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Donoho-Stark's and Hardy's uncertainty principles for the short-time quaternion offset linear canonical transform

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Abstract. The quaternion offset linear canonical transform (QOLCT) which is time-shifted and frequency-modulated version of the quaternion linear canonical transform (QLCT) provides a more general framework of most existing signal processing tools. For the generalized QOLCT, the classical Heisenberg's and Lieb's uncertainty principles have been studied recently. In this paper, we first define the short-time quaternion offset linear canonical transform (ST-QOLCT) and derive its relationship with the quaternion Fourier transform (QFT). The crux of the paper lies in the generalization of several well known uncertainty principles for the ST-QOLCT, including Donoho-Stark's uncertainty principle, Hardy's uncertainty principle, Beurling's uncertainty principle, and Logarithmic uncertainty principle.

1. Introduction

The linear canonical transform (LCT) with four parameters (a, b, c, d) has been generalized to a six parameter transform (a, b, c, d, p, q) known as offset linear canonical transform (OLCT). Due to the time shifting 'p' and frequency modulation 'q' parameters, the OLCT has gained more flexibility over classical LCT. Hence has found wide applications in image and signal processing. The quaternion offset linear canonical transform (QOLCT) which is time-shifted and frequency-modulated version of the quaternion linear canonical transform (QLCT) provides a more general framework of most existing signal processing tools. For more details we refer to [1]-[7] and references therein.

Because of its wide applications in signal analysis, image processing and optics the quaternion offset linear canonical transform (QOLCT) has attained much universality in recent years. However, the QOLCT is inadequate for localizing the QOLCT-frequency of non-transient signals, as such, it is indispensable to introduce an eccentric localized transform coined as the short-time quaternion offset linear canonical transform (ST-QOLCT), which can effectively reveal the local QOLCT-frequency content of such signals. The ST-QOLCT enjoys high resolution, provides local Information and eliminates cross terms. The chrip signals can be better analysed through ST-QOLCT. We refer to [8–11] for more details.

Let us now move to the side of uncertainty inequality. Uncertainty principle was introduced by German physicists Heisenberg [12] in 1927 which is known as the heart of any signal processing tool. With the

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passage of time researchers further extended the uncertainty principle to different types of new uncertainty principles associated with the Fourier transform, for instance Heisenberg's uncertainty principle, Logarithmic uncertainty principle, Hardy's uncertainty principle and Beurling's uncertainty principle. Later these uncertainty principles were extended to Quaternion domain. In [13, 14] authors proposed uncertainty principles associated with the OLCT and in [15–17] authors establish uncertainty principles for the windowed linear canonical transform (WLCT) and windowed offset linear canonical transform (WOLCT). Recently, the uncertainty principles associated with the QOLCT were proposed in [18, 19]. Also Gao and Li [20] recently developed uncertainty principles for two sided windowed linear canonical transform. Later Bhat and Dar [21] establish uncertainty principles for 2D Gabor quaternion offset linear canonical transform. Where as Lieb's uncertainty principle has been established in [19]. However, Donoho-Stark's uncertainty principle, Hardy's uncertainty principle and Beurling's uncertainty principle have not been established for ST-QOLCT. Taking this opportunity, we shall study these uncertainty principles for the ST-QOLCT domain.

The rest of paper is organised as follows. In Section 2, we provide some preliminaries needed for subsequent sections. In Section 3, we establish a relationship of ST-QOLCT with QOLCT and QFT. In Section 4, we develop some novel uncertainty principles like Donoho-Stark's, Hardy's and Beurling's. Finally we establish Logarithmic uncertainty principle using Pitt's Inequality.

2. Preliminaries

In this section, we collect some basic facts on the quaternion algebra and the QFT, which will be needed throughout the paper.

2.1. Quaternion algebra

In 1834 W. R. Hamilton introduced quaternion algebra by extension of the complex number to an associative non-commutative 4D algebra. Denoted by $\mathbb{H}$ in his honor where every element of $\mathbb{H}$ has a Cartesian form given by

$$\mathbb{H} = \{q | q := [q]_0 + i[q]_1 + j[q]_2 + k[q]_3, [q]_i \in \mathbb{R}, i = 0, 1, 2, 3\} \tag{1}$$

where i, j, k are imaginary units obeying Hamilton's multiplication rules:

$$i^2 = j^2 = k^2 = -1, \tag{2}$$

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j. \tag{3}$$

Let $[q]_0$ and $vec(q) = i[q]_1 + j[q]_2 + k[q]_3$ denote the real scalar part and the vector part of quaternion number $q = [q]_0 + i[q]_1 + j[q]_2 + k[q]_3$, respectively. Then, from [22], the real scalar part has a cyclic multiplication symmetry

$$[pql]_0 = [qlp]_0 = [lpq]_0, \qquad \forall q, p, l \in \mathbb{H}, \tag{4}$$

the conjugate of a quaternion q is defined by $\overline{q} = [q]_0 - i[q]_1 - j[q]_2 - k[q]_3$, and the norm of $q \in \mathbb{H}$ defined as

$$\left|q\right| = \sqrt{q\overline{q}} = \sqrt{[q]_0^2 + [q]_1^2 + [q]_2^2 + [q]_3^2}. \tag{5}$$

It is easy to verify that

$$\overline{pq} = \overline{q}\,\overline{p}, \qquad |qp| = |q||p|, \quad \forall q, p \in \mathbb{H}. \tag{6}$$

In this paper, we will study the quaternion-valued signal $f : \mathbb{R}^2 \to \mathbb{H}$, f which can be expressed as $f = f_0 + if_1 + jf_2 + kf_3$, with $f_m : \mathbb{R}^2 \to \mathbb{R}$ $for\ m = 0, 1, 2, 3$. The quaternion inner product for quaternion valued signals $f, g : \mathbb{R}^2 \to \mathbb{H}$, as follows:

$$\langle f, g\rangle = \int_{\mathbb{R}^2} f(\mathbf{x})\,\overline{g(\mathbf{x})}\,d\mathbf{x} \tag{7}$$

where $\mathbf{x} = (x_1, x_2)$, $f(\mathbf{x}) = f(x_1, x_2)$, $\mathbf{x} = dx_1dx_2$, and so on.
Hence, the natural norm is given by

$$\left|f\right|_2 = \sqrt{< f, f >} = (\int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x})^{\frac{1}{2}}, \tag{8}$$

and the quaternion module $L^2(\mathbb{R}^2, \mathbb{H})$, is given by

$$L^2(\mathbb{R}^2, \mathbb{H}) = \{f : \mathbb{R}^2 \to \mathbb{H}, \left|f\right|_2 < \infty\}. \tag{9}$$

Lemma 2.1. *If $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then the Cauchy-Schwarz inequality holds[25]*

$$\left|\langle f, g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}\right|^2 \leq \|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|g\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{10}$$

If and only if $f = ig$ for some quaternionic parameter $i \in \mathbb{H}$, the equality holds.

