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Radius of Ma-Minda starlikeness of certain normalised analytic functions

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Abstract. We find the radius of Ma-Minda starlikeness of normalised analytic functions of the form $g(z) = z(f'(z))^\alpha$, $\alpha > 0$ where f is in the class $\mathcal{CV}[A,B]$ of Janowski convex functions and $g(z) = z(zf'(z)/f(z))^\alpha$, $\alpha > 0$ where f is in the class $\mathcal{CV}'$ defined. As particular cases, we obtain criteria for these functions to belong to certain Ma-Minda classes.

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of analytic functions defined on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, normalised by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbb{D}$. A function $f \in \mathcal{A}$ is starlike if f maps $\mathbb{D}$ onto a domain which is starlike with respect to the origin or equivalently if $\operatorname{Re}(zf'(z)/f(z)) > 0$ for all $z \in \mathbb{D}$. Similarly, a function $f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is convex or equivalently if $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ for all $z \in \mathbb{D}$. The class of all starlike functions $f \in \mathcal{A}$ is denoted by $\mathcal{ST}$ and that of all convex functions $f \in \mathcal{A}$ is denoted by $\mathcal{CV}$. There are several subclasses of starlike and convex functions and they can be unified by using the concept of subordination. For two analytic functions f and g, we say that the function f is subordinate to the function g, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists a function $w \in \mathcal{B}$ such that $f = g \circ w$, where $\mathcal{B}$ is the class of all analytic functions $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$. If the function g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Ma and Minda [14] used subordination to define the classes $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$ as

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \tag{1.1}$$

and

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\} \tag{1.2}$$

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respectively, where $\varphi : \mathbb{D} \to \mathbb{C}$ is an analytic function with positive real part, $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0) = 1$ and is symmetric about the real axis and $\varphi'(0) > 0$. For different choices of the function φ in (1.1) and (1.2), different subclasses of the class of starlike and convex functions respectively are obtained. For example, when $\varphi(z) = (1 + Az)/(1 + Bz)$, where $-1 \leqslant B < A \leqslant 1$, the classes $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$ are respectively denoted as $\mathcal{ST}[A, B]$ and $\mathcal{CV}[A, B]$. The class $\mathcal{ST}[A, B]$ is called the class of Janowski starlike functions [7] and $\mathcal{CV}[A, B]$, the class of Janowski convex functions. For $A = 1-2\alpha \quad (0 \leqslant \alpha < 1)$ and $B = -1$, the classes $\mathcal{ST}[A, B]$ and $\mathcal{CV}[A, B]$ respectively reduces to $\mathcal{ST}(\alpha)$, the class of starlike functions of order α and $\mathcal{CV}(\alpha)$, the class of convex functions of order α.

In this paper, we are interested in the classes $C_1^\alpha[A, B]$ and C_2^α respectively defined by

$$C_1^\alpha[A, B] := \{g \in \mathcal{A} : g(z) = z(f'(z))^\alpha, f \in \mathcal{CV}[A, B], \alpha > 0\}$$

and

$$C_2^\alpha := \left\{g \in \mathcal{A} : g(z) = z\left(\frac{zf'(z)}{f(z)}\right)^\alpha, f \in \mathcal{CV}', \alpha > 0\right\},$$

where the class $\mathcal{CV}'$ is defined as

$$\mathcal{CV}' := \left\{f \in \mathcal{A} : \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{z}{1-z}\right\}.$$

For small values of α, the functions behave like the identity function and so will belong to the classes of our interest. However, for $B = -1$, the range of $zg'(z)/g(z)$ is unbounded and hence these classes are not be contained in various subclasses that are obtained for special choices of the function φ. We are particularly interested in the classes $\mathcal{ST}_e := \mathcal{ST}(e^z)$, $\mathcal{ST}_C := \mathcal{ST}(1 + (4/3)z + (2/3)z^2)$, $\mathcal{ST}_{Ne} := \mathcal{ST}(1 + z - (z^3/3))$, $\mathcal{ST}_R := \mathcal{ST}(1 + (z^2 + kz)/(k^2 - kz)), k = 1 + \sqrt{2}$, $\mathcal{ST}_{SG} := \mathcal{ST}(2/(1 + e^{-z}))$, $\mathcal{ST}_{sin} := \mathcal{ST}(1 + \sin z)$, $\mathcal{ST}_{\leftmoon} := \mathcal{ST}(z + \sqrt{1 + z^2})$, $\mathcal{ST}_\wp := \mathcal{ST}(1 + ze^z)$ and $\mathcal{ST}_h := \mathcal{ST}(1 + \sinh^{-1}(z))$.

When the inclusion does not hold, we shall be interested in the corresponding radius problem. Recall that for two subclasses $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$, the largest number $\mathcal{R} \in (0, 1]$ such that for $0 < r < \mathcal{R}$, $f(rz)/r \in \mathcal{F}$ for every $f \in \mathcal{G}$ is called the $\mathcal{F}$-radius of the class $\mathcal{G}$ and is denoted by $\mathcal{R}_\mathcal{F}(\mathcal{G})$. Radius problems have been explored and studied extensively recently in [1, 8, 12, 13, 15, 19]. In this paper, we find the radii constants for functions in the classes $C_1^\alpha[A, B]$ and C_2^α to belong to various classes like the class of Janowski starlike functions, $\mathcal{ST}_e$, $\mathcal{ST}_C$, $\mathcal{ST}_{Ne}$ and so on, by finding the largest positive number $\mathcal{R}$ less than 1 such that the image of the disc $\mathbb{D}_\mathcal{R} := \{z \in \mathbb{C} : |z| < \mathcal{R}\}$ under the mapping $zg'(z)/g(z)$ for g in the classes defined, lie inside the image of the corresponding superordinate functions. The radii obtained are sharp. By the Alexander's Theorem [3, Thm 2.12], the class $C_3^\alpha[A, B]$ defined by

$$C_3^\alpha[A, B] := \left\{g \in \mathcal{A} : g(z) = z\left(\frac{f(z)}{z}\right)^\alpha, f \in \mathcal{ST}[A, B], \alpha > 0\right\}$$

satisfies $C_1^\alpha[A, B] = C_3^\alpha[A, B]$ and, therefore, the radius results obtained in this paper for the class $C_1^\alpha[A, B]$ gives the corresponding results for the class $C_3^\alpha[A, B]$.

2. Radius of starlikeness associated with the Janowski starlike functions

In this section, we discuss condition for the classes $C_1^\alpha[A, B]$ and C_2^α to be contained in the class $\mathcal{ST}[C, D]$ of Janowski starlike functions and find the radius of Janowski starlikeness when the condition fails. We shall make use of the following theorem.

Theorem 2.1. *For $|B| \leqslant 1$, $A \neq B$ and $|D| \leqslant 1$, $C \neq D$, the class $\mathcal{ST}[C, D]$ is contained in the class $\mathcal{ST}[A, B]$ if and only if $|AD - BC| \leqslant |A - B| - |C - D|$.*

Proof. With the restriction that $-1 \leqslant B < A \leqslant 1$ and $-1 \leqslant D < C \leqslant 1$, this was proved by Silverman and Silvia [21]. The general case follows easily from the proof of Theorem 2.3 of [5]. □

We first give a condition for the inclusion $\mathcal{C}_1^\alpha[A,B] \subset \mathcal{ST}[C,D]$ to hold.

Theorem 2.2. *For $-1 \leqslant D < C \leqslant 1$, the class $\mathcal{C}_1^\alpha[A,B]$ is contained in the class $\mathcal{ST}[C,D]$, if and only if*

$$|BC - D(B + \alpha(A - B))| \leqslant C - D - \alpha(A - B).$$

Proof. Let the function $g \in \mathcal{C}_1^\alpha[A,B]$. Then a calculation readily shows that

$$\frac{zg'(z)}{g(z)} = 1 - \alpha + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

Since $f \in \mathcal{CV}[A,B]$, we get

$$\frac{zg'(z)}{g(z)} \prec \frac{1 + (B + \alpha(A - B))z}{1 + Bz}$$

or equivalently $g \in \mathcal{ST}[B + \alpha(A - B), B]$. Therefore, by Theorem 2.1, the class

$$\mathcal{ST}[B + \alpha(A - B), B] \subset \mathcal{ST}[C,D]$$

if and only if the inequality

$$|BC - D(B + \alpha(A - B))| \leqslant C - D - \alpha(A - B)$$

holds. □

If the condition in Theorem 2.2 does not hold, then the following theorem gives the radius of Janowski starlikeness for the class $\mathcal{C}_1^\alpha[A,B]$.

Theorem 2.3. *Let $\alpha > 0$, $-1 \leqslant B < A \leqslant 1$ and $-1 \leqslant D < C \leqslant 1$. If the condition in Theorem 2.2 does not hold, then the radius of starlikeness associated with the class $\mathcal{ST}[C,D]$ for the class $\mathcal{C}_1^\alpha[A,B]$ is given by*

$$\mathcal{R}_{\mathcal{ST}[C,D]}(\mathcal{C}_1^\alpha[A,B]) = \frac{C - D}{\alpha(A - B) + |BC - D(B + \alpha(A - B))|}.$$

Proof. The function $g \in \mathcal{C}_1^\alpha[A,B]$ implies that $g \in \mathcal{ST}[B+\alpha(A-B), B]$. Define the functions $P(z) := (1+Cz)/(1+Dz)$ and $Q(z) := (1 + (B + \alpha(A - B))z)/(1 + Bz)$. We have to determine ρ such that $0 < \rho \leqslant 1$ and $Q(\rho z) \prec P(z)$ for $z \in \mathbb{D}$. Define the function $H := P^{-1} \circ Q$. Then we can see that

$$H(z) = \frac{\alpha(A - B)z}{(C - D) + (BC - D(B + \alpha(A - B)))z}.$$

For $|z| = r$, we get

$$\begin{aligned} |H(z)| &= \frac{\alpha(A - B)|z|}{|(C - D) + (BC - D(B + \alpha(A - B)))z|} \\ &\leqslant \frac{\alpha(A - B)r}{(C - D) - |BC - D(B + \alpha(A - B))|\, r} \end{aligned}$$

and hence $|H(z)| \leqslant 1$ for

$$r \leqslant \frac{C - D}{\alpha(A - B) + |BC - D(B + \alpha(A - B))|} =: \rho.$$

Therefore, the radius of starlikeness associated with the class $\mathcal{ST}[C,D]$ for the class $C_1^{\alpha}[A,B]$ is at least ρ. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}[A,B]$ given by

$$\tilde{f}(z)=\begin{cases}\dfrac{1}{A}\left((1+Bz)^{\left(\frac{A}{B}\right)}-1\right) & \text{if} \quad A\neq 0, B\neq 0\\ \dfrac{1}{A}\left(e^{Az}-1\right) & \text{if} \quad A\neq 0, B=0\\ \dfrac{\log(1+Bz)}{B} & \text{if} \quad A=0, B\neq 0.\end{cases} \tag{2.1}$$

For the above function $\tilde{f}$, the corresponding function $\tilde{g}\in C_1^{\alpha}[A,B]$ given by $\tilde{g}(z)=z(\tilde{f}'(z))^{\alpha}$ satisfies

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)}=1+\frac{\alpha(A-B)z}{1+Bz}. \tag{2.2}$$

Case(i): $BC-D(B+\alpha(A-B))\geqslant 0$. In this case, we have

$$\rho=\frac{C-D}{\alpha(A-B)+BC-D(B+\alpha(A-B))}$$

and for $z=-\rho$, (2.2) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)}=\frac{(-\rho)\tilde{g}'(-\rho)}{\tilde{g}(-\rho)}=\frac{1-C}{1-D},$$

thus proving the sharpness for ρ.

Case(ii): $BC-D(B+\alpha(A-B))\leqslant 0$. In this case, we have

$$\rho=\frac{C-D}{\alpha(A-B)-BC+D(B+\alpha(A-B))}$$

and for $z=\rho$, (2.2) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)}=\frac{(\rho)\tilde{g}'(\rho)}{\tilde{g}(\rho)}=\frac{1+C}{1+D},$$

which proves the sharpness for ρ. □

For $C=1$ and $D=-1$ in Theorem 2.3, we get the following corollary.

Corollary 2.4. *The radius of starlikeness for the class $C_1^{\alpha}[A,B]$ is*

$$\mathcal{R}_{\mathcal{ST}}(C_1^{\alpha}[A,B])=\frac{2}{\alpha(A-B)+|2B+\alpha(A-B)|}.$$

The following theorem gives the inclusion result for the class C_2^{α}.

Theorem 2.5. *For $-1\leqslant D<C\leqslant 1$, the class C_2^{α} is contained in the class $\mathcal{ST}[C,D]$, if*

$$|C+D(\alpha-1)|\leqslant C-D-\alpha.$$

Proof. *Let the function $g\in C_2^{\alpha}$. Then we get*

$$\frac{zg'(z)}{g(z)}=1+\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right).$$

Since $f\in\mathcal{CV}'$, we get

$$\frac{zg'(z)}{g(z)}\prec\frac{1+(\alpha-1)z}{1-z}$$

or equivalently $g\in\mathcal{ST}[\alpha-1,-1]$. Therefore, by Theorem 2.1, the class $\mathcal{ST}[\alpha-1,-1]$ is contained in the class $\mathcal{ST}[C,D]$ if and only if the condition $|C+D(\alpha-1)|\leqslant C-D-\alpha$ holds. □

It should be noted that the condition $|C + D(\alpha - 1)| \leqslant C - D - \alpha$ holds only for $D = -1$ and $C \geqslant \alpha - 1$. The radius of starlikeness associated with the Janowski starlike functions for the class C_2^α is given in the following theorem.

