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Approximation by nonlinear Bernstein-Chlodowsky operators of Kantorovich type

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Abstract. In this study, we give the monotonicity of the Bernstein-Chlodowsky max product operator. Then, we introduce Bernstein-Chlodowsky-Kantorovich operators of max-product type and obtain this operator preserves quasi-concavity. Also, we give some approximation properties of Lipschitz functions by max-product kind of Bernstein-Chlodowsky-Kantorovich operators.

1. Introduction

The properties of approximation for linear positive operators especially Bernstein operators have a significant impact on the approximation theory ([20]-[24]). Bernstein operators and its generalizations have an substantial role in Computer-Aided Geometric Design (CAGD) to introduce surfaces and curves and have been investigated in many papers (see [17]-[19]). The numerical solution of partial differential equations, CAGD, 3D modeling and font design are some areas of application.

In recent years, positive nonlinear operators have been introduced instead of positive linear operators. These nonlinear operators have better approximation behavior to the linear operator. The nonlinear positive operators introduced by Bede et al. in [1]. In ([1]-[16]) ”max-product type operators” were introduced by using maximum on the behalf of sum in usual linear operators and gave Jackson-type error estimate according to modulus of continuity. Since max-product kind of approximation theory is a very rich and useful phenomena of approximating continuous functions, researchers have turned to this new field in recent years. The max-product sampling type operators, the neural network max-product type operators, and several others have been studied ([25]-[28]). In [29], the authors showed that the Bernstein operators of max-product type can convenient in approximating fuzzy numbers.

In this study, our aim is that max-product kind Bernstein-Chlodowsky operator preserves quasi-concavity, firstly. Then, we give the construction of max-product Bernstein-Chlodowsky operators of Kantorovich type. Also, we obtain quantitative approximation conclusions in the uniform norm and give some shape preserving properties.

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2. Preliminaries

Here, it is emphasized some general notations about the nonlinear operators of maximum product (max-product) kind. We handle the operations $\bigvee$ and $\cdot$ which means is maximum and product respectively. Then $(\mathbb{R}_+, \bigvee, \cdot)$ is called Max-Product algebra over the set of positive reals, has a semiring structure and it is called as. Let $\mathtt{A} \subset \mathbb{R}$ be a bounded or unbounded interval, and $CB_+(\mathtt{A})$ be a space of continuous and bounded functions on $\mathtt{A}$. The general form of max product approximation operator $L_n : CB_+(\mathtt{A}) \to CB_+(\mathtt{A})$ defined as follows:

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k\in \mathtt{A}_n} \wp_{n,k}(x) \cdot f(x_{n,k})}{\bigvee_{k\in \mathtt{A}_n} \wp_{n,k}(x)} \tag{1}$$

where $n \in \mathbb{N}$, $f \in CB_+(\mathtt{A})$ and $x_k \in \mathtt{A}$, for all k, finite or infinite families of indices are $\mathtt{A}_n$. These operators are nonlinear, positive operators and satisfy a pseudo linearity condition of the form

$$L_n^{(M)}(\rho \cdot\ r \vee \varrho \cdot\ s)(x) = \rho \cdot\ L_n^{(M)}(r)(x) \vee \varrho \cdot\ L_n^{(M)}(s)(x), \forall \rho, \varrho \in \mathbb{R}_+ \text{ and } r, s : \mathtt{A} \to \mathbb{R}_+.$$

In order to give some properties of the operators $L_n^{(M)}$, we present the following auxiliary Lemma.

Lemma 2.1. *([2]) Let us take $\mathtt{A} \subset \mathbb{R}$ and $L_n^{(M)} : CB_+(\mathtt{A}) \to CB_+(\mathtt{A})$, $n \in \mathbb{N}$ be a sequence of operators satisfying the following properties :*

i. (Monotonicity) If $r, s \in CB_+(\mathtt{A})$ provide $r \le s$ then $L_n^{(M)}(r) \le L_n^{(M)}(s)$ for all $n \in \mathbb{N}$;

ii. (Subadditivity)If $L_n^{(M)}(r + s) \le L_n^{(M)}(r) + L_n^{(M)}(s)$ for all $r, s \in CB_+(I)$,

then we get

$$|L_n^{(M)}(r)(x) - L_n^{(M)}(s)(x)| \le L_n^{(M)}(|r - s|)(x),$$

for all $r, s \in CB_+(\mathtt{A})$, $n \in \mathbb{N}$ and $x \in \mathtt{A}$.