2.2. *The two-sided QFT*

The QFT belongs to the family of Clifford Fourier transformations.

It is a generalization of the classical Fourier transform (CFT) [23].Some useful properties, and theorems of this transform are generalizations of the corresponding properties and theorems of the classical Fourier transform with some modifications. There are three different types of QFT, the left-sided QFT , the right-sided QFT, and two-sided QFT [24]. In this paper our focus shall be on two-sided QFT. So from here on by QFT we mean two-sided quaternion Fourier transform. Let us begin with definition of the two-sided QFT and provide some properties used in the sequel.

Definition 2.2. *(Two-sided QFT.)*
For $f \in L^1\left(\mathbb{R}^2, \mathbb{H}\right)$ the two-sided QFT with respect to unit quaternions i; j is given by

$$\mathcal{F}^{i,j}[f](\mathbf{w}) = \int_{\mathbb{R}^2} e^{-iw_1x_1}\, f(\mathbf{x})\, e^{-jw_2x_2}dt, \quad \textit{where } \mathbf{x}, \mathbf{w} \in \mathbb{R}^2. \tag{11}$$

We define the modulus of $\mathcal{F}[f]^{i,j}$ as follows :

$$\left|\mathcal{F}^{i,j}[f]\right| := \sqrt{\sum_{m=0}^{m=3} \left|\mathcal{F}^{i,j}\left[f_m\right]\right|^2}. \tag{12}$$

Furthermore, we define a new L^2-norm of $\mathcal{F}[f]$ as follows :

$$\left\|\mathcal{F}^{i,j}[f]\right\|_2 := \sqrt{\int_{\mathbb{R}^2} \left|\mathcal{F}^{i,j}\left[f\right](y)\right|^2 dy}. \tag{13}$$

Lemma 2.3. (Dilation property)
Let k_1, k_2 be a positive scalar constants, we have

$$\mathcal{F}^{i,j}\left[f(x_1, x_2)\right]\left(\frac{w_1}{k_1}, \frac{w_2}{k_1}\right) = k_1k_2\mathcal{F}^{i,j}\left[f(k_1x_1, k_2x_2)\right](w_1, w_2). \tag{14}$$

we can also write it as

$$\mathcal{F}^{i,j}\left[f(\mathbf{x})\right]\left(\frac{\mathbf{w}}{\mathbf{k}}\right) = \mathbf{k}\mathcal{F}^{i,j}\left[f(\mathbf{k}\mathbf{x})\right](\mathbf{w}). \tag{15}$$

Lemma 2.4. (QFT Plancherel)
Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then

$$\int_{\mathbb{R}^2} \left|\mathcal{F}^{i,j}\left[f\right](\mathbf{w})\right|^2 d\mathbf{w} = 4\pi^2 \int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x}. \tag{16}$$

Lemma 2.5. (Inverse QFT)
If $f \in L^1\left(\mathbb{R}^2, \mathbb{H}\right)$, and $\mathcal{F}^{i,j}[f] \in L^1\left(\mathbb{R}^2, \mathbb{H}\right)$, then the two-sided QFT is an invertible transform and its inverse is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{iw_1x_1} \mathcal{F}^{i,j}[f(\mathbf{x})](\mathbf{w}) e^{jw_2x_2} d\mathbf{w}. \tag{17}$$

2.3. Quaternion offset linear canonical transform (QOLCT)

The quaternion linear canonical transform(QLCT) is a generalization of the linear canonical transform(LCT) firstly defined by Kou, et al. [1, 2] . Later in [3] Hitzer.E generalized the definitions of Kou, etl to introduce two-sided QLCT. In this paper, we mainly focus on the two-sided QLCT.

Definition 2.6. (Quaternion Linear Canonical Transform.)
Let $A_s = \begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter satisfying $\det(A_s) = 1$, for $s = 1, 2$. The two-sided QLCT of signal $f \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$ is defined by

$$\mathcal{L}_{A_1,A_2}[f](\mathbf{w}) = \int_{\mathbb{R}^2} K^i_{A_1}(x_1, \omega_1) f(\mathbf{x}) K^j_{A_2}(x_2, \omega_2) \mathrm{d}\mathbf{x}, \tag{18}$$

where $\mathbf{w} = (\omega_1, \omega_2) \in \mathbb{R}^2$ is regarded as the QLCT domain, and the kernel signals $K^i_{A_1}(x_1, \omega_1)$, $K^j_{A_2}(x_2, \omega_2)$ are respectively given by

$$K^i_{A_1}(x_1, \omega_1) := \begin{cases} \frac{1}{\sqrt{2\pi i b_1}} e^{i\left(\frac{a_1}{2b_1}x_1^2 - \frac{x_1\omega_1}{b_1} + \frac{d_1}{2b_1}\omega_1^2\right)}, & b_1 \neq 0 \\ \sqrt{d_1} e^{i\frac{c_1d_1}{2}\omega_1^2} \delta(x_1 - d_1 w_1), & b_1 = 0 \end{cases} \tag{19}$$

and

$$K^j_{A_2}(x_2, \omega_2) := \begin{cases} \frac{1}{\sqrt{2\pi j b_2}} e^{j\left(\frac{a_2}{2b_2}x_2^2 - \frac{x_2\omega_2}{b_2} + \frac{d_2}{2b_2}\omega_2^2\right)}, & b_2 \neq 0 \\ \sqrt{d_2} e^{j\frac{c_2d_2}{2}\omega_2^2} \delta(x_2 - d_2 w_2), & b_2 = 0 \end{cases} \tag{20}$$

where $\delta(x)$ representing the Dirac function.

Here we note that for $b_s = 0, s = 1, 2$ the QLCT of a signal boils down to chirp multiplication operations, and it is of no particular interest for our objective in this work. So without loss of generality, we set $b_s \neq 0$ in rest of paper.