Theorem 2.6. *Let $\alpha > 0$ and $-1 \leqslant D < C \leqslant 1$. If the condition in Theorem 2.5 does not hold, then the radius of starlikeness associated with the class $\mathcal{ST}[C, D]$ for the class C_2^α is given by*

$$\mathcal{R}_{\mathcal{ST}[C,D]}(C_2^\alpha) = \frac{C - D}{\alpha + |C + D(\alpha - 1)|}.$$

Proof. The function $g \in C_2^\alpha$ implies that $g \in \mathcal{ST}[\alpha - 1, -1]$. Define $P(z) := (1 + Cz)/(1 + Dz)$ and $Q(z) := (1 + (\alpha - 1)z)/(1 - z)$. We have to determine ρ such that $0 < \rho \leqslant 1$ and $Q(\rho z) \prec P(z)$ for $z \in \mathbb{D}$. Define the function $H := P^{-1} \circ Q$. Then it can be seen that

$$H(z) = \frac{\alpha z}{(C - D) - (C + D(\alpha - 1))z}.$$

Observe that, for $|z| = r$,

$$\begin{aligned} |H(z)| &= \frac{\alpha |z|}{|(C - D) - (C + D(\alpha - 1))z|} \\ &\leqslant \frac{\alpha r}{(C - D) - |C + D(\alpha - 1)|\, r}. \end{aligned}$$

Therefore, it follows that $|H(z)| \leqslant 1$ for

$$r \leqslant (C - D)/(\alpha + |C + D(\alpha - 1)|) =: \rho.$$

Thus, the radius of starlikeness associated with the class $\mathcal{ST}[C, D]$ for the class C_2^α is at least ρ. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}'$ given by $\tilde{f}(z) = z/(1 - z)$. Then for the corresponding function $\tilde{g} \in C_2^\alpha$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = 1 + \frac{\alpha z}{1 - z}. \tag{2.3}$$

Case(i): $C + D(\alpha - 1) \geqslant 0$. In this case, $\rho = (C - D)/(\alpha + C + D(\alpha - 1))$ and for $z = \rho$, (2.3) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho)\tilde{g}'(\rho)}{\tilde{g}(\rho)} = \frac{1 + C}{1 + D},$$

thus proving the sharpness for ρ.

Case(ii): $C + D(\alpha - 1) \leqslant 0$. Here $\rho = (C - D)/(\alpha - C - D(\alpha - 1))$ and for $z = -\rho$, (2.3) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho)\tilde{g}'(-\rho)}{\tilde{g}(-\rho)} = \frac{1 - C}{1 - D},$$

which proves the sharpness for ρ. □

For $C = 1$ and $D = -1$ in Theorem 2.6, we get the following corollary.

Corollary 2.7. *The radius of starlikeness for the class C_2^α is $2/(\alpha + |2 - \alpha|)$.*

3. Radius of starlikeness associated with the exponential function

The class $\mathcal{ST}_e = \mathcal{ST}(e^z)$, which was introduced by Mendiratta et al. [16], consists of all functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec e^z$ or equivalently $\left|\log(zf'(z)/f(z))\right| < 1$. The following lemmas are used to find the radius of starlikeness associated with the exponential function for the classes $C_1^\alpha[A, B]$ and C_2^α.

Lemma 3.1. *[16] For $1/e < a < e$, let r_a be given by*

$$r_a = \begin{cases} a - \dfrac{1}{e} & \text{if } \dfrac{1}{e} < a \leqslant \dfrac{e+e^{-1}}{2} \\ e - a & \text{if } \dfrac{e+e^{-1}}{2} \leqslant a < e. \end{cases}$$

Then $\{w : |w - a| < r_a\} \subset \Omega_e := \{w : |\log w| < 1\}$, where Ω_e is the image of the unit disc $\mathbb{D}$ under the exponential function.

For $-1 \leqslant B < A \leqslant 1$ and $p(z) = 1 + c_1 z + c_2 z^2 + ...$, we say that $p \in \mathcal{P}[A, B]$ if

$$p(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{D}).$$

Note that $f \in \mathcal{ST}[A, B]$ if and only if $zf'(z)/f(z) \in \mathcal{P}[A, B]$.

Lemma 3.2. *[18] If $p \in \mathcal{P}[A, B]$, then*

$$\left| p(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leqslant \frac{(A - B)r}{1 - B^2r^2} \quad (|z| \leqslant r < 1).$$

The above lemmas are used to prove the following inclusion result.

Theorem 3.3. *The inclusion $C_1^\alpha[A, B] \subset \mathcal{ST}_e$ holds if either*

1. $(-\alpha B(A - B))/(1 - B^2) \leqslant (e + e^{-1} - 2)/2$ *and* $(\alpha(A - B))/(1 - B) \leqslant (e - 1)/e$ *or*
2. $(-\alpha B(A - B))/(1 - B^2) \geqslant (e + e^{-1} - 2)/2$ *and* $(\alpha(A - B))/(1 + B) \leqslant e - 1$.

Proof. *We have already seen that the function $g \in C_1^\alpha[A, B]$ implies that $g \in \mathcal{ST}[B + \alpha(A - B), B]$. Therefore by using Lemma 3.2 we get,*

$$\left| \frac{zg'(z)}{g(z)} - \frac{1 - (B^2 + \alpha B(A - B))r^2}{1 - B^2r^2} \right| \leqslant \frac{\alpha(A - B)r}{1 - B^2r^2} \quad (|z| \leqslant r < 1). \tag{3.1}$$

The centre and radius of the disc given in (3.1) *are*

$$c_1(\alpha, A, B)(r) := \frac{1 - (B^2 + \alpha B(A - B))r^2}{1 - B^2r^2}$$

and

$$a_1(\alpha, A, B)(r) := \frac{\alpha(A - B)r}{1 - B^2r^2}$$

respectively. Note that

$$c_1(\alpha, A, B)'(r) = \frac{-2\alpha B(A - B)r}{(1 - B^2r^2)^2},$$

which shows that $c_1(\alpha, A, B)(r)$ is an increasing function of r if $B < 0$ and is a decreasing function of r if $B > 0$. Also it can be seen that $c_1(\alpha, A, B)(r) \geqslant 1$ if $B \leqslant 0$ and $c_1(\alpha, A, B)(r) \leqslant 1$ if $B \geqslant 0$.

Now, assume that (1) holds. The inequality $(-\alpha B(A-B))/(1-B^2) \leqslant (e+e^{-1}-2)/2$ is equivalent to $c_1(\alpha, A, B)(1) \leqslant (e + e^{-1})/2$. The result follows from Lemma 3.1, since the inequality $a_1(\alpha, A, B)(1) \leqslant c_1(\alpha, A, B)(1) - 1/e$ follows from $(\alpha(A - B))/(1 - B) \leqslant (e - 1)/(e)$.

Assume that $(-\alpha B(A - B))/(1 - B^2) \geqslant (e + e^{-1} - 2)/2$ and $(\alpha(A - B))/(1 + B) \leqslant e - 1$. The first inequality reduces to $c_1(\alpha, A, B)(1) \geqslant (e + e^{-1})/2$. The result will follow from Lemma 3.1 if $a_1(\alpha, A, B)(1) \leqslant e - c_1(\alpha, A, B)(1)$ which directly follows from the inequality $(\alpha(A - B))/(1 + B) \leqslant e - 1$.

□

When the conditions in Theorem 3.3 fail to hold, then we discuss about the radius of starlikeness associated with the exponential function for the class $\mathcal{C}_1^\alpha[A,B]$ which is stated in the following theorems.

Theorem 3.4. *Let $\alpha > 0$, $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. If neither condition (1) nor condition (2) of Theorem 3.3 holds, then the radius of starlikeness associated with the exponential function for the class $\mathcal{C}_1^\alpha[A,B]$ is given by*

$$\mathcal{R}_{\mathcal{ST}_e}(\mathcal{C}_1^\alpha[A,B]) = \begin{cases} \dfrac{e-1}{e\alpha(A-B)+(e-1)B} & \text{if} \quad \alpha(A-B) \geqslant 2|B| \\ \dfrac{e-1}{\alpha(A-B)-(e-1)B} & \text{if} \quad \alpha(A-B) \leqslant 2|B|. \end{cases}$$

Proof. We prove the theorem by showing that the disc $\mathbb{D}(c_1(\alpha,A,B)(r); a_1(\alpha,A,B)(r))$ given in (3.1) is contained in Ω_e for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_e}(\mathcal{C}_1^\alpha[A,B])$. Let

$$\rho_2 := \frac{e-1}{e\alpha(A-B)+(e-1)B}$$

and

$$\rho_3 := \frac{e-1}{\alpha(A-B)-(e-1)B}.$$

Since $B \leqslant 0$, the centre $c_1(\alpha,A,B)(r) \geqslant 1$. We can see that ρ_2 is the root of the polynomial

$$\xi(r) := ((e\alpha - e + 1)B^2 - e\alpha AB)r^2 - (e\alpha(A-B))r + (e-1),$$

where $\xi(r) = 0$ is equivalent to the equation $a_1(\alpha,A,B)(r) = c_1(\alpha,A,B)(r) - (1/e)$. Note that $\xi(0) = e - 1 > 0$ and $\xi(1) = (e\alpha - e + 1)B^2 - e\alpha AB - e\alpha(A-B) + e - 1 < 0$, since the condition (1) in Theorem 3.3 does not hold. Hence $\rho_2 \in (0,1)$. Similarly, ρ_3 is the positive root of the polynomial

$$\psi(r) := ((\alpha + e - 1)B^2 - \alpha AB)r^2 + (\alpha(A-B))r + (1-e),$$

where $\psi(r) = 0$ is equivalent to the equation $a_1(\alpha,A,B)(r) = e - c_1(\alpha,A,B)(r)$. Clearly $\psi(0) = 1 - e < 0$ and since the condition (2) in Theorem 3.3 does not hold, $\psi(1) = (\alpha + e - 1)B^2 - \alpha AB + \alpha(A-B) + 1 - e > 0$ and thus $\rho_3 \in (0,1)$. The number

$$\rho_1 := \sqrt{\frac{e + e^{-1} - 2}{2\alpha|B|(A-B) + (e + e^{-1} - 2)B^2}}$$

is the positive root of the polynomial

$$\tau(r) := ((e + e^{-1} - 2)B^2 + 2\alpha|B|(A-B))r^2 + 2 - (e + e^{-1}).$$

Observe that $\tau(r) = 0$ is equivalent to the equation $c_1(\alpha,A,B)(r) = (e + e^{-1})/2$. Comparing ρ_2 and ρ_1, we get $\rho_2 \leqslant \rho_1$ if and only if $\alpha(A-B) \geqslant 2|B|$.

Case(i): $\alpha(A-B) \geqslant 2|B|$. When $\alpha(A-B) \geqslant 2|B|$, $\rho_2 \leqslant \rho_1$ and since $c_1(\alpha,A,B)(r)$ is an increasing function of r, this implies that $c_1(\alpha,A,B)(\rho_2) \leqslant c_1(\alpha,A,B)(\rho_1) = (e + e^{-1})/2$. Therefore by Lemma 3.1, the radius of starlikeness associated with the exponential function for the class $\mathcal{C}_1^\alpha[A,B]$ is at least ρ_2. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}[A,B]$ given by (2.1). For the above function $\tilde{f}$ and the corresponding function $\tilde{g} \in \mathcal{C}_1^\alpha[A,B]$, we get the expression for $z\tilde{g}'(z)/\tilde{g}(z)$ as in (2.2). Then for $z = -\rho_2$,

$$\left|\log\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right)\right| = \left|\log\left(\frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)}\right)\right| = \left|\log\left(\frac{1}{e}\right)\right| = 1,$$

thus proving the sharpness for ρ_2.

Case(ii): $\alpha(A-B) \leqslant 2|B|$. When $\alpha(A-B) \leqslant 2|B|$, $\rho_1 \leqslant \rho_2$ and hence $(e+e^{-1})/2 = c_1(\alpha, A, B)(\rho_1) \leqslant c_1(\alpha, A, B)(\rho_2)$. Hence by Lemma 3.1, the radius of starlikeness associated with the exponential function for the class $C_1^\alpha[A, B]$ is at least ρ_3. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}[A, B]$ given by (2.1). *It can be seen that for $z = \rho_3$,*

$$\left|\log\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right)\right| = \left|\log\left(\frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)}\right)\right| = \left|\log e\right| = 1,$$

thus proving the sharpness for ρ_3. □

The result in the case when $0 < B < 1$ is similar, which we state in the following theorem without proof.

Theorem 3.5. *Let $\alpha > 0$, $0 < B < 1$ and $B < A \leqslant 1$. If neither condition (1) nor condition (2) of Theorem 3.3 holds, then the radius of starlikeness associated with the exponential function for the class $C_1^\alpha[A, B]$ is given by*

$$\mathcal{R}_{\mathcal{ST}_e}(C_1^\alpha[A, B]) = \frac{e-1}{e\alpha(A-B) + (e-1)B}.$$

We now turn our attention to finding the radius of starlikeness associated with the exponential function for the class C_2^α, which is stated in the following theorem.

Theorem 3.6. *Let $\alpha > 0$. Then the radius of starlikeness associated with the exponential function for the class C_2^α is given by*

$$\mathcal{R}_{\mathcal{ST}_e}(C_2^\alpha) = \begin{cases} \dfrac{e-1}{e(\alpha-1)+1} & \text{if} \quad \alpha \geqslant 2 \\ \dfrac{e-1}{\alpha+e-1} & \text{if} \quad \alpha \leqslant 2. \end{cases}$$

Proof. *It is already seen that the function $g \in C_2^\alpha$ implies that $g \in \mathcal{ST}[\alpha-1, -1]$. Therefore by using Lemma 3.2 we get,*

$$\left|\frac{zg'(z)}{g(z)} - \frac{1+(\alpha-1)r^2}{1-r^2}\right| \leqslant \frac{\alpha r}{1-r^2} \quad (|z| \leqslant r < 1). \tag{3.2}$$

The centre and radius of the disc given in (3.2) *are*

$$c_2(\alpha)(r) := \frac{1+(\alpha-1)r^2}{1-r^2},$$

and

$$a_2(\alpha)(r) := \frac{\alpha r}{1-r^2}$$

respectively. Note that

$$c_2(\alpha)'(r) = \frac{2\alpha r}{(1-r^2)^2},$$

which shows that $c_2(\alpha)(r)$ is an increasing function of r. Also it can be seen that $c_2(\alpha)(r) > 1$.