Remark 2.2. *Max-product operators defined by (1) verify the conditions in Lemma 2.1, (i), (ii). In fact, instead of (i) it satisfies the stronger condition*

$$L_n^{(M)}(r \vee s)(x) = L_n^{(M)}(r)(x) \vee L_n^{(M)}(s)(x), r, s \in CB_+(\mathtt{A}).$$

Actually, taking in the above equality $r \le s, r, s \in CB_+(\mathtt{A})$, it easily follows $L_n^{(M)}(r)(x) \le L_n^{(M)}(s)(x)$.

Furthermore, the operators of max-product type is positive homogenous, that is $L_n^{(M)}(\lambda r) = \lambda L_n^{(M)}(r)$ for all $\lambda \ge 0$.

Corollary 2.3. *([2]) Let $L_n^{(M)} : CB_+(\mathtt{A}) \to CB_+(\mathtt{A})$, $n \in \mathbb{N}$ be a sequence of operators providing the circumstances (i)-(ii) in Lemma 2.1 and in addition be a positive homogenous operator. Then for all $r \in CB_+(\mathtt{A})$, $n \in \mathbb{N}$ and $x \in \mathtt{A}$ we have*

$$|f(x) - L_n^{(M)}(r)(x)| \le \left[\frac{1}{\delta} L_n^{(M)}(\varphi_x)(x) + L_n^{(M)}(e_0)(x)\right] \omega_1(r;\delta)_I + r(x) \cdot\ \left|L_n^{(M)}(e_0)(x) - 1\right|,$$

where $\delta > 0$, $e_0(t) = 1$ for all $t \in I$, $\varphi_x(t) = |t - x|$ for all $t \in \mathtt{A}, x \in \mathtt{A}$. Here, $\omega_1(r;\delta)_{\mathtt{A}} = \max_{\substack{x,y\in \mathtt{A}\\ |x-y|\le\delta}} |r(x) - r(y)|$ is the first modulus of continuity. If $\mathtt{A}$ is unbounded interval then we suppose that there exists $L_n^{(M)}(\varphi_x)(x) \in \mathbb{R}_+ \bigcup\{+\infty\}$, for any $x \in \mathtt{A}, n \in \mathbb{N}$.

Corollary 2.4. *([2]) Assume that in addition to the conditions in Corollary 2.2, the sequence $(L_n^{(M)})_n$ satisfies $L_n^{(M)}(e_0) = e_0$, for all $n \in \mathbb{N}$. Then for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in \mathtt{A}$ we have*

$$|r(x) - L_n^{(M)}(r)(x)| \le \left[1 + \frac{1}{\delta} L_n^{(M)}(\varphi_x)(x)\right] \omega_1(r;\delta)_I.$$

In [8], authors defined Kantorovich variant of each max-product operator $L_n^{(M)}(f)$ as follows:

$$LK_n^{(M)}(f)(x) = \frac{\bigvee_{k\in A_n} \wp_{n,k}(x)\cdot (1/(x_{n,k+1}-x_{n,k}))\cdot \int_{x_{n,k}}^{x_{n,k+1}} f(t)dt}{\bigvee_{k\in A_n} \wp_{n,k}(x)}. \tag{2}$$

In that paper, the authors presented that Kantorovich type of various max-product operators are subadditive, monotone and positively homogenous, also proved that quantitative estimates, shape preserving properties and localization results for these operators. Here, we want to present structure of Bernstein-Chlodowsky-Kantorovich operators of max-product kind and obtain quantitative approximation results. But, firstly we need some consequences about Bernstein-Chlodowsky max-product operators.

