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The structure of F^2 as an associative algebra via quadratic forms

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Abstract. Let F be a totally ordered field and $\omega \in \overline{F}$ (a field extension of F) be a solution to the equation $x^2 = ax + b \in F[x]$, where a and b are fixed with $b \neq 0$. With the help of this idea, we convert the F-vector space F^2 into an associative F-algebra. As far as F^2 can even be converted into a field. In the next step, based on a quadratic form, we define an inner product on F^2 with values in F and call it the F-inner product. The defined inner product is mostly studied for its various properties. In particular, when $F = \mathbb{R}$, we show that $\mathbb{R}^2$ with the defined product satisfies well-known inequalities such as the Cauchy-Schwarz and the triangle inequality. Under certain conditions, the reverse of recent inequalities is established. Some interesting properties of quadratic forms on F^2 such as the invariant property are presented. In the sequel, we let $SL(2,\mathbb{R})$ denote the subgroup of $M(2,\mathbb{R})$ that consists of matrices with determinant 1 and set $\mathbb{G} = SL(2,\mathbb{R}) \cap \mathbb{M}_{\mathbb{R}}$, where $\mathbb{M}_{\mathbb{R}}$ is the matrix representation of $\mathbb{R}^2$. We then verify the coset space $\frac{SL(2,\mathbb{R})}{\mathbb{G}}$ with the quotient topology is homeomorphic to H (the upper-half complex plane) with the usual topology. Finally, we determine some families of functions in $C(H,\mathbb{C})$, the ring consisting of complex-valued continuous functions on H; related to elements of $\mathbb{G}$ for which the functional equation $f \circ g = g \circ f$ is satisfied.

1. Introduction and preliminary results

A *partially ordered set* (in brief, *poset*) is a set together with a partial order relation $\leq$ satisfying reflexive, antisymmetric, and transitive properties. A *totally ordered set* is a poset in which every pair of elements x, y are comparable, i.e., $x \leq y$ or $y \leq x$. Hence, a totally ordered set is often referred to as a chain. The notions $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of positive integers, rational numbers, real numbers, and complex numbers, respectively. A *totally ordered ring* is a *partially ordered ring* (see [4, 0.19]) that is ordered by its ordering relation. So each element is comparable with 0. A totally ordered field F is a *lattice ordered ring* that means if $x, y \in F$, then $x \vee y := \sup\{x, y\} \in F$ (note, the supremum is x or y). Also, $x \wedge y := \inf\{x, y\} = -(-x \vee -y) \in F$. In particular, $|x| := x \vee -x \in F$. Whenever F is referred to as a topological space, its topology is *the interval topology*, i.e., the family of all rays $\{x : x > c\}$ and $\{x : x < d\}$ $(c, d \in F)$ is a subbase for the open sets in F. Hence, the family of all the open intervals $(x, y) := \{z \in F : x < z < y\}$ is a base for the topology. The topological concepts that we need can be found in [2] and [23]. Throughout the paper, F is a totally ordered field with the interval topology, and note that F contains a copy of $\mathbb{Q}$ (Proposition 1.1). For example, $\mathbb{R}$ and every countable subfield of $\mathbb{R}$ are totally ordered fields. If for $0 < y \in F$ there is $x \in F$ such that $y = x^n$, then x is called *the n^{th} root* of y and denoted by $\sqrt[n]{y}$ or $y^{\frac{1}{n}}$ (i.e., $x = y^{\frac{1}{n}}$). Recall that $\mathbb{Q}$ does not satisfy the

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property of 2^{th} root for all $y > 0$. But in $\mathbb{R}$, all nonnegative elements have the same number of square roots. If $0 < x < y$, then $x^n < y^n$, where $n \in \mathbb{N}$. Hence, a positive element has at most one positive n^{th} root (see Proposition 1.1). A mapping Q of an R-module M to R is called a *quadratic form*, if $Q(rx) = r^2Q(x)$ for each $r \in R$ and $x \in M$; and the mapping $B : M \times M \to R$ defined by $B(x, y) = Q(x + y) - Q(x) - Q(y)$ is a bilinear symmetric form (see [8, 1.2]). For a deeper discussion of quadratic forms, we refer the reader to [5], [8], [9], [11], [12], [14] and [15].

The paper is organized as follows: In Section 2, with the help of a solution $\omega \in \overline{F}$ of the equation $x^2 = ax + b \in F[x]$, we convert F^2 into an associative F-algebra. As far as F^2 can even be converted into a field. In the next step, based on a quadratic form, we define an inner product on F^2 with values in F and call it the F-inner product. The defined product is mostly studied for its various properties. In particular, we focus on the case of $F = \mathbb{R}$ and show that $\mathbb{R}^2$ with this product satisfies well-known inequalities such as *the Cauchy-Schwarz* and *the triangle inequality*. Under certain conditions, the reverse of recent inequalities is established. In the sequel, we let $SL(2, \mathbb{R})$ denote the subgroup of $M(2, \mathbb{R})$ that consists of matrices with determinant 1. The best general references here are [6] and [10]. Set $\mathbb{G} = SL(2, \mathbb{R}) \cap \mathbb{M}_{\mathbb{R}}$. We then show that the coset space $\frac{SL(2,\mathbb{R})}{\mathbb{G}}$ with the quotient topology is homeomorphic to H (the upper-half complex plane) with the usual topology. In Section 3, we determine some families of functions in $C(H, \mathbb{C})$, the ring consisting of complex-valued continuous functions on H (actually, from H to H); related to elements of $\mathbb{G}$ for which the functional equation $f \circ g = g \circ f$ is satisfied.

