^{k-1}\cos\left(\frac{(n-j)\pi}{n}\right)+1\right]^2}{n}},$$

by Lemma 4, we can get specific result.
For another, using Theorem 1, and for the matrices $Q_1, Q_2, Q_3, \cdots, Q_{n-1}$ as mentioned above, we have

$$\begin{aligned}
B &= \begin{pmatrix} \cos(\frac{0\pi}{n}) & \cos(\frac{1\pi}{n}) & \cdots & \cos(\frac{(n-2)\pi}{n}) & \cos(\frac{(n-1)\pi}{n}) \\ r\cos(\frac{(n-1)\pi}{n}) & m_1 & \cdots & \cos(\frac{(n-3)\pi}{n}) & \cos(\frac{(n-2)\pi}{n}) \\ r\cos(\frac{(n-2)\pi}{n}) & m_2 & \cdots & \cos(\frac{(n-4)\pi}{n}) & \cos(\frac{(n-3)\pi}{n}) \\ \vdots & \vdots & & \vdots & \vdots \\ r\cos(\frac{1\pi}{n}) & r\cos(\frac{2\pi}{n}) + r\cos(\frac{1\pi}{n}) & \cdots & m_2 & m_1 \end{pmatrix}_{n\times n} \\
&= \begin{pmatrix} \cos(\frac{0\pi}{n}) & \cos(\frac{1\pi}{n}) & \cdots & \cos(\frac{(n-2)\pi}{n}) & \cos(\frac{(n-1)\pi}{n}) \\ r\cos(\frac{(n-1)\pi}{n}) & \cos\left(\frac{0\pi}{n}\right) & \cdots & \cos\left(\frac{(n-3)\pi}{n}\right) & \cos\left(\frac{(n-2)\pi}{n}\right) \\ r\cos(\frac{(n-2)\pi}{n}) & r\cos\left(\frac{(n-1)\pi}{n}\right) & \cdots & \cos\left(\frac{(n-4)\pi}{n}\right) & \cos\left(\frac{(n-3)\pi}{n}\right) \\ \vdots & \vdots & & \vdots & \vdots \\ r\cos(\frac{1\pi}{n}) & r\cos(\frac{2\pi}{n}) & \cdots & r\cos\left(\frac{(n-1)\pi}{n}\right) & \cos\left(\frac{0\pi}{n}\right) \end{pmatrix} \\
&+ r\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \cos\left(\frac{(n-1)\pi}{n}\right) & \cdots & 0 & 0 \\ 0 & \cos\left(\frac{(n-2)\pi}{n}\right) & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \cos(\frac{1\pi}{n}) & \cdots & \cos\left(\frac{(n-2)\pi}{n}\right) & \cos\left(\frac{(n-1)\pi}{n}\right) \end{pmatrix} \\
&= C_{r_1} + rT_1.
\end{aligned}$$

Obviously, C_{r_1} is $r-$circulant matrix, T_1 is lower triangular matrix, by the Lemma 3, Theorem 1 and the identities of matrix norms [12], we can get

$$\|B\|_2 = \|C_{r_1} + rT_1\|_2 \le \|C_{r_1}\|_2 + |r|\|T_1\|_2.$$

when n is even,
(i)$|r| > 1$,

$$\|B\|_2 \le 2|r|\sum_{j=0}^{n-1}\left|\cos\left(\frac{j\pi}{n}\right)\right| - |r| = |r|\left(2\cot\left(\frac{\pi}{2n}\right) - 1\right),$$

(ii)$|r| \le 1$,

$$\|B\|_2 \le (|r|+1)\sum_{j=0}^{n-1}\left|\cos\left(\frac{j\pi}{n}\right)\right| - |r| = (|r|+1)\cot\left(\frac{\pi}{2n}\right) - |r|.$$

When n is odd,
(i)$|r| > 1$,

$$\|B\|_2 \le 2|r|\sum_{j=0}^{n-1}\left|\cos\left(\frac{j\pi}{n}\right)\right| - |r| = |r|\left(2\csc\left(\frac{\pi}{2n}\right) - 1\right),$$