In the paper [12], it was proved that the $C_n^{(M)}(f)$ max-product operator preserves the quasi-convexity. In the following theorem, we want to determine the monotonicity of $C_n^{(M)}(f)$ on $[0, b_n]$ as we will need some results for the following theorems.

Theorem 2.5. *Let us take the function $f : [0, b_n] \to \mathbb{R}_+$ and let us fix $n \in \mathbb{N}$, $n \geq 1$. Now, we assume that there exists $c \in [0, b_n]$ such that f is non-decreasing on $[0, c]$ and non-increasing on $[c, b_n]$. Then, there exists $c' \in [0, b_n]$ such that $C_n^{(M)}(f)$ is non-decreasing on $[0, c']$ and non-increasing on $[c', b_n]$. We have also $\mid c - c' \mid \leq \frac{b_n}{n+1}$ and $\mid C_n^{(M)}(f)(c) - f(c) \mid \leq \omega_1\left(f; \frac{b_n}{n+1}\right)$.*

Proof. For the proof, we will show that the monotonicity on each interval of the form $\left[b_n\frac{j}{n+1}, b_n\frac{j+1}{n+1}\right]$, $j \in \{0, 1, \cdots, n\}$. Let us take $j_c \in \{0, 1, \cdots, n\}$ such that $c \in \left[b_n\frac{j_c}{n+1}, b_n\frac{j_c+1}{n+1}\right]$. Then, we will be able to determine the monotonicity of $C_n^{(M)}(f)$ on $[0, b_n]$ by the continuity of $C_n^{(M)}(f)$.

Let us choose arbitrary $j \in \{0, 1, \cdots, j_c - 1\}$ and $x \in \left[b_n\frac{j}{n+1}, b_n\frac{j+1}{n+1}\right]$. Using the monotonicity of f, we have $f\left(b_n\frac{j}{n}\right) \geq f\left(b_n\frac{j-1}{n}\right) \geq \cdots \geq f(0)$. By Lemma3 in [12], it is easy to see that $f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq \cdots \geq f_{0,n,j}(x)$. Then we can say that $C_n^{(M)}(f)(x) = \bigvee_{k=j}^{n} f_{k,n,j}(x)$. Because $C_n^{(M)}(f)$ is defined as the maximum of non-decreasing functions, it follows that it is non-decreasing on $\left[b_n\frac{j}{n+1}, b_n\frac{j+1}{n+1}\right]$. Considering the continuity of $C_n^{(M)}(f)$, f is non-decreasing on $\left[0, b_n\frac{j_c+1}{n+1}\right]$. Let us consider arbitrary $j \in \{j_c + 1, j_c + 2, \cdots, n\}$ and $x \in \left[b_n\frac{j}{n+1}, b_n\frac{j+1}{n+1}\right]$. By the monotonicity of f, we get $f\left(b_n\frac{j}{n}\right) \geq f\left(b_n\frac{j+1}{n}\right) \geq \cdots \geq f(1)$. By Lemma3 in [12], it is easy to see that $f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq \cdots \geq f_{n,n,j}(x)$. Then, we can say that $C_n^{(M)}(f)(x) = \bigvee_{k=0}^{j} f_{k,n,j}(x)$.Because $C_n^{(M)}(f)$ is defined as the maximum of non-increasing functions, it follows that it is non-increasing on $\left[b_n\frac{j}{n+1}, b_n\frac{j+1}{n+1}\right]$. Considering the continuity of $C_n^{(M)}(f)$, f is non-increasing on $\left[b_n\frac{j_c+1}{n+1}, b_n\right]$.