Lemma 2.7. *Suppose $f \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$, then the inversion of the QLCT of f is given by*

$$\begin{aligned} f(\mathbf{x}) &= \mathcal{L}^{-1}_{A_1,A_2}[\mathcal{L}_{A_1,A_2}[f]](\mathbf{x}) \\ &= \int_{\mathbb{R}^2} K^{-i}_{A_1}(x_1, \omega_1) \mathcal{L}_{A_1,A_2}\left\{f\right\}(\mathbf{w}) K^{-j}_{A_2}(x_2, \omega_2) \mathrm{d}\mathbf{w}. \end{aligned} \tag{21}$$

We now generalize the definitions of [4, 5] as follows:

Definition 2.8. (QOLCT.) *Let* $A_s = \begin{bmatrix} a_s & b_s & | & p_s \\ c_s & d_s & | & q_s \end{bmatrix}$, *be a matrix parameter such that* $a_s, b_s, c_s, d_s, p_s, q_s \in \mathbb{R}$, $b_s \neq 0$ *and* $a_sd_s - b_sc_s = 1$, *for* $s = 1, 2$. *The two-sided quaternion offset linear canonical transform of any quaternion valued function* $f \in L^2(\mathbb{R}^2, \mathbb{H})$, *is given by*

$$\mathcal{O}^{i,j}_{A_1,A_2}\Big[f(\mathbf{x})\Big](\mathbf{w}) = \int_{\mathbb{R}^2} K^i_{A_1}(x_1, w_1)\, f(\mathbf{x})\, K^j_{A_2}(x_2, w_2) d\mathbf{x} \tag{22}$$

where $\mathbf{x} = (x_1, x_2)$, $w = (w_1, w_2)$ *and the kernel signals* $K^i_{A_1}(x_1, w_1)$ *and* $K^j_{A_2}(x_2, w_2)$ *are respectively given by*

$$K^i_{A_1}(x_1, w_1) = \frac{1}{\sqrt{2\pi b_1 i}}\, e^{\frac{i}{2b_1}\left[a_1x_1^2 - 2x_1(w_1 - p_1) - 2w_1(d_1p_1 - b_1q_1) + d_1(w_1^2 + p_1^2)\right]}, b_1 \neq 0 \tag{23}$$

$$K^j_{A_2}(x_2, w_2) = \frac{1}{\sqrt{2\pi b_2 j}}\, e^{\frac{j}{2b_2}\left[a_2x_2^2 - 2x_2(w_2 - p_2) - 2w_2(d_2p_2 - b_2q_2) + d_2(w_2^2 + p_2^2)\right]}, b_2 \neq 0 \tag{24}$$

Note: The left-sided and right-sided QOLCT can be defined correspondingly by placing the two above kernels both on the left or on the right, respectively.

Lemma 2.9. *Suppose* $f \in L^2(\mathbb{R}^2, \mathbb{H})$ *then the inversion of two-sided QOLCT is given by*

$$f(\mathbf{x}) = \int_{\mathbb{R}^2} K^{-i}_{A_1}(x_1, w_1)\, \mathcal{O}^{i,j}_{A_1,A_2}\Big[f\Big](\mathbf{w})\, K^{-j}_{A_2}(x_2, w_2)\, d\mathbf{w}.$$

Lemma 2.10. *(Plancherel for QOLCT) Every two dimensional quaternion valued function* $f \in L^2(\mathbb{R}^2, \mathbb{H})$ *and its two-sided QOLCT are related to the Plancherel identity in the following way:*

$$\left\|\mathcal{O}^{i,j}_{A_1,A_2}\Big[f\Big]\right\|_2 = \left\|f\right\|_2. \tag{25}$$

3. Short-time Quaternion Offset Linear Canonical Transform(ST-QOLCT)

In this section, we shall formally introduce the notion of the two-sided Short-time quaternion offset linear canonical transform (ST-QOLCT) then establish some properties of the proposed transform.

Definition 3.1. (ST-QOLCT.) *Let* $A_s = \begin{bmatrix} a_s & b_s & | & p_s \\ c_s & d_s & | & q_s \end{bmatrix}$, *be a matrix parameter such that* $a_s, b_s, c_s, d_s, p_s, q_s \in \mathbb{R}$, $b_s \neq 0$ *and* $a_sd_s - b_sc_s = 1$, *for* $s = 1, 2$. *The two-sided short-time quaternion offset linear canonical transform of any quaternion valued function* $f \in L^2(\mathbb{R}^2, \mathbb{H})$, *with respect window function* $\phi \in L^2(\mathbb{R}^2, \mathbb{H})$ *is given by*

$$\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](\mathbf{w}, \mathbf{u}) = \int_{\mathbb{R}^2} K^{\mathbf{i}}_{A_1}(x_1, w_1)\, f(\mathbf{x})\overline{\phi(\mathbf{x} - \mathbf{u})}\, K^j_{A_2}(x_2, w_2) dt \tag{26}$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{w} = (w_1, w_2)$, $\mathbf{u} = (u_1, u_2)$ *and the quaternion kernels* $K^i_{A_1}(x_1, w_1)$ *and* $K^j_{A_2}(x_2, w_2)$ *are given by equation 2.23 and 2.24 respectively.*

Note: It is worth to note that the quaternion ST-OLCT(3.1), boils down to various linear integral transforms such as:

- Short-time versions of quaternion linear canonical transform when matrices parameters $A_s = \left[\begin{array}{cc|c} a_s & b_s & 0 \\ c_s & d_s & 0 \end{array}\right]$,

- Quaternion short-time fractional Fourier transform when $A_s = \left[\begin{array}{cc|c} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{array}\right]$,

- Quaternion short-time Fourier transform when $A_s = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$.

First of all we define the relation between ST-QOLCT and QOLCT, we begin as:
Since $b_s \neq 0$, $s = 1, 2$, as in other cases proposed transform reduces to a chrip multiplications. Thus for fixed $\mathbf{u}$ we have

$$\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](\mathbf{w},\mathbf{u}) = O^{i,j}_{A_1,A_2}\Big[f(\mathbf{x})\overline{\phi(\mathbf{x}-\mathbf{u})}\Big](\mathbf{w}) \tag{27}$$

$$= \int_{\mathbb{R}^2} K^i_{A_1}(x_1,w_1)\, f(\mathbf{x})\overline{\phi(\mathbf{x}-\mathbf{u})}\, K^j_{A_2}(x_2,w_2)d\mathbf{x} \tag{28}$$

Applying the inverse QOLCT to (27), we have

$$f\overline{\Theta}_{\mathbf{u}}(\mathbf{x}) = f(\mathbf{x})\overline{\phi(\mathbf{x}-\mathbf{u})} = \{O^{i,j}_{A_1,A_2}\}^{-1}\Big[\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](\mathbf{w},\mathbf{u})\Big](\mathbf{x}) \tag{29}$$

$$= \int_{\mathbb{R}^2} K^{-i}_{A_1}(x_1,w_1)\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](\mathbf{w},\mathbf{u})K^{-j}_{A_2}(x_2,w_2)d\mathbf{w} \tag{30}$$

where $f\overline{\Theta}_{\mathbf{u}}(\mathbf{x})$ is known as modified signal.