Our aim is to show that the disc $\mathbb{D}(c_2(\alpha)(r); a_2(\alpha)(r))$ given in (3.2) *is contained in Ω_e for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_e}(C_2^\alpha)$. Let*

$$\rho_2 := \frac{e-1}{e(\alpha-1)+1}$$

and

$$\rho_3 := \frac{e-1}{\alpha+e-1}.$$

Here $c_2(\alpha)(r) > 1$*. For* $\alpha \geqslant 2$*, it can be seen that* ρ_2 *is the positive root of the polynomial*

$$\xi(r) := (e(\alpha-1)+1)r^2 - (e\alpha)r + (e-1)$$

that is less than 1 and $\xi(r) = 0$ *is equivalent to the equation* $a_2(\alpha)(r) = c_2(\alpha)(r) - (1/e)$*. Similarly,* ρ_3 *is the positive root of the polynomial*

$$\psi(r) := (\alpha+e-1)r^2 + \alpha r + (1-e)$$

and is less than 1 since $\alpha > 0$*. Also the equation* $\psi(r) = 0$ *is equivalent to the equation* $a_2(\alpha)(r) = e - c_2(\alpha)(r)$*. The number*

$$\rho_1 := \sqrt{\frac{e+e^{-1}-2}{2\alpha+e+e^{-1}-2}}$$

is the positive root of the polynomial

$$\tau(r) := (2\alpha+e+e^{-1}-2)r^2 + 2 - (e+e^{-1})$$

where $\tau(r) = 0$ *is equivalent to the equation* $c_2(\alpha)(r) = (e+e^{-1})/2$*. Note that* $\rho_2 \leqslant \rho_1$ *if and only if* $\alpha \geqslant 2$*.*

Case(i): $\alpha \geqslant 2$*. When* $\alpha \geqslant 2$*,* $\rho_2 \leqslant \rho_1$ *and since* $c_2(\alpha)(r)$ *is an increasing function of* r*,* $c_2(\alpha)(\rho_2) \leqslant c_2(\alpha)(\rho_1) = (e+e^{-1})/2$*. Therefore by Lemma 3.1, the radius of starlikeness associated with the exponential function for the class* C_2^α *is at least* ρ_2*. To prove the sharpness, consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}'$ *given by* $\tilde{f}(z) = z/(1-z)$*. Then for the corresponding function* $\tilde{g} \in C_2^\alpha$ *and for* $z = -\rho_2$*, (2.3) gives*

$$\left|\log\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right)\right| = \left|\log\left(\frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)}\right)\right| = \left|\log\left(\frac{1}{e}\right)\right| = 1,$$

thus proving the sharpness for ρ_2*.*

Case(ii): $\alpha \leqslant 2$*. In this case,* $\rho_1 \leqslant \rho_2$ *and since* $c_2(\alpha)(r)$ *is an increasing function of* r*,* $(e+e^{-1})/2 = c_2(\alpha)(\rho_1) \leqslant c_2(\alpha)(\rho_2)$*. Therefore by Lemma 3.1, the radius of starlikeness associated with the exponential function for the class* C_2^α *is at least* ρ_3*. To prove the sharpness, consider the function* $\tilde{f}(z) = z/(1-z)$ *from the class* $\mathcal{CV}'$*. It can be seen that for* $z = \rho_3$*,*

$$\left|\log\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right)\right| = \left|\log\left(\frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)}\right)\right| = \left|\log e\right| = 1.$$

□

4. Radius of starlikeness associated with the class $\mathcal{ST}_C$

The class $\mathcal{ST}_C = \mathcal{ST}(\varphi_C)$, where $\varphi_C(z) = 1 + (4/3)z + (2/3)z^2$, was studied by Sharma et al. [20]. The boundary of $\varphi_C(\mathbb{D})$ is a cardiod.

Lemma 4.1. *[20] For* $1/3 < a < 3$*, let* r_a *be given by*

$$r_a = \begin{cases} a - \dfrac{1}{3} & \text{if } \dfrac{1}{3} < a \leqslant \dfrac{5}{3} \\[2ex] 3 - a & \text{if } \dfrac{5}{3} \leqslant a < 3. \end{cases}$$

Then $\{w : |w-a| < r_a\} \subset \varphi_C(\mathbb{D}) = \Omega_C$*, where* Ω_C *is the region bounded by the cardiod* $\{x+iy : (9x^2+9y^2-18x+5)^2 - 16(9x^2+9y^2-6x+1) = 0\}$*.*

Theorem 4.2. *The inclusion* $C_1^\alpha[A,B] \subset \mathcal{ST}_C$ *holds if either*

1. $(-\alpha B(A-B))/(1-B^2) \leqslant 2/3$ *and* $(\alpha(A-B))/(1-B) \leqslant 2/3$
   or
2. $(-\alpha B(A-B))/(1-B^2) \geqslant 2/3$ *and* $(\alpha(A-B))/(1+B) \leqslant 2$.

Proof. *Assume that (1) holds. The inequality* $(-\alpha B(A-B))/(1-B^2) \leqslant 2/3$ *is equivalent to* $c_1(\alpha,A,B)(1) \leqslant 5/3$. *Since the inequality* $a_1(\alpha,A,B)(1) \leqslant c_1(\alpha,A,B)(1) - 1/3$ *follows from* $(\alpha(A-B))/(1-B) \leqslant 2/3$, *the result follows from Lemma 4.1.*

Assume that $(-\alpha B(A-B))/(1-B^2) \geqslant 2/3$ *and* $(\alpha(A-B))/(1+B) \leqslant 2$. *The first inequality reduces to* $c_1(\alpha,A,B)(1) \geqslant 5/3$. *The result will follow from Lemma 4.1 if* $a_1(\alpha,A,B)(1) \leqslant 3 - c_1(\alpha,A,B)(1)$ *which directly follows from the inequality* $(\alpha(A-B))/(1+B) \leqslant 2$.

□

When the conditions in Theorem 4.2 do not hold, then the results which are stated in the following theorems have a scope of discussion.

Theorem 4.3. *Let* $\alpha > 0$, $-1 \leqslant B \leqslant 0$ *and* $B < A \leqslant 1$. *If neither condition (1) nor condition (2) of Theorem 4.2 holds, then the radius of starlikeness associated with the cardiod* φ_C *for the class* $C_1^\alpha[A,B]$ *is given by*

$$\mathcal{R}_{\mathcal{ST}_C}(C_1^\alpha[A,B]) = \begin{cases} \dfrac{2}{3\alpha(A-B)+2B} & \text{if} \quad \alpha(A-B) \geqslant 2|B| \\ \dfrac{2}{\alpha(A-B)-2B} & \text{if} \quad \alpha(A-B) \leqslant 2|B|. \end{cases}$$

Proof. *The theorem is proved by showing that the disc* $\mathbb{D}(c_1(\alpha,A,B)(r); a_1(\alpha,A,B)(r))$ *given in* (3.1) *is contained in* Ω_C *for all* $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_C}(C_1^\alpha[A,B])$. *Let*

$$\rho_2 := \frac{2}{3\alpha(A-B)+2B}$$

and

$$\rho_3 := \frac{2}{\alpha(A-B)-2B}.$$

Here the centre $c_1(\alpha,A,B)(r) \geqslant 1$ *since* $B \leqslant 0$. *It can be seen that* ρ_2 *is the root of the polynomial*

$$\xi(r) := (2B^2 + 3\alpha B(A-B))r^2 + 3\alpha(A-B)r - 2$$

and a simple calculation shows that $\xi(r) = 0$ *is equivalent to the equation* $a_1(\alpha,A,B)(r) = c_1(\alpha,A,B)(r) - (1/3)$. *It can be easily shown that* ρ_2 *lies between* 0 *and* 1 *as* $\xi(0) = -2 < 0$ *and* $\xi(1) = 2B^2 + 3\alpha B(A-B) + 3\alpha(A-B) - 2 > 0$, *since the condition (1) in Theorem 4.2 does not hold. Similarly,* ρ_3 *is the positive root of the polynomial*

$$\psi(r) := (2B^2 - \alpha B(A-B))r^2 + \alpha(A-B)r - 2.$$

Observe that $\psi(r) = 0$ *is equivalent to the equation* $a_1(\alpha,A,B)(r) = 3 - c_1(\alpha,A,B)(r)$. *Clearly* $\psi(0) = -2 < 0$ *and since the condition (2) in Theorem 4.2 does not hold,* $\psi(1) = 2B^2 - \alpha B(A-B) + \alpha(A-B) - 2 > 0$ *which shows that* $\rho_3 \in (0,1)$. *The number*

$$\rho_1 := \sqrt{\frac{2}{3\alpha|B|(A-B)+2B^2}}$$

is the positive root of the polynomial

$$\tau(r) := (2B^2 + 3\alpha|B|(A-B))r^2 - 2,$$

where $\tau(r) = 0$ *is equivalent to the equation* $c_1(\alpha, A, B)(r) = 5/3$. *It can be seen that* $\rho_2 \leqslant \rho_1$ *if and only if* $\alpha(A - B) \geqslant 2|B|$. *Therefore we consider the following cases separately.*

Case(i): $\alpha(A-B) \geqslant 2|B|$. *When* $\alpha(A-B) \geqslant 2|B|$, $\rho_2 \leqslant \rho_1$ *and thus* $c_1(\alpha, A, B)(\rho_2) \leqslant c_1(\alpha, A, B)(\rho_1) = 5/3$, *due to the increasing nature of* $c_1(\alpha, A, B)(r)$. *Therefore the radius of starlikeness associated with the cardiod* φ_C *for the class* $C_1^\alpha[A, B]$ *is at least* ρ_2, *by using Lemma 4.1. To prove the sharpness, consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}[A, B]$ *given by* (2.1). *For the above function* $\tilde{f}$ *and the corresponding function* $\tilde{g} \in C_1^\alpha[A, B]$, *we get the expression for* $z\tilde{g}'(z)/\tilde{g}(z)$ *as in* (2.2). *Thus for* $z = -\rho_2$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = \frac{1}{3} = \varphi_C(-1),$$

which proves the sharpness for ρ_2.

Case(ii): $\alpha(A - B) \leqslant 2|B|$. *When* $\alpha(A - B) \leqslant 2|B|$, $\rho_1 \leqslant \rho_2$, *which gives* $5/3 = c_1(\alpha, A, B)(\rho_1) \leqslant c_1(\alpha, A, B)(\rho_2)$. *Therefore by Lemma 4.1, the radius of starlikeness associated with the cardiod* φ_C *for the class* $C_1^\alpha[A, B]$ *is at least* ρ_3. *To prove the sharpness, consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}[A, B]$ *given by* (2.1). *It can be seen from* (2.2) *that for* $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 3 = \varphi_C(1),$$

thus proving the sharpness for ρ_3. □

The following theorem is for the case when $0 < B < 1$, which we state without proof.

Theorem 4.4. *Let* $\alpha > 0$, $0 < B < 1$ *and* $B < A \leqslant 1$. *If neither condition (1) nor condition (2) of Theorem 4.2 holds, then the radius of starlikeness associated with the cardiod* φ_C *for the class* $C_1^\alpha[A, B]$ *is given by*

$$\mathcal{R}_{\mathcal{ST}_C}(C_1^\alpha[A, B]) = \frac{2}{2B + 3\alpha(A - B)}.$$

The following theorem gives the radius of starlikeness associated with the function φ_C for the class C_2^α.

Theorem 4.5. *Let* $\alpha > 0$. *Then the radius of starlikeness associated with the cardiod* φ_C *for the class* C_2^α *is given by*

$$\mathcal{R}_{\mathcal{ST}_C}(C_2^\alpha) = \begin{cases} \dfrac{2}{3\alpha - 2} & \text{if} \quad \alpha \geqslant 2 \\ \dfrac{2}{\alpha + 2} & \text{if} \quad \alpha \leqslant 2. \end{cases}$$

Proof. *We will show that the disc* $\mathbb{D}(c_2(\alpha)(r); a_2(\alpha)(r))$ *given in* (3.2) *is contained in* Ω_C *for all* $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_C}(C_2^\alpha)$. *Here* $c_2(\alpha)(r) > 1$. *Let*

$$\rho_2 := \frac{2}{3\alpha - 2}$$

and

$$\rho_3 := \frac{2}{\alpha + 2}.$$

It can be seen that ρ_2 *is the root of the polynomial*

$$\xi(r) := (3\alpha - 2)r^2 - (3\alpha)r + 2$$

which lies in the interval $(0, 1)$ *if* $\alpha \geqslant 2$ *and* $\xi(r) = 0$ *is equivalent to the equation* $a_2(\alpha)(r) = c_2(\alpha)(r) - (1/3)$. *Also,* ρ_3 *is the positive root of the polynomial*

$$\psi(r) := (\alpha + 2)r^2 + \alpha r - 2$$

and is less than 1 since $\alpha > 0$. Note that the equation $\psi(r) = 0$ is equivalent to the equation $a_2(\alpha)(r) = 3 - c_2(\alpha)(r)$. The number

$$\rho_1 := \sqrt{\frac{2}{3\alpha + 2}}$$

is the positive root of the polynomial

$$\tau(r) := (3\alpha + 2)r^2 - 2$$

where $\tau(r) = 0$ is equivalent to the equation $c_2(\alpha)(r) = 5/3$. A calculation shows that $\rho_2 \leqslant \rho_1$ if and only if $\alpha \geqslant 2$, which leads us to consider the following cases.

Case(i): $\alpha \geqslant 2$. When $\alpha \geqslant 2$, $\rho_2 \leqslant \rho_1$ and since $c_2(\alpha)(r)$ is increasing in nature, $c_2(\alpha)(\rho_2) \leqslant c_2(\alpha)(\rho_1) = 5/3$. Therefore by Lemma 4.1, the radius of starlikeness associated with the cardiod φ_C for the class $\mathcal{C}_2^\alpha$ is at least ρ_2. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}'$ given by $\tilde{f}(z) = z/(1 - z)$. Then for the corresponding function $\tilde{g} \in \mathcal{C}_2^\alpha$, $z\tilde{g}'(z)/\tilde{g}(z)$ is given by (2.3). Hence for $z = -\rho_2$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = \frac{1}{3} = \varphi_C(-1),$$

thus proving the sharpness for ρ_2.

