Filomat 37:14 (2023), 4725–4742
https://doi.org/10.2298/FIL2314725A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A best proximity point theorem for C-class-proximal non-self mappings and applications to an integro-differential system of equations

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Abstract. In this paper, we propose a best proximity point theorem for a novel class of non-self-mappings by using the definition of a pair $(\mathcal{F}, h)$ of upper class of type II and the concept of C-class functions. Several consequences (including the case of self-mappings) of our obtained results are suggested. We support our obtained results by concrete examples. In the end, we consider an integro-differential system of equations. We ensure the existence of an optimal solution, which turns to be an exact solution of the system when boundary conditions are equal.

1. Introduction

The Banach contraction principle (in abbreviation, BCP) [1] has been considered the beginning of metric fixed point theory. Several authors generalized and improved it in different directions. In 2003, Kirk et al. [2] gave a new generalization of the BCP via a new notion of cyclic mappings. In their result, the cyclic mapping may not be continuous, unlike the Banach's result. This is the important issue of their result. Later, many researchers gave some fixed point results for cyclic mappings. On the other hand, one of the nice generalizations of the BCP has been considered by taking into account non-self mappings. Let P be a nonempty subset of a metric space (X, d) and $T : P \to X$ be a mapping. The solutions of the equation $Tx = x$ are fixed points of T. Consequently, $T(P) \cap P \neq \emptyset$ is a necessary (not sufficient) condition for the existence of a fixed point for the operator T. If this necessary condition does not hold, then $d(x, Tx) > 0$ for any $x \in P$ and the mapping $T : P \to X$ does not have any fixed point, that is, the equation $Tx = x$ has no solution. Subsequently, one targets to determine an element x that is in some sense close proximity to Tx. As a result, a best proximity point theorem furnishes sufficient conditions for the existence of an optimal approximate solution x, known as a best proximity point of the mapping T, satisfying the condition that $d(x, Tx) = d(P, Q)$. The concept of best proximity points for a non-self-mapping $T : P \to Q$ was first introduced by Basha in [3].

2020 *Mathematics Subject Classification*. 47H10; 54H25; 46S40.
Keywords. Best proximity point, $(\mathcal{F}, h, f, \alpha - \lambda - a - b)$-proximal non-self-mapping, C- class function, fractional equation.
Received: 04 August 2022; Revised: 23 October 2022; Accepted: 08 November 2022
Communicated by Erdal Karapınar
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These points are the optimum solutions for the equation $d(x,Tx) = d(P,Q)$. Best proximity point theorems for some proximal contractions in the context of complete metric spaces have been investigated in [4]-[27].

The paper aims to present a new best proximity theorem for a non-self-mapping $T : P \to Q$ involving a pair $(\mathcal{F}, h)$ of upper class of type II and a C- function f. Here, we require that the pair (P,Q) satisfies the $\mathcal{P}$- property and the non-self-mapping T is $\alpha - \lambda - a - b$-proximal sub-admissible. We also suggest some consequences and examples in order to show the effectiveness of the obtained results. Finally, an application on a system of fractional differential equations is presented. Here, we ensure the existence of an optimal solution related to the system. This optimal solution is an exact solution of the system when considering a particular case on boundary conditions.

2. Preliminaries and Definitions

Let (P,Q) be a pair of nonempty subsets of a metric space (X,d). We adopt the following notations:
$d(P,Q) := \inf\{d(a,b) : a \in P, b \in Q\}$;
$P_0 := \{a \in P :$ there exists $b \in Q$ such that $d(a,b) = d(P,Q)\}$;
$Q_0 := \{b \in Q :$ there exists $a \in P$ such that $d(a,b) = d(P,Q)\}$.

Definition 2.1. *[4] Let $S : P \to Q$ be a non-self-mapping. An element x_* is said to be a best proximity point of S if $d(x_*, Sx_*) = d(P,Q)$.*

In 2011, Raj [8] introduced the $\mathcal{P}$-property as follows:

Definition 2.2. *[8] Let (P,Q) be a pair of nonempty subsets of a metric space (X,d) such that P_0 is nonempty. Then, the pair (P,Q) is said to have the $\mathcal{P}$-property if and only if $d(x_1,y_1) = d(x_2,y_2) = d(P,Q) \Longrightarrow d(x_1,x_2) = d(y_1,y_2)$ where $x_1,x_2 \in P$ and $y_1,y_2 \in Q$.*

By using this notion, some best proximity point results were proved for various classes of non-self-mappings. To enrich more the fixed point theory, Ansari and Shukla [28] initiated the concept of functions of subclass of type I.

Definition 2.3. *[28] We say that $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I, if $x \geq 1 \Longrightarrow h(1,y) \leq h(x,y)$ for all $y \in \mathbb{R}^+$.*

As examples, we cite

(a) $h(x,y) = (y+l)^x, l > 1$;
(b) $h(x,y) = (x+l)^y, l > 1$;
(c) $h(x,y) = y$;
(d) $h(x,y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) y,\ n \in \mathbb{N}$;
(e) $h(x,y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1,\ n \in \mathbb{N}$

for all $x,y \in \mathbb{R}^+$.
Going in same direction, we state the following definition due to Ansari and Shukla [28].

Definition 2.4. *[28] Let $h,\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type I, if h is a function of subclass of type I and:*
(i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s,t) \leq \mathcal{F}(1,t)$,
(ii) $h(1,y) \leq \mathcal{F}(s,t) \Longrightarrow y \leq st$ for all $t,y \in \mathbb{R}^+$.

As examples, we take

(a) $h(x,y) = (y+l)^x, l > 1$ and $\mathcal{F}(s,t) = st + l$;
(b) $h(x,y) = (x+l)^y, l > 1$ and $\mathcal{F}(s,t) = (1+l)^{st}$;

(c) $h(x, y) = x^m y$, $m \in \mathbb{N}$ and $\mathcal{F}(s, t) = st$;
(d) $h(x, y) = y$ and $\mathcal{F}(s, t) = t$;
(e) $h(x, y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) y$, $n \in \mathbb{N}$ and $\mathcal{F}(s, t) = st$;
(f) $h(x, y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1$, $n \in \mathbb{N}$ and $\mathcal{F}(s, t) = (1 + l)^{st}$

for all $x, y, s, t \in \mathbb{R}^+$.

Definition 2.5. *[28] We say that $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type II, if $x, y \geq 1 \Longrightarrow h(1, 1, z) \leq h(x, y, z)$ for all $z \in \mathbb{R}^+$.*

As examples, we choose

(a) $h(x, y, z) = (z + l)^{xy}, l > 1$;
(b) $h(x, y, z) = (xy + l)^z, l > 1$;
(c) $h(x, y, z) = z$;
(d) $h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N}$;
(e) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}$

for all $x, y, z \in \mathbb{R}^+$.

Definition 2.6. *[28] Let $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type II, if h is a subclass of type II and: (i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$, (ii) $h(1, 1, z) \leq \mathcal{F}(s, t) \Longrightarrow z \leq st$ for all $s, t, z \in \mathbb{R}^+$.*

As examples, we take

(a) $h(x, y, z) = (z + l)^{xy}, l > 1, \mathcal{F}(s, t) = st + l$;
(b) $h(x, y, z) = (xy + l)^z, l > 1, \mathcal{F}(s, t) = (1 + l)^{st}$;
(c) $h(x, y, z) = z, \mathcal{F}(s, t) = st$;
(d) $h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N}, \mathcal{F}(s, t) = s^p t^p$
(e) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}, \mathcal{F}(s, t) = s^k t^k$

for all $x, y, z, s, t \in \mathbb{R}^+$. Here, the pair $(\mathcal{F}, h)$ is an upper class of type II.

In 2014, the concept of C-class functions was introduced by Ansari [29]. By using this concept, many fixed point theorems in the literature have been generalized.