Proposition 1.1. ([4, 0.20]) *Let D be a totally ordered integral domain. If $0 < x < y$, then $x^n < y^n$, where $n \in \mathbb{N}$. Hence, a positive element has at most one positive n^{th} root. D contains a natural copy of $\mathbb{N}$. If D is a totally ordered field, then D contains a copy of $\mathbb{Q}$.*

Proposition 1.2. *Let R be a totally ordered commutative ring and $0 < x, y \in R$. Then $x < y$ if and only if $x^n < y^n$ for each $n \in \mathbb{N}$.*

Proof. Since R is commutative, we conclude that $x^n - y^n = (x - y)\Big(\sum_{i=1}^{i=n} x^{n-i}y^{i-1}\Big)$. Moreover, the last sum is positive. Therefore, $x - y < 0$ gives $x^n - y^n < 0$ and vice versa. Actually, $x - y$ and $x^n - y^n$ have the same sign. It means both are positive or both are negative, and we are done □

2. The structure of F^2 as an associative F-algebra and some of its properties

Let F be a totally ordered field and $a, b \in F$ be fixed with $b \neq 0$. Suppose ω satisfies the equation $x^2 = ax + b$, i.e., $\omega^2 = a\omega + b$. If $x^2 = ax + b$ has a zero in F, then $\omega \in F$. Otherwise, we may assume that ω belongs to a field extension (not necessarily totally ordered) $\overline{F}$ of F. For example, $x^2 = -1$ with coefficients in $\mathbb{R}$ ($a = 0$, $b = -1$) has $\omega = i \in \mathbb{C}$ as a zero. Also, $\omega = \sqrt{2} \in \mathbb{R}$ satisfies $x^2 = 2$ with coefficients in $\mathbb{Q}$ ($a = 0$, $b = 2$). Now, we define Δ as follows and refer to it often because it plays a crucial role in most results.

$$\Delta = a^2 + 4b.$$

Let $F^2 = \{(x, y) : x, y \in F\}$. Then F^2 with the pointwise addition and the scalar multiplication is a vector space over F. Also, F^2 can be identified by the set $\{x + y\omega : x, y \in F\}$ via the map $(x, y) \mapsto x + y\omega$. Our goal in this part is to convert the vector space F^2 into an associative algebra. For $X = (x, y), Y = (x', y') \in F^2$, we put $X = x + y\omega$ and $Y = x' + y'\omega$. Therefore, $X + Y = x + x' + (y + y')\omega$ and $\lambda X = \lambda x + \lambda y\omega$ ($\lambda \in F$). Also, $X \times Y$ represents the product of X and Y and is defined by the following relation,

$$\begin{aligned} X \times Y = (x + y\omega)(x' + y'\omega) &= xx' + (xy' + yx')\omega + yy'\omega^2 \\ &= xx' + byy' + (xy' + yx' + ayy')\omega \\ &= (xx' + byy',\ xy' + yx' + ayy'). \qquad (*) \end{aligned}$$

Note that $X \times Y = Y \times X$. In particular, if $F = \mathbb{R}$, $a = 0$ and $b = -1$, then the above multiplication agrees with the multiplication in $\mathbb{C}$. From now on, the matrix $\begin{bmatrix} x & y \end{bmatrix}$ is used instead of the ordered pair (x, y).

Proposition 2.1. *Let $M(2,F)$ be the ring of all 2-square matrices over F and $SL(2,F) = \{A \in M(2,F) : \det(A) = 1\}$. Then $H := \left\{\begin{bmatrix}1 & c\\ 0 & 1\end{bmatrix} : c \in F\right\}$ and $K := \left\{\begin{bmatrix}1 & 0\\ d & 1\end{bmatrix} : d \in F\right\}$ are abelian subgroups of $SL(2,F)$. Furthermore, $H \cong F \cong K$ as groups.*

Theorem 2.2. *Let $\mathbb{M}_F = \left\{\begin{bmatrix}x & y\\ by & x+ay\end{bmatrix} : x, y \in F\right\}$. Then $\mathbb{M}_F$ is an F-subalgebra of $M(2,F)$. Moreover, $F^2 \cong \mathbb{M}_F$ as algebras.*

Proof. Let $A = \begin{bmatrix}x & y\\ by & x+ay\end{bmatrix}, B = \begin{bmatrix}x' & y'\\ by' & x'+ay'\end{bmatrix} \in \mathbb{M}_F$. Then $-A, A+B \in \mathbb{M}_F$. Also,

$$AB = \begin{bmatrix}xx' + byy' & xy' + yx' + ayy' \\ \\ b(yx' + xy' + ayy') & xx' + byy' + a(xy' + yx' + ayy')\end{bmatrix} \in \mathbb{M}_F.$$

Moreover, $1_{\mathbb{M}_F} = 1_{M(2,F)} = I$, the identity matrix. Therefore, $\mathbb{M}_F$ is a subring of $M(2,F)$. Since $rA \in \mathbb{M}_F$, for every $r \in F$, we infer that $\mathbb{M}_F$ is an F-subalgebra of $M(2,F)$. Now, let us define

$$\varphi : F^2 \to \mathbb{M}_F \text{ by } \varphi(X) = \begin{bmatrix}x & y\\ by & x+ay\end{bmatrix}, \text{where } X = \begin{bmatrix}x & y\end{bmatrix}.$$

Then $\varphi(X+Y) = \varphi(X) + \varphi(Y)$ and $\varphi(rX) = r\varphi(X)$, where $Y = \begin{bmatrix}x' & y'\end{bmatrix}$ and $r \in F$. Moreover,

$$\varphi(X \times Y) = \begin{bmatrix}xx' + byy' & xy' + yx' + ayy' \\ \\ b(yx' + xy' + ayy') & xx' + byy' + a(xy' + yx' + ayy')\end{bmatrix} \big(\text{see } (*)\big). \text{ We also have}$$

$$\begin{aligned}\varphi(X)\varphi(Y) &= \begin{bmatrix}x & y\\ by & x+ay\end{bmatrix}\begin{bmatrix}x' & y'\\ by' & x'+ay'\end{bmatrix} \\ &= \begin{bmatrix}xx' + byy' & xy' + yx' + ayy' \\ \\ b(yx' + xy' + ayy') & xx' + byy' + a(xy' + yx' + ayy')\end{bmatrix}.\end{aligned}$$

So $\varphi(X \times Y) = \varphi(X)\varphi(Y)$. Furthermore, φ is one-to-one and onto. This yields φ is an algebra isomorphism. □

Remark 2.3. For $X = \begin{bmatrix}x & y\end{bmatrix} \in F^2$, we may let $\det(X) = \det(\varphi(X)) = x^2 + axy - by^2$. It easily follows that $\det(X)$ is a quadratic form (see [8, 1.2]). In the sequel, we will need $\det(X)$ as in the following form.

$$\det(X) = x^2 + axy - by^2 = (x + \frac{ay}{2})^2 - \frac{1}{4}(a^2 + 4b)y^2 \tag{1}$$

$$= (x + \frac{ay}{2})^2 - \frac{1}{4}\Delta y^2. \tag{2}$$

The next result is now immediate.