Case(ii): $\alpha \leqslant 2$. Here $\rho_1 \leqslant \rho_2$ and thus $5/3 = c_2(\alpha)(\rho_1) \leqslant c_2(\alpha)(\rho_2)$. Therefore by Lemma 4.1, the radius of starlikeness associated with the cardiod φ_C for the class $\mathcal{C}_2^\alpha$ is at least ρ_3. To prove the sharpness, consider the function $\tilde{f}(z) = z/(1 - z)$ from the class $\mathcal{CV}'$. It can be seen from (2.3) that for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 3 = \varphi_C(1).$$

□

5. Radius of starlikeness associated with the class $\mathcal{ST}_R$

The class $\mathcal{ST}_R = \mathcal{ST}(\varphi_R)$ of starlike functions associated with the rational function $\varphi_R(z) = 1 + ((z^2 + kz)/k^2 - kz)$ for $k = \sqrt{2} + 1$, was introduced by Kumar and Ravichandran [11].

Lemma 5.1. *[11] For $2(\sqrt{2} - 1) < a < 2$, let r_a be given by*

$$r_a = \begin{cases} a - 2(\sqrt{2} - 1) & \text{if } 2(\sqrt{2} - 1) < a \leqslant \sqrt{2} \\ 2 - a & \text{if } \sqrt{2} \leqslant a < 2. \end{cases}$$

Then $\{w : |w - a| < r_a\} \subset \varphi_R(\mathbb{D})$.

Theorem 5.2. *The inclusion $\mathcal{C}_1^\alpha[A, B] \subset \mathcal{ST}_R$ holds if either*

1. $(-\alpha B(A - B))/(1 - B^2) \leqslant \sqrt{2} - 1$ *and* $(\alpha(A - B))/(1 - B) \leqslant 3 - 2\sqrt{2}$
   or
2. $(-\alpha B(A - B))/(1 - B^2) \geqslant \sqrt{2} - 1$ *and* $(\alpha(A - B))/(1 + B) \leqslant 1$.

Proof. Assume that $(-\alpha B(A - B))/(1 - B^2) \leqslant \sqrt{2} - 1$ and $(\alpha(A - B))/(1 - B) \leqslant 3 - 2\sqrt{2}$. The inequality $(-\alpha B(A - B))/(1 - B^2) \leqslant \sqrt{2} - 1$ is equivalent to $c_1(\alpha, A, B)(1) \leqslant \sqrt{2}$. The result follows from Lemma 5.1 since the inequality $a_1(\alpha, A, B)(1) \leqslant c_1(\alpha, A, B)(1) - 2(\sqrt{2} - 1)$ follows from $(\alpha(A - B))/(1 - B) \leqslant 3 - 2\sqrt{2}$.

Now assume that (2) holds. The first inequality of (2) reduces to $c_1(\alpha, A, B)(1) \geqslant \sqrt{2}$. The result will follow from Lemma 5.1 as the condition $a_1(\alpha, A, B)(1) \leqslant 2 - c_1(\alpha, A, B)(1)$ directly follows from the inequality $(\alpha(A - B))/(1 + B) \leqslant 1$.

□

When the conditions in Theorem 5.2 do not hold, then we discuss about the radius problem which is stated in the following theorems.

Theorem 5.3. *Let $\alpha > 0$, $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. If neither condition (1) nor condition (2) of Theorem 5.2 holds, then the radius of starlikeness associated with the rational function φ_R for the class $C_1^\alpha[A, B]$ is given by*

$$\mathcal{R}_{\mathcal{ST}_R}(C_1^\alpha[A,B]) = \begin{cases} \dfrac{3-2\sqrt{2}}{\alpha(A-B)+(3-2\sqrt{2})B} & \text{if} \quad \alpha(A-B) \geqslant (\sqrt{2}-1)|B| \\ \dfrac{1}{\alpha(A-B)-B} & \text{if} \quad \alpha(A-B) \leqslant (\sqrt{2}-1)|B|. \end{cases}$$

Proof. We aim to show that the disc $\mathbb{D}(c_1(\alpha, A, B)(r); a_1(\alpha, A, B)(r))$ given in (3.1) is contained in $\varphi_R(\mathbb{D})$ for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_R}(C_1^\alpha[A, B])$. Let

$$\rho_2 := \frac{3-2\sqrt{2}}{\alpha(A-B)+(3-2\sqrt{2})B}$$

and

$$\rho_3 := \frac{1}{\alpha(A-B)-B}.$$

As we have seen before, the centre $c_1(\alpha, A, B)(r) \geqslant 1$ since $B \leqslant 0$. The polynomial

$$\xi(r) := ((\alpha + 2\sqrt{2} - 3)B^2 - \alpha AB)r^2 - \alpha(A - B)r + 3 - 2\sqrt{2},$$

satisfies the condition that $\xi(r) = 0$ is equivalent to the equation $a_1(\alpha, A, B)(r) = c_1(\alpha, A, B)(r) - 2(\sqrt{2} - 1)$. Note that $\xi(0) = 3 - 2\sqrt{2} > 0$ and $\xi(1) = (\alpha + 2\sqrt{2} - 3)B^2 - \alpha AB - \alpha(A - B) + 3 - 2\sqrt{2} < 0$, since the condition (1) in Theorem 5.2 does not hold. Hence there exists a root of the polynomial $\xi(r)$ in the interval $(0, 1)$ which is precisely ρ_2. Now consider the polynomial

$$\psi(r) := ((\alpha + 1)B^2 - \alpha AB)r^2 + (\alpha(A - B))r - 1,$$

which has ρ_3 as its positive root. Observe that $\psi(r) = 0$ is equivalent to the equation $a_1(\alpha, A, B)(r) = 2 - c_1(\alpha, A, B)(r)$. Clearly $\psi(0) = -1 < 0$ and since the condition (2) in Theorem 5.2 does not hold, $\psi(1) = (\alpha+1)B^2 - \alpha AB + \alpha(A-B) - 1 > 0$ and thus $\rho_3 \in (0, 1)$. Let

$$\rho_1 := \sqrt{\frac{\sqrt{2}-1}{\alpha|B|(A-B)+(\sqrt{2}-1)B^2}}.$$

Then it can be seen that ρ_1 is the positive root of the polynomial

$$\tau(r) := (\alpha|B|(A - B) + (\sqrt{2} - 1)B^2)r^2 + 1 - \sqrt{2}$$

and $\tau(r) = 0$ is equivalent to the equation $c_1(\alpha, A, B)(r) = \sqrt{2}$. A comparison on ρ_2 and ρ_1 shows that $\rho_2 \leqslant \rho_1$ if and only if $\alpha(A - B) \geqslant (\sqrt{2} - 1)|B|$.

Case(i): $\alpha(A - B) \geqslant (\sqrt{2} - 1)|B|$. In this case, since $\rho_2 \leqslant \rho_1$ and since $c_1(\alpha, A, B)(r)$ is an increasing function of r, we get $c_1(\alpha, A, B)(\rho_2) \leqslant c_1(\alpha, A, B)(\rho_1) = \sqrt{2}$. Therefore by Lemma 5.1, the radius of starlikeness associated with the rational function φ_R for the class $C_1^\alpha[A, B]$ is at least ρ_2. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}[A, B]$ given by (2.1). For the above function $\tilde{f}$, the corresponding function $\tilde{g} \in C_1^\alpha[A, B]$ and for $z = -\rho_2$, (2.2) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = 2(\sqrt{2} - 1) = \varphi_R(-1),$$

thus proving the sharpness for ρ_2.

Case(ii): $\alpha(A - B) \leqslant (\sqrt{2} - 1)|B|$. *When* $\alpha(A - B) \leqslant (\sqrt{2} - 1)|B|$, $\rho_1 \leqslant \rho_2$ *and hence* $\sqrt{2} = c_1(\alpha, A, B)(\rho_1) \leqslant c_1(\alpha, A, B)(\rho_2)$. *Therefore Lemma 5.1 shows that the radius of starlikeness associated with the rational function* φ_R *for the class* $C_1^\alpha[A, B]$ *is at least* ρ_3. *To prove the sharpness, the function* $\tilde{f}$ *from the class* $\mathcal{CV}[A, B]$ *given by* (2.1) *is considered. From* (2.2) *it follows that for* $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 2 = \varphi_R(1),$$

which proves the sharpness for ρ_3. □

The case when $0 < B < 1$ has a similar proof, hence we state the result in the following theorem without proof.

Theorem 5.4. *Let* $\alpha > 0$, $0 < B < 1$ *and* $B < A \leqslant 1$. *If neither condition (1) nor condition (2) of Theorem 5.2 holds, then the radius of starlikeness associated with the rational function* φ_R *for the class* $C_1^\alpha[A, B]$ *is given by*

$$\mathcal{R}_{\mathcal{ST}_R}(C_1^\alpha[A, B]) = \frac{3 - 2\sqrt{2}}{\alpha(A - B) + (3 - 2\sqrt{2})B}.$$

Our next theorem gives the radius of starlikeness associated with the function φ_R for the class C_2^α.

Theorem 5.5. *Let* $\alpha > 0$. *Then the radius of starlikeness associated with the rational function* φ_R *for the class* C_2^α *is given by*

$$\mathcal{R}_{\mathcal{ST}_R}(C_2^\alpha) = \begin{cases} \dfrac{3 - 2\sqrt{2}}{\alpha - (3 - 2\sqrt{2})} & \text{if} \quad \alpha \geqslant \sqrt{2} - 1 \\ \dfrac{1}{\alpha + 1} & \text{if} \quad \alpha \leqslant \sqrt{2} - 1. \end{cases}$$

Proof. *The theorem is proved by showing that the disc* $\mathbb{D}(c_2(\alpha)(r); a_2(\alpha)(r))$ *given in* (3.2) *is contained in* $\varphi_R(\mathbb{D})$ *for all* $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_R}(C_2^\alpha)$. *Here* $c_2(\alpha)(r) > 1$. *Consider the polynomial*

$$\xi(r) := (\alpha + 2\sqrt{2} - 3)r^2 - \alpha r + 3 - 2\sqrt{2}.$$

Then, for $\alpha \geqslant \sqrt{2} - 1$, ρ_2 *is the positive root of* $\xi(r)$ *that is less than 1, where*

$$\rho_2 := \frac{3 - 2\sqrt{2}}{\alpha - (3 - 2\sqrt{2})}.$$

Note that $\xi(r) = 0$ *is equivalent to the equation* $a_2(\alpha)(r) = c_2(\alpha)(r) - (2(\sqrt{2} - 1))$. *Let*

$$\rho_3 := \frac{1}{\alpha + 1}.$$

Then, ρ_3 *is the positive root of the polynomial*

$$\psi(r) := (\alpha + 1)r^2 + \alpha r - 1$$

and is less than 1 since $\alpha > 0$. *Also the equation* $\psi(r) = 0$ *is equivalent to the equation* $a_2(\alpha)(r) = 2 - c_2(\alpha)(r)$. *The number*

$$\rho_1 := \sqrt{\frac{\sqrt{2} - 1}{\alpha + \sqrt{2} - 1}}$$

is the positive root of the polynomial

$$\tau(r) := (\alpha + \sqrt{2} - 1)r^2 + 1 - \sqrt{2}$$

where $\tau(r) = 0$ *is equivalent to the equation* $c_2(\alpha)(r) = \sqrt{2}$*. Note that* $\rho_2 \leqslant \rho_1$ *if and only if* $\alpha \geqslant \sqrt{2} - 1$*.*

Case(i): $\alpha \geqslant \sqrt{2} - 1$*. In this case,* $\rho_2 \leqslant \rho_1$ *and thus* $c_2(\alpha)(\rho_2) \leqslant c_2(\alpha)(\rho_1) = \sqrt{2}$*, since* $c_2(\alpha)(r)$ *is an increasing function of r. Therefore by Lemma 5.1, the radius of starlikeness associated with the rational function* φ_R *for the class* C_2^α *is at least* ρ_2*. To prove the sharpness, consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}'$ *given by* $\tilde{f}(z) = z/(1-z)$*. Then for the corresponding function* $\tilde{g} \in C_2^\alpha$ *and for* $z = -\rho_2$*, (2.3) shows that*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = 2(\sqrt{2} - 1) = \varphi_R(-1),$$

thus proving the sharpness for ρ_2*.*

Case(ii): $\alpha \leqslant \sqrt{2} - 1$*. In this case,* $\sqrt{2} = c_2(\alpha)(\rho_1) \leqslant c_2(\alpha)(\rho_2)$ *since* $\rho_1 \leqslant \rho_2$ *and* $c_2(\alpha)(r)$ *is an increasing function of r. Therefore by Lemma 5.1, the radius of starlikeness associated with the rational function* φ_R *for the class* C_2^α *is at least* ρ_3*. To prove the sharpness, consider the function* $\tilde{f}(z) = z/(1-z)$ *from the class* $\mathcal{CV}'$*. Then, (2.3) shows that for* $z = \rho_3$*,*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 2 = \varphi_R(1).$$

□

6. Radius of starlikeness associated with the class $\mathcal{ST}_{Ne}$

The class of starlike functions associated with a nephroid domain, given by $\mathcal{ST}_{Ne} = \mathcal{ST}(\varphi_{Ne})$ where $\varphi_{Ne}(z) = 1 + z - (z^3/3)$ was studied by Wani and Swaminathan [23]. The function φ_{Ne} maps the unit circle onto a 2-cusped curve,

$$\left((u-1)^2 + v^2 - \frac{4}{9}\right)^3 - \frac{4v^2}{3} = 0.$$

The radius problems for the functions associated with the nephroid domain was discussed by Wani and Swaminathan [22] and proved the following lemma.