Definition 2.7. *[29] A mapping $f : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following axioms:*
(1) $f(s, t) \leq s$;
(2) $f(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, \infty)$.

For such a function f, we have $f(0, 0) = 0$. Denote by C the C-class functions.

Example 2.8. *[29] The following functions $f : [0, \infty)^2 \to \mathbb{R}$ are elements of C for all $s, t \in [0, \infty)$:*
(1) $f(s, t) = s - t$;
(2) $f(s, t) = ms$ with $0<m<1$;
(3) $f(s, t) = \frac{s}{(1+t)^r}$ with $r \in (0, \infty)$;
(4) $f(s, t) = \log(t + a^s)/(1 + t)$ with $a > 1$;
(5) $f(s, t) = \ln(1 + a^s)/2$ with $a > e$;
(6) $f(s, t) = (s + l)^{(1/(1+t)^r)} - l$ with $l > 1$ and $r \in (0, \infty)$;
(7) $f(s, t) = s \log_{t+a} a$ with $a > 1$;
(8) $f(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$;
(9) $f(s, t) = s\beta(s)$ where $\beta : [0, \infty) \to [0, 1)$ is continuous;

(10) $f(s,t) = s - \frac{t}{k+t}$;
(11) $f(s,t) = s - \varphi(t)$ *where* $\varphi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\varphi(t) = 0 \Leftrightarrow t = 0$;
(12) $f(s,t) = sh(s,t)$ *where* $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is a continuous function such that* $h(t,s) < 1$ *for all* $t, s > 0$;
(13) $f(s,t) = s - (\frac{2+t}{1+t})t$;
(14) $f(s,t) = \sqrt[n]{\ln(1+s^n)}$;
(15) $f(s,t) = \phi(s)$ *where* $\phi : [0,\infty) \to [0,\infty)$ *is an upper semicontinuous function such that* $\phi(0) = 0$ *and* $\phi(t) < t$ *for each* $t > 0$,
(16) $f(s,t) = \frac{s}{(1+s)^r}$ *with* $r \in (0,\infty)$.

3. Main results

First, we introduce the following concepts.

Definition 3.1. *Consider a non-self-mapping* $S : P \to Q$ *and given* $\alpha, \lambda, a, b : P \longrightarrow [0,+\infty)$. *S is said to be* $\alpha - \lambda - a - b$*-proximal sub-admissible if*
$\alpha(x_1) \geq 1, a(x_1) \leq 1, \lambda(x_2) \geq 1, b(x_2) \leq 1$ *and* $d(u_1, Sx_1) = d(u_2, Sx_2) = d(P,Q) \Longrightarrow \alpha(u_1) \geq 1, a(u_1) \leq 1, \lambda(u_2) \geq 1$ *and* $b(u_2) \leq 1$ *for all* $x_1, x_2, u_1, u_2 \in P$.

Example 3.2. *Endow the space* $X = \mathbb{R}$ *with the metric* $d(x,y) = |x-y|$. *Let* $P = (-\infty,-1]$ *and* $Q = [\frac{5}{4},+\infty)$. *Consider the non-self mapping* $S : P \longrightarrow Q$ *given as*

$$S(x) = \begin{cases} -x^3 + \frac{5}{4} & \text{if} \quad x \in (-\infty,-2]. \\ -\frac{1}{4}x + 1 & \text{if} \quad x \in [-2,-1]. \end{cases}$$

Take α, λ, a, b: $P \longrightarrow [0,+\infty)$ *sas*

$$\alpha(x) = \lambda(x) = \begin{cases} 3 & \text{if} \quad x \in [-2,-1]. \\ 0 & \text{otherwise,} \end{cases}$$

and

$$a(x) = b(x) = \begin{cases} 0 & \text{if} \quad x \in [-2,-1]. \\ 3 & \text{otherwise.} \end{cases}$$

Note that $d(P,Q) = \frac{9}{4}$. *The mapping S is* $\alpha - \lambda - a - b$*-proximal sub-admissible. Indeed, let* $x_1, x_2, u_1, u_2 \in P$ *such that* $\alpha(x_1) \geq 1, a(x_1) \leq 1, \lambda(x_2) \geq 1, b(x_2) \leq 1$ *and* $d(u_1, Sx_1) = d(u_2, Sx_2) = d(P,Q) = \frac{9}{4}$. *We have*

$$\begin{cases} a(x_1) \leq 1; b(x_2) \leq 1 \\ d(u_1, Sx_1) = \frac{9}{4} \\ d(u_2, Sx_2) = \frac{9}{4}. \end{cases}$$

Then

$$\begin{cases} x_1, x_2 \in [-2,-1] \\ d(u_1, Sx_1) = \frac{9}{4} \\ d(u_2, Sx_2) = \frac{9}{4}. \end{cases}$$

Note that $Sx \in [\frac{5}{4}, \frac{3}{2}]$ *for all* $x \in [-2,-1]$. *So* $Sx_1, Sx_2 \in [\frac{5}{4}, \frac{3}{2}]$, *and since we have* $d(u_1, Sx_1) = d(u_2, Sx_2) = \frac{9}{4}$, *necessarily* $u_1 = u_2 = -1$. *That is,* $\alpha(u_1) = 3 \geq 1, a(u_1) = 0 \leq 1, \lambda(u_2) = 3 \geq 1$, *and* $b(u_2) = 0 \leq 1$.

A particular case of Definition 3.1 is as follows:

Definition 3.3. *Consider a non-self-mapping* $S : P \to Q$ *and given* $\alpha, \lambda : P \longrightarrow [0,+\infty)$. *The mapping S is said to be* $\alpha - \lambda$*-proximal admissible if*
$\alpha(x_1) \geq 1, \lambda(x_2) \geq 1$ *and* $d(u_1, Sx_1) = d(u_2, Sx_2) = d(P,Q) \Longrightarrow \alpha(u_1) \geq 1$, *and* $\lambda(u_2) \geq 1$ *for all* $x_1, x_2, u_1, u_2 \in P$.

Definition 3.4. *Let (X,d) be a metric space and (P,Q) be a pair of nonempty subsets of X. A non-self-mapping $S : P \to Q$ is called $(\mathcal{F},h,f,\alpha-\lambda-a-b)$-proximal, where $\alpha,\lambda,a,b : P \longrightarrow [0,+\infty)$ and the pair $(\mathcal{F},h)$ is an upper class of type II, if there exists a C-function f such that*

$$h(\alpha(x),\lambda(y),d(Sx,Sy)) \leq \mathcal{F}(a(x)b(y), f(d(x,y),d(x,y))), \tag{3.1}$$

for all $x,y \in P$.

Example 3.5. *Endow $X = \mathbb{R}^2$ with the metric $d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$. Let $P = \{(0,x) : x \geq 1\}$ and $Q = \{(2,y) : y \geq 1\}$. Given a non-self mapping $S : P \longrightarrow Q$ as $S((0,x)) = (2,\ln(2+x))$. Let $a((0,x)) = b((0,x)) = \alpha((0,x)) = \lambda((0,x)) = 1$ for all $x \geq 1$. We consider the C-class function $f(s,t) = \frac{1}{2}s$. Take the upper class of type II as the pair $(\mathcal{F},h)$ defined by $\mathcal{F}(s,t) = st$ and $h(x,y,z) = z$. For all $(0,x),(0,y) \in P$, there exists $c \in]x,y[$ so that*

$$\begin{aligned} h(1,1,d(S((0,x)),S((0,y)))) &= |\ln(2+x)-\ln(2+y)| \\ &== \frac{1}{2+c}\left|x-y\right| \\ &\leq \frac{1}{2}\left|x-y\right| \\ &= \mathcal{F}(1, f(d((0,x),(0,y)),d((0,x),(0,y)))). \end{aligned}$$

Then S is an $(\mathcal{F},h,f,1-1-1-1)$-proximal mapping.

Now, we propose the following best proximity point result.