Corollary 2.4. *For every $A, B \in \mathbb{M}_F$, we have $AB = BA$. So every subring of $\mathbb{M}_F$ is commutative and every subgroup of $\mathbb{M}_F$ with multiplication is abelian. In particular, $\overline{\mathbb{G}} := \{M \in \mathbb{M}_F : \det(M) \neq 0\}$, and $\mathbb{G} := \{M \in \mathbb{M}_F : \det(M) = 1\}$ are subgroups of $\mathbb{M}_F$. Moreover, $\mathbb{G}$ is a normal subgroup of $\overline{\mathbb{G}}$, and further, $\mathbb{G} = SL(2,F) \cap \mathbb{M}_F$.*

Proposition 2.5. *Let $\mathbb{G}$ and $\overline{\mathbb{G}}$ be as defined in Corollary 2.4, and let $F = \mathbb{R}$. Then the following hold:*

(*i*) *If $\Delta = a^2 + 4b < 0$, then $\frac{\overline{\mathbb{G}}}{\mathbb{G}} \cong \mathbb{R}^+ \cong \frac{\mathbb{R}\setminus\{0\}}{\{1,-1\}}$.*

(*ii*) *If $\Delta > 0$, then $\frac{\overline{\mathbb{G}}}{\mathbb{G}} \cong \mathbb{R} \setminus \{0\}$.*

Proof. First, we note that $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^+ := \{r \in \mathbb{R} : r > 0\}$ are multiplicative abelian groups with the same identity element 1.

(*i*) Define $\psi_1 : \overline{\mathbb{G}} \to \mathbb{R}^+$ with $\psi_1(M) = \det(M)$ (note, $\Delta < 0$ gives $\det(M) > 0$). So ψ_1 is a homomorphism and $\ker(\psi_1) = \mathbb{G}$. Now, if $r > 0$ and $X = \begin{bmatrix} \sqrt{r} & 0 \end{bmatrix}$, then $\psi_1(X) = r$, i.e., ψ_1 is onto. For the second assertion, consider $\psi : \mathbb{R} \setminus \{0\} \to \mathbb{R}^+$ with $\psi(r) = |r|$. Hence, ψ is onto and $\ker(\psi) = \{-1, 1\}$.

(*ii*) Define $\psi_2 : \overline{\mathbb{G}} \to \mathbb{R} \setminus \{0\}$ like ψ_1, i.e., $\psi_2(M) = \det(M)$. So ψ_2 is a homomorphism and $\ker(\psi_2) = \mathbb{G}$. If $r > 0$, then we choose X as the same matrix in part (*i*), and if $r < 0$, then we take $x + \frac{1}{2}ay = 0$ and $\frac{-1}{4}\Delta y^2 = r$, and let $X = \begin{bmatrix} x & y \end{bmatrix}$ for which x and y satisfy the latter equation. Then in both cases $\psi_2(X) = r$, i.e., ψ_2 is onto, which completes the proof. □

Theorem 2.6. *Let $\Delta < 0$. Then $\mathbb{M}_F$ (and hence F^2) is a field.*

Proof. According to Corollary 2.4, $\mathbb{M}_F$ (and hence F^2) is a commutative ring. Let $0 \neq M = \begin{bmatrix} x & y \\ by & x + ay \end{bmatrix} \in \mathbb{M}_F$. We claim that $\det(M) \neq 0$ (and thus by (2), $\det(M) > 0$). Otherwise, $\det(M) = 0$ gives

$$0 \leq (x + \frac{ay}{2})^2 = \frac{1}{4}\Delta y^2 \leq 0.$$

Therefore, $y = 0$ and hence $x = 0$, which implies that $M = 0$. So every non-zero element of $\mathbb{M}_F$ is invertible, meaning that $\mathbb{M}_F$, as well as F^2, is a field. □

Definition 2.7. Let V be an F-vector space and a map $\varphi : V \times V \to F$ provides all the requirements of an inner product. Then, we call the pair (V, φ) or simply V an *F-inner product space* over F and φ the *F-inner product*.

Example 2.8. Let $P : F^2 \times F^2 \to F$ be defined by

$$P(X, Y) = \frac{1}{2}\Big[2x_1x_2 + a(x_1y_2 + y_1x_2) - 2by_1y_2\Big], \text{ where } X = \begin{bmatrix} x_1 & y_1 \end{bmatrix}, \text{ and } Y = \begin{bmatrix} x_2 & y_2 \end{bmatrix}. \tag{3}$$

If $\Delta < 0$, then it is easy to verify that P is an F-inner product, and thus F^2 is an F-inner product space.

Theorem 2.9. *Let P be as defined in Example 2.8. Then the following hold:*
(*i*) $P(X + Y, X + Y) + P(X - Y, X - Y) = 2P(X, X) + 2P(Y, Y)$.
(*ii*) $P(X + Y, X + Y) - P(X - Y, X - Y) = 4P(X, Y)$.
(*iii*) *If $\Delta = a^2 + 4b \leq 0$, then $P(X, X) = \det(X) \geq 0$.*
(*iv*) *If $\Delta \leq 0$, then $P^2(X, Y) \leq P(X, X)P(Y, Y)$.*
(*v*) *If $\Delta \geq 0$, then $P^2(X, Y) \geq P(X, X)P(Y, Y)$.*
(*vi*) *$P^2(X, Y) = P(X, X)\,P(Y, Y)$ if and only if $\Delta = 0$ or $X = \lambda Y$ for some $\lambda \in F$.*

Proof. (*i*)-(*ii*). Since the mapping P is bilinear, it follows that

$$P(X + Y, X + Y) = P(X, X) + P(Y, Y) + 2P(X, Y), \text{ and} \tag{4}$$
$$P(X - Y, X - Y) = P(X, X) + P(Y, Y) - 2P(X, Y). \tag{5}$$

The results are now obtained by adding and subtracting recent expressions respectively.

(*iii*) It follows from (2).