Lemma 6.1. *[22] For* $1/3 < a < 5/3$*, let* r_a *be given by*

$$r_a = \begin{cases} a - \dfrac{1}{3} & \text{if } \dfrac{1}{3} < a \leqslant 1 \\ \dfrac{5}{3} - a & \text{if } 1 \leqslant a < \dfrac{5}{3}. \end{cases}$$

Then $\{w : |w - a| < r_a\} \subset \varphi_{Ne}(\mathbb{D}) = \Omega_{Ne}$*, where* Ω_{Ne} *is the region bounded by the nephroid* φ_{Ne}*, that is*

$$\Omega_{Ne} := \left\{\left((u-1)^2 + v^2 - \frac{4}{9}\right)^3 - \frac{4v^2}{3} < 0\right\}.$$

Theorem 6.2. *The inclusion* $C_1^\alpha[A, B] \subset \mathcal{ST}_{Ne}$ *holds if either*

1. $B \geqslant 0$ *and* $(\alpha(A - B))/(1 - B) \leqslant 2/3$
   or
2. $B \leqslant 0$ *and* $(\alpha(A - B))/(1 + B) \leqslant 2/3$*.*

Proof. *Assume that (1) holds. The inequality $B \geqslant 0$ is equivalent to $c_1(\alpha, A, B)(1) \leqslant 1$. The result follows from Lemma 6.1, since the inequality $a_1(\alpha, A, B)(1) \leqslant c_1(\alpha, A, B)(1) - (1/3)$ follows from $(\alpha(A-B))/(1-B) \leqslant 2/3$.*

Now assume that $B \leqslant 0$ and $(\alpha(A-B))/(1+B) \leqslant 2/3$. The first inequality reduces to $c_1(\alpha, A, B)(1) \geqslant 1$. As the condition $a_1(\alpha, A, B)(1) \leqslant (5/3) - c_1(\alpha, A, B)(1)$ directly follows from the inequality $(\alpha(A-B))/(1+B) \leqslant 2/3$, the result follows from Lemma 6.1.

□

Our next theorem gives the radius of starlikeness associated with the function φ_{Ne} for the class $C_1^\alpha[A, B]$, when the conditions in Theorem 6.2 do not hold.

Theorem 6.3. *Let $\alpha > 0$ and $-1 \leqslant B < A \leqslant 1$. If neither condition (1) nor condition (2) of Theorem 6.2 holds, then the radius of starlikeness associated with the nephroid φ_{Ne} for the class $C_1^\alpha[A, B]$ is given by*

$$\mathcal{R}_{\mathcal{ST}_{Ne}}(C_1^\alpha[A, B]) = \frac{2}{3\alpha(A-B) + 2|B|}.$$

Proof. *Proving the containment of the disc $\mathbb{D}(c_1(\alpha, A, B)(r); a_1(\alpha, A, B)(r))$ given in (3.1) in $\varphi_{Ne}(\mathbb{D})$ for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_{Ne}}(C_1^\alpha[A, B])$ gives the required result. We prove the theorem by considering the cases $B \geqslant 0$ and $B \leqslant 0$ separately. Let*

$$\rho_2 := \frac{2}{3\alpha(A-B) + 2B}$$

and

$$\rho_3 := \frac{2}{3\alpha(A-B) - 2B}.$$

Consider the case when $B \geqslant 0$. Then the centre $c_1(\alpha, A, B)(r) \leqslant 1$ and $\rho_2 = 2/(3\alpha(A-B) + 2|B|)$. We can see that ρ_2 is the root of the polynomial

$$\xi(r) := (2B^2 + 3B\alpha(A-B))r^2 + 3\alpha(A-B)r - 2,$$

where $\xi(r) = 0$ is equivalent to the equation $a_1(\alpha, A, B)(r) = c_1(\alpha, A, B)(r) - (1/3)$. Here $\xi(0) = -2 < 0$ and $\xi(1) = 2B^2 + 3B\alpha(A-B) + 3\alpha(A-B) - 2 > 0$, since the condition (1) in Theorem 6.2 does not hold, which shows that $\rho_2 \in (0, 1)$. Therefore by Lemma 6.1, the radius of starlikeness associated with the nephroid φ_{Ne} is at least ρ_2. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}[A, B]$ given by (2.1) and the corresponding function $\tilde{g} \in C_1^\alpha[A, B]$. Then for $z = -\rho_2$, (2.2) shows that

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = \frac{1}{3} = \varphi_{Ne}(-1),$$

thus proving the sharpness for ρ_2.

Similarly, consider the case when $B \leqslant 0$. This implies that the centre $c_1(\alpha, A, B)(r) \geqslant 1$ and we can see that $\rho_3 = 2/(3\alpha(A-B) + 2|B|)$. Note that ρ_3 is the positive root of the polynomial

$$\psi(r) := (2B^2 - 3B\alpha(A-B))r^2 + 3\alpha(A-B)r - 2,$$

where $\psi(r) = 0$ is equivalent to the equation $a_1(\alpha, A, B)(r) = (5/3) - c_1(\alpha, A, B)(r)$. Clearly $\rho_3 \in (0, 1)$ as $\psi(0) = -2 < 0$ and $\psi(1) = 2B^2 - 3B\alpha(A-B) + 3\alpha(A-B) - 2 > 0$, since the condition (2) in Theorem 6.2 does not hold. Hence by Lemma 6.1, the radius of starlikeness associated with the nephroid φ_{Ne} for the class $C_1^\alpha[A, B]$ is at least ρ_3. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}[A, B]$ given by (2.1). By using (2.2) it can be seen that for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = \frac{5}{3} = \varphi_{Ne}(1),$$

which proves the sharpness for ρ_3. □

The radius of starlikeness associated with the function φ_{Ne} for the class C_2^α is stated in the following theorem.

Theorem 6.4. *Let $\alpha > 0$. Then the radius of starlikeness associated with the nephroid φ_{Ne} for the class C_2^α is given by*

$$\mathcal{R}_{\mathcal{ST}_{Ne}}(C_2^\alpha) = \frac{2}{3\alpha + 2}.$$

Proof. *Our aim is to show that the disc $\mathbb{D}(c_2(\alpha)(r); a_2(\alpha)(r))$ given in* (3.2) *is contained in $\varphi_{Ne}(\mathbb{D})$ for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_{Ne}}(C_2^\alpha)$. Here $c_2(\alpha)(r) > 1$. Let*

$$\rho_3 := \frac{2}{3\alpha + 2}.$$

Consider the polynomial

$$\psi(r) := (3\alpha + 2)r^2 + 3\alpha r - 2.$$

Then ρ_3 is the positive root of the polynomial $\psi(r)$ and is less than 1 since $\alpha > 0$ and $\psi(r) = 0$ is equivalent to the equation $a_2(\alpha)(r) = (5/3) - c_2(\alpha)(r)$. Therefore by Lemma 6.1, the radius of starlikeness associated with the nephroid φ_{Ne} for the class C_2^α is at least ρ_3. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}'$ given by $\tilde{f}(z) = z/(1 - z)$ and the corresponding function $\tilde{g} \in C_2^\alpha$. Then for $z = \rho_3$, by (2.3)*,*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = \frac{5}{3} = \varphi_{Ne}(1),$$

thus proving the sharpness for ρ_3. □

7. Radius of starlikeness associated with the class $\mathcal{ST}_{\leftmoon}$

Raina and Sokól [17] considered the class $\mathcal{ST}_{\leftmoon} = \mathcal{ST}(\varphi_{\leftmoon})$, where $\varphi_{\leftmoon}(z) = z + \sqrt{1 + z^2}$ and proved that $f \in \mathcal{ST}_{\leftmoon}$ if and only if $zf'(z)/f(z) \in \Omega_{\leftmoon} := \{w \in \mathbb{C} : |w^2 - 1| < 2|w|\}$ which is the interior of a lune. The following lemma due to Gandhi and Ravichandran [4] is used to find the radius of starlikeness associated with the function $\varphi_{\leftmoon}$ for the classes $C_1^\alpha[A, B]$ and C_2^α.

Lemma 7.1. *[4] For $\sqrt{2} - 1 < a \leqslant \sqrt{2} + 1$, let r_a be given by*

$$r_a = 1 - |\sqrt{2} - a|.$$

Then $\{w : |w - a| < r_a\} \subset \Omega_{\leftmoon} := \{w : \left|w^2 - 1\right| < 2|w|\}$.

Theorem 7.2. *The inclusion $C_1^\alpha[A, B] \subset \mathcal{ST}_{\leftmoon}$ holds if either*

1. $(-\alpha B(A - B))/(1 - B^2) \leqslant \sqrt{2} - 1$ *and* $(\alpha(A - B))/(1 - B) \leqslant 2 - \sqrt{2}$
   or
2. $(-\alpha B(A - B))/(1 - B^2) \geqslant \sqrt{2} - 1$ *and* $(\alpha(A - B))/(1 + B) \leqslant \sqrt{2}$.

Proof. *Assume that (1) holds. The inequality $(-\alpha B(A - B))/(1 - B^2) \leqslant \sqrt{2} - 1$ is equivalent to $c_1(\alpha, A, B)(1) \leqslant \sqrt{2}$. The result follows from Lemma 7.1, since the inequality $a_1(\alpha, A, B)(1) \leqslant c_1(\alpha, A, B)(1) - (\sqrt{2} - 1)$ follows from $(\alpha(A - B))/(1 - B) \leqslant 2 - \sqrt{2}$.*

Now assume that $(-\alpha B(A - B))/(1 - B^2) \geqslant \sqrt{2} - 1$ and $(\alpha(A - B))/(1 + B) \leqslant \sqrt{2}$. The first inequality reduces to $c_1(\alpha, A, B)(1) \geqslant \sqrt{2}$. The result will follow from Lemma 7.1 as the condition $a_1(\alpha, A, B)(1) \leqslant \sqrt{2} + 1 - c_1(\alpha, A, B)(1)$ directly follows from the inequality $(\alpha(A - B))/(1 + B) \leqslant \sqrt{2}$.

□

The results stated in the next two theorems are discussed when the conditions in Theorem 7.2 do not hold.

Theorem 7.3. *Let $\alpha > 0$, $-1 \leqslant B \leqslant 0$ and $B < A \leqslant 1$. If neither condition (1) nor condition (2) of Theorem 7.2 holds, then the radius of starlikeness associated with the lune $\varphi_{\mathcal{C}}$ for the class $C_1^\alpha[A,B]$ is given by*

$$\mathcal{R}_{\mathcal{ST}_{\mathcal{C}}}(C_1^\alpha[A,B]) = \begin{cases} \dfrac{2-\sqrt{2}}{\alpha(A-B)+(2-\sqrt{2})B} & \text{if} \quad \alpha(A-B) \geqslant 2|B| \\ \dfrac{\sqrt{2}}{\alpha(A-B)-\sqrt{2}B} & \text{if} \quad \alpha(A-B) \leqslant 2|B|. \end{cases}$$

Proof. *We aim to show that the disc $\mathbb{D}(c_1(\alpha,A,B)(r); a_1(\alpha,A,B)(r))$ given in* (3.1) *is contained in $\Omega_{\mathcal{C}}$ for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_{\mathcal{C}}}(C_1^\alpha[A,B])$. Let*

$$\rho_2 := \frac{2-\sqrt{2}}{\alpha(A-B)+(2-\sqrt{2})B}$$

and

$$\rho_3 := \frac{\sqrt{2}}{\alpha(A-B)-\sqrt{2}B}.$$

The centre $c_1(\alpha,A,B)(r) \geqslant 1$ since $B \leqslant 0$. We can see that ρ_2 is the root of the polynomial

$$\xi(r) := ((2-\sqrt{2})B^2 + \alpha B(A-B))r^2 + \alpha(A-B)r - 2 + \sqrt{2},$$

where $\xi(r) = 0$ is equivalent to the equation $a_1(\alpha,A,B)(r) = c_1(\alpha,A,B)(r) - (\sqrt{2}-1)$. Note that $\rho_2 \in (0,1)$ as $\xi(0) = -2+\sqrt{2} < 0$ and $\xi(1) = (2-\sqrt{2})B^2 + \alpha B(A-B) + \alpha(A-B) - 2 + \sqrt{2} > 0$, since the condition (1) in Theorem 7.2 does not hold. Similarly, ρ_3 is the positive root of the polynomial

$$\psi(r) := ((\sqrt{2})B^2 - \alpha B(A-B))r^2 + (\alpha(A-B))r - \sqrt{2},$$

where $\psi(r) = 0$ is equivalent to the equation $a_1(\alpha,A,B)(r) = \sqrt{2}+1-c_1(\alpha,A,B)(r)$. Clearly $\psi(0) = -\sqrt{2} < 0$ and the condition (2) in Theorem 7.2 does not hold implies that $\psi(1) = (\sqrt{2})B^2 - \alpha B(A-B) + \alpha(A-B) - \sqrt{2} > 0$. Hence $\rho_3 \in (0,1)$. Let

$$\rho_1 := \sqrt{\frac{\sqrt{2}-1}{\alpha|B|(A-B)+(\sqrt{2}-1)B^2}}.$$

Then ρ_1 is the positive root of the polynomial

$$\tau(r) := (\alpha|B|(A-B) + (\sqrt{2}-1)B^2)r^2 + 1 - \sqrt{2}.$$

Observe that $\tau(r) = 0$ is equivalent to the equation $c_1(\alpha,A,B)(r) = \sqrt{2}$. A readily calculation shows that $\rho_2 \leqslant \rho_1$ if and only if $\alpha(A-B) \geqslant 2|B|$.

Case(i): $\alpha(A-B) \geqslant 2|B|$. In this case, $\rho_2 \leqslant \rho_1$ and since $c_1(\alpha,A,B)(r)$ is an increasing function of r, we get $c_1(\alpha,A,B)(\rho_2) \leqslant c_1(\alpha,A,B)(\rho_1) = \sqrt{2}$. Therefore by Lemma 7.1, the radius of starlikeness associated with the lune $\varphi_{\mathcal{C}}$ for the class $C_1^\alpha[A,B]$ is at least ρ_2. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}[A,B]$ given by (2.1) *and the corresponding function $\tilde{g} \in C_1^\alpha[A,B]$. Then for $z = -\rho_2$, by* (2.2) *we get*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = \sqrt{2}-1 = \varphi_{\mathcal{C}}(-1),$$

thus proving the sharpness for ρ_2.