(ii)$|r| \le 1$,

$$\|B\|_2 \le (|r|+1)\sum_{j=0}^{n-1}\left|\cos\left(\frac{j\pi}{n}\right)\right| - |r| = (|r|+1)\csc\left(\frac{\pi}{2n}\right) - |r|.$$

Thus the proof is completed.
Theorem 3 Let $C = RFPrLrRcircfr_r\left(\sin\left(\frac{0\cdot\pi}{n}\right), \sin\left(\frac{1\cdot\pi}{n}\right), \sin\left(\frac{2\cdot\pi}{n}\right), \cdots, \sin\left(\frac{(n-1)\cdot\pi}{n}\right)\right)$ be a $n\times n$ RFPrLrR circulant

matrix, we have three lower bounds

$$\|C\|_2 \geq \begin{cases} \sqrt{\frac{\xi}{n}}, \\ \left|\cot\left(\frac{\pi}{2n}\right) + \frac{2r-1}{n}\sum_{k=1}^{n-1} k\sin\left(\frac{k\pi}{n}\right)\right|, \\ \sqrt{\frac{\cot^2\left(\frac{\pi}{2n}\right) + \sum_{k=2}^{n}\left[(2r-1)\sum_{j=1}^{k-1}\sin\left(\frac{(n-j)\pi}{n}\right) + \cot\left(\frac{\pi}{2n}\right)\right]^2}{n}} \end{cases}$$

$$\|C\|_2 \leq \begin{cases} 2|r|\cot\left(\frac{\pi}{2n}\right), & |r| > 1; \\ (|r|+1)\cot\left(\frac{\pi}{2n}\right), & |r| \leq 1. \end{cases}$$

where

$$\xi = \frac{n^2}{2} + \frac{(2r^2-1)n^2}{4} + 2r^2\cos\left(\frac{\pi}{n}\right)\left(\frac{n^2-n}{4}\right).$$

Proof.

$$C = \begin{pmatrix} \sin(\frac{0\pi}{n}) & \sin(\frac{1\pi}{n}) & \cdots & \sin(\frac{(n-2)\pi}{n}) & \sin(\frac{(n-1)\pi}{n}) \\ r\sin(\frac{(n-1)\pi}{n}) & s_1 & \cdots & \sin(\frac{(n-3)\pi}{n}) & \sin(\frac{(n-2)\pi}{n}) \\ r\sin(\frac{(n-2)\pi}{n}) & s_2 & \cdots & \sin(\frac{(n-4)\pi}{n}) & \sin(\frac{(n-3)\pi}{n}) \\ \vdots & \vdots & & \vdots & \vdots \\ r\sin(\frac{1\pi}{n}) & r\sin(\frac{2\pi}{n}) + r\sin(\frac{1\pi}{n}) & \cdots & s_2 & s_1 \end{pmatrix}_{n\times n},$$

where $s_1 = \sin\left(\frac{0\pi}{n}\right) + r\sin\left(\frac{(n-1)\pi}{n}\right)$, $s_2 = r\sin\left(\frac{(n-1)\pi}{n}\right) + r\sin\left(\frac{(n-2)\pi}{n}\right)$.
Using the Frobenius norms and Theorem 1, $c_j = \sin\left(\frac{j\pi}{n}\right)$, we have

$$\begin{aligned} \|C\|_F^2 &= \sum_{j=0}^{n-1} c_j^2 + r^2\sum_{j=1}^{n-1} c_j^2 + r^2\sum_{j=1}^{n-2} j(c_j + c_{j+1})^2 + \sum_{j=1}^{n-1}(n-j-1)c_j^2 + (n-1)(c_0 + rc_{n-1})^2 \\ &= \sum_{j=0}^{n-1} c_j^2 + n\sum_{j=1}^{n-1} c_j^2 + (2r^2-1)\sum_{j=1}^{n-1} jc_j^2 + 2r^2\sum_{j=1}^{n-2} jc_jc_{j+1} + (n-1)(c_0 + rc_{n-1})^2 - r^2(n-1)c_{n-1}^2 \\ &= n\sum_{j=0}^{n-1} c_j^2 + (2r^2-1)\sum_{k=1}^{n-1}\sum_{j=n-k}^{n-1} c_j^2 + 2r^2\sum_{k=1}^{n-2}\sum_{j=n-k-1}^{n-2} c_jc_{j+1} + 2(n-1)rc_0c_{n-1} \\ &= n\sum_{j=0}^{n-1} c_j^2 + (2r^2-1)\sum_{k=1}^{n-1}\left(\sum_{j=0}^{n-1} c_j^2 - \sum_{j=0}^{n-k-1} c_j^2\right) + 2r^2\sum_{k=1}^{n-2}\left(\sum_{j=0}^{n-2} c_jc_{j+1} - \sum_{j=0}^{n-k-2} c_jc_{j+1}\right) \\ &+ 2(n-1)rc_0c_{n-1}, \end{aligned}$$