(*iv*) First, we let

$$\mathcal{A} = P(X, X)P(Y, Y) - P^2(X, Y), \tag{6}$$

and then calculate as follows:

$$\begin{aligned}
\mathcal{A} &= \left(x_1^2 + ax_1y_1 - by_1^2\right)\left(x_2^2 + ax_2y_2 - by_2^2\right) \\
&\quad - \frac{1}{4}\Big[2x_1x_2 + a(x_1y_2 + y_1x_2) - 2by_1y_2\Big]^2 \\
&= x_1^2x_2^2 + ax_1^2x_2y_2 - bx_1^2y_2^2 + ax_1y_1x_2^2 \\
&\quad + a^2x_1y_1x_2y_2 - abx_1y_1y_2^2 - by_1^2x_2^2 - aby_1^2x_2y_2 \\
&\quad + b^2y_1^2y_2^2 - \frac{1}{4}\Big[4x_1^2x_2^2 + a^2x_1^2y_2^2 + a^2y_1^2x_2^2 \\
&\quad + 4b^2y_1^2y_2^2 + 4ax_1^2x_2y_2 + 4ax_1y_1x_2^2 - 4bx_1y_1x_2y_2 \\
&\quad + 2a^2x_1y_1x_2y_2 - 4abx_1y_1y_2^2 - 4aby_1^2x_2y_2\Big] \\
&= -(x_1y_2 - y_1x_2)^2\Delta.
\end{aligned}$$

We summarize the above calculations as follows:

$$\mathcal{A} = -(x_1y_2 - y_1x_2)^2\Delta. \tag{7}$$

Now, if $\Delta \leq 0$, then $\mathcal{A} \geq 0$ and we reach the claim.

(*v*) Reusing (7), we obtain $\mathcal{A} \leq 0$ when $\Delta \geq 0$.

(*vi*) ($\Rightarrow$) If $\Delta \neq 0$, then $x_1y_2 = y_1x_2$. We can assume that $Y \neq 0$. If $y_2 \neq 0$, then we take $\lambda = y_1y_2^{-1}$, and if $x_2 \neq 0$, then we take $\lambda = x_1x_2^{-1}$. So $X = \lambda Y$.

($\Leftarrow$) It is obvious. □

Remark 2.10. If F is $\mathbb{R}$ or $\mathbb{C}$, then every inner product induces a norm, called its canonical norm, that is defined in the natural way, by $\|x\| = \sqrt{\langle x\ x\rangle}$. With this norm, every inner product space becomes a normed vector space. So, every general property of normed vector spaces applies to inner product spaces. But in general, it is not true that every F-inner product induces a norm because the square root $\sqrt{\langle x\ x\rangle}$ does not necessarily belong to F (for example $F = \mathbb{Q}$).

For $X, Y \in \mathbb{R}^2$, we put $P(X, Y) = X \cdot Y$ and $P(X, X) = X \cdot X = \|X\|^2$.

Theorem 2.11. *Let* $X = \begin{bmatrix} x_1 & y_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 & y_2 \end{bmatrix} \in \mathbb{R}^2$. *Then the following hold:*

(*i*) $\|X + Y\|^2 + \|X - Y\|^2 = 2(X \cdot X) + 2(Y \cdot Y)$.

(*ii*) $\|X + Y\|^2 - \|X - Y\|^2 = 4(X \cdot Y)$.

(*iii*) *If* $\Delta \leq 0$, *then* $\|X\|^2 = X \cdot X = \det(X) \geq 0$.

(*iv*) *If* $\Delta \leq 0$, *then* $|X \cdot Y| \leq \|X\|\,\|Y\|$. *(the Cauchy-Schwarz inequality)*

(*v*) *If* $\Delta \geq 0$, *then* $|X \cdot Y| \geq \|X\|\,\|Y\|$. *(the reverse of Cauchy-Schwarz inequality)*

(*vi*) *If* $\Delta \leq 0$, *then* $\|X + Y\| \leq \|X\| + \|Y\|$. *(the triangle inequality)*

(*vii*) *If* $\Delta \geq 0$ *and* $X \cdot Y \geq 0$, *then* $\|X + Y\| \geq \|X\| + \|Y\|$. *(the reverse of triangle inequality)*

(*viii*) $\|X\|\,\|Y\| = |X \cdot Y|$ *if and only if* $\Delta = 0$ *or* $X = \lambda Y$, *for some* $\lambda \in \mathbb{R}$.

Proof. (*i*)-(*v*) and (*viii*) are obtained by (*i*)-(*v*) and (*vi*) in Theorem 2.9, respectively.

(*vi*) From (4) we get

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2 + 2(X \cdot Y), \text{ and } \|X - Y\|^2 = \|X\|^2 + \|Y\|^2 - 2(X \cdot Y). \tag{8}$$

Also, Theorem 2.9(*iv*) implies that

$$(X \cdot Y)^2 \leq \|X\|^2\|Y\|^2, \text{ and therefore } |X \cdot Y| = \sqrt{(X \cdot Y)^2} \leq \sqrt{\|X\|^2\|Y\|^2} = \|X\|\,\|Y\|.$$

Hence, we obtain

$$\begin{aligned}||X+Y||^2 = ||X||^2+||Y||^2+2(X\cdot Y) &\leq ||X||^2+||Y||^2+2|X\cdot Y|\\ &\leq ||X||^2+||Y||^2+2||X||\,||Y||\\ &= (||X||+||Y||)^2.\end{aligned}$$

So $||X+Y|| \leq ||X||+||Y||$, meaning that triangle inequality is established.

(*vii*) Using the assumptions, $\Delta \geq 0$, $X\cdot Y \geq 0$, and Theorem 2.9(v), we get $X\cdot Y = |X\cdot Y| \geq ||X||\,||Y||$. The result is now obtained by replacing $\leq$ with $\geq$ in the above calculation, and we are done. □

Corollary 2.12. *If $\Delta < 0$, then the mapping $||\cdot|| : \mathbb{R}^2 \to \mathbb{R}$ which $X \mapsto ||X|| = \sqrt{det(X)}$ turns $\mathbb{R}^2$ into a normed space.*

Remark 2.13. Similar to real-valued functions on $\mathbb{R}^2$, the gradient vector (gradient) of $f : F^2 \to F$ at a point X_0 is the vector $\nabla f(X_0) = \left(\frac{\partial f}{\partial x}(X_0), \frac{\partial f}{\partial y}(X_0)\right)$, for brevity, $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, obtained by evaluating the partial derivatives of f at X_0. If the second derivations of f exist at X_0, then we let $J := d^2f = \begin{bmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2}\end{bmatrix}$ and call it the *Jacobean matrix* of f.