Let us evaluate the case $j = j_c$. If $b_n\frac{j}{n} \leq c$, then by the monotonicity of f it follows that $f\left(b_n\frac{j_c}{n}\right) \geq f\left(b_n\frac{j_c-1}{n}\right) \geq \cdots \geq f(0)$. Hence, in this case we get that f is non-decreasing on $\left[b_n\frac{j_c}{n+1}, b_n\frac{j_c+1}{n+1}\right]$. It follows that f is non-decreasing on $\left[0, b_n\frac{j_c+1}{n+1}\right]$ and non-increasing on $\left[b_n\frac{j_c+1}{n+1}, b_n\right]$. Also, $c' = \frac{j_c+1}{n+1}$ is the maximum point of $C_n^{(M)}(f)$ and $\mid c - c' \mid \leq \frac{b_n}{n+1}$. If $b_n\frac{j}{n} \geq c$, then by the monotonicity of f it follows that $f\left(b_n\frac{j_c}{n}\right) \geq f\left(b_n\frac{j_c+1}{n}\right) \geq \cdots \geq f(1)$. Hence, in this case we get that f is non-increasing on $\left[b_n\frac{j_c}{n+1}, b_n\frac{j_c+1}{n+1}\right]$. It follows that f is non-decreasing on $\left[0, b_n\frac{j_c}{n+1}\right]$ and non-increasing on $\left[b_n\frac{j_c}{n+1}, b_n\right]$. Also, $c' = \frac{j_c}{n+1}$ is the maximum point of $C_n^{(M)}(f)$ and $\mid c - c' \mid \leq \frac{b_n}{n+1}$.

Now,let take into account that $C_n^{(M)}(x) \leq f(c)$ for all $x \in [0, b_n]$. Actually, this is by the description of $C_n^{(M)}(f)$ and according to c is the global maximum point of f. Therefore, we have

$$\begin{aligned} \mid C_n^{(M)}(f)(c) - f(c) \mid &= f(c) - C_n^{(M)}(f)(c) = f(c) - \bigvee_{k=0}^{n} f_{k,n,j}(c) \\ &\leq f(c) - f_{j_c,n,j_c}(c) = f(c) - f\left(b_n\frac{j_c}{n}\right). \end{aligned} \tag{3}$$

As $c, b_n \frac{j_c}{n} \in \left[b_n \frac{j_c}{n+1}, b_n \frac{j_c+1}{n+1}\right]$, we can obtain $f(c) - f\left(b_n \frac{j_c}{n}\right) \leq \omega_1\left(f; \frac{1}{n+1}\right)$. □

Now, let us construct of max-product kind Bernstein-Chlodowsky operators of Kantorovich type as follows;

$$CK_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} s_{n,k}(x) \frac{n}{b_n} \int_{\frac{b_n k}{n}}^{\frac{b_n(k+1)}{n}} f(t)dt}{\bigvee_{k=0}^{n} s_{n,k}(x)} \tag{4}$$

with

$$s_{n,k}(x) = \binom{n}{k}\left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k},$$

where $x \in [0, b_n]$ and (b_n) is a sequence of positive real numbers such that $\lim_{n\to\infty} b_n = \infty$, and $\lim_{n\to\infty} \frac{b_n}{n} = 0$.

3. Shape preserving properties

Here, we give some shape preserving properties by the Bernstein-Chlodowsky-Kantorovich operators $CK_n^{(M)}$.

Theorem 3.1. *For $f \in C_+([0, b_n])$,*

1. *Let f is non-decreasing (non-increasing) on $[0, b_n]$ then for $\forall n \in \mathbb{N}$ $CK_n^{(M)}$ is non-decreasing (non-increasing) on $[0, b_n]$,*
2. *Let f is quasi-convex on $[0, b_n]$ then $\forall n \in \mathbb{N}$ $CK_n^{(M)}$ is quasi-convex on $[0, b_n]$.*

Proof. (1) Since we have $LK_n^{(M)}$ given in (2), we can write $CK_n^{(M)}$ as follows;

$$CK_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} \binom{n}{k}\left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} f(\xi_{n,k})}{\bigvee_{k=0}^{n} \binom{n}{k}\left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}},$$

where $\xi_{n,k} \in (x_{n,k}, x_{n,k+1})$ for all $k = 0, \cdots, n$. Taking into account the proofs of the paper [12] for the Bernstein-Chlodowsky max-product operators, Bernstein-Chlodowsky-Kantorovich operators proofs will be based on the following functions properties

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \left(\frac{x}{b_n - x}\right)^{k-j} f(\xi_{n,k}).$$

From [12] in Shape-Preserving properties section,one can see that the functions act identically for the function $f_{k,n,j}(x)$. Therefore, one can get the required results.