Case(ii): $\alpha(A-B)\leqslant 2|B|$. *When* $\alpha(A-B)\leqslant 2|B|$, $\rho_1\leqslant\rho_2$ *and hence* $\sqrt{2}=c_1(\alpha,A,B)(\rho_1)\leqslant c_1(\alpha,A,B)(\rho_2)$. *Therefore by Lemma 7.1, the radius of starlikeness associated with the lune* $\varphi_{\mathbb{C}}$ *for the class* $C_1^\alpha[A,B]$ *is at least* ρ_3. *To prove the sharpness, we consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}[A,B]$ *given by* (2.1). *It can be seen that for* $z=\rho_3$, *by using* (2.2) *we get*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)}=\frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)}=1+\sqrt{2}=\varphi_{\mathbb{C}}(1),$$

thus proving the sharpness for ρ_3. □

The following theorem gives the radius result when $0<B<1$, which we state without proof.

Theorem 7.4. *Let* $\alpha>0$, $0<B<1$ *and* $B<A\leqslant 1$. *If neither condition (1) nor condition (2) of Theorem 7.2 holds, then the radius of starlikeness associated with the lune* $\varphi_{\mathbb{C}}$ *for the class* $C_1^\alpha[A,B]$ *is given by*

$$\mathcal{R}_{\mathcal{ST}_{\mathbb{C}}}(C_1^\alpha[A,B])=\frac{2-\sqrt{2}}{\alpha(A-B)+(2-\sqrt{2})B}.$$

Our next theorem gives the radius of starlikeness associated with the function $\varphi_{\mathbb{C}}$ for the class C_2^α.

Theorem 7.5. *Let* $\alpha>0$. *Then the radius of starlikeness associated with the lune* $\varphi_{\mathbb{C}}$ *for the class* C_2^α *is given by*

$$\mathcal{R}_{\mathcal{ST}_{\mathbb{C}}}(C_2^\alpha)=\begin{cases}\dfrac{2-\sqrt{2}}{\alpha-(2-\sqrt{2})} & \text{if}\quad \alpha\geqslant 2\\[2mm] \dfrac{\sqrt{2}}{\alpha+\sqrt{2}} & \text{if}\quad \alpha\leqslant 2.\end{cases}$$

Proof. *Our aim is to show that the disc* $\mathbb{D}(c_2(\alpha)(r);a_2(\alpha)(r))$ *given in* (3.2) *is contained in* $\Omega_{\mathbb{C}}$ *for all* $0<r\leqslant\mathcal{R}_{\mathcal{ST}_{\mathbb{C}}}(C_2^\alpha)$. *We have* $c_2(\alpha)(r)>1$. *Let*

$$\rho_2:=\frac{2-\sqrt{2}}{\alpha-(2-\sqrt{2})}\quad\text{and}\quad\rho_3:=\frac{\sqrt{2}}{\alpha+\sqrt{2}}.$$

For $\alpha\geqslant 2$, *it can be seen that* ρ_2 *is the positive root of the polynomial*

$$\xi(r):=(\alpha+\sqrt{2}-2)r^2-\alpha r+2-\sqrt{2}$$

that is less than 1 and $\xi(r)=0$ *is equivalent to the equation* $a_2(\alpha)(r)=c_2(\alpha)(r)-(\sqrt{2}-1)$. *In a similar manner, we can see that* ρ_3 *is the positive root of the polynomial*

$$\psi(r):=(\alpha+\sqrt{2})r^2+\alpha r-\sqrt{2}$$

and is less than 1 since $\alpha>0$. *Also the equation* $\psi(r)=0$ *is equivalent to the equation* $a_2(\alpha)(r)=\sqrt{2}+1-c_2(\alpha)(r)$. *The number*

$$\rho_1:=\sqrt{\frac{\sqrt{2}-1}{\alpha+\sqrt{2}-1}}$$

is the positive root of the polynomial

$$\tau(r):=(\alpha+\sqrt{2}-1)r^2+1-\sqrt{2}$$

where $\tau(r) = 0$ *is equivalent to the equation* $c_2(\alpha)(r) = \sqrt{2}$. *Note that* $\rho_2 \leqslant \rho_1$ *if and only if* $\alpha \geqslant 2$.

Case(i): $\alpha \geqslant 2$. *When* $\alpha \geqslant 2$, $\rho_2 \leqslant \rho_1$ *and since* $c_2(\alpha)(r)$ *is an increasing function of* r, $c_2(\alpha)(\rho_2) \leqslant c_2(\alpha)(\rho_1) = \sqrt{2}$. *Therefore by Lemma 7.1, the radius of starlikeness associated with the lune* $\varphi_{\mathbb{C}}$ *for the class* C_2^α *is at least* ρ_2. *To prove the sharpness, consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}'$ *given by* $\tilde{f}(z) = z/(1-z)$. *Then for the corresponding function* $\tilde{g} \in C_2^\alpha$, *for* $z = -\rho_2$, (2.3) *shows that*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = \sqrt{2} - 1 = \varphi_{\mathbb{C}}(-1),$$

thus proving the sharpness for ρ_2.

Case(ii): $\alpha \leqslant 2$. *In this case,* $\rho_1 \leqslant \rho_2$ *and since* $c_2(\alpha)(r)$ *is an increasing function of* r, $\sqrt{2} = c_2(\alpha)(\rho_1) \leqslant c_2(\alpha)(\rho_2)$. *Therefore by Lemma 7.1, the radius of starlikeness associated with the lune* $\varphi_{\mathbb{C}}$ *for the class* C_2^α *is at least* ρ_3. *To prove the sharpness, consider the function* $\tilde{f}(z) = z/(1-z)$ *from the class* $\mathcal{CV}'$. *From* (2.3) *it can be seen that, for* $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 1 + \sqrt{2} = \varphi_{\mathbb{C}}(1).$$

□

8. Radius of starlikeness associated with the class $\mathcal{ST}_{\wp}$

Kumar and Kamaljeet [10] defined the class $\mathcal{ST}_{\wp} = \mathcal{ST}(\varphi_{\wp})$, where $\varphi_{\wp}(z) = 1 + ze^z$. The boundary of $\varphi_{\wp}(\mathbb{D})$ is a cardiod. The following lemma is due to them.

Lemma 8.1. *[10] For* $1 - (1/e) < a < 1 + e$, *let* r_a *be given by*

$$r_a = \begin{cases} (a-1) + \dfrac{1}{e} & \text{if } 1 - \dfrac{1}{e} < a \leqslant 1 + \dfrac{e - e^{-1}}{2} \\ e - (a-1) & \text{if } 1 + \dfrac{e - e^{-1}}{2} \leqslant a < 1 + e. \end{cases}$$

Then $\{w : |w - a| < r_a\} \subset \varphi_{\wp}(\mathbb{D})$.

Theorem 8.2. *The inclusion* $C_1^\alpha[A,B] \subset \mathcal{ST}_{\wp}$ *holds if either*

1. $(-\alpha B(A-B))/(1-B^2) \leqslant (e - e^{-1})/2$ *and* $(\alpha(A-B))/(1-B) \leqslant 1/e$
   or
2. $(-\alpha B(A-B))/(1-B^2) \geqslant (e - e^{-1})/2$ *and* $(\alpha(A-B))/(1+B) \leqslant e$.

Proof. *Assume that* $(-\alpha B(A-B))/(1-B^2) \leqslant (e - e^{-1})/2$ *and* $(\alpha(A-B))/(1-B) \leqslant 1/e$. *The first inequality is equivalent to* $c_1(\alpha, A, B)(1) \leqslant 1 + (e - e^{-1})/2$. *The required result follows from Lemma 8.1 as the inequality* $a_1(\alpha, A, B)(1) \leqslant c_1(\alpha, A, B)(1) - 1 + 1/e$ *is obtained directly from* $(\alpha(A-B))/(1-B) \leqslant 1/e$.

Assume that $(-\alpha B(A-B))/(1-B^2) \geqslant (e - e^{-1})/2$ *and* $(\alpha(A-B))/(1+B) \leqslant e$. *The first inequality reduces to* $c_1(\alpha, A, B)(1) \geqslant 1 + (e - e^{-1})/2$. *As the condition* $a_1(\alpha, A, B)(1) \leqslant e - (c_1(\alpha, A, B)(1) - 1)$ *is obtained from the inequality* $(\alpha(A-B))/(1+B) \leqslant e$, *the result follows from Lemma 8.1.*

□

When the conditions in Theorem 8.2 do not hold, then we discuss about the radius problem which is stated in the next two theorems.

Theorem 8.3. *Let* $\alpha > 0$, $-1 \leqslant B \leqslant 0$ *and* $B < A \leqslant 1$. *If neither condition (1) nor condition (2) of Theorem 8.2 holds, then the radius of starlikeness associated with the cardiod* $\varphi_{\wp}$ *for the class* $C_1^\alpha[A,B]$ *is given by*

$$\mathcal{R}_{\mathcal{ST}_{\wp}}(C_1^\alpha[A,B]) = \begin{cases} \dfrac{1}{e\alpha(A-B) + B} & \text{if } \alpha(A-B)(e - e^{-1}) \geqslant 2|B| \\ \dfrac{e}{\alpha(A-B) - eB} & \text{if } \alpha(A-B)(e - e^{-1}) \leqslant 2|B|. \end{cases}$$

Proof. *We prove the theorem by showing that the disc* $\mathbb{D}(c_1(\alpha, A, B)(r); a_1(\alpha, A, B)(r))$ *given in* (3.1) *is contained in* $\varphi_\wp(\mathbb{D})$ *for all* $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_\wp}(C_1^\alpha[A, B])$*. Since* $B \leqslant 0$*, the centre* $c_1(\alpha, A, B)(r) \geqslant 1$*. Consider the polynomial*

$$\xi(r) := (B^2 + e\alpha B(A - B))r^2 + e\alpha(A - B)r - 1.$$

Then we can see that ρ_2 *is the root of the polynomial* $\xi(r)$*, where*

$$\rho_2 := \frac{1}{e\alpha(A - B) + B}$$

and note that $\xi(r) = 0$ *is equivalent to the equation* $a_1(\alpha, A, B)(r) = c_1(\alpha, A, B)(r) - 1 + (1/e)$*. As* $\xi(0) = -1 < 0$ *and* $\xi(1) = B^2 + e\alpha B(A - B) + e\alpha(A - B) - 1 > 0$*, since the condition (1) in Theorem 8.2 does not hold, we get* $\rho_2 \in (0, 1)$*. Similarly, let*

$$\rho_3 := \frac{e}{\alpha(A - B) - eB}.$$

Then ρ_3 *is the positive root of the polynomial*

$$\psi(r) := (eB^2 - \alpha B(A - B))r^2 + \alpha(A - B)r - e,$$

where $\psi(r) = 0$ *is equivalent to the equation* $a_1(\alpha, A, B)(r) = e - (c_1(\alpha, A, B)(r) - 1)$*. Clearly* $\psi(0) = -e < 0$ *and since the condition (2) in Theorem 8.2 does not hold,* $\psi(1) = eB^2 - \alpha B(A - B) + \alpha(A - B) - e > 0$ *and thus* $\rho_3 \in (0, 1)$*. The number*

$$\rho_1 := \sqrt{\frac{e - e^{-1}}{2\alpha|B|(A - B) + (e - e^{-1})B^2}}$$

is the positive root of the polynomial

$$\tau(r) := ((e - e^{-1})B^2 + 2\alpha|B|(A - B))r^2 - e + e^{-1}.$$

Observe that $\tau(r) = 0$ *is equivalent to the equation* $c_1(\alpha, A, B)(r) = 1 + (e - e^{-1})/2$*. Comparing* ρ_2 *and* ρ_1*, we get the relation that* $\rho_2 \leqslant \rho_1$ *if and only if* $\alpha(A - B)(e - e^{-1}) \geqslant 2|B|$*.*

Case(i): $\alpha(A - B)(e - e^{-1}) \geqslant 2|B|$*. Here* $c_1(\alpha, A, B)(\rho_2) \leqslant c_1(\alpha, A, B)(\rho_1) = 1 + (e - e^{-1})/2$ *as* $\rho_2 \leqslant \rho_1$*. Therefore by Lemma 8.1, the radius of starlikeness associated with the cardiod* $\varphi_\wp$ *for the class* $C_1^\alpha[A, B]$ *is at least* ρ_2*. To prove the sharpness, consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}[A, B]$ *given by* (2.1) *and the corresponding function* $\tilde{g} \in C_1^\alpha[A, B]$*. Then for* $z = -\rho_2$*, by using* (2.2)*, we get*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = 1 - e^{-1} = \varphi_\wp(-1),$$

thus proving the sharpness for ρ_2*.*

Case(ii): $\alpha(A-B)(e-e^{-1}) \leqslant 2|B|$*. In this case,* $\rho_1 \leqslant \rho_2$ *and hence* $1+(e-e^{-1})/2 = c_1(\alpha, A, B)(\rho_1) \leqslant c_1(\alpha, A, B)(\rho_2)$*. Therfore, Lemma 8.1 guarantees that the radius of starlikeness associated with the cardiod* $\varphi_\wp$ *for the class* $C_1^\alpha[A, B]$ *is at least* ρ_3*. The function* $\tilde{f}$ *from the class* $\mathcal{CV}[A, B]$ *given by* (2.1) *is considered to prove the sharpness. It can be seen from* (2.2) *that for* $z = \rho_3$*,*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 1 + e = \varphi_\wp(1),$$

thus proving the sharpness for ρ_3*.* □

The result in the case when $0 < B < 1$ is similar, which we state in the following theorem without proof.

Theorem 8.4. *Let $\alpha > 0$, $0 < B < 1$ and $B < A \leqslant 1$. If neither condition (1) nor condition (2) of Theorem 8.2 holds, then the radius of starlikeness associated with the cardiod $\varphi_\wp$ for the class $C_1^\alpha[A, B]$ is given by*

$$\mathcal{R}_{\mathcal{ST}_\wp}(C_1^\alpha[A, B]) = \frac{1}{e\alpha(A - B) + B}.$$

The following theorem gives the radius of starlikeness associated with function $\varphi_\wp$ for the class C_2^α.