Proposition 2.14. *If $X, Y \in F^2$ and Y^t denote the transpose of Y, then $2P(X,Y) = XJY^t$ (see (3)). In particular, $2||X||^2 = XJX^t$.*

Proof. Let $f : F^2 \to F$ be defined by $f(X) = \det(X) = x^2 + axy - by^2$, where $X = \begin{bmatrix}x & y\end{bmatrix}$ (Remark 2.3). Then $J = \begin{bmatrix}2 & a\\ a & -2b\end{bmatrix}$. Let $X = \begin{bmatrix}x_1 & y_1\end{bmatrix}, Y = \begin{bmatrix}x_2 & y_2\end{bmatrix} \in F^2$. Then, with a simple calculation, we get

$$XJY^t = 2x_1x_2 + a(x_1y_2 + y_1x_2) - 2by_1y_2 = 2P(X,Y),$$

which gives the result. Also, $XJX^t = 2P(X,X) = 2||X||^2$. □

Proposition 2.15. *Let $M \in M(2,F)$ and $\Delta = a^2 + 4b \neq 0$.*

(i) If $M \in \mathbb{M}_F$ (Theorem 2.2), then $MJM^t = \det(M)J$.

(ii) $M \in \mathbb{G} = \left\{M \in \mathbb{M}_F : \det(M) = 1\right\}$ (Corollary 2.4) if and only if $\det(M) > 0$ and $MJM^t = J$.

Proof. (i) It is straightforward, so we eliminate the proof.

(ii) ($\Rightarrow$) It follows from (i).

($\Leftarrow$) Let $M = \begin{bmatrix}x & y\\ z & t\end{bmatrix} \in M(2,F)$ such that $MJM^t = J$. Then the assumption, $\det(J) = -\Delta \neq 0$, gives $\det^2(M) = 1$ and thus $\det(M) = 1$. Moreover, from equation $MJ = J(M^t)^{-1}$ we obtain $z = by$ and $t = x + ay$, i.e., $M \in \mathbb{G}$. □

Definition 2.16. Let $X = \begin{bmatrix}x_1 & y_1\end{bmatrix}, Y = \begin{bmatrix}x_2 & y_2\end{bmatrix} \in F^2$. Then we define

$$\begin{aligned}&P_1(X,Y) = \frac{1}{2}\Big(2x_1x_2 + a(x_1y_2 + x_2y_1) - 2by_1y_2\Big),\\ &P_2(X,Y) = \frac{1}{2}\Big(2x_1x_2 + \frac{a}{b}(x_1y_2 + x_2y_1) - \frac{2}{b}y_1y_2\Big),\\ &Q_1(X) = P_1(X,X) = x_1^2 + ax_1y_1 - by_1^2, \text{ and}\\ &Q_2(X) = P_2(X,X) = x_1^2 + \frac{a}{b}x_1y_1 - \frac{y_1^2}{b}.\end{aligned}$$

Notice that the mappings $P_1, P_2 : F^4 \to F$, and, $Q_1, Q_2 : F^2 \to F$ are continuous. Moreover, if $\Delta < 0$, then P_1 (= P, in Example 2.8) and P_2 are F-inner products. Also, Q_1 and Q_2 are quadratic forms. Furthermore, the gradient vectors of Q_1 and Q_2 are as follows.

$$\nabla Q_1(X) = \Big(2x_1 + ay_1,\ ax_1 - 2by_1\Big), \text{ and } \nabla Q_1(Y) = \Big(2x_2 + ay_2,\ ax_2 - 2by_2\Big).$$
$$\nabla Q_2(X) = \Big(2x_1 + \frac{a}{b}y_1,\ \frac{a}{b}x_1 - \frac{2}{b}y_1\Big), \text{ and } \nabla Q_2(Y) = \Big(2x_2 + \frac{a}{b}y_2,\ \frac{a}{b}x_2 - \frac{2}{b}y_2\Big).$$

The relations between P_1 and P_2 as well as Q_1 and Q_2 are presented in the next two theorems.

A quadratic form Q on F^2 is called *G-invariant*, if for all $A \in G$ and $X \in F^2$; we have $Q(AX^t) = Q(X)$, where G is a family of invertible elements of $M(2, F)$ and X^t is the transpose of X.

Theorem 2.17. *Let* $G_1 = \left\{N = \begin{bmatrix} x & y \\ \frac{y}{b} & x + \frac{a}{b}y \end{bmatrix} : det(N) \neq 0\right\}$ *and* $X = \begin{bmatrix} x_1 & y_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 & y_2 \end{bmatrix} \in F^2$. *Then the following hold:*

(i) $P_1\Big(\nabla Q_2(X), \nabla Q_2(Y)\Big) = \frac{1}{2b}\Delta P_2(X, Y)$.

(ii) $Q_1\Big(\nabla Q_2(X)\Big) = \frac{1}{2b}\Delta Q_2(X)$.

(iii) $Q_1(NX^t) = \det(N)Q_1(X)$. *In particular, if* $det(N) = 1$, *then* Q_1 *is* G_1*-invariant.*

Proof. (i)

$$\begin{aligned} P_1\Big(\nabla Q_2(X), \nabla Q_2(Y)\Big) &= \frac{1}{2}\Big[2(2x_1 + \frac{a}{b}y_1)(2x_2 + \frac{a}{b}y_2) + a[(2x_1 + \frac{a}{b}y_1) \\ &\quad \times (\frac{a}{b}x_2 - \frac{2}{b}y_2) + (2x_2 + \frac{a}{b}y_2)(\frac{a}{b}x_1 - \frac{2}{b}y_1)] \\ &\quad - 2b(\frac{a}{b}x_1 - \frac{2}{b}y_1)(\frac{a}{b}x_2 - \frac{2}{b}y_2)\Big] \\ &= \frac{1}{2}\Big[(8 + \frac{2a^2}{b} + \frac{2a^2}{b} - \frac{2a^2}{b})x_1x_2 + (\frac{4a}{b} - \frac{4a}{b} \\ &\quad + \frac{a^3}{b^2} + \frac{4a}{b})x_1y_2 + (\frac{4a}{b} + \frac{a^3}{b^2} + \frac{-4a}{b} + \frac{4a}{b})x_2y_1 \\ &\quad + (\frac{2a^2}{b^2} + \frac{-2a^2}{b^2} + \frac{-2a^2}{b^2} + \frac{-8}{b})y_1y_2\Big] \\ &= \frac{1}{2}(\frac{a^2 + 4b}{b})\Big[2x_1x_2 + (x_1y_2 + y_1x_2)\frac{a}{b} - \frac{2}{b}y_1y_2\Big] \\ &= \frac{1}{2}(\frac{a^2 + 4b}{b})P_2(X, Y) \\ &= \frac{1}{2b}\Delta P_2(X, Y). \end{aligned}$$

(ii) By (i), we have

$$\begin{aligned} Q_1\Big(\nabla Q_2(X)\Big) &= P_1\Big(\nabla Q_2(X), \nabla Q_2(X)\Big) \\ &= \frac{1}{2b}\Delta P_2(X, X) = \frac{1}{2b}\Delta Q_2(X). \end{aligned}$$

(iii) If X^t is the transpose of X, then $NX^t = \begin{bmatrix} xx_1 + yy_1 \\ \frac{y}{b}x_1 + xy_1 + \frac{a}{b}yy_1 \end{bmatrix}$, and $Q_1(X^t) = Q_1(X) = x_1^2 + ax_1y_1 - by_1^2$.