by Lemma 1, $\sum_{j=0}^{n-1}\sin^2\left(\frac{j\pi}{n}\right) = \frac{n}{2}$. Using the identities

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, e(x) = e^{2\pi ix},$$

$$\sum_{k=1}^{n-1}\sum_{j=0}^{n-k-1}\sin^2\left(\frac{j\pi}{n}\right)=\frac{n(n-2)}{4}.$$

$$\begin{aligned}\sum_{j=0}^{n-2}\sin\left(\frac{j\pi}{n}\right)\sin\left(\frac{(j+1)\pi}{n}\right) &= \frac{1}{2}\sum_{j=0}^{n-2}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{(2j+1)\pi}{n}\right)\right)\\ &= \frac{n}{2}\cos\left(\frac{\pi}{n}\right),\end{aligned}$$

by Theorem 1, we can get

$$\begin{aligned}&\sum_{k=1}^{n-2}\sum_{j=0}^{n-k-2}\sin\left(\frac{j\pi}{n}\right)\sin\left(\frac{(j+1)\pi}{n}\right)\\ =\ &\sum_{k=1}^{n-2}\frac{n-k-1}{2}\cos\left(\frac{\pi}{n}\right)-\frac{1}{2}\sum_{k=1}^{n-2}\sum_{j=0}^{n-k-2}\cos\left(\frac{(2j+1)\pi}{n}\right)\\ =\ &\frac{n^2-3n+2}{4}\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n}\right)\\ =\ &\frac{n^2-3n}{4}\cos\left(\frac{\pi}{n}\right).\end{aligned}$$

Therefore,

$$\|C\|_F^2=\frac{n^2}{2}+\frac{(2r^2-1)n^2}{4}+2r^2\cos\left(\frac{\pi}{n}\right)\left(\frac{n^2-n}{4}\right)=\xi.$$

We can get $\|C\|_2\geq\frac{1}{\sqrt{n}}\|C\|_F=\sqrt{\frac{\xi}{n}}$, where $\xi=\frac{n^2}{2}+\frac{(2r^2-1)n^2}{4}+2r^2\cos\left(\frac{\pi}{n}\right)\left(\frac{n^2-n}{4}\right)$.
By virtue of Lemma 8, we have

$$\begin{aligned}&r_1=\sum_{k=0}^{n-1}\sin\left(\frac{k\pi}{n}\right)=\cot\left(\frac{\pi}{2n}\right),\\ &r_2=(2r-1)\sin\left(\frac{(n-1)\pi}{n}\right)+\cot\left(\frac{\pi}{2n}\right),\\ &r_3=(2r-1)\left[\sin\left(\frac{(n-2)\pi}{n}\right)+\sin\left(\frac{(n-1)\pi}{n}\right)\right]+\cot\left(\frac{\pi}{2n}\right),\\ &r_4=(2r-1)\left[\sin\left(\frac{(n-3)\pi}{n}\right)+\sin\left(\frac{(n-2)\pi}{n}\right)+\sin\left(\frac{(n-1)\pi}{n}\right)\right]+\cot\left(\frac{\pi}{2n}\right),\\ &\vdots\\ &r_n=(2r-1)\left[\sin\left(\frac{1\pi}{n}\right)+\sin\left(\frac{2\pi}{n}\right)+\cdots+\sin\left(\frac{(n-1)\pi}{n}\right)\right]+\cot\left(\frac{\pi}{2n}\right)=2r\cot\left(\frac{\pi}{2n}\right),\\ &\|C\|_2\geq\frac{|r_1+r_2+\cdots+r_n|}{n}=\left|\cot\left(\frac{\pi}{2n}\right)+\frac{2r-1}{n}\sum_{k=1}^{n-1}k\sin\left(\frac{k\pi}{n}\right)\right|.\end{aligned}$$