Theorem 8.5. *Let $\alpha > 0$. Then the radius of starlikeness associated with the cardiod $\varphi_\wp$ for the class C_2^α is given by*

$$\mathcal{R}_{\mathcal{ST}_\wp}(C_2^\alpha) = \begin{cases} \dfrac{1}{e\alpha - 1} & \text{if} \quad \alpha \geqslant \dfrac{2}{e - e^{-1}} \\ \dfrac{e}{\alpha + e} & \text{if} \quad \alpha \leqslant \dfrac{2}{e - e^{-1}}. \end{cases}$$

Proof. *Our aim is to show that the disc $\mathbb{D}(c_2(\alpha)(r); a_2(\alpha)(r))$ given in* (3.2) *is contained in $\varphi_\wp(\mathbb{D})$ for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_\wp}(C_2^\alpha)$. Let*

$$\rho_2 := \frac{1}{e\alpha - 1}$$

and

$$\rho_3 := \frac{e}{\alpha + e}.$$

Here $c_2(\alpha)(r) > 1$. It can be seen that for $\alpha \geqslant 2/(e - e^{-1})$, ρ_2 is the positive root of the polynomial

$$\xi(r) := (e\alpha - 1)r^2 - e\alpha r + 1$$

that is less than 1 and $\xi(r) = 0$ is equivalent to the equation $a_2(\alpha)(r) = c_2(\alpha)(r) - 1 + (1/e)$. Similarly, ρ_3 is the positive root of the polynomial

$$\psi(r) := (\alpha + e)r^2 + \alpha r - e$$

and is less than 1 since $\alpha > 0$. Also the equation $\psi(r) = 0$ is equivalent to the equation $a_2(\alpha)(r) = e - (c_2(\alpha)(r) - 1)$. The number

$$\rho_1 := \sqrt{\frac{e - e^{-1}}{2\alpha + e - e^{-1}}}$$

is the positive root of the polynomial

$$\tau(r) := (2\alpha + e - e^{-1})r^2 - e + e^{-1}$$

where $\tau(r) = 0$ is equivalent to the equation $c_2(\alpha)(r) = 1 + (e - e^{-1})/2$. Comparing ρ_1 and ρ_2, we get that $\rho_2 \leqslant \rho_1$ if and only if $\alpha \geqslant 2/(e - e^{-1})$. Therefore we consider the following cases.

Case(i): $\alpha \geqslant 2/(e - e^{-1})$. When $\alpha \geqslant 2/(e - e^{-1})$, $\rho_2 \leqslant \rho_1$ and since $c_2(\alpha)(r)$ is an increasing function of r, $c_2(\alpha)(\rho_2) \leqslant c_2(\alpha)(\rho_1) = 1 + (e - e^{-1})/2$. Therefore by Lemma 8.1, the radius of starlikeness associated with the cardiod $\varphi_\wp$ for the class C_2^α is at least ρ_2. To prove the sharpness, consider the function $\tilde{f}$ from the class $\mathcal{CV}'$ given by $\tilde{f}(z) = z/(1 - z)$ and the corresponding function $\tilde{g} \in C_2^\alpha$. Then by (2.3) we can see that, for $z = -\rho_2$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = 1 - e^{-1} = \varphi_\wp(-1),$$

thus proving the sharpness for ρ_2.

Case(ii): $\alpha \leqslant 2/(e - e^{-1})$. When $\alpha \leqslant 2/(e - e^{-1})$, $\rho_1 \leqslant \rho_2$ and since $c_2(\alpha)(r)$ is an increasing function of r, $1 + (e - e^{-1})/2 = c_2(\alpha)(\rho_1) \leqslant c_2(\alpha)(\rho_2)$. Therefore by Lemma 8.1, the radius of starlikeness associated with the cardioid $\varphi_\wp$ for the class C_2^α is at least ρ_3. To prove the sharpness, consider the function $\tilde{f}(z) = z/(1-z)$ from the class $\mathcal{CV}'$. Then for $z = \rho_3$, (2.3) shows that

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 1 + e = \varphi_\wp(1).$$

□

9. Radius of starlikeness associated with the class $\mathcal{ST}_{SG}$

The class $\mathcal{ST}_{SG} = \mathcal{ST}(\varphi_{SG})$ where $\varphi_{SG}(z) = 2/(1 + e^{-z})$ was introduced by Goel and Kumar [6]. The boundary of $\varphi_{SG}(\mathbb{D})$ is a modified sigmoid. They proved the following lemma.

Lemma 9.1. *[6] For $2/(1 + e) < a < 2e/(1 + e)$, let r_a be given by*

$$r_a = \frac{e-1}{e+1} - |a - 1|.$$

Then $\{w : |w - a| < r_a\} \subset \varphi_{SG}(\mathbb{D}) = \Delta_{SG} := \{w : |\log w/(2 - w)| < 1\}$.

Theorem 9.2. *The inclusion $C_1^\alpha[A, B] \subset \mathcal{ST}_{SG}$ holds if either*

1. *$B \geqslant 0$ and $(\alpha(A - B))/(1 - B) \leqslant (e - 1)/(e + 1)$*
   or
2. *$B \leqslant 0$ and $(\alpha(A - B))/(1 + B) \leqslant (e - 1)/(e + 1)$.*

Proof. *Assume that $B \geqslant 0$ and $(\alpha(A-B))/(1-B) \leqslant (e-1)/(e+1)$. The inequality $B \geqslant 0$ is equivalent to $c_1(\alpha, A, B)(1) \leqslant 1$. Since the inequality $a_1(\alpha, A, B)(1) \leqslant c_1(\alpha, A, B)(1)+(e-1)/(e+1)-1$ follows from $(\alpha(A-B))/(1-B) \leqslant (e-1)/(e+1)$, the result follows from Lemma 9.1.*

Now assume that $B \leqslant 0$ and $(\alpha(A-B))/(1+B) \leqslant (e-1)/(e+1)$. The first inequality reduces to $c_1(\alpha, A, B)(1) \geqslant 1$. The result will follow from Lemma 9.1 if $a_1(\alpha, A, B)(1) \leqslant (e - 1)/(e + 1) + 1 - c_1(\alpha, A, B)(1)$ which directly follows from the inequality $(\alpha(A - B))/(1 + B) \leqslant (e - 1)/(e + 1)$.
□

When the conditions in Theorem 9.2 do not hold, then we have the result giving the radius of starlikeness associated with the function φ_{SG} for the class $C_1^\alpha[A, B]$, which is stated in the following theorem.

Theorem 9.3. *Let $\alpha > 0$ and $-1 \leqslant B < A \leqslant 1$. If neither condition (1) nor condition (2) of Theorem 9.2 holds, then the radius of starlikeness associated with the modified sigmoid function φ_{SG} for the class $C_1^\alpha[A, B]$ is given by*

$$\mathcal{R}_{\mathcal{ST}_{SG}}(C_1^\alpha[A, B]) = \frac{e-1}{(e+1)\alpha(A-B) + (e-1)|B|}.$$

Proof. *Consider the disc $\mathbb{D}(c_1(\alpha, A, B)(r); a_1(\alpha, A, B)(r))$ given in (3.1). We prove the theorem by proving the containment of this disc in Δ_{SG} for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_{SG}}(C_1^\alpha[A, B])$. Consider the case when $B \geqslant 0$. Then the centre $c_1(\alpha, A, B)(r) \leqslant 1$. Let*

$$\rho_2 := \frac{e-1}{(e+1)\alpha(A-B) + (e-1)B}.$$

Note that $\rho_2 = (e - 1)/((e + 1)\alpha(A - B) + (e - 1)|B|)$ and ρ_2 is the root of the polynomial

$$\xi(r) := ((e - 1)B^2 + (e + 1)B\alpha(A - B))r^2 + (e + 1)\alpha(A - B)r + 1 - e,$$

where $\xi(r) = 0$ *is equivalent to the equation* $a_1(\alpha, A, B)(r) = c_1(\alpha, A, B)(r) + (e-1)/(e+1) - 1$. *As* $\xi(0) = 1 - e < 0$ *and* $\xi(1) = (e-1)B^2 + (e+1)B\alpha(A-B) + (e+1)\alpha(A-B) + 1 - e > 0$, *since the condition (1) in Theorem 9.2 does not hold, the belongingness of* ρ_2 *in the interval* $(0,1)$ *is guaranteed. Therefore by Lemma 9.1, the radius of starlikeness associated with the modified sigmoid function* φ_{SG} *is at least* ρ_2. *To prove the sharpness, consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}[A,B]$ *given by* (2.1). *For the above function* $\tilde{f}$ *and the corresponding function* $\tilde{g} \in C_1^\alpha[A,B]$, *by using* (2.2), *for* $z = -\rho_2$, *we get*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = \frac{2}{1+e} = \varphi_{SG}(-1),$$

thus proving the sharpness for ρ_2.

Similarly, when $B \leqslant 0$, *the centre* $c_1(\alpha, A, B)(r) \geqslant 1$. *Note that* ρ_3 *is the positive root of the polynomial*

$$\psi(r) := ((e-1)B^2 - (e+1)B\alpha(A-B))r^2 + (e+1)\alpha(A-B)r + 1 - e,$$

where

$$\rho_3 := \frac{e-1}{(e+1)\alpha(A-B) - (e-1)B}.$$

Observe that $\rho_3 = (e-1)/((e+1)\alpha(A-B) + (e-1)|B|)$ *and* $\psi(r) = 0$ *is equivalent to the equation* $a_1(\alpha, A, B)(r) = (e-1)/(e+1) + 1 - c_1(\alpha, A, B)(r)$. *Clearly* $\psi(0) = 1 - e < 0$ *and since the condition (2) in Theorem 9.2 does not hold,* $\psi(1) = (e-1)B^2 - (e+1)B\alpha(A-B) + (e+1)\alpha(A-B) + 1 - e > 0$ *and thus* $\rho_3 \in (0,1)$. *Hence by Lemma 9.1, the radius of starlikeness associated with the modified sigmoid function* φ_{SG} *for the class* $C_1^\alpha[A,B]$ *is at least* ρ_3. *To prove the sharpness, consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}[A,B]$ *given by* (2.1). *It can be seen from* (2.2) *that for* $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = \frac{2e}{e+1} = \varphi_{SG}(1),$$

thus proving the sharpness for ρ_3. □

The following theorem gives the radius result associated with the function φ_{SG} corresponding to the class C_2^α.

Theorem 9.4. *Let* $\alpha > 0$. *Then the radius of starlikeness associated with the modified sigmoid function* φ_{SG} *for the class* C_2^α *is given by*

$$\mathcal{R}_{\mathcal{ST}_{SG}}(C_2^\alpha) = \frac{e-1}{(e+1)\alpha + (e-1)}.$$

Proof. Our aim is to show that the disc $\mathbb{D}(c_2(\alpha)(r); a_2(\alpha)(r))$ *given in* (3.2) *is contained in* Δ_{SG} *for all* $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_{SG}}(C_2^\alpha)$. *Let*

$$\rho_3 := \frac{e-1}{(e+1)\alpha + (e-1)}.$$

Here $c_2(\alpha)(r) > 1$. *It can be seen that* ρ_3 *is the positive root of the polynomial*

$$\psi(r) := ((e+1)\alpha + e - 1)r^2 + (e+1)\alpha r + 1 - e$$

and is less than 1 since $\alpha > 0$ *and* $\psi(r) = 0$ *is equivalent to the equation* $a_2(\alpha)(r) = (e-1)/(e+1) + 1 - c_2(\alpha)(r)$. *Therefore by Lemma 9.1, the radius of starlikeness associated with the modified sigmoid function* φ_{SG} *for the class* C_2^α *is at least* ρ_3. *To prove the sharpness, consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}'$ *given by* $\tilde{f}(z) = z/(1-z)$. *Then for the corresponding function* $\tilde{g} \in C_2^\alpha$, *for* $z = \rho_3$, (2.3) *shows that*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = \frac{2e}{e+1} = \varphi_{SG}(1),$$

thus proving the sharpness for ρ_3. □

10. Radius of starlikeness associated with the class $\mathcal{ST}_{\sin}$

Cho et al. [2] introduced the class $\mathcal{ST}_{\sin} = \mathcal{ST}(\varphi_{\sin})$, where $\varphi_{\sin}(z) = 1 + \sin z$ and proved the following lemma.

Lemma 10.1. *[2] For $1 - \sin 1 < a < 1 + \sin 1$, let r_a be given by*

$$r_a = \sin 1 - |a - 1|.$$

Then $\{w : |w - a| < r_a\} \subset \varphi_{\sin}(\mathbb{D})$.

Theorem 10.2. *The inclusion $C_1^\alpha[A, B] \subset \mathcal{ST}_{\sin}$ holds if either*

1. *$B \geqslant 0$ and $(\alpha(A - B))/(1 - B) \leqslant \sin 1$*
   or
2. *$B \leqslant 0$ and $(\alpha(A - B))/(1 + B) \leqslant \sin 1$.*

Proof. *Assume that (1) holds. The inequality $B \geqslant 0$ is equivalent to $c_1(\alpha, A, B)(1) \leqslant 1$. The condition $a_1(\alpha, A, B)(1) \leqslant c_1(\alpha, A, B)(1) + (\sin 1) - 1$ follows from $(\alpha(A - B))/(1 - B) \leqslant \sin 1$ and hence by Lemma 10.1, the result follows.*

Similarly, if $B \leqslant 0$ and $(\alpha(A - B))/(1 + B) \leqslant \sin 1$, then the first inequality reduces to $c_1(\alpha, A, B)(1) \geqslant 1$ and the condition $a_1(\alpha, A, B)(1) \leqslant (\sin 1) + 1 - c_1(\alpha, A, B)(1)$ directly follows from the inequality $(\alpha(A - B))/(1 + B) \leqslant \sin 1$. Therefore, the result follows from Lemma 10.1.

□

Our next theorem gives the radius of starlikeness associated with the function $\varphi_{\sin}$ for the class $C_1^\alpha[A, B]$, when the conditions in Theorem 10.2 do not hold.