Next we give the relation between two-sided ST-QOLCT and two-sided QFT, for that we have following lemma. It will be useful for our analysis of the ST-QOLCT.

Lemma 3.2. *The two-sided ST-QOLCT(3.1) of a signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ can be reduced to the two-sided QFT(11) as*

$$\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](w,u) = \frac{1}{\sqrt{2\pi i b_1}} e^{i[-\frac{1}{b_1}w_1(d_1p_1-b_1s_1)+\frac{d_1}{2b_1}(w_1^2+p_1^2)]}\mathcal{F}^{i,j}(h)\left(\frac{\mathbf{w}}{\mathbf{b}},\mathbf{u}\right)$$
$$\times\frac{1}{\sqrt{2\pi j b_2}} e^{j[-\frac{1}{b_2}w_2(d_2p_2-b_2s_2)+\frac{d_2}{2b_2}(w_2^2+p_2^2)]}$$

where

$$h(\mathbf{x},\mathbf{u}) = e^{i[\frac{a_1}{2b_1}x_1^2+\frac{1}{b_1}x_1p_1]} f\overline{\Theta}_{\mathbf{u}}(\mathbf{x}) e^{j[\frac{a_2}{2b_2}x_2^2+\frac{1}{b_2}x_2p_2]} \tag{31}$$

and $\mathbf{b} = (b_1, b_2)$, $\mathcal{F}^{i,j}(h)$ is the QFT of signal h given by (11).

Proof. From the definition of the ST-QOLCT, we have

$$\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](\mathbf{w},\mathbf{u})$$
$$= \int_{\mathbb{R}^2} K^i_{A_1}(x_1,w_1)f(x)\overline{\phi(x-u)}\,K^j_{A_2}(x_2,w_2)dx$$
$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 i}} e^{\frac{i}{2b_1}\Big[a_1x_1^2-2x_1(w_1-p_1)-2w_1(d_1p_1-b_1q_1)+d_1(w_1^2+p_1^2)\Big]} f(x)\overline{\phi(x-u)}$$
$$\times e^{\frac{j}{2b_2}\Big[a_2x_2^2-2x_2(w_2-p_2)-2w_2(d_2p_2-b_2q_2)+d_2(w_2^2+p_2^2)\Big]}dx$$
$$= \frac{1}{\sqrt{2\pi i b_1}} e^{i[-\frac{1}{b_1}w_1(d_1p_1-b_1s_1)+\frac{d_1}{2b_1}(w_1^2+p_1^2)]} \int_{\mathbb{R}^2} e^{-i\frac{1}{b_1}x_1w_1}\Big(e^{i[\frac{a_1}{2b_1}x_1^2+\frac{1}{b_1}x_1p_1]}f(x)\overline{\phi(\mathbf{x}-\mathbf{u})}$$
$$\times\, e^{j[\frac{a_2}{2b_2}x_2^2+\frac{1}{b_2}x_2p_2]}\Big)e^{-j\frac{1}{b_2}x_2w_2}dx\frac{1}{\sqrt{2\pi j b_2}}e^{j[-\frac{1}{b_2}w_2(d_2p_2-b_2s_2)+\frac{d_2}{2b_2}(w_2^2+p_2^2)]}$$

on setting $h(\mathbf{x},\mathbf{u}) = e^{i[\frac{a_1}{2b_1}x_1^2+\frac{1}{b_1}x_1p_1]} f\overline{\Theta}_{\mathbf{u}}(\mathbf{x})e^{j[\frac{a_2}{2b_2}x_2^2+\frac{1}{b_2}x_2p_2]}$ in the above equationwe get the desired result.

$$\mathbb{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](\mathbf{w},\mathbf{u}) = \frac{1}{\sqrt{2\pi i b_1}}e^{i[-\frac{1}{b_1}w_1(d_1p_1-b_1s_1)+\frac{d_1}{2b_1}(w_1^2+p_1^2)]}\mathcal{F}^{i,j}(h)\left(\frac{\mathbf{w}}{\mathbf{b}},\mathbf{u}\right)$$
$$\times\frac{1}{\sqrt{2\pi j b_2}}e^{j[-\frac{1}{b_2}w_2(d_2p_2-b_2s_2)+\frac{d_2}{2b_2}(w_2^2+p_2^2)]}$$

□

3.1. Some properties of ST-QOLCT

Theorem 3.3. *Let $f,\phi \in L^2(\mathbb{R}^2,\mathbb{H})$. Then its ST-QOLCT satisfies:*
(i)The map $f \longrightarrow \mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f]$ is real linear.
(ii)$\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f]$ is is uniformly continuous and bounded on the time–frequency plane $\mathbb{R}^2\times\mathbb{R}^2$ and satisfies :

$$|\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](\mathbf{w},\mathbf{u})| \le \frac{1}{2\pi\sqrt{|b_1b_2|}}||f||_{L^2(R^2,H)}||\phi||_{L^2(\mathbb{R}^2,\mathbb{H})} \tag{32}$$

Proof. *The proof of (i) follows by definition(3.1) and (ii) is proved in ([21], Thm 1).* □

Theorem 3.4. (Moyal's formula). *Let $\phi,\psi \in L^2(\mathbb{R}^2,\mathbb{H})$ be a fixed non-zero window functions and $f,g \in L^2\left(\mathbb{R}^2,\mathbb{H}\right)$ then*

$$\left\langle \mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u}), \mathcal{S}^{\mathbb{H}}_{\psi,A_1,A_2}[g](\mathbf{w},\mathbf{u})\right\rangle = [\langle f,g\rangle\left\langle\phi,\psi\right\rangle] \tag{33}$$

Proof. *The proof is already present in [21].* □

Consequences of theorem 3.4.