(2) Because of Corollary 4 in [12], the properties from the previous condition (i), we can obtain the desired result. □

Now, we will present that Bernstein-Chlodowsky-Kantorovich operators $CK_n^{(M)}$ preserves quasi-concavity. This qualification is provided for the operator $C_n^{(M)}$ in [12]. Since it is difficult to apply this proof to the Kantorovich variant, it is planned to find a direct relationship between these two operators and preserve quasi-concavity property.

Let us take the arbitrary function $f \in C_+([0, b_n])$ and consider

$$f_n(x) = \frac{x}{b_n} \int_{nx/(n+1)}^{(nx+b_n)/(n+1)} f(t)dt. \tag{5}$$

Hence, the operator $CK_n^{(M)}$ can be introduced from the operator $C_n^{(M)}$. In other words, we can write $C_n^{(M)}(f_n)(x) = CK_n^{(M)}(f)(x)$ for all $x \in [0, b_n]$. Here, $f_n \in C_+([0, b_n])$ and when f is strictly positive then so is f_n. A function $f : [\alpha, \beta] \to \mathbb{R}$ is quasi-concave when f is quasi-convex. Once f is continuous, quasi-concavity equivalently means that there exists $c \in [\alpha, \beta]$ such that f is non-decreasing on $[\alpha, c]$ and non-increasing on $[c, \beta]$.

Now, let us show that the operator $CK_n^{(M)}$ preserves quasi-concavity.

Theorem 3.2. *Let the function $f : [0, b_n] \to \mathbb{R}_+$ is continuous and quasi-concave on $[0, b_n]$ then $CK_n^{(M)}(f)$ is quasi-concave on $[0, b_n]$, for all $n \in \mathbb{N}$.*

Proof. Let us take the function f_n defined in 5 and take $c \in [0, 1]$ such that f is non-decreasing on $[0, c]$ and non-increasing on $[c, 1]$. Then, let $j(c) \in \{0, ..., n\}$ such that $\frac{b_n j(c)}{n+1} \leq c \leq \frac{b_n(j(c)+1)}{n+1}$.

Now, we evaluate the function h_n which interpolates f_n at all knots $\frac{kb_n}{n}$, $k = 0, 1, \cdots, n$ and which is contiuous on $[0, b_n]$ and affine on any interval $\left[\frac{kb_n}{n}, \frac{(k+1)b_n}{n}\right]$. It means that h_n is the continuous polygonal line which interpolates f_n at all the knots $\frac{kb_n}{n}$, $k = 0, 1, \cdots, n$. It means that

$$C_n^{(M)}(f_n)(x) = C_n^{(M)}(h_n)(x),\ x \in [0, b_n],$$

therefore we get

$$CK_n^{(M)}(f)(x) = C_n^{(M)}(h_n)(x),\ x \in [0, b_n].$$

Let us take $0 \leq k_1 < k_2 \leq j(c) - 1$. We get

$$h_n\left(k_1 \frac{b_n}{n}\right) = \frac{n+1}{b_n} \int_{k_1 \frac{b_n}{n+1}}^{(k_1+1)\frac{b_n}{n+1}} f(t)dt,$$
$$h_n\left(k_2 \frac{b_n}{n}\right) = \frac{n+1}{b_n} \int_{k_2 \frac{b_n}{n+1}}^{(k_2+1)\frac{b_n}{n+1}} f(t)dt.$$