Theorem 10.3. *Let $\alpha > 0$ and $-1 \leqslant B < A \leqslant 1$. If neither condition (1) nor condition (2) of Theorem 10.2 holds, then the radius of starlikeness associated with the function $\varphi_{\sin}$ for the class $C_1^\alpha[A, B]$ is given by*

$$\mathcal{R}_{\mathcal{ST}_{\sin}}(C_1^\alpha[A, B]) = \frac{\sin 1}{\alpha(A - B) + (\sin 1)|B|}.$$

Proof. *By proving that the disc $\mathbb{D}(c_1(\alpha, A, B)(r); a_1(\alpha, A, B)(r))$ given in (3.1) is contained in $\varphi_{\sin}(\mathbb{D})$ for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_{\sin}}(C_1^\alpha[A, B])$, the required result will follow. Let*

$$\rho_2 := \frac{\sin 1}{\alpha(A - B) + (\sin 1)B}$$

and

$$\rho_3 := \frac{\sin 1}{\alpha(A - B) - (\sin 1)B}.$$

If $B \geqslant 0$, the centre $c_1(\alpha, A, B)(r) \leqslant 1$ and $\rho_2 = (\sin 1)/(\alpha(A - B) + (\sin 1)|B|)$. We can see that ρ_2 is the root of the polynomial

$$\xi(r) := ((\sin 1)B^2 + \alpha B(A - B))r^2 + \alpha(A - B)r - \sin 1,$$

where $\xi(r) = 0$ is equivalent to the equation $a_1(\alpha, A, B)(r) = c_1(\alpha, A, B)(r) + (\sin 1) - 1$. Note that $\xi(0) = -\sin 1 < 0$ and $\xi(1) = (\sin 1)B^2 + \alpha B(A - B) + \alpha(A - B) - \sin 1 > 0$, since the condition (1) in Theorem 10.2 does not hold. Hence $\rho_2 \in (0, 1)$. Therefore by Lemma 10.1, the radius of starlikeness associated with the function $\varphi_{\sin}$ is at least ρ_2. Now consider the function $\tilde{f}$ from the class $\mathcal{CV}[A, B]$ given by (2.1) and the corresponding function $\tilde{g} \in C_1^\alpha[A, B]$. Then, for $z = -\rho_2$, by using (2.2), we get

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = 1 - \sin 1 = \varphi_{\sin}(-1),$$

which proves the sharpness for ρ_2.

Similarly, if $B \leqslant 0$*, the centre* $c_1(\alpha, A, B)(r) \geqslant 1$ *and* $\rho_3 = (\sin 1)/(\alpha(A-B) + (\sin 1)|B|)$*. Note that* ρ_3 *is the positive root of the polynomial*

$$\psi(r) := ((\sin 1)B^2 - \alpha B(A-B))r^2 + \alpha(A-B)r - \sin 1,$$

where $\psi(r) = 0$ *is equivalent to the equation* $a_1(\alpha, A, B)(r) = (\sin 1) + 1 - c_1(\alpha, A, B)(r)$*. Clearly* $\psi(0) = -\sin 1 < 0$ *and since the condition (2) in Theorem 10.2 does not hold,* $\psi(1) = (\sin 1)B^2 - \alpha B(A-B) + \alpha(A-B) - \sin 1 > 0$ *and thus* $\rho_3 \in (0, 1)$*. Hence by Lemma 10.1, the radius of starlikeness associated with the function* $\varphi_{\sin}$ *for the class* $C_1^\alpha[A, B]$ *is at least* ρ_3*. The sharpness can be proved by considering the function* $\tilde{f}$ *from the class* $\mathcal{CV}[A, B]$ *given by* (2.1)*. Then if* $\tilde{g}$ *is the corresponding function in* $C_1^\alpha[A, B]$*, by using* (2.2)*, for* $z = \rho_3$*, we get*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 1 + \sin 1 = \varphi_{\sin}(1),$$

thus proving the sharpness for ρ_3. □

We now turn our attention to finding the radius of starlikeness associated with the function $\varphi_{\sin}$ for the class C_2^α which is stated in the following theorem.

Theorem 10.4. *Let* $\alpha > 0$*. Then the radius of starlikeness associated with the function* $\varphi_{\sin}$ *for the class* C_2^α *is given by*

$$\mathcal{R}_{\mathcal{ST}_{\sin}}(C_2^\alpha) = \frac{\sin 1}{\alpha + \sin 1}.$$

Proof. *Our aim is to show that the disc* $\mathbb{D}(c_2(\alpha)(r); a_2(\alpha)(r))$ *given in* (3.2) *is contained in* $\varphi_{\sin}(\mathbb{D})$ *for all* $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_{\sin}}(C_2^\alpha)$*. Here* $c_2(\alpha)(r) > 1$*. Consider the polynomial*

$$\psi(r) := (\alpha + \sin 1)r^2 + \alpha r - \sin 1$$

and let

$$\rho_3 := \frac{\sin 1}{\alpha + \sin 1}.$$

It can be seen that ρ_3 *is the positive root of the polynomial* $\psi(r)$ *and is less than 1 since* $\alpha > 0$ *and* $\psi(r) = 0$ *is equivalent to the equation* $a_2(\alpha)(r) = (\sin 1) + 1 - c_2(\alpha)(r)$*. Therefore by Lemma 10.1, the radius of starlikeness associated with the function* $\varphi_{\sin}$ *for the class* C_2^α *is at least* ρ_3*. Considering the function* $\tilde{f}$ *from the class* $\mathcal{CV}'$ *given by* $\tilde{f}(z) = z/(1-z)$ *and the corresponding function* $\tilde{g} \in C_2^\alpha$*, by using* (2.3)*, for* $z = \rho_3$*, we get*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 1 + \sin 1 = \varphi_{\sin}(1),$$

which proves the sharpness for ρ_3. □

11. Radius of starlikeness associated with the class $\mathcal{ST}_h$

Kumar and Arora [9] defined the class $\mathcal{ST}_h = \mathcal{ST}(\varphi_h)$ where $\varphi_h(z) = 1 + \sinh^{-1}(z)$. The boundary of $\varphi_h(\mathbb{D})$ is petal shaped. The following lemma is due to them.

Lemma 11.1. *[9] For* $1 - \sinh^{-1}(1) < a < 1 + \sinh^{-1}(1)$*, let* r_a *be given by*

$$r_a = \begin{cases} a - (1 - \sinh^{-1}(1)) & \text{if} \quad 1 - \sinh^{-1}(1) < a \leqslant 1 \\ 1 + \sinh^{-1}(1) - a & \text{if} \quad 1 \leqslant a < 1 + \sinh^{-1}(1). \end{cases}$$

Then $\{w : |w - a| < r_a\} \subset \varphi_h(\mathbb{D}) = \Omega_h := \{w \in \mathbb{C} : |\sinh(w-1)| < 1\}$.

Theorem 11.2. *The inclusion $\mathcal{C}_1^\alpha[A,B] \subset \mathcal{ST}_h$ holds if either*

1. $B \geqslant 0$ *and* $(\alpha(A-B))/(1-B) \leqslant \sinh^{-1}(1)$
   or
2. $B \leqslant 0$ *and* $(\alpha(A-B))/(1+B) \leqslant \sinh^{-1}(1)$.

Proof. *Assume that* $B \geqslant 0$ *and* $(\alpha(A-B))/(1-B) \leqslant \sinh^{-1}(1)$. *The inequality* $B \geqslant 0$ *is equivalent to* $c_1(\alpha,A,B)(1) \leqslant 1$. *The result follows from Lemma 11.1, since the inequality* $a_1(\alpha,A,B)(1) \leqslant c_1(\alpha,A,B)(1) + (\sinh^{-1}(1)) - 1$ *follows from* $(\alpha(A-B))/(1-B) \leqslant \sinh^{-1}(1)$.

If $B \leqslant 0$ *and* $(\alpha(A-B))/(1+B) \leqslant \sinh^{-1}(1)$, *then it can be seen that the first inequality reduces to* $c_1(\alpha,A,B)(1) \geqslant 1$. *The result will follow from Lemma 11.1 if* $a_1(\alpha,A,B)(1) \leqslant 1 + (\sinh^{-1}(1)) - c_1(\alpha,A,B)(1)$ *which directly follows from the inequality* $(\alpha(A-B))/(1+B) \leqslant \sinh^{-1}(1)$.

□

The result stated in the following theorem can be discussed if the conditions in Theorem 11.2 do not hold.

Theorem 11.3. *Let* $\alpha > 0$ *and* $-1 \leqslant B < A \leqslant 1$. *If neither condition (1) nor condition (2) of Theorem 11.2 holds, then the radius of starlikeness associated with the function* φ_h *for the class* $\mathcal{C}_1^\alpha[A,B]$ *is given by*

$$\mathcal{R}_{\mathcal{ST}_h}(\mathcal{C}_1^\alpha[A,B]) = \frac{\sinh^{-1}(1)}{\alpha(A-B) + (\sinh^{-1}(1))|B|}.$$

Proof. *We aim to show that the disc* $\mathbb{D}(c_1(\alpha,A,B)(r); a_1(\alpha,A,B)(r))$ *given in* (3.1) *is contained in* Ω_h *for all* $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_h}(\mathcal{C}_1^\alpha[A,B])$. *Let*

$$\rho_2 := \frac{\sinh^{-1}(1)}{\alpha(A-B) + (\sinh^{-1}(1))B}$$

and

$$\rho_3 := \frac{\sinh^{-1}(1)}{\alpha(A-B) - (\sinh^{-1}(1))B}.$$

Consider the case when $B \geqslant 0$. *In this case, the centre* $c_1(\alpha,A,B)(r) \leqslant 1$ *and note that* $\rho_2 = (\sinh^{-1}(1))/(\alpha(A-B) + (\sinh^{-1}(1))|B|)$. *We can see that* ρ_2 *is the root of the polynomial*

$$\xi(r) := ((\sinh^{-1}(1))B^2 + \alpha B(A-B))r^2 + \alpha(A-B)r - \sinh^{-1}(1),$$

where $\xi(r) = 0$ *is equivalent to the equation* $a_1(\alpha,A,B)(r) = c_1(\alpha,A,B)(r) + (\sinh^{-1}(1)) - 1$. *Observe that* $\xi(0) = -\sinh^{-1}(1) < 0$ *and* $\xi(1) = (\sinh^{-1}(1))B^2 + \alpha B(A-B) + \alpha(A-B) - \sinh^{-1}(1) > 0$, *since the condition (1) in Theorem 11.2 does not hold. Hence* $\rho_2 \in (0,1)$. *Therefore by Lemma 11.1, the radius of starlikeness associated with the function* φ_h *is at least* ρ_2. *To prove the sharpness, consider the function* $\tilde{f}$ *from the class* $\mathcal{CV}[A,B]$ *given by* (2.1) *and the corresponding function* $\tilde{g} \in \mathcal{C}_1^\alpha[A,B]$. *Then for* $z = -\rho_2$, *by using* (2.2) *we get*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(-\rho_2)\tilde{g}'(-\rho_2)}{\tilde{g}(-\rho_2)} = 1 - \sinh^{-1}(1) = \varphi_h(-1),$$

thus proving the sharpness for ρ_2.

Similarly, considering the case when $B \leqslant 0$, *we get that the centre* $c_1(\alpha,A,B)(r) \geqslant 1$ *and* $\rho_3 = (\sinh^{-1}(1))/(\alpha(A-B) + (\sinh^{-1}(1))|B|)$. *Consider the polynomial*

$$\psi(r) := ((\sinh^{-1}(1))B^2 - \alpha B(A-B))r^2 + \alpha(A-B)r - \sinh^{-1}(1)$$

and note that ρ_3 is the positive root of the polynomial $\psi(r)$. A calculation readily shows that $\psi(r) = 0$ is equivalent to the equation $a_1(\alpha, A, B)(r) = 1 + (\sinh^{-1}(1)) - c_1(\alpha, A, B)(r)$. Clearly $\psi(0) = -\sinh^{-1}(1) < 0$ and since the condition (2) in Theorem 11.2 does not hold, $\psi(1) = (\sinh^{-1}(1))B^2 - \alpha B(A-B) + \alpha(A-B) - \sinh^{-1}(1) > 0$ and thus $\rho_3 \in (0, 1)$. Hence by Lemma 11.1, the radius of starlikeness associated with the function φ_h for the class $C_1^\alpha[A, B]$ is at least ρ_3. Now consider the function $\tilde{f}$ from the class $\mathcal{CV}[A, B]$ given by (2.1). *It can be seen from* (2.2) *that for $z = \rho_3$,*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 1 + \sinh^{-1}(1) = \varphi_h(1),$$

which proves the sharpness for ρ_3. □

The radius of starlikeness associated with the function φ_h for the class C_2^α is discussed in the following theorem.

Theorem 11.4. *Let $\alpha > 0$. Then the radius of starlikeness associated with the function φ_h for the class C_2^α is given by*

$$\mathcal{R}_{\mathcal{ST}_h}(C_2^\alpha) = \frac{\sinh^{-1}(1)}{\alpha + \sinh^{-1}(1)}.$$

Proof. *We prove the theorem by showing that the disc $\mathbb{D}(c_2(\alpha)(r); a_2(\alpha)(r))$ given in* (3.2) *is contained in Ω_h for all $0 < r \leqslant \mathcal{R}_{\mathcal{ST}_h}(C_2^\alpha)$. Here the centre of the disc $c_2(\alpha)(r) > 1$. Let*

$$\rho_3 := \frac{\sinh^{-1}(1)}{\alpha + \sinh^{-1}(1)}.$$

Then ρ_3 is the positive root of the polynomial

$$\psi(r) := (\alpha + \sinh^{-1}(1))r^2 + \alpha r - \sinh^{-1}(1)$$

and since $\alpha > 0$, ρ_3 is less than 1. Also $\psi(r) = 0$ is equivalent to the equation $a_2(\alpha)(r) = 1 + (\sinh^{-1}(1)) - c_2(\alpha)(r)$. Therefore by Lemma 11.1, the radius of starlikeness associated with the function φ_h for the class C_2^α is at least ρ_3. Considering the function $\tilde{f}$ from the class $\mathcal{CV}'$ given by $\tilde{f}(z) = z/(1-z)$ and the corresponding function $\tilde{g} \in C_2^\alpha$, (2.3) *shows that for $z = \rho_3$,*

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{(\rho_3)\tilde{g}'(\rho_3)}{\tilde{g}(\rho_3)} = 1 + \sinh^{-1}(1) = \varphi_h(1),$$

thus proving the sharpness for ρ_3. □

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