(i) If $\phi=\psi$, then

$$\langle \mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u}), \mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f_2](\mathbf{w},\mathbf{u})\rangle = ||\phi||^2_{L^2(\mathbb{R}^2,\mathbb{H})}\langle f,g\rangle \tag{34}$$

(ii) If $f=g$, then

$$\langle \mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u}), \mathcal{S}^{\mathbb{H}}_{\psi,A_1,A_2}[f](\mathbf{w},\mathbf{u})\rangle = \langle\phi,\psi\rangle||f||^2_{L^2(\mathbb{R}^2,\mathbb{H})} \tag{35}$$

(iii) If $f = g$ and $\phi = \psi$, then

$$\langle \mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u}), \mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})\rangle = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})|^2 dudw = ||\phi||^2_{L^2(\mathbb{R}^2,\mathbb{H})}||f||^2_{L^2(\mathbb{R}^2,\mathbb{H})} \tag{36}$$

Note (36) is known as the energy preserving relation for the proposed ST-QOLCT.

Remark 3.5. (Isometry). *For $||\phi||^2_{L^2(\mathbb{R}^2,\mathbb{H})} = 1$ (36) reduces to*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathcal{S}^{A_1,A_2}_{\phi}[f](\mathbf{w},\mathbf{u})|^2 dudw = ||f||^2_{L^2(\mathbb{R}^2,\mathbb{H})} \tag{37}$$

i.e the proposed ST-QOLCT(3.1) becomes an isometry from $L^2(\mathbb{R}^2,\mathbb{H})$ into $L^2(\mathbb{R}^2,\mathbb{H})$.In other words, the total energy of a quaternion-valued signal computed in the in the quaternion short-time offset linear canonical domain is equal to the total energy computed in the spatial domain.

Theorem 3.6. (Reconstruction formula). *Every 2D quaternion signal $f \in L^2(\mathbb{R}^2,\mathbb{H})$ can be fully reconstructed by the formula*

$$f(\mathbf{x}) = \frac{1}{||\phi||^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K^{-i}_{A_1}(x_1,w_1)\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})K^{-j}_{A_2}(x_2,w_2)\phi(\mathbf{x}-\mathbf{u})d\mathbf{w}d\mathbf{u}. \tag{38}$$

Proof. *Already proved in [21]* □

4. Uncertainty principles for the QWLCT

In this section we study several different kinds of uncertainty principles associated with ST-QOLCT.

4.1. Donoho-Stark's uncertainty principle

In this subsection, according to the relationship between the ST-QOLCT and the QFT, we present a exquisite uncertainty principle on $\mathbb{R}^2$ concerning to the Donoho-Stark's uncertainty principle. First we revisit the concept of $\epsilon-$*concentrate* of a quaternion valued signal on a measurable set $M \subseteq \mathbb{R}^2$,. Let us begin with the following definition.

Definition 4.1. *For $\epsilon \geq 0$, a quaternion valued signal $f \in L^2(\mathbb{R}^2,\mathbb{H})$ is said to be $\epsilon -$ concentrated on a measurable set $M \subseteq \mathbb{R}^2$, if*

$$\left(\int_{\mathbb{R}^2\setminus M} |f(\mathbf{x})|^2 d\mathbf{x}\right)^{\frac{1}{2}} \leq \epsilon ||f||_2 \tag{39}$$

If $0 \leq \epsilon \leq \frac{1}{2}$, then the most of energy is concentrated on M, and M is indeed the essential support of f, if $\epsilon = 0$, then D is the exact support of f. Similarly, we say that its $\mathcal{F}^{i,j}$ is $\epsilon -$ concentrated on a measurable set $N \subseteq \mathbb{R}^2$, if

$$\left(\int_{\mathbb{R}^2\setminus N} |\mathcal{F}^{i,j}[f(\mathbf{x})](\mathbf{w})|^2 d\mathbf{w}\right)^{\frac{1}{2}} \leq \epsilon ||\mathcal{F}^{i,j}[f]||_2 \tag{40}$$

Lemma 4.2. (Donoho-Stark's uncertainty principle for QFT[25]) *Let $f \in L^2(\mathbb{R}^2,\mathbb{H})$ with $f \neq 0$ is ϵ_M-concentrated on $M \subseteq \mathbb{R}^2$, and $\mathcal{F}^{i,j}(f)$ is ϵ_N-concentrated on $N \subseteq \mathbb{R}^2$. Then*

$$|M||N| \geq 2\pi(1 - \epsilon_M - \epsilon_N)^2. \tag{41}$$

where $|M|$ and $|N|$ are the measures of the sets M and N.

Definition 4.3. *Let $f, \phi \in L^2(\mathbb{R}^2, \mathbb{H})$ where ϕ is a non zero window function then $\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})$ is ϵ_N-concentrated on $N \subseteq \mathbb{R}^2$, if*

$$\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2\setminus N} |\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f(\mathbf{x})](\mathbf{w},\mathbf{u})|^2 d\mathbf{w}d\mathbf{u}\right)^{\frac{1}{2}} \leq \epsilon_N \|\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f]\|_2 \tag{42}$$

where $\|\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f]\|_2 = \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f(\mathbf{x})](\mathbf{w},\mathbf{u})|^2 d\mathbf{w}d\mathbf{u}\right)^{\frac{1}{2}}$

Theorem 4.4. (Donoho-Stark's uncertainty principle for ST-QOLCT). *Let ϕ be the nonzero quaternion window function and $f \neq 0$ be a quaternion signal function in $L^2(\mathbb{R}^2, \mathbb{H})$ is ϵ_M-concentrated on measurable set $M \subseteq \mathbb{R}^2$, and $\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})$ is ϵ_N-concentrated on $N \subseteq \mathbb{R}^2$. Then*

$$|M||N| \geq 2\pi b_1 b_2 (1 - \epsilon_M - \epsilon_N)^2. \tag{43}$$

Proof. *We have from Lemma(3.2)*

$$\mathbb{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](\mathbf{w},\mathbf{u}) = \frac{1}{\sqrt{2\pi i b_1}} e^{i[-\frac{1}{b_1}w_1(d_1p_1 - b_1 s_1) + \frac{d_1}{2b_1}(w_1^2+p_1^2)]}\mathcal{F}^{i,j}(h)\left(\frac{\mathbf{w}}{\mathbf{b}},\mathbf{u}\right) \times \frac{1}{\sqrt{2\pi j b_2}} e^{j[-\frac{1}{b_2}w_2(d_2p_2 - b_2 s_2) + \frac{d_2}{2b_2}(w_2^2+p_2^2)]}$$

where

$$h(\mathbf{x},\mathbf{u}) = e^{i[\frac{a_1}{2b_1}x_1^2 + \frac{1}{b_1}x_1p_1]} f\overline{\Theta}_{\mathbf{u}}(\mathbf{x}) e^{j[\frac{a_2}{2b_2}x_2^2 + \frac{1}{b_2}x_2p_2]} \tag{44}$$

For $\mathbf{u} = \mathbf{x}$, we have $|h(\mathbf{x})| = |f(\mathbf{x})\overline{\phi(0)}|$, with $|\overline{\phi(0)}| > 0$, since f is ϵ_M-concentrated on measurable set $M \subseteq \mathbb{R}^2$, then by definition(4.1)

$$\left(\int_{\mathbb{R}^2\setminus M} |f(\mathbf{x})|^2 d\mathbf{x}\right)^{\frac{1}{2}} \leq \epsilon_M \|f\|_2 \Rightarrow \left(\int_{\mathbb{R}^2\setminus M} |h(\mathbf{x})|^2 d\mathbf{x}\right)^{\frac{1}{2}} \leq \epsilon_M \|f\|_2$$

i.e. $h(\mathbf{x})$ is ϵ_M-concentrated on measurable set $M \subseteq \mathbb{R}^2$.