Theorem 3.6. *Let (P,Q) be a pair of nonempty closed subsets of a complete metric space (X,d) such that P_0 is nonempty. Given $\alpha,\lambda,a,b : P \longrightarrow [0,+\infty)$ and $f \in C$. Consider an $(\mathcal{F},h,f,\alpha-\lambda-a-b)$-proximal non-self-mapping $S : P \longrightarrow Q$ satisfying the following assertions:*

1. *$S(P_0) \subseteq Q_0$ and the pair (P,Q) verifies the $\mathcal{P}$-property;*
2. *S is $\alpha-\lambda-a-b$-proximal sub-admissible;*
3. *there exist elements $x_0,x_1 \in P$ with $d(x_1,Sx_0) = d(P,Q)$ such that $\alpha(x_0) \geq 1$, $\lambda(x_1) \geq 1$, $a(x_0) \leq 1$ and $b(x_1) \leq 1$;*
4. *if $\{x_n\}$ a sequence in P such that $\alpha(x_n) \geq 1$, $a(x_n) \leq 1$, and $\lim_{n\longrightarrow+\infty} x_n = x_* \in P$, then $\lambda(x_*) \geq 1$ and $b(x_*) \leq 1$.*

Then S has a unique best proximity point $x_* \in P$ such that $d(x_*,Sx_*) = d(P,Q)$. If $y_* \in P$ is another best proximity point for S with $\alpha(y_*) \geq 1$ and $b(y_*) \leq 1$, then $x_* = y_*$.

Proof. Thanks to condition (3), there exist $x_0,x_1 \in P$ such that
$d(x_1,Sx_0) = d(P,Q)$, $\alpha(x_0) \geq 1, a(x_0) \leq 1$; $\lambda(x_1) \geq 1$ and $b(x_1) \leq 1$. As $T(P_0) \subseteq Q_0$, there exits $x_2 \in P_0$ such that $d(x_2,Sx_1) = d(P,Q)$.
The mapping S is $\alpha-\lambda-a-b$-proximal sub-admissible and since $\alpha(x_0) \geq 1, a(x_0) \leq 1, \lambda(x_1) \geq 1, b(x_1) \leq 1$ and $d(x_1,Sx_0) = d(x_2,Sx_1) = d(P,Q)$, this implies that $\alpha(x_1) \geq 1, a(x_1) \leq 1$; $\lambda(x_2) \geq 1$ and $b(x_2) \leq 1$. In a similar manner, by induction, we can construct a sequence $\{x_n\} \subset P_0$ such that for all $n \geq 0$,

$$d(x_{n+1},Sx_n) = d(P,Q) \quad \text{and} \quad \alpha(x_n) \geq 1, \lambda(x_{n+1}) \geq 1, a(x_n) \leq 1 \ \text{ and } \ b(x_{n+1}) \leq 1. \tag{3.2}$$

Our next step is to prove that $\{x_n\}$ is a Cauchy sequence. Let us first prove that $\lim_{n\longrightarrow+\infty} d(x_n,x_{n-1}) = 0$. Since $d(x_{n+1},Sx_n) = d(P,Q)$ and $d(x_n,Tx_{n-1}) = d(P,Q)$, using the $\mathcal{P}$-property, we get

$$d(x_n,x_{n+1}) = d(Sx_{n-1},Sx_n). \tag{3.3}$$

As S is an $(\mathcal{F}, h, f, \alpha - \lambda - a - b)$-proximal non-self mapping and $\alpha(x_{n-1}) \geq 1, a(x_{n-1}) \leq 1, \lambda(x_n) \geq 1, b(x_n) \leq 1$, then

$$\begin{aligned} h(1,1,d(Sx_{n-1},Sx_n)) &\leq h(\alpha(x_{n-1}),\lambda(x_n),d(Sx_{n-1},Sx_n)) \\ &\leq \mathcal{F}(a(x_{n-1})b(x_n), f(d(x_{n-1},x_n),d(x_{n-1},x_n))) \\ &\leq \mathcal{F}(1, f(d(x_{n-1},x_n),d(x_{n-1},x_n))). \end{aligned}$$

Using (3.3) and the definition of h, we obtain

$$d(x_{n+1},x_n) \leq f(d(x_{n-1},x_n),d(x_{n-1},x_n)) \leq d(x_{n-1},x_n) \quad \text{for all } n \geq 1. \tag{3.4}$$

Let $r = \lim_{n\longrightarrow+\infty} d(x_{n-1},x_n)$. Using equation (3.4) and letting $n \longrightarrow +\infty$, we obtain that

$$r \leq f(r,r) \leq r.$$

Using the definition of the function f, we conclude that

$$\lim_{n\longrightarrow+\infty} d(x_{n-1},x_n) = 0. \tag{3.5}$$

Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that, for all positive integers k such that $m(k) > n(k) > k$ with $\alpha(x_{m(k)}) \geq 1$, $\lambda(x_{n(k)}) \geq 1$, $d(x_{m(k)},x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)},x_{n(k)-1}) < \epsilon$. Using the triangular inequality, we get

$$\begin{aligned} \epsilon \leq d(x_{m(k)},x_{n(k)}) &\leq d(x_{m(k)},x_{n(k)-1}) + d(x_{n(k)-1},x_{n(k)}) \\ &\leq \epsilon + d(x_{n(k)-1},x_{n(k)}). \end{aligned} \tag{3.6}$$

Taking limit as $k \longrightarrow +\infty$ in the above inequality (3.6) and using (3.5), we conclude that

$$\lim_{n\longrightarrow+\infty} d(x_{m(k)},x_{n(k)}) = \epsilon. \tag{3.7}$$

Using again the triangular inequality,

$$d(x_{m(k)},x_{n(k)}) \leq d(x_{m(k)},x_{m(k)+1}) + d(x_{m(k)+1},x_{n(k)+1}) + d(x_{n(k)+1},x_{n(k)}). \tag{3.8}$$

Using inequality (3.8), one writes

$$\epsilon \leq d(x_{m(k)+1},x_{n(k)+1}) \leq d(x_{m(k)+1},x_{m(k)}) + d(x_{m(k)},x_{n(k)}) + d(x_{n(k)},x_{n(k)+1}). \tag{3.9}$$

Letting $k \longrightarrow +\infty$ and using (3.5) and (3.7), we get

$$\lim_{n\longrightarrow+\infty} d(x_{m(k)+1},x_{n(k)+1}) = \epsilon. \tag{3.10}$$

Since S is an $(\mathcal{F}, h, f, \alpha - \lambda - a - b)$-proximal mapping, we obtain

$$\begin{aligned} h(1,1,d(Sx_{m(k)},Sx_{n(k)})) &\leq h(\alpha(x_{m(k)}),\lambda(x_{n(k)}),d(Sx_{m(k)},Sx_{n(k)})) \\ &\leq \mathcal{F}(a(x_{m(k)})b(x_{n(k)}), f(d(x_{m(k)},x_{n(k)}),d(x_{m(k)},x_{n(k)}))) \\ &\leq \mathcal{F}(1, f(d(x_{m(k)},x_{n(k)}),d(x_{m(k)},x_{n(k)}))). \end{aligned}$$

That is,

$$d(x_{m(k)+1},x_{n(k)+1}) \leq f(d(x_{m(k)},x_{n(k)}),d(x_{m(k)},x_{n(k)})). \tag{3.11}$$

Letting $k \longrightarrow +\infty$, we conclude that

$$\epsilon \leq f(\epsilon,\epsilon) \leq \epsilon.$$

Hence,

$$\lim_{k\longrightarrow+\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon = 0.$$

It is a contradiction.

Thus, $\{x_n\}$ is a Cauchy sequence in the closed subset P of the metric space (X, d).