Therefore,

$$\begin{aligned}
Q_1(NX^t) &= (xx_1 + yy_1)^2 + a(xx_1 + yy_1)\,(\frac{y}{b}x_1 + xy_1 + \frac{a}{b}yy_1) \\
&\quad - b(\frac{y}{b}x_1 + xy_1 + \frac{a}{b}yy_1)^2 \\
&= (xx_1)^2 + (yy_1)^2 + 2xx_1yy_1 + a\Big[xx_1^2\frac{y}{b} + x^2x_1y_1 + \frac{a}{b}xx_1yy_1 \\
&\quad + \frac{y^2}{b}x_1y_1 + xyy_1^2 + \frac{a}{b}y^2y_1^2\Big] - b\Big[x_1^2\frac{y^2}{b^2} + x^2y_1^2 + \frac{a^2}{b^2}y^2y_1^2 \\
&\quad + \frac{2}{b}xx_1yy_1 + \frac{2a}{b^2}x_1y_1y^2 + \frac{2a}{b}xyy_1^2\Big] \\
&= \Big[x^2 + \frac{a}{b}xy - \frac{y^2}{b}\Big]x_1^2 + \Big[2xy + ax^2 + \frac{a^2}{b}xy + \frac{a}{b}y^2 - 2xy \\
&\quad - \frac{2a}{b}y^2\Big]x_1y_1 + \Big[y^2 + axy + \frac{a^2}{b}y^2 - bx^2 - \frac{a^2}{b}y^2 - 2axy\Big]y_1^2 \\
&= \Big(x^2 + \frac{a}{b}xy - \frac{y^2}{b}\Big)\Big(x_1^2 + ax_1y_1 - by_1^2\Big) \\
&= \det(N)Q_1(X).
\end{aligned}$$

The second assertion is now obvious. □

Theorem 2.18. *Let* $G_2 = \Big\{M = \begin{bmatrix} x & y \\ by & x+ay \end{bmatrix} : det(M) \neq 0\Big\}$ *and* $X = \begin{bmatrix} x_1 & y_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 & y_2 \end{bmatrix} \in F^2$. *Then the following hold:*

(*i*) $P_2\Big(\nabla Q_1(X), \nabla Q_1(Y)\Big) = \frac{1}{2b}\Delta P_1(X, Y)$.

(*ii*) $Q_2\Big(\nabla Q_1(X)\Big) = \frac{1}{2b}\Delta Q_1(X)$.

(*iii*) $Q_2(MX^t) = \det(M)Q_2(X)$. *In particular, if* $det(M) = 1$, *then* Q_2 *is* G_2*-invariant.*

Proof. The proof is exactly the same as the proof of Theorem 2.17, so the details are omitted. □

The next result is an application of Theorem 2.17.

Corollary 2.19. *Let* $F = \mathbb{R}$. *Then the tangent line to the curve* $Q_1(X) = x^2 + axy - by^2 = 1$ *at the point* $X_0 = (x_0, y_0)$ *(belonging to the curve) is* $P_1(X, X_0) = 1$.

Proof. The tangent line to the curve at X_0 is the line through X_0 whose slope is

$$m = \frac{\frac{-\partial Q_1}{\partial x}(X_0)}{\frac{\partial Q_1}{\partial y}(X_0)} = \frac{(2x_0 + ay_0)}{(2by_0 - ax_0)}.$$

Calculations give an equation to the tangent line that is $P_1(X, X_0) = 1$. □

In the remainder of this section, we focus on the case of $F = \mathbb{R}$, and to obtain the main result (Theorem 2.24), we will use [3, Chapter I], [6], and [10]. Remember that by Corollary 2.4, we have

$$\mathbb{G} = \left\{M = \begin{bmatrix} x & y \\ by & x+ay \end{bmatrix} : x, y \in \mathbb{R}, \det(M) = 1\right\} = SL(2, \mathbb{R}) \cap \mathbb{M}_{\mathbb{R}}. \tag{9}$$

Remark 2.20. Remember that the roots of the equation $x^2 = ax + b$ are $\omega_1, \omega_2 = \frac{a \pm \sqrt{\Delta}}{2}$, where $\Delta = a^2 + 4b$. If $\Delta < 0$, then $\omega_1, \omega_2 = \frac{a \pm \sqrt{-\Delta} i}{2}$. Let $\lambda_1 = \frac{1}{\omega_1}$ and $\lambda_2 = \frac{1}{\omega_2}$. Then $\lambda_1, \lambda_2 = \frac{2}{a \pm \sqrt{-\Delta} i} = u \mp vi$, where u and v are defined in (10). From now on, we let

$$\omega := \omega_2, \text{ and } \lambda := \lambda_2 = u + vi, \text{ where, } u = \frac{2a}{a^2 - \Delta} = \frac{a}{-2b}, \text{ and } v = \frac{2\sqrt{-\Delta}}{a^2 - \Delta} = \frac{\sqrt{-\Delta}}{-2b}. \tag{10}$$

Since $\Delta < 0$ and $b \neq 0$; we get $b < 0$. So $\mathrm{Im}(\lambda) = v > 0$, i.e., $\lambda \in H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, the upper-half plane.

Definition 2.21. Let $M = \begin{bmatrix} x & y \\ r & t \end{bmatrix} \in SL(2, \mathbb{R})$ and consider a mapping $T_M : H \to \mathbb{C}$ which $z \mapsto M(z) := \frac{xz+y}{rz+t}$ and let $z_0 \in H$ be fixed. Then we call M a *stabilizer* of z_0, or equivalently, z_0 is a *fixed point* of M, if $M(z_0) = z_0$.

In the next result, we show that the stabilizers of λ (see (10)) are precisely the elements of $\mathbb{G}$.