Also it is given that $\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})$ is ϵ_N-concentrated on $N \subseteq \mathbb{R}^2$ and we have $|\mathbb{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](\mathbf{w},\mathbf{u})| = |\frac{1}{\sqrt{2\pi \mathbf{b}_1}}\mathcal{F}^{i,j}(h)\left(\frac{\mathbf{w}}{\mathbf{b}},\mathbf{u}\right)\frac{1}{\sqrt{2\pi \mathbf{b}_1}}|$, which implies $\mathcal{F}^{i,j}(h)\left(\frac{\mathbf{w}}{\mathbf{b}},\mathbf{u}\right)$ is ϵ_N-concentrated on $N \subseteq \mathbb{R}^2$, that is to say, is $\mathcal{F}^{i,j}(h)\,(\mathbf{w},\mathbf{u})$ is ϵ_N-concentrated on $\frac{N}{\mathbf{b}} \subseteq \mathbb{R}^2$. Hence, applying Lemma (4.2) to the function h, we obtain

$$|M|\left|\frac{N}{\mathbf{b}}\right| \geq 2\pi(1 - \epsilon_M - \epsilon_N)^2. \tag{45}$$

Which gives

$$|M||N| \geq 2\pi b_1 b_2 (1 - \epsilon_M - \epsilon_N)^2. \tag{46}$$

□

which completes the proof.

Corollary 4.5. *If $f\overline{\Theta}_{\mathbf{u}}(\mathbf{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$, supp$f\overline{\Theta}_{\mathbf{u}}(\mathbf{x}) \subseteq M$ and supp$\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\Big[f\Big](\mathbf{w},\mathbf{u}) \subseteq N$, then*

$$|M||N| \geq 2\pi\mathbf{b}. \tag{47}$$

Proof. *It is clear from definition (4.1) that $f(\mathbf{x})$ is $0-$concentrated on M iff supp$(f) = M$. Therefore if we take $\epsilon_M = \epsilon_N = 0$ in theorem (4.4) we get desired result.* □

4.2. Hardy's uncertainty principle

G.H Hardy introduced Hardy's uncertainty principle[26] in 1933 which is qualitative in nature, it states that it is impossible for a non zero signal function and its Fourier transform to decrease very rapidly simultaneously. We first present the Hardy's UP for the Two-sided QFT[27].

Lemma 4.6. *Let α and β are positive constants .Suppose $f \in L^2(\mathbb{R}^2, \mathbb{H})$ with*

$$|f(\mathbf{x})| \leq Ce^{-\alpha|\mathbf{x}|^2}, \ \mathbf{x} \in \mathbb{R}^2. \tag{48}$$

$$|\mathcal{F}^{i,j}\{f\}(\mathbf{w})| \leq C'e^{-\beta|\mathbf{w}|^2}, \ \mathbf{w} \in \mathbb{R}^2. \tag{49}$$

for some positive constants C, C'.Then, three cases can occur :

- *If $\alpha\beta > \frac{1}{4}$, then $f = 0$.*
- *If $\alpha\beta = \frac{1}{4}$, then $f(t) = Ae^{-\alpha|\mathbf{x}|^2}$, whit A is a quaternion constant.*
- *If $\alpha\beta < \frac{1}{4}$, then there are infinitely many such functions f.*

By using Lemma3.2 and Lemma 4.6 , we derive Hardy's uncertainty principle for the ST-QOLCT..

Theorem 4.7. *Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H})$ be a non zero window function . Suppose $f \in L^2(\mathbb{R}^2, \mathbb{H})$ with*

$$\left|f(\mathbf{x})\right| \leq Ce^{-\alpha|\mathbf{x}|^2}, \mathbf{x} \in \mathbb{R}^2. \tag{50}$$

$$\left|\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2} f(\mathbf{bw} + \mathbf{p}, \mathbf{u})\right| \leq C'e^{-\beta|\mathbf{w}|^2}, \ \mathbf{w} \in \mathbb{R}^2. \tag{51}$$

for some constants $\alpha, \beta > 0$ and C, C' are positive constants ,then:

- *If $\alpha\beta > \frac{1}{4}$, then $f = 0$.*
- *If $\alpha\beta = \frac{1}{4}$, then $f(\mathbf{x}) = e^{-i\frac{a_1}{2b_1}x_1^2 - i\frac{1}{b_1}x_1p_1} \frac{A}{\overline{\phi(0)}} e^{-\alpha|\mathbf{x}|^2} e^{-j\frac{a_2}{2b_2}x_2^2 - j\frac{1}{b_2}x_2p_2}$, where A is a quaternion constant.*
- *If $\alpha\beta < \frac{1}{4}$, then there are infinitely many f.*

Proof. *On substituting $\mathbf{u} = \mathbf{x}$ in (44),we have*

$$h(\mathbf{x}) = e^{i[\frac{a_1}{2b_1}x_1^2 + \frac{1}{b_1}x_1p_1]} f(\mathbf{x})\overline{\phi(0)} e^{j[\frac{a_2}{2b_2}x_2^2 + \frac{1}{b_2}x_2p_2]}$$

clearly RHS of above equation belongs to $L^2(\mathbb{R}^2, \mathbb{H})$ and $|\overline{\phi(0)}|$ is a positve quantity and

$$|h(\mathbf{x})| = |f(\mathbf{x})||\overline{\phi(0)}| \leq |\overline{\phi(0)}|Ce^{-\alpha|x|^2} = C_1e^{-\alpha|\mathbf{x}|^2} \tag{4.3}$$