Since (X, d) is complete and P is closed, there is $x_* \in P$ so that the sequence $\{x_n\}$ converges to x_*. Using the hypothesis (4) of the theorem, $\lambda(x_*) \geq 1$. Since S is an $(\mathcal{F}, h, f, a, b, \alpha - \lambda)$-proximal Geraghty mapping, we have

$$\begin{aligned} h(1, 1, d(Sx_n, Sx_*) &\leq h(\alpha(x_n), \lambda(x_*), d(Sx_n, Sx_*)) \\ &\leq \mathcal{F}(a(x_n)b(x_*), f(d(x_n, x_*), d(x_n, x_*))). \\ &\leq \mathcal{F}(1, f(d(x_n, x_*), d(x_n, x_*))). \end{aligned}$$

Hence,

$$d(Sx_n, Sx_*) \leq f(d(x_n, x_*)), d(x_n, x_*)), \quad \forall n.$$

By triangular inequality and (3.2), we have

$$\begin{aligned} d(x_*, Sx_*) &\leq d(x_*, x_{n+1}) + d(x_{n+1}, Tx_n) + d(Sx_n, Sx_*) \\ &= d(x_*, x_{n+1}) + d(P, Q) + d(Sx_n, Sx_*). \end{aligned} \tag{3.12}$$

We obtain for all n,

$$d(Tx_n, Sx_*) \geq d(x_*, Sx_*) - d(x_*, x_{n+1}) - d(M, N). \tag{3.13}$$

Using (3.12) and (3.13) , we get

$$\begin{aligned} d(x_n, x_*) &\geq f(d(x_n, x_*)), d(x_n, x_*)) \\ &\geq d(x_*, Sx_*) - d(x_*, x_{n+1}) - d(P, Q). \end{aligned}$$

As $n \longrightarrow +\infty$, we get

$$d(x_*, Sx_*) - d(P, Q) \leq 0.$$

So we deduce that $d(x_*, Sx_*) = d(P, Q)$. Therefore, x_* is a best proximity point for S.

For the uniqueness, suppose that there are two distinct best proximity points for S such that $x_* \neq y_*$. Thus, $\rho = d(x_*, y_*) > 0$. Since $d(Sx_*, x_*) = d(Sy_*, y_*) = d(P, Q)$, using The P- property, we conclude that $\rho = d(Sx_*, Sy_*)$.

Since S is an $(\mathcal{F}, h, f, \alpha - \lambda - a - b)$-proximal mapping, we obtain

$$\begin{aligned} h(1, 1, d(Sy_*, Sx_*) &\leq h(\alpha(y_*), \lambda(x_*), d(Sy_*, Sx_*)) \\ &\leq \mathcal{F}(a(y_*)b(x_*), f(d(y_*, x_*), d(y_*, x_*))) \\ &\leq \mathcal{F}(1, f(d(y_*, x_*), d(y_*, x_*))). \end{aligned}$$

That is,

$$\rho \leq f(\rho, \rho) \leq \rho.$$

Thus, $\rho = 0$, which is a contradiction. □

Example 3.7. *Consider the complete Euclidian space $X = \mathbb{R}$ endowed with the metric $d(x, y) = |x - y|$. Let $P = [2, 3]$ and $Q = [\frac{1}{2}, 1]$. Consider the non-self mapping $S : P \longrightarrow Q$ given as $S(x) = \frac{x}{x+1}$. $S(P) \subset Q$ also P and Q are closed subset on the complete space $(\mathbb{R}, d)$. It is easy to see that the couple (P, Q) satisfies the $\mathcal{P}$-property. Choose $a(x) = b(x) = \alpha(x) = 1 = \lambda(x)$ for all $x, y \in P$. We have*

$$d(S(\frac{1 + \sqrt{5}}{2}), \frac{1 + \sqrt{5}}{2}) = 1 = d(P, Q).$$

For all $x, y \in P = [2,3]$, *we can prove that*

$$d(S(x), S(y)) = |\frac{x}{x+1} - \frac{y}{y+1}| = \frac{|x-y|}{(1+x)(1+y)} \leq \frac{1}{9}\left|x-y\right|.$$

That is, S *is an* $(\mathcal{F}, h, f, 1-1-1-1)$*-proximal Geraghty mapping. This inequality is true for* $x = y$. *If* $x \neq y$ *we have for all* $x, y \in P = [2,3]$, $|x-y| \leq 1$ *meanwhile* $9 \leq (1+x)(1+y) \leq 16$. *So* $\frac{1}{16} \leq \frac{1}{(x+1)(y+1)} \leq \frac{1}{9}$ *and* $\frac{|x-y|}{(1+x)(y+1)} \leq \frac{1}{9}\left|x-y\right|$.

Thus, the main hypotheses of Theorem 3.6 are satisfied for the functions $h(x,y,z) = z, \mathcal{F}(s,t) = st$ *and* $f(s,t) = s - \frac{t}{1+t}$. *So there is a unique best proximity point of the mapping* S, *which is* $\frac{1+\sqrt{5}}{2}$. *It is the only solution of the equation* $|x - \frac{x}{x+1}| = 1 = d(P,Q)$.

4. Consequences

As consequences of Theorem 3.6, we state the following corollaries:

Corollary 4.1. *Let* (P,Q) *be a pair of nonempty closed subsets of a complete metric space* (X,d) *such that* P_0 *is nonempty. Given* $\alpha, \lambda, a, b : P \longrightarrow [0, +, \infty)$. *Consider a non-self-mapping* $S : P \longrightarrow Q$ *satisfying the following assertions:*

1. $S(P_0) \subseteq Q_0$ *and the pair* (P,Q) *satisfies the* $\mathcal{P}$*-property;*
2. S *is* $\alpha - \lambda - a - b$*-proximal sub-admissible;*
3. *there exist elements* $x_0, x_1 \in P$ *such that* $d(x_1, Sx_0) = d(P,Q)$ *such that* $\alpha(x_0) \geq 1$ *and* $\lambda(x_1) \geq 1$;$a(x_0) \leq 1$ *and* $b(x_1) \leq 1$;
4. *if* $\{x_n\}$ *a sequence in* P *such that* $\alpha(x_n) \geq 1$;$a(x_n) \leq 1$,*and* $\lim_{n \longrightarrow +\infty} x_n = x_* \in P$, *then* $\lambda(x_*) \geq 1$ *and* $b(x_*) \leq 1$;
5. *there exists a pair* $(\mathcal{F}, h)$ *of upper class of type II, such that*

$$h(\alpha(x), \lambda(y), d(Sx, Sy)) \leq \mathcal{F}(a(x)b(y), 0)$$

for all $x, y \in P$.

Then, S *has a unique best proximity point* $x_* \in P$ *such that* $d(x_*, Sx_*) = d(P,Q)$. *If* $y_* \in Q$ *is another best proximity point for* S *with* $\alpha(y_*) \geq 1$ *and* $b(y_*) \leq 1$ *then* $x_* = y_*$.

Proof. It is an immediate consequence of Theorem 3.6 with $f(s,t) = s - t$. □

Remark 4.2.

$$h(\alpha(x), \lambda(y), d(Sx, Sy)) \leq \mathcal{F}(a(x)b(y), 0).$$

-For the case $h = (z+l)^{xy}, \mathcal{F}(s,t) = st + l$, *the above condition is:*

$$(d(Tx, Ty) + l)^{\alpha(x)\lambda(y)} \leq l$$

for $l > 1$. *Corollary 4.1 can be written in a more elegant way.*

Corollary 4.3. *Let* (P,Q) *be a pair of nonempty closed subsets of a complete metric space* (X,d) *such that* P_0 *is nonempty. Given* $\alpha, \lambda, a, b : P \longrightarrow [0, +\infty)$. *Consider a non-self-mapping* $S : P \longrightarrow Q$ *satisfying the following assertions:*

1. $S(P_0) \subseteq Q_0$ *and the pair* (P,Q) *satisfies the* $\mathcal{P}$*-property;*
2. S *is* $\alpha - \lambda - a - b$*-proximal sub-admissible;*

3. *there exist elements* $x_0, x_1 \in P$ *such that* $d(x_1, Sx_0) = d(P, Q)$ *such that* $\alpha(x_0) \geq 1$ *and* $\lambda(x_1) \geq 1$;$a(x_0) \leq 1$ *and* $b(x_1) \leq 1$;
4. *if* $\{x_n\}$ *a sequence in* P *such that* $\alpha(x_n) \geq 1$;$a(x_n) \leq 1$,*and* $\lim\limits_{n\longrightarrow+\infty} x_n = x_* \in P$, *then* $\lambda(x_*) \geq 1$ *and* $b(x_*) \leq 1$;
5. $$(d(Sx, Sy) + l)^{\alpha(x)\lambda(y)} \leq l$$
   for all $x, y \in P$ *and* $l > 1$.