Lemma 2.22. *λ is a fixed point of a matrix $M \in SL(2, \mathbb{R})$ if and only if $M \in \mathbb{G}$.*

Proof. ($\Rightarrow$) Let $M = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in SL(2, \mathbb{R})$ such that $M(\lambda) = \frac{x\lambda+y}{z\lambda+t} = \lambda$. Then

$$\frac{x+y\omega}{z+t\omega} = \frac{1}{\omega}, \text{ and so } x\omega + y\omega^2 = z + t\omega.$$

Replacing ω^2 with $a\omega + b$ (since $\omega^2 = a\omega + b$) gives $z = by$, and $t = x + ay$. Hence, $M \in \mathbb{G}$.

($\Leftarrow$) Let $M \in \mathbb{G}$. Since $\det(M) = 1$, it follows that $x + ay$ and by cannot be zero at the same time. Now,

$$M(\lambda) = \lambda \Leftrightarrow \frac{x + y\omega}{by + (x + ay)\omega} = \frac{1}{\omega} \Leftrightarrow x\omega + y\omega^2 = by + (x + ay)\omega.$$

The last equality holds because $\omega^2 = a\omega + b$. Thus, λ is a fixed point of M, and we are done. □

Proposition 2.23. *The mapping $p : SL(2, \mathbb{R}) \to H$ which $M = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \mapsto M(\lambda)$ is onto and continuous.*

Proof. First, we claim that $(zu + t) + zvi \neq 0$. Otherwise, $z = 0$ and therefore $t = 0$ which is absurd, since $\det(M) = 1$. Also, $M(\lambda) = \frac{x\lambda+y}{z\lambda+t} = \frac{(xu+y)+xvi}{(zu+t)+zvi} \in H$ because $\mathrm{Im}(M(\lambda)) = \frac{v}{(zu+t)^2+(zv)^2} > 0$. Next, for $z = x + yi \in H$ (i.e., $y > 0$), we set

$$M_X = \begin{bmatrix} \sqrt{\frac{y}{v}} & x\sqrt{\frac{v}{y}} - u\sqrt{\frac{y}{v}} \\ 0 & \sqrt{\frac{v}{y}} \end{bmatrix}, \text{ where } X = \begin{bmatrix} x & y \end{bmatrix}. \tag{11}$$

Now, it is easy to check that $M_X(\lambda) = z$. Thus $p(M_X) = z$, i.e., p is onto. Remember that $SL(2, \mathbb{R})$ is a subspace of $\mathbb{R}^4$ with the usual topology and $p(M) = M(\lambda)$. Therefore, p is continuous. □

For a topological space X and a set Y with an onto mapping $\pi : X \to Y$, a topology can be induced on Y, which is called *the quotient topology*. The space Y is called a *quotient space* of X and π a *quotient map*. Hence, V is open in Y if and only if $\pi^{-1}(V)$ is open in X. Now, consider $SL(2, \mathbb{R})$ as a subspace of $\mathbb{R}^4$ with the usual topology, and let $\frac{SL(2,\mathbb{R})}{\mathbb{G}}$ denote the family of all cosets of $\mathbb{G}$ (see (9)) as a quotient space of $SL(2, \mathbb{R})$. Note that $\mathbb{G}$ is not necessarily a normal subgroup of $SL(2, \mathbb{R})$. Hence, $\frac{SL(2,\mathbb{R})}{\mathbb{G}}$ is regarded as a set of cosets of $\mathbb{G}$.

In [3, Chapter I], the upper-half complex plane H with the usual topology is described as a coset space, by $H \sim \frac{SL(2,\mathbb{R})}{SO(2,\mathbb{R})}$, where the special orthogonal group $SO(2, \mathbb{R})$ is the stabilizer of i (where $i^2 = -1$). Here, in the next theorem, we present a new description of H, where $\mathbb{G}$ is the stabilizer of λ, see (10).

Another main result in this section is as follows:

Theorem 2.24. *Suppose that $\frac{SL(2,\mathbb{R})}{\mathbb{G}}$ is equipped with the quotient topology. Then $\frac{SL(2,\mathbb{R})}{\mathbb{G}} \sim H$.*

Proof. Define $\varphi : \frac{SL(2,\mathbb{R})}{\mathbb{G}} \to H$ with $\varphi(M\mathbb{G}) = M(\lambda)$. If $M\mathbb{G} = N\mathbb{G}$, then $N^{-1}M\mathbb{G} = \mathbb{G}$ and thus $N^{-1}M \in \mathbb{G}$. By Lemma 2.22, $N^{-1}M(\lambda) = \lambda$ and hence $M(\lambda) = N(\lambda)$, so φ is well-defined. Now, suppose that $\varphi(M\mathbb{G}) = \varphi(N\mathbb{G})$. Hence, $M(\lambda) = N(\lambda)$ and thus $N^{-1}M(\lambda) = \lambda$. Reusing Lemma 2.22 we get $N^{-1}M \in \mathbb{G}$. Therefore, $M\mathbb{G} = N\mathbb{G}$, i.e., φ is one-to-one. To show that φ is onto, for $z = x + yi \in H$, it suffices to choose M_X the same matrix as defined in (11). Hence, $\varphi(M_X\mathbb{G}) = M_X(\lambda) = z$. Let V be an open set in H. Then $p^{-1}(V)$ is open in $SL(2, \mathbb{R})$

(Proposition 2.23) and hence $\frac{p^{-1}(V)}{\mathbb{G}}$ is open in $\frac{SL(2,\mathbb{R})}{\mathbb{G}}$. Furthermore, $\varphi^{-1}(V) = \left\{M\mathbb{G} : M(\lambda) \in V\right\} = \frac{p^{-1}(V)}{\mathbb{G}}$. Therefore, φ is continuous. Now, let us define

$$\psi : H \to \frac{SL(2,\mathbb{R})}{\mathbb{G}} \text{ with } \psi(x+yi) = M_X\mathbb{G}, \text{ where } M_X \text{ is defined in (11).} \tag{12}$$

Also, let $\pi : SL(2,\mathbb{R}) \to \frac{SL(2,\mathbb{R})}{\mathbb{G}}$ be the quotient map and $\psi' : H \to SL(2,\mathbb{R})$ be defined by $\psi'(x+yi) = M_X$. Then $\psi = \pi \circ \psi'$, and it is continuous because both π and ψ' are continuous. Remember that $\varphi(M\mathbb{G}) = M(\lambda) = x + yi \in H$. On the other hand, we have $M_X(\lambda) = x + yi$. So $M\mathbb{G} = M_X\mathbb{G}$ (Lemma 2.22). Thus,

$$(\psi \circ \varphi)(M\mathbb{G}) = (\psi \circ \varphi)(M_X\mathbb{G}) = \psi(x+yi) = M_X\mathbb{G} = M\mathbb{G}, \text{ and}$$
$$(\varphi \circ \psi)(x+yi) = \varphi(M_X\mathbb{G}) = M_X(\lambda) = x + yi.$$

This yields both $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity maps. So $\psi = \varphi^{-1}$ and hence φ is a homeomorphism. □

3. Finding some solutions for the functional equation $f \circ g = g \circ f$

Let $f, g : H \to H$ be continuous functions and $f \circ g$ represents their composition. In this section, we are going to introduce some families of continuous functions f, g from H to H which satisfy the functional equation $f \circ g = g \circ f$. Evidently, the invertible continuous functions f and f^{-1} are the solutions to the equation. Below, we prove that these types of functions are closely related to elements of $\mathbb{G}$ (see (9)). This is also generalized in Theorem 3.3.