Now applying(3.2) and (51),we have

$$|\mathcal{F}^{i,j}[h(\mathbf{x})](\mathbf{w})| = \sqrt{b_1b_2}\left|\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2} f(\mathbf{bw} + p, \mathbf{u})\right| \leq \sqrt{\mathbf{b}}C_0e^{-\beta|\mathbf{w}|^2} \tag{52}$$

Therefore, it follows from Lemma4.6 that,
If $\alpha\beta > \frac{1}{4}$ then $h = 0$, so $f = 0$.
If $\alpha\beta = \frac{1}{4}$ then
$f(\mathbf{x}) = Ae^{-\alpha|\mathbf{x}|^2}$, for some constant A.
Hence

$$f(t) = e^{-i\frac{a_1}{2b_1}x_1^2 - i\frac{1}{b_1}x_1p_1} \frac{A}{\overline{\phi(0)}} e^{-\alpha|\mathbf{x}|^2} e^{-j\frac{a_2}{2b_2}x_2^2 - j\frac{1}{b_2}x_2p_2}.$$

If $\alpha\beta < \frac{1}{4}$, then there are infinitely many such functions f, that verify (50) *and (* (51).
This completes the proof. □

It follows from theorem 4.7 that it is impossible for a signal f and its two-sided ST-QOLCT to both decrease very rapidly.

4.3. Beurling's uncertainty principle

Beurling's uncertainty principle [29], [28] is a mutant of Hardy's uncertainty principle. The following Lemma is the Beurling's uncertainty principle for the Two-sided QOLCT ([18] Cor. 4.7) .

Lemma 4.8. *Let $f \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$ and $d \geq 0$ satisfy*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\left|f(\mathbf{x})\right| \left|O^{i,j}_{A_1,A_2}\left[f\right](\mathbf{w})\right|}{(1+|\mathbf{x}|+|\mathbf{w}|)^d} e^{|\mathbf{x}||\mathbf{w}|}\, d\mathbf{x}d\mathbf{w} < \infty, \tag{53}$$

Then
$f(\mathbf{x}) = P(\mathbf{x})e^{-a|\mathbf{x}|^2}$, *a.e.*

Where $a > 0$ and $P(\mathbf{x})$ is a quaternion polynomial of degree $< \frac{d-2}{2}$.
In particular, $f = 0$ a.e. when $d \leq 2$.

On the basis of lemma 4.8, we give the Beurlings' uncertainty principle associated with ST-QOLCT domains.

Theorem 4.9. *Let $\phi, f \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$ where ϕ be a non zero quaternion window function and $d \geq 0$ satisfy*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\left|f(\mathbf{x})\right| |\overline{\phi(\mathbf{x}-\mathbf{u})}| \left|\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\left[f\right](\mathbf{w},\mathbf{u})\right|}{(1+|\mathbf{x}|+|w|)^d} e^{|\mathbf{x}||\mathbf{w}|}\, dxdw < \infty \tag{54}$$

Then
$f(\mathbf{x}) = \frac{P(\mathbf{x})}{\overline{\phi(\mathbf{x}-\mathbf{u})}} e^{-a|\mathbf{x}|^2}$, *a.e.*

Where $a > 0$ and P is a quaternion polynomial of degree $< \frac{d-2}{2}$.
In particular, $f = 0$ a.e. when $d \leq 2$.

Proof. From (29) we have $f\overline{\Theta}_{\mathbf{u}}(\mathbf{x}) = f(x)\overline{\phi(x-u)} \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\left|f\overline{\Theta}_{\mathbf{u}}(\mathbf{x})\right| \left|O^{i,j}_{A_1,A_2}\left[f\Theta_{\mathbf{u}}(\mathbf{x})\right](\mathbf{w})\right|}{(1+|\mathbf{x}|+|\mathbf{w}|)^d} e^{|\mathbf{x}||\mathbf{w}|}\, d\mathbf{x}d\mathbf{w}\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\left|f(\mathbf{x})\right| |\overline{\phi(\mathbf{x}-\mathbf{u})}| \left|O^{i,j}_{A_1,A_2}\left[f\overline{\Theta}_{\mathbf{u}}(\mathbf{x})\right](\mathbf{w})\right|}{(1+|\mathbf{x}|+|\mathbf{w}|)^d} e^{|\mathbf{x}||\mathbf{w}|}\, d\mathbf{x}d\mathbf{w}\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\left|f(\mathbf{x})\right| |\overline{\phi(\mathbf{x}-\mathbf{u})}| \left|\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\left[f\right](\mathbf{w},\mathbf{u})\right|}{(1+|\mathbf{x}|+|\mathbf{w}|)^d} e^{|\mathbf{x}||\mathbf{w}|}\, d\mathbf{x}\mathbf{w} < \infty.
\end{aligned}$$

Therefore by Lemma 4.8, we have $f\overline{\Theta}_{\mathbf{u}}(\mathbf{x}) = P(x)e^{-a|x|^2}$, a.e where $a > 0$ and $P(\mathbf{x})$ is a quaternion polynomial of degree $< \frac{d-2}{2}$.
i.e.$f(\mathbf{x}) = \frac{P(\mathbf{x})}{\overline{\phi(\mathbf{x}-\mathbf{u})}} e^{-a|\mathbf{x}|^2}$
In particular, $f = 0$ a.e. when $d \leq 2$ on account of $f\overline{\Theta}_{\mathbf{u}}(\mathbf{x}) = 0$ □

4.4. Logarithmic uncertainty principle

In this subsection we derive logarithmic uncertainty principle for ST-QOLCT by using Pitt's inequality for ST-QOLCT. Prior to that we derive Pitt's inequality for ST-QOLCT by using the Lemma 3.2 and Pitt's inequality for the QFT.

Lemma 4.10. (Pitt's inequality for the two-sided QFT [30])
For $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$, and $0 \leq \alpha < 2$,

$$\int_{\mathbb{R}^2} |\mathbf{w}|^{-\alpha} \left\| \mathcal{F}^{i,j} \{f(\mathbf{x})\} (\mathbf{w}) \right\|^2 d\mathbf{w} \ \leq C_\alpha \int_{\mathbb{R}^2} |\mathbf{x}|^\alpha \left| f(\mathbf{x}) \right|^2 d\mathbf{x}. \tag{55}$$

With $C_\alpha := \frac{4\pi^2}{2^\alpha}[\Gamma(\frac{2-\alpha}{4})/\Gamma(\frac{2+\alpha}{4})]^2$, and $\Gamma(.)$ is the Gamma function and $\mathcal{S}(\mathbb{R}^2, \mathbb{H})$ denotes the Schwartz space.