Then, S *has a unique best proximity point* $x_* \in P$ *such that* $d(x_*, Sx_*) = d(P, Q)$. *If* $y_* \in P$ *is another best proximity point for* S *with* $\alpha(y_*) \geq 1$ *and* $b(y_*) \leq 1$ *then* $x_* = y_*$.

- For the case $h(x, y, z) = (xy + l)^z, \mathcal{F}(s, t) = (1 + l)^{st}$, *we have* $(\alpha(x)\lambda(y) + l)^{d(Sx,Sy)} \leq 1$. *Corollary 4.1 can be expressed in more explicit way.*

Corollary 4.4. *Let* (P, Q) *be a pair of nonempty closed subsets of a complete metric space* (X, d) *such that* P_0 *is nonempty. Given* $\alpha, \lambda, a, b : P \longrightarrow [0, +, \infty)$. *Consider a non-self-mapping* $S : P \longrightarrow Q$ *satisfying the following assertions:*

1. $S(P_0) \subseteq Q_0$ *and the pair* (P, Q) *satisfies the* $\mathcal{P}$*-property;*
2. S *is* $\alpha - \lambda - a - b$*-proximal sub-admissible;*
3. *there exist elements* $x_0, x_1 \in P$ *such that* $d(x_1, Sx_0) = d(P, Q)$ *such that* $\alpha(x_0) \geq 1$ *and* $\lambda(x_1) \geq 1$;$a(x_0) \leq 1, b(x_1) \leq 1$;
4. *if* $\{x_n\}$ *a sequence in* P *such that* $\alpha(x_n) \geq 1$;$a(x_n) \leq 1$, *and* $\lim_{n\longrightarrow+\infty} x_n = x_* \in P$, *then* $\lambda(x_*) \geq 1, b(x_*) \leq 1$.
5. $$(\alpha(x)\lambda(y) + l)^{d(Sx,Sy)} \leq 1$$
   for all $x, y \in P$ *and* $l > 1$.

Then, S *has a unique best proximity point* $x_* \in P$ *such that* $d(x_*, Sx_*) = d(P, Q)$. *If* $y_* \in P$ *is another best proximity point for* S *with* $\alpha(y_*) \geq 1$ *and* $b(y_*) \leq 1$ *then* $x_* = y_*$.

Corollary 4.5. *Let* (P, Q) *be a pair of nonempty closed subsets of a complete metric space* (X, d) *such that* P_0 *is nonempty. Given* $\alpha, \lambda, a, b : P \longrightarrow [0, +, \infty)$. *Consider a non-self-mapping* $S : P \longrightarrow Q$ *satisfying the following assertions:*

1. $S(P_0) \subseteq Q_0$ *and the pair* (P, Q) *satisfies the* $\mathcal{P}$*-property;*
2. S *is* $\alpha - \lambda - a - b$*-proximal sub-admissible;*
3. *there exist elements* $x_0, x_1 \in P$ *such that* $d(x_1, Sx_0) = d(P, Q)$ *such that* $\alpha(x_0) \geq 1$ *and* $\lambda(x_1) \geq 1$;$a(x_0) \leq 1, b(x_1) \leq 1$;
4. *if* $\{x_n\}$ *a sequence in* P *such that* $\alpha(x_n) \geq 1$;$a(x_n) \leq 1$, *and* $\lim_{n\longrightarrow+\infty} x_n = x_* \in P$, *then* $\lambda(x_*) \geq 1, b(x_*) \leq 1$;
5. *there exists a pair* $(\mathcal{F}, h)$ *of upper class of type II, such that*
   $$h(\alpha(x), \lambda(y), d(Sx, Sy)) \leq \mathcal{F}(a(x)b(y), \frac{d(x, y)}{(1 + d(x, y))^r})$$
   for all $x, y \in P$ *and* $r > 0$.

Then, S *has a unique best proximity point* $x_* \in P$ *such that* $d(x_*, Sx_*) = d(P, Q)$. *If* $y_* \in P$ *is another best proximity point for* S *with* $\alpha(y_*) \geq 1$ *and* $b(y_*) \leq 1$ *then* $x_* = y_*$.

Proof. It is an immediate consequence of Theorem 3.6 with $f(s, t) = \frac{s}{(1+t)^r}$. □

Corollary 4.6. *Let* (P, Q) *be a pair of nonempty closed subsets of a complete metric space* (X, d) *such that* P_0 *is nonempty. Given* $\alpha, \lambda, a, b : P \longrightarrow [0, +, \infty)$. *Consider a non-self-mapping* $S : P \longrightarrow Q$ *satisfying the following assertions:*

1. *$S(P_0) \subseteq Q_0$ and the pair (P,Q) satisfies the $\mathcal{P}$-property;*
2. *S is $\alpha-\lambda-a-b$-proximal sub-admissible;*
3. *there exist elements $x_0, x_1 \in P$ such that $d(x_1, Tx_0) = d(M,N)$ such that $\alpha(x_0) \geq 1$ and $\lambda(x_1) \geq 1; a(x_0) \leq 1, b(x_1) \leq 1$;*
4. *if $\{x_n\}$ a sequence in P such that $\alpha(x_n) \geq 1$;$a(x_n) \leq 1$, and $\lim_{n\longrightarrow+\infty} x_n = x_* \in P$, then $\lambda(x_*) \geq 1, b(x_*) \leq 1$;*
5. *there exists a pair $(\mathcal{F}, h)$ of upper class of type II, such that*

$$h(\alpha(x), \lambda(y), d(Sx,Sy)) \leq \mathcal{F}(a(x)b(y), \frac{log(d(x,y) + a^{d(x,y)})}{1+d(x,y)}$$

   for all $x, y \in P$ and $a > 1$.

Then, S has a unique best proximity point $x_* \in P$ such that $d(x_*, Tx_*) = d(P,Q)$. If $y_* \in P$ is another best proximity point for T with $\alpha(y_*) \geq 1$ and $b(y_*) \leq 1$ then $x_* = y_*$.

Proof. It is an immediate consequence of Theorem 3.6 with $f(s,t) = \frac{log(t+a^s)}{1+t}$. □

Corollary 4.7. *Let (P,Q) be a pair of nonempty closed subsets of a complete metric space (X,d) such that P_0 is nonempty. Given $\alpha, \lambda, a, b : M \longrightarrow [0,+,\infty)$. Consider a non-self-mapping $S : P \longrightarrow Q$ satisfying the following assertions:*

1. *$S(P_0) \subseteq Q_0$ and the pair (P,Q) satisfies the $\mathcal{P}$-property;*
2. *S is $\alpha-\lambda-a-b$-proximal sub-admissible;*
3. *there exist elements $x_0, x_1 \in P$ such that $d(x_1, Sx_0) = d(P,Q)$ such that $\alpha(x_0) \geq 1$ and $\lambda(x_1) \geq 1; a(x_0) \leq 1, b(x_1) \leq 1$;*
4. *if $\{x_n\}$ a sequence in P such that $\alpha(x_n) \geq 1$;$a(x_n) \leq 1$, and $\lim_{n\longrightarrow+\infty} x_n = x_* \in P$, then $\lambda(x_*) \geq 1, b(x_*) \leq 1$;*
5. *there exists a pair $(\mathcal{F}, h)$ of upper class of type II, such that*

$$h(\alpha(x), \lambda(y), d(Sx,Sy)) \leq \mathcal{F}(a(x)b(y), \frac{\ln(1 + a^{d(x,y)})}{2}$$

   for all $x, y \in P$ and $a > e$.