Proposition 3.1. *Let $T_M : H \to \mathbb{C}$ be defined by $T_M(z) = M(z) = \frac{xz+y}{byz+x+ay}$, where $M = \begin{bmatrix} x & y \\ by & x+ay \end{bmatrix} \in \mathbb{G}$ be fixed (in fact, $T_M : H \to H$) and let $\mathbb{T} = \left\{T_M : M \in \mathbb{G}\right\}$. Then $\mathbb{T}$ with the composition of functions is an abelian group. Moreover, $\mathbb{T} \cong \frac{\mathbb{G}}{\mathbb{N}_0}$, where $\mathbb{N}_0 = \{I, -I\}$.*

Proof. First, we note that if $y = 0$, then $x = \pm 1$ because $\det(M) = 1$. Hence, $T_M(z) = z$. Otherwise, it easily follows that $T_M(z) \in H$ (i.e., $\mathrm{Im}(T_M(z)) > 0$) and further T_M is continuous. Let $N = \begin{bmatrix} x' & y' \\ by' & x'+ay' \end{bmatrix} \in \mathbb{G}$. Then

$$(T_M \circ T_N)(z) = T_M\Big(\frac{x'z+y'}{by'z+x'+ay'}\Big) = \frac{x\Big(\frac{x'z+y'}{by'z+x'+ay'}\Big) + y}{by\Big(\frac{x'z+y'}{by'z+x'+ay'}\Big) + x + ay}$$
$$= \frac{(xx'+byy')z + xy' + x'y + ayy'}{(byx'+bxy'+abyy')z + byy' + xx' + ayx' + axy' + a^2yy'} = T_{MN}(z).$$

So $\mathbb{T}$ is closed under the composition of functions. Moreover, it is easily seen that $(T_M)^{-1} = T_{M^{-1}}$, and the identity element in $\mathbb{T}$ is T_I. Therefore, $\mathbb{T}$ is a group. Now, since $\mathbb{G}$ is abelian, we have $MN = NM$ and hence

$$T_M \circ T_N = T_{MN} = T_{NM} = T_N \circ T_M.$$

This yields $\mathbb{T}$ is abelian. To establish the second assertion, consider the mapping $\varphi : \mathbb{G} \to \mathbb{T}$ with $\varphi(M) = T_M$. So φ is an epimorphism. Let $M \in \ker(\varphi)$. Since T_M is the identity map, we obtain $\frac{xz+y}{byz+x+ay} = z$. From $b \neq 0$, we get $y = 0$. Now, $\det(M) = 1$ yields $x = \pm 1$ and thus $M = \pm I$. Hence, $\ker(\varphi) \subseteq \mathbb{N}_0$. Moreover, $\mathbb{N}_0 \subseteq \ker(\varphi)$. Therefore, $\frac{\mathbb{G}}{\mathbb{N}_0} \cong \mathbb{T}$. □

Corollary 3.2. *Every two elements of $\mathbb{T}$ are solutions for the equation $f \circ g = g \circ f$.*

The main result of this section is the next theorem, which generalizes Proposition 3.1.

Theorem 3.3. *Let $u : H \to H$ be an invertible continuous function and T_M, T_N be as defined in Proposition 3.1. Define $f(z) = u^{-1}\big(T_M(u(z))\big)$, briefly, $f = u^{-1}\big(T_M(u)\big)$, and $g = u^{-1}\big(T_N(u)\big)$. Then $f \circ g = g \circ f$.*

Proof. By Proposition 3.1, T_M and T_N, and therefore, f and g are continuous. Now, the proof is as follows:

$$\begin{aligned}
(f \circ g)(z) &= f\Big(g(z)\Big) = f\Big(u^{-1}(T_N(u))\Big) \\
&= f\Big(u^{-1}(\frac{x'u + y'}{by'u + x' + ay'})\Big) \\
&= u^{-1}\Big(\frac{xu(u^{-1}(\frac{x'u+y'}{by'u+x'+ay'})) + y}{byu(u^{-1}(\frac{x'u+y'}{by'u+x'+ay'})) + x + ay}\Big) \\
&= u^{-1}\Big(\frac{x(\frac{x'u+y'}{by'u+x'+ay'}) + y}{by(\frac{x'u+y'}{by'u+x'+ay'}) + x + ay}\Big) \\
&= u^{-1}\Big(\frac{xx'u + xy' + byy'u + x'y + ayy'}{by'xu + byy' + bxy'u + abyy'u + xx' + ax'y + axy' + a^2yy'}\Big) \\
&= u^{-1}\Big(\frac{x'(\frac{xu+y}{byu+x+ay}) + y'}{by'(\frac{xu+y}{byu+x+ay}) + x' + ay'}\Big) \\
&= u^{-1}\Big(\frac{x'u(u^{-1}(\frac{xu+y}{byu+x+ay})) + y'}{by'u(u^{-1}(\frac{xu+y}{byu+x+ay})) + x' + ay'}\Big) \\
&= u^{-1}\Big(T_N(f(z))\Big) \\
&= (g \circ f)(z).
\end{aligned}$$

□

An immediate consequence of the above theorem is given below:

Corollary 3.4. *Let $u : H \to H$ be an invertible continuous function and let*

$$\mathbb{T}_u = \Big\{u^{-1}\Big(T_M(u)\Big) : M \in \mathbb{G}\Big\}.$$

Then every pair of elements of $\mathbb{T}_u$ satisfy the equation $f \circ g = g \circ f$. In particular, if u is the identity map, then $\mathbb{T}_u = \mathbb{T}$ (Proposition 3.1).

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