Theorem 4.11. (Pitt's inequality of the ST-QOLCT.)*Under the assumptions of lemma 4.10, we have*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{w}|^{-\alpha} |\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})|^2 \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{w} \leq \frac{1}{4\pi^2 |\mathrm{b}_1\mathrm{b}_2|^\alpha} \mathrm{C}_\alpha \|\phi\|^2_{\mathrm{L}^2(\mathbb{R}^2,\mathbb{H})} \int_{\mathbb{R}^2} |\mathbf{x}|^\alpha |\mathrm{f}(\mathbf{x})|^2 \mathrm{d}\mathbf{x}, \tag{56}$$

Proof. *By lemma (3.2), we have*

$$\begin{aligned} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{w}|^{-\alpha} |\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})|^2 \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{w} &= \frac{1}{4\pi^2 |b_1 b_1|} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{w}|^{-\alpha} |\mathcal{F}^{i,j}(h)\left(\frac{\mathbf{w}}{\mathbf{b}},\mathbf{u}\right)|^2 \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{w} \\ &= \frac{1}{4\pi^2 |b_1 b_1|} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{z}\mathbf{b}|^{-\alpha} |\mathcal{F}^{i,j}(h)\,(\mathbf{z},\mathbf{u})\,|^2 |\mathbf{b}| \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{z} \end{aligned} \tag{57}$$

where last equation is obtained by taking $\mathbf{z}\mathbf{b} = \mathbf{w}$.

Since $h(\mathbf{x},\mathbf{u}) = e^{i[\frac{a_1}{2b_1}x_1^2 + \frac{1}{b_1}x_1p_1]} f\overline{\Theta}_{\mathbf{u}}(\mathbf{x}) e^{j[\frac{a_2}{2b_2}x_2^2 + \frac{1}{b_2}x_2p_2]}$ *therefore by applying Lemma 4.10, we obtain from (57)*

$$\begin{aligned} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{w}|^{-\alpha} |\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})|^2 \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{w} &\leq \frac{1}{4\pi^2 |b_1 b_2|^\alpha} C_\alpha \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{x}|^\alpha |h(\mathbf{x})|^2 \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{x} \\ &= \frac{1}{4\pi^2 |b_1 b_2|^\alpha} C_\alpha \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{x}|^\alpha |f\overline{\Theta}_{\mathbf{u}}(\mathbf{x})|^2 \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{x} \\ &= \frac{1}{4\pi^2 |b_1 b_2|^\alpha} C_\alpha \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{x}|^\alpha |f(\mathbf{x})\overline{\phi((x-u))}|^2 \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{x} \\ &= \frac{1}{4\pi^2 |b_1 b_2|^\alpha} C_\alpha \int_{\mathbb{R}^2} |\mathbf{x}|^\alpha |f(\mathbf{x})|^2 \int_{\mathbb{R}^2} |\overline{\phi((x-u))}|^2 \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{x} \\ &= \frac{1}{4\pi^2 |b_1 b_2|^\alpha} C_\alpha \|\phi\|^2 \int_{\mathbb{R}^2} |\mathbf{x}|^\alpha |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \end{aligned}$$

□

Theorem 4.12. (Logarithmic UP for the ST-QOLCT)
Let $f, \phi \in \mathcal{S}\left(\mathbb{R}^2, \mathbb{H}\right)$ where ϕ is a non zero window function, then

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} ln(\,|\mathbf{w}|)\ \left|\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}\{f\}(\mathbf{w},\mathbf{u})\right|^2 d\mathbf{w}d\mathbf{u} + \frac{\|\phi\|^2}{4\pi^2} \int_{\mathbb{R}^2} \ln(|\mathbf{x}|)\ \left|f(\mathbf{x})\right|^2 d\mathbf{x} \tag{58}$$

$$\geq\ \frac{(A + \ln|b_1 b_2|)}{4\pi^2} \|\phi\|^2 \int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x}, \tag{59}$$

with $A = \ln(2) + \Gamma'\left(\frac{1}{2}\right)/\Gamma(\frac{1}{2})$.

Proof. *Based on Pitt's inequality, Logarithmic uncertainty principle for the two sided ST-QOLCT can be proved by taking a function Ψ as*

$$\Psi(\alpha) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{w}|^{-\alpha} |\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})|^2 \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{w} - \frac{\mathrm{D}_\alpha}{|\mathrm{b}_1\mathrm{b}_2|^\alpha} \|\phi\|^2_{\mathrm{L}^2(\mathbb{R}^2,\mathbb{H})} \int_{\mathbb{R}^2} |\mathbf{x}|^\alpha |\mathrm{f}(\mathbf{x})|^2 \mathrm{d}\mathbf{x}$$

where $D_\alpha = \frac{C_\alpha}{4\pi^2}$

Implies

$$\Psi'(\alpha) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{w}|^{-\alpha} \ln|\mathbf{w}||\mathcal{S}^{\mathbb{H}}_{\phi,A_1,A_2}[f](\mathbf{w},\mathbf{u})|^2 d\mathbf{u}d\mathbf{w} - D'_\alpha \|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} \int_{\mathbb{R}^2} \left|\frac{\mathbf{x}}{\mathbf{b}}\right|^\alpha |f(\mathbf{x})|^2 d\mathbf{x}$$
$$-D_\alpha \|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} \int_{\mathbb{R}^2} \left|\frac{\mathbf{x}}{\mathbf{b}}\right|^\alpha \ln\left|\frac{\mathbf{x}}{\mathbf{b}}\right| |f(\mathbf{x})|^2 d\mathbf{x}$$

Now following the procedure of theorem 4.11 in [18] we will get desired result. □

Remark 4.13. *Another way to prove Logarithmic uncertainty principle for the two sided ST-QOLCT is by using relation between QFT and ST-QOLCT in Logarithmic uncertainty principle for the QFT (see[20] Logarithmic uncertainty principle for the QWLCT).*

5. Conclusions

In this paper, first we establish a relation between two-sided QFT and two-sided ST-QOLCT. Second, we established some basic properties of the two-sided ST-QOLCT including the Moyal's formula which are proved in [21]. These results are very important for their applications in digital signal and image processing. Finally, the uncertainty principles for the ST-QOLCT such as Donoho-Stark's uncertainty principle, Hardy's uncertainty principle, Beurling's uncertainty principle, and Logarithmic uncertainty principle are obtained. In our future works, we will discuss the physical significance and engineering background of this paper. Moreover, we will formulate convolution and correlation theorems for the ST-QOLCT.

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