Then, S has a unique best proximity point $x_* \in P$ such that $d(x_*, Sx_*) = d(P,Q)$. If $y_* \in P$ is another best proximity point for S with $\alpha(y_*) \geq 1$ and $b(y_*) \leq 1$ then $x_* = y_*$.

Proof. It is an immediate consequence of Theorem 3.6 with $f(s,t) = \frac{1+a^s}{2}$. □

Corollary 4.8. *Let (P,Q) be a pair of nonempty closed subsets of a complete metric space (X,d) such that P_0 is nonempty. Given $\alpha, \lambda, a, b : P \longrightarrow [0,+,\infty)$. Consider a non-self-mapping $S : P \longrightarrow Q$ satisfying the following assertions:*

1. *$S(P_0) \subseteq Q_0$ and the pair (P,Q) satisfies the $\mathcal{P}$-property;*
2. *S is $\alpha-\lambda-a-b$-proximal sub-admissible;*
3. *there exist elements $x_0, x_1 \in P$ such that $d(x_1, Tx_0) = d(P,Q)$ such that $\alpha(x_0) \geq 1$ and $\lambda(x_1) \geq 1; a(x_0) \leq 1, b(x_1) \leq 1$;*
4. *if $\{x_n\}$ a sequence in P such that $\alpha(x_n) \geq 1$;$a(x_n) \leq 1$, and $\lim_{n\longrightarrow+\infty} x_n = x_* \in P$, then $\lambda(x_*) \geq 1, b(x_*) \leq 1$;*
5. *there exists a pair $(\mathcal{F}, h)$ of upper class of type II, such that*

$$h(\alpha(x), \lambda(y), d(Sx,Sy)) \leq \mathcal{F}(a(x)b(y), d(x,y) - \frac{d(x,y)}{2+d(x,y)})$$

   for all $x, y \in P$.

Then, S has a unique best proximity point $x_* \in P$ such that $d(x_*, Tx_*) = d(P, Q)$. If $y_* \in P$ is another best proximity point for T with $\alpha(y_*) \geq 1$ and $b(y_*) \leq 1$ then $x_* = y_*$.

Proof. It is an immediate consequence of Theorem 3.6 with

$$f(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t}).$$

□

Corollary 4.9. *Let (P, Q) be a pair of nonempty closed subsets of a complete metric space (X, d) such that P_0 is nonempty. Given $\alpha, \lambda, a, b : P \longrightarrow [0, +\infty)$. Consider a non-self-mapping $S : P \longrightarrow Q$ satisfying the following assertions:*

1. *$S(P_0) \subseteq Q_0$ and the pair (P, Q) satisfies the $\mathcal{P}$-property;*
2. *S is $\alpha - \lambda - a - b$-proximal sub-admissible;*
3. *there exist elements $x_0, x_1 \in P$ such that*
   $d(x_1, Sx_0) = d(P, Q)$ such that $\alpha(x_0) \geq 1$ and $\lambda(x_1) \geq 1; a(x_0) \leq 1, b(x_1) \leq 1$;
4. *if $\{x_n\}$ a sequence in P such that $\alpha(x_n) \geq 1$;$a(x_n) \leq 1$,and $\lim_{n \longrightarrow +\infty} x_n = x_* \in P$, then $\lambda(x_*) \geq 1, b(x_*) \leq 1$;*
5. *there exists a pair $(\mathcal{F}, h)$ of upper class of type II, such that*

$$h(\alpha(x), \lambda(y), d(Sx, Sy)) \leq \mathcal{F}(a(x)b(y), \theta(d(x, y)))$$

for all $x, y \in P$ and θ is the generalized Mizoguchi-Takahashi type function.

Then, S has a unique best proximity point $x_* \in P$ such that $d(x_*, Tx_*) = d(P, Q)$. If $y_* \in P$ is another best proximity point for S with $\alpha(y_*) \geq 1$ and $b(y_*) \leq 1$ then $x_* = y_*$.

Proof. It is an immediate consequence of Theorem 3.6 with $f(s, t) = \theta(s)$. □

Corollary 4.10. *Let (P, Q) be a pair of nonempty closed subsets of a complete metric space (X, d) such that P_0 is nonempty. Given $\alpha, \lambda, a, b : P \longrightarrow [0, +, \infty)$. Consider a non-self-mapping $S : P \longrightarrow Q$ satisfying the following assertions:*

1. *$S(P_0) \subseteq Q_0$ and the pair (P, Q) satisfies the $\mathcal{P}$-property;*
2. *S is $\alpha - \lambda - a - b$-proximal sub-admissible;*
3. *there exist elements $x_0, x_1 \in P$ such that*
   $d(x_1, Sx_0) = d(P, Q)$ such that $\alpha(x_0) \geq 1$ and $\lambda(x_1) \geq 1; a(x_0) \leq 1, b(x_1) \leq 1$;
4. *if $\{x_n\}$ a sequence in P such that $\alpha(x_n) \geq 1$;$a(x_n) \leq 1$,and $\lim_{n \longrightarrow +\infty} x_n = x_* \in M$, then $\lambda(x_*) \geq 1, b(x_*) \leq 1$;*
5. *there exists a pair $(\mathcal{F}, h)$ of upper class of type II, such that*

$$h(\alpha(x), \lambda(y), d(Sx, Sy)) \leq \mathcal{F}(a(x)b(y), \frac{d(x,y)}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{e^{-u}}{\sqrt{u} + d(x,y)} du)$$

for all $x, y \in P$.

Then, S has a unique best proximity point $x_* \in P$ such that $d(x_*, Sx_*) = d(P, Q)$. If $y_* \in P$ is another best proximity point for S with $\alpha(y_*) \geq 1$ and $b(y_*) \leq 1$ then $x_* = y_*$.

Proof. It is an immediate consequence of Theorem 3.6 with $f(s, t) = \dfrac{s}{\Gamma(\frac{1}{2})} \displaystyle\int_0^\infty \frac{e^{-u}}{\sqrt{u} + t} du$. □

Remark 4.11. *For the case $h(x, y, z) = z, \mathcal{F}(s, t) = st$ and $\alpha = \lambda = a = b = 1$ and $p = 1$, the definition of $(\mathcal{F}, h, f, 1 - 1 - 1 - 1)$-proximal concept, was studied first by Ayari et al. [30] when considering the following definition.*

Definition 4.12. *[30] A non-self mapping $S : P \longrightarrow Q$ is C- proximal contractive if and only if there exists $f \in C$ such that*

$$d(Sx, Sy) \leq f(d(x, y), d(x, y)),$$

for all $x, y \in P$.

Remark 4.13. *In the particular case $f(s, t) = \beta(s)t$, we obtain the definition of non-self Geraghty mappings introduced by Caballero et. [31]. Also, we reach the result of Komal et al. [32]. Note that the difference of approaches is that the authors in [32] introduced the notion of α-proximal admissible non-self mappings, but in our current paper we introduced the concept of $\alpha - \lambda - a - b$-proximal sub-admissible non-self mappings. Our main contraction is also more general than the one in [32].*

5. A fixed point result: The self-mapping case

Definition 5.1. *Let (P, d) be a metric space. A self-mapping $T : P \to P$ is called $(\mathcal{F}, h, f)$-contractive, where the pair $(\mathcal{F}, h)$ is an upper class of type II, if there exists a C-function f such that*

$$h(1, 1, d(Tx, Ty)) \leq \mathcal{F}(1, f(d(x, y), d(x, y))), \tag{5.14}$$

for all $x, y \in P$.