Thus we have,

$$
\begin{aligned}
||C||_2 &\geq \sqrt{\frac{|r_1|^2+|r_2|^2+\cdots+|r_n|^2}{n}} \\
&= \sqrt{\frac{\cot^2\left(\frac{\pi}{2n}\right)+\sum_{k=2}^{n}\left[(2r-1)\sum_{j=1}^{k-1}\sin\left(\frac{(n-j)\pi}{n}\right)+\cot\left(\frac{\pi}{2n}\right)\right]^2}{n}}.
\end{aligned}
$$

For another, using Theorem 1, and for the matrices $Q_1, Q_2, Q_3, \cdots, Q_{n-1}$ as mentioned above, we have

$$
C = \begin{pmatrix}
\sin(\frac{0\pi}{n}) & \sin(\frac{1\pi}{n}) & \cdots & \sin(\frac{(n-2)\pi}{n}) & \sin(\frac{(n-1)\pi}{n}) \\
r\sin(\frac{(n-1)\pi}{n}) & s_1 & \cdots & \sin(\frac{(n-3)\pi}{n}) & \sin(\frac{(n-2)\pi}{n}) \\
r\sin(\frac{(n-2)\pi}{n}) & s_2 & \cdots & \sin(\frac{(n-4)\pi}{n}) & \sin(\frac{(n-3)\pi}{n}) \\
\vdots & \vdots & & \vdots & \vdots \\
r\sin(\frac{1\pi}{n}) & r\sin(\frac{2\pi}{n})+r\sin(\frac{1\pi}{n}) & \cdots & s_2 & s_1
\end{pmatrix}_{n\times n}
$$

$$
\begin{aligned}
C &= \begin{pmatrix}
\sin(\frac{0\pi}{n}) & \sin(\frac{1\pi}{n}) & \cdots & \sin(\frac{(n-2)\pi}{n}) & \sin(\frac{(n-1)\pi}{n}) \\
r\sin(\frac{(n-1)\pi}{n}) & \sin\left(\frac{0\pi}{n}\right) & \cdots & \sin\left(\frac{(n-3)\pi}{n}\right) & \sin\left(\frac{(n-2)\pi}{n}\right) \\
r\sin(\frac{(n-2)\pi}{n}) & r\sin\left(\frac{(n-1)\pi}{n}\right) & \cdots & \sin\left(\frac{(n-4)\pi}{n}\right) & \sin\left(\frac{(n-3)\pi}{n}\right) \\
\vdots & \vdots & & \vdots & \vdots \\
r\sin(\frac{1\pi}{n}) & r\sin(\frac{2\pi}{n}) & \cdots & r\sin\left(\frac{(n-1)\pi}{n}\right) & \sin\left(\frac{0\pi}{n}\right)
\end{pmatrix} \\
&+ r\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & \sin\left(\frac{(n-1)\pi}{n}\right) & \cdots & 0 & 0 \\
0 & \sin\left(\frac{(n-2)\pi}{n}\right) & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & \sin(\frac{1\pi}{n}) & \cdots & \sin\left(\frac{(n-2)\pi}{n}\right) & \sin\left(\frac{(n-1)\pi}{n}\right)
\end{pmatrix} \\
&= C_{r_2} + rT_2.
\end{aligned}
$$

By the Lemma 1,2,3, Theorem 1 and the reference [12], we can get

$$||C||_2 = ||C_{r_2} + rT_2||_2 \leq ||C_{r_2}||_2 + |r|||T_2||_2.$$

Thus,

$$|r| > 1, ||C||_2 \leq 2|r|\cot\left(\frac{\pi}{2n}\right),$$

$$|r| \leq 1, ||C||_2 \leq (|r|+1)\cot\left(\frac{\pi}{2n}\right).$$

So the proof is completed.