As $\frac{k_1+1}{n+1} \leq \frac{k_2}{n+1}$ and f is increasing on $\left[0, (k_2+1)\frac{b_n}{n+1}\right]$, we get $h_n\left(k_1 \frac{b_n}{n}\right) \leq h_n\left(k_2 \frac{b_n}{n}\right)$ by applying the mean value theorem. The description of h_n implies that h_n is increasing on $\left[0, \frac{j(c)-1}{n} b_n\right]$. Similarly, we also get h_n is decreasing on $\left[\frac{j(c)+1}{n} b_n, b_n\right]$. Now, let us assume that $f\left(\frac{j(c)b_n}{n+1}\right) \geq f\left(\frac{(j(c)+1)b_n}{n+1}\right)$. As f is a quasi-concav function, $f(x) \geq f\left((j(c)+1)\frac{b_n}{n+1}\right)$ for any $x \in \left[j(c)\frac{b_n}{n+1}, (j(c)+1)\frac{b_n}{n+1}\right]$. Because there exists $x_0 \in \left[j(c)\frac{b_n}{n+1}, (j(c)+1)\frac{b_n}{n+1}\right]$ such that

$$\frac{n}{b_n} \int_{j(c)b_n/n}^{(j(c)+1)b_n/n} = f(x_0) = h_n\left(j(c)\frac{b_n}{n}\right),$$

and taking into account that f is increasing on $\left[(j(c)+1)\frac{b_n}{n}, b_n\right]$ we obtain $f\left(\frac{(j(c)+1)b_n}{n+1}\right) \geq h_n\left(\frac{(j(c)+1)b_n}{n}\right)$. Then we have h_n is decreasing on $\left[j(c)\frac{b_n}{n}, (j(c)+1)\frac{b_n}{n}\right]$. But f is affine on $\left[\frac{(j(c)-1)b_n}{n}, \frac{j(c)b_n}{n}\right]$. This implies that h_n is either increasing on $\left[0, \frac{(j(c)-1)b_n}{n}\right]$ and decreasing on $\left[\frac{(j(c)-1)b_n}{n}, b_n\right]$ or it is increasing on $\left[0, \frac{j(c)b_n}{n}\right]$ and decreasing on $\left[\frac{j(c)b_n}{n}, b_n\right]$. It means that h_n is quasi-concave on $[0, b_n]$. When $f\left(\frac{j(c)b_n}{n+1}\right) \leq f\left(\frac{(j(c)+1)b_n}{n+1}\right)$, we obtain the same results by similar reasoning. By [12], one can easily see that $C_n^{(M)}(h_n)$ is quasi-concave on $[0, b_n]$. Since $C_n^{(M)}(h_n) = CK_n^{(M)}(f)$, it follows that $CK_n^{(M)}(f)$ is quasi-concave on $[0, b_n]$. □

Now,we obtain an upper bound for the approximation of f by the uniform norm with f_n by taking into account the function f_n given in 5. For some $x \in [o, b_n]$, there exists $\xi_x \in \left[\frac{nx}{n+1}, \frac{nx+b_n}{n+1}\right]$ such that $f_n(x) = f(\xi_x)$ by using mean value theorem. We can easily see that $|\ \xi_x - x\ | \leq \frac{b_n}{n+1}$ that means

$$|\ f(x) - f_n(x)\ | \leq \omega_1\left(f; b_n/(n+1)\right),\ x \in \mathbb{R},\ n \in \mathbb{N}. \tag{6}$$

In the following theorem, we present the order of approximation b_n/n in the approximation by the operator $CK_n^{(M)}$ for the class of Lipschitz function, hence a similar conclusion which holds for the operator $C_n^{(M)}$.

Theorem 3.3. *Let us take f is Lipschitz on $[0, b_n]$ with Lipschitz constant C and assume that $m_f > 0$ is the lower bound of f. Then, we obtain*

$$\| CK_n^{(M)}(f) - f \| \leq 2C\left(\frac{C}{m_f} + 5\right)\frac{b_n}{n},\ n \geq 1.$$

Proof. The proof is easily seen from [12], the authors give the result that when $f \in C_+([0, b_n])$ is concave on $[0, b_n]$, then they get $| CK_n^{(M)}(f)(x) - f(x) | \leq 2\omega_1\left(f; \frac{b_n}{n}\right)$ and by the estimation the above result and also taking into account that $\omega_1\left(f; \frac{b_n}{n}\right) \leq C.\frac{b_n}{n}$. □

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