By considering $P = Q$ in Theorem 3.6, we obtain the following fixed point result.

Theorem 5.2. *Let (P, d) a complete metric space. If a self-mapping $S : P \to P$ is $(\mathcal{F}, h, f)$-contractive, then S possesses a unique fixed point.*

An immediate consequence is the Banach principle by considering $h(x, y, z) = z$, $\mathcal{F}(s, t) = st$ and $f(s, t) = ks$ where $k \in (0, 1)$.

Corollary 5.3. *Let (P, d) a complete metric space. Let consider a self-mapping $S : P \to P$ satisfying:*

$$d(Sx, Sy) \leq f(d(x, y), d(x, y))$$

for all $x, y \in P$, where f is a C-function. Then S possesses a unique fixed point.

Proof. This can be obtained by taking $h(x, y, z) = z$ and $\mathcal{F}(s, t) = st$ in Theorem 5.2. □

6. An application to fractional differential equations

A fractional order differential equation (FODE) is a generalized form of an integer order differential equation. The FODE is useful in many areas, e.g., for the depiction of a physical model of various phenomena in pure and applied science. Fractional differential equations have attracted much attention last decades and have been widely used in engineering, physics, chemistry, biology, and other fields. Many nonlinear fractional diffusion equations have no exact solution, the approximate solution or numerical solution may be a good approach. For more related details, see [33–37]. In this section, take $\delta \in]0, 1[$ and $t_1 \in \mathbb{R}$. Choose $I_\delta = [t_1 - \delta, t_1 + \delta]$. Denote by D^α the Fractional Caputo-Fabrizio derivative (CFD) of order $0 < \alpha < 1$. Note that

$$D^\alpha x(t) = \frac{1}{1-\alpha} \int_{t_1}^{t} x'(t) e^{\frac{-\alpha}{1-\alpha}(t-s)} ds,$$

where x is a function defined on I_δ with values in $\mathbb{R}$. The corresponding fractional integral is introduced by

$$J^\alpha x(t) = (1-\alpha)x(t) + \alpha \int_{t_1}^{t} x(s) ds.$$

Let $\mu > 0$ and $x_1, y_1 \in \mathbb{R}$. Consider $J = [x_1 - \mu, x_1 + \mu]$ and $K = [y_1 - \mu, y_1 + \mu]$. Take

$$P = \{x : I_\delta \longrightarrow J \text{ continuous such that } \quad x(t_1) = x_1\}$$

and

$$Q = \{y : I_\delta \longrightarrow K \text{ continuous such that } \quad y(t_1) = y_1\}.$$

Note that P and Q are nonempty, bounded, closed and convex sets. We have

$$d(P,Q) =: \inf\{\left\|x - y\right\|_\infty : x \in P, y \in Q\} = \left|x_1 - y_1\right|.$$

As an application, we consider the following integro-differential system of equations:

$$\begin{cases} D^\alpha x(t) = f_1(t, x(t)) : x(t_1) = x_1 \\ D^\alpha y(t) = f_2(t, y(t)) : y(t_1) = y_1, \end{cases} \tag{6.15}$$

where $f_1; f_2$ are integrable functions from $I_\delta \times P$ and $I_\delta \times Q$, respectively. Now, we give the definition of a pair (Φ, Ψ) to be an optimal solution of the system (6.15) (in the case it does not posses a solution). We say that the pair (Φ, Ψ) is an optimal solution (6.15 if the following hold for all $T \in I_\delta$:

$$\left|x_1 + (1-\alpha)f_1(t, \Phi(t)) + \alpha \int_{t_1}^{t} f_1(s, \Phi(s))ds - \Phi(t)\right| \le \left|x_1 - y_1\right|, \tag{6.16}$$

and

$$\left|y_1 + (1-\alpha)f_1(t, \Psi(t)) + \alpha \int_{t_1}^{t} f_1(s, \Psi(s))ds - \Psi(t)\right| \le \left|x_1 - y_1\right|. \tag{6.17}$$

Remark 6.1. *It is to be noted that if $y_1 = x_1$, then the pair (Φ, Ψ) verifying (6.16) and (6.17) (so it is an optimal solution) turns out to be an exact solution of the system (6.15).*

For $x \in P$ and $y \in Q$, consider

$$\|x\|_\infty = \sup\{|x(t)| : t \in I_\delta\},$$
$$\left\|f_1\right\|_\infty = \sup\left\{\left|f_1(t, x(t))\right| : t \in I_\delta\right\},$$
$$\left\|f_2\right\|_\infty = \sup\left\{\left|f_2(t, y(t))\right| : t \in I_\delta\right\},$$
$$\left\|f_1 - f_2\right\|_\infty = \sup\left\{\left|f_1(t, x(t)) - f_2(t, y(t))\right| : t \in I_\delta\right\}.$$

The following lemma is needful in the sequel. We omit its proof.

Lemma 6.2. *For $\alpha, \delta, \epsilon \in]0,1[$, the function Ψ defined by*

$$\Psi(\sigma) = \sigma(1-\alpha) + \alpha\sigma\delta + \epsilon - \epsilon\sigma + \alpha\sigma\epsilon - \alpha\delta\epsilon\sigma$$

is increasing on the interval $]0,1[$ with values in $\Psi(]0,1[) =]\epsilon, 1 - \alpha(1-\delta)(1-\epsilon)[\subset]0,1[$.

Consider the following assumptions:

(A1) There is $\sigma > 0$ such that
$\left\|f_1 - f_2\right\|_\infty \le \sigma\left(\|u - v\|_\infty - \left|x_1 - y_1\right|\right)$ for all $u \in P$ and $v \in Q$ such that $\|u - v\|_\infty > \left|x_1 - y_1\right|$,
$\left|f_1(t, r(t)) - f_1(t, s(t))\right| \le \sigma \|r - s\|_\infty, \ \forall t \in I_\delta; \forall r, s \in P$,
$\left|f_2(t, q(t)) - f_2(t, w(t))\right| \le \sigma \left\|q - w\right\|_\infty, \ \forall t \in I_\delta; \forall q, w \in Q;$

(A2) $f_2(t_1, y_1) = 0$ and $f_1(t_1, x_1) = 0$;

(A3) $\max\left\{\left\|f_1\right\|_\infty ; \left\|f_2\right\|_\infty\right\} < \mu$.

Define the functions S and H by

$$\begin{aligned} S: \quad & P \longrightarrow Q \\ & x \longmapsto Sx: \quad I_\delta \longrightarrow K \\ & \qquad\qquad t \longmapsto Sx(t) = y_1 + (1-\alpha) f_2(t, x(t)) + \alpha \int_{t_1}^t f_2(s, x(s)) ds, \end{aligned}$$

and

$$\begin{aligned} H: \quad & Q \longrightarrow P \\ & y \longmapsto Hy: \quad I_\delta \longrightarrow J \\ & \qquad\qquad t \longmapsto Hy(t) = x_1 + (1-\alpha) f_1(t, y(t)) + \alpha \int_{t_1}^t f_1(s, y(s)) ds. \end{aligned}$$

We define the operator $T : P \cup Q \longrightarrow C(I_\delta, \mathbb{R})$ by

$$Tx(t) = \begin{cases} Sx(t) : \text{if} \quad x \in P \\ Hx(t) : \text{if} \quad x \in Q. \end{cases}$$

The essential result of this section is as follows:

Theorem 6.3. *Assume that the assertions $(A_1), (A_2)$ and (A_3) hold. Then the system (6.15) has an optimal solution $(\Phi, \Psi) \in P \times Q$, whenever $\sigma \in]0, 1[$.*

Proof. To prove that the system (6.15) has an optimal solution $(\Phi, \Psi) \in P \times Q$, it suffices to ensure that S has Φ as a best proximity point and H has Ψ as a best proximity.