3. Conclusion

In the present paper, we obtained some bounds for the spectral norms of the RFPrLrR circulant matrices with the exponential forms and trigonometric functions by using a different methods. Based on this article, the following matrix can be defined with the help of geometric circulant matrix, similar to the relationship of $r-$circulant and RFPrLrR circulant matrices for further research. Moreover, various linear algebraic

properties of such matrices can be studied by journal readers.
Let F_{r^*} be an $n\times n$ matrix. The geometric RFPrLrR circulant matrix can be defined as follows:

$$F_{r^*}=\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1}\\ ra_{n-1} & a_0+ra_{n-1} & a_1 & \cdots & a_{n-3} & a_{n-2}\\ r^2a_{n-2} & ra_{n-1}+ra_{n-2} & a_0+ra_{n-1} & \cdots & a_{n-4} & a_{n-3}\\ r^3a_{n-3} & r^2a_{n-2}+ra_{n-3} & ra_{n-1}+ra_{n-2} & \cdots & a_{n-5} & a_{n-4}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ r^{n-1}a_1 & r^{n-2}a_2+ra_1 & r^{n-3}a_3+ra_2 & \cdots & ra_{n-1}+ra_{n-2} & a_0+ra_{n-1}\end{pmatrix}_{n\times n},$$

then we can get some properties for F_{r^*}. For example,

$$\begin{aligned} F_{r^*} &= \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1}\\ ra_{n-1} & a_0+ra_{n-1} & a_1 & \cdots & a_{n-3} & a_{n-2}\\ r^2a_{n-2} & ra_{n-1}+ra_{n-2} & a_0+ra_{n-1} & \cdots & a_{n-4} & a_{n-3}\\ r^3a_{n-3} & r^2a_{n-2}+ra_{n-3} & ra_{n-1}+ra_{n-2} & \cdots & a_{n-5} & a_{n-4}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ r^{n-1}a_1 & r^{n-2}a_2+ra_1 & r^{n-3}a_3+ra_2 & \cdots & ra_{n-1}+ra_{n-2} & a_0+ra_{n-1}\end{pmatrix}\\ &= \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1}\\ ra_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2}\\ r^2a_{n-2} & ra_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3}\\ r^3a_{n-3} & r^2a_{n-2} & ra_{n-1} & \cdots & a_{n-5} & a_{n-4}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ r^{n-1}a_1 & r^{n-2}a_2 & r^{n-3}a_3 & \cdots & ra_{n-1} & a_0\end{pmatrix}\\ &+ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & ra_{n-1} & 0 & \cdots & 0 & 0\\ 0 & ra_{n-2} & ra_{n-1} & \cdots & 0 & 0\\ 0 & ra_{n-3} & ra_{n-2} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & ra_1 & ra_2 & \cdots & ra_{n-2} & ra_{n-1}\end{pmatrix}\\ &= C_{r^*}+rT, \end{aligned}$$

then we have $\|F_{r^*}\|_2=\|C_{r^*}+rT\|_2\le\|C_r^*\|_2+|r|\|T\|_2$, the spectral norm of $\|C_r^*\|_2$, we can see reference [18], thus we can get the upper bound of $\|F_{r^*}\|_2$.

$$\|F_{r^*}\|_E^2=|a_0|^2+(n-1)|a_0+ra_{n-1}|^2+\sum_{k=1}^{n-1}|a_k|^2+\sum_{k=1}^{n-1}|r^{n-k}|^2|a_k|^2+\sum_{k=2}^{n-1}(k-1)|r^{n-k}a_k+ra_{k-1}|^2.$$

For the row sum $r_1,r_2,\cdots,r_n$, $r_1=\sum_{k=0}^{n-1}a_k$,

$$r_k=\sum_{i=2}^{k}(r^{i-1}a_{n+1-i}+ra_{n+1-i})+\sum_{i=0}^{n-k}a_i,\ (k=2,\cdots,n),$$

so by Lemma 8, we can get two lower bounds,

$$\|F_{r^*}\|_2\ge\frac{|r_1+r_2+\cdots+r_n|}{n},\|F_{r^*}\|_2\ge\sqrt{\frac{|r_1|^2+|r_2|^2+\cdots+|r_n|^2}{n}}.$$

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