First, we claim that T is a cyclic operator. Let $x \in P$, then by assumptions (A2) and (A3) we obtain $Tx(t_1) = y_1$. Also,

$$\begin{aligned} \left|Tx(t) - y_1\right| &= \left|(1-\alpha) f_2(t, x(t)) + \alpha \int_{t_1}^t f_2(s, x(s)) ds\right| \\ &\leq (1-\alpha)\left\|f_2\right\| + \alpha\delta\left\|f_2\right\| \\ &\leq \mu. \end{aligned}$$

Therefore, $Tx(t) \in K$ and by definition of Q, we deduce $Tx \in Q$. In the same manner, by assumption (A2), we have for $y \in Q$, $Ty(t_1) = x_1$. Also,

$$\begin{aligned} \left|Ty(t) - x_1\right| &= \left|(1-\alpha) f_1(t, y(t)) + \alpha \int_{t_1}^t f_1(s, y(s)) ds\right| \\ &\leq (1-\alpha)\left\|f_1\right\| + \alpha\delta\left\|f_1\right\| \\ &\leq \mu. \end{aligned}$$

Therefore, $Ty(t) \in J$ and by definition of P, we deduce $Ty \in P$. We conclude that T is cyclic.

We state the following cases:

Case 1. Let $(u, v) \in P \times Q$ such that $\|u - v\|_\infty > \left|x_1 - y_1\right|$
Then we will show that it suffices to prove that T is an $(\mathbf{F}, h, f, \alpha - \lambda - a - b)$-proximal mapping for any $(u, v) \in P \times Q$ such that $\|u - v\|_\infty > \left|x_1 - y_1\right|$

Since we have $\|u-v\|_\infty > \left|x_1-y_1\right|$, one writes $\frac{|x_1-y_1|}{\|u-v\|_\infty} < 1$, and so there exists $\epsilon \in]0,1[$ such that $\frac{d(P,Q)}{\|u-v\|_\infty} \leq \epsilon < 1$. Therefore, $d(P,Q) \leq \epsilon.\,\|u-v\|_\infty$

$$\begin{aligned}
|Tu(t)-Tv(t)| &= |y_1+(1-\alpha)f_2(t,u(t))+\alpha\int_{t_1}^{t} f_2(s,u(s))ds - x_1-(1-\alpha)f_1(t,v(t)) \\
&- \alpha\int_{t_1}^{t} f_1(s,v(s))ds| \\
&\leq \left|x_1-y_1\right|+(1-\alpha)\left\|f_1-f_2\right\|_\infty+\alpha\left|\int_{t_1}^{t}\left\|f_1-f_2\right\|_\infty ds\right| \\
&\leq \left|x_1-y_1\right|+(1-\alpha)\sigma\left(\|u-v\|_\infty-\left|x_1-y_1\right|\right)+\alpha\sigma\left(\|u-v\|_\infty-\left|x_1-y_1\right|\right)|t-t_1| \\
&\leq \left|x_1-y_1\right|+(1-\alpha)\sigma\left(\|u-v\|_\infty-\left|x_1-y_1\right|\right)+\alpha\sigma\delta\left(\|u-v\|_\infty-\left|x_1-y_1\right|\right) \\
&\leq (\sigma(1-\alpha)+\alpha\sigma\delta)\,\|u-v\|_\infty+(1-\sigma(1-\alpha+\alpha\delta))\left|x_1-y_1\right|.
\end{aligned}$$

For $\sigma, \delta \in]0,1[$, we have $1-\sigma(1-\alpha+\alpha\delta) \geq 0$, and we obtain

$$\begin{aligned}
|Tu(t)-Tv(t)| &\leq (\sigma(1-\alpha)+\alpha\sigma\delta+\epsilon(1-\sigma(1-\alpha+\alpha\delta)))\,\|u-v\|_\infty \\
&\leq (\sigma(1-\alpha)+\sigma\alpha\delta+\epsilon-\epsilon\sigma+\alpha\sigma\epsilon-\sigma\alpha\epsilon\delta)\,\|u-v\|_\infty \\
&= \Psi(\sigma)\,\|u-v\|_\infty \\
&\leq \max(\sigma,\Psi(\sigma))\,\|u-v\|_\infty.
\end{aligned}$$

By Lemma (6.2), for $\epsilon, \delta, \alpha \in]0,1[$, the function $\sigma \longmapsto \Psi(\sigma)$ is strictly increasing function on the interval $]0,1[$ with values in $\Psi(]0,1[) =]\epsilon, 1-\alpha(1-\delta)(1-\epsilon)[\subset]0,1[$. This implies that $0 < \Psi(\sigma) < 1$.

Case 2. For $(u,v) \in P\times P$, we have

$$\begin{aligned}
|Tu(t)-Tv(t)| &= \left|(1-\alpha)f_2(t,u(t))+\alpha\int_{t_1}^{t} f_2(s,u(s))ds-(1-\alpha)f_2(t,v(t))-\alpha\int_{t_1}^{t} f_2(s,v(s))ds\right| \\
&\leq (1-\alpha)\left\|f_2(.,u)-f_2(.,v)\right\|_\infty+\alpha\left|\int_{t_1}^{t}\left\|f_2(.,u)-f_2(.,v\right\|_\infty ds\right| \\
&\leq (1-\alpha)\sigma(\|u-v\|_\infty+\alpha\sigma\,\|u-v\|_\infty\,|t-t_1| \\
&= [(1-\alpha)\sigma+\alpha\sigma\,|t-t_1|]\,\|u-v\|_\infty \\
&\leq ((1-\alpha)\sigma+\alpha\delta\sigma)\,\|u-v\|_\infty \\
&\leq \sigma.\,\|u-v\|_\infty \\
&\leq \max(\sigma,\Psi(\sigma))\,\|u-v\|_\infty.
\end{aligned}$$

Case 3. Let $(u,v) \in Q\times Q$. We have the same work as in case 2.

From the above three cases, one deduce that $S :\to Q$ and $H : Q \to P$ are $(\mathbf{F}, h, f, \alpha-\lambda-a-b)$-proximal mappings, with

$$h(x,y,z)=z; \mathbf{F}(s,t)=st; f(s,t)=\max(\sigma,\Psi(\sigma)).s; \quad \alpha=\lambda=a=b=1.$$

Now, we show that $T(P_0) \subset Q_0$.
Let $x \in P_0$, then there exists $y \in Q$ such that $\left\|x-y\right\|_\infty = d(P,Q)$.
Suppose that $Tx \notin Q_0$, then for all $z \in P$ we have $\|Tx-z\|_\infty > \left|x_1-y_1\right| = d(P,Q)$. For $z = Ty$ ($Ty \in P$ because T is a cyclic mapping), we have

$$\left|x_1-y_1\right| < \left\|Tx-Ty\right\|_\infty \leq \max(\sigma,\Psi(\sigma))\left|x_1-y_1\right| < \left|x_1-y_1\right|.$$

It is a contradiction. Therefore, $T(P_0) \subset Q_0$.

Let $(x,z) \in P^2$ and $(y,t) \in Q^2$ such that $\left\|x-y\right\| = \|z-t\| = d(P,Q)$.
Suppose that $\|x-z\| \neq \left\|y-t\right\|$. Without loss of generality we can assume that $\|x-z\| > \left\|y-t\right\|$. P is a convex set, then

$$[x,z] := \{\lambda x + (1-\lambda)z : \lambda \in [0,1]\} \subset P.$$

Let $w \in [x,z[$ such that $\|w-t\|_\infty = d(t,[x,z])$. Then $\|z-t\|_\infty > d(t,[x,z])$, and so

$$\|z-t\|_\infty > \|w-t\|_\infty = d(t,[x,z]) \geq d(t,P) \geq d(P,Q) = \left\|x-y\right\|_\infty .$$

We obtain a contradiction. Therefore, the pair (P,Q) satisfies the $\mathcal{P}$-property.

Let $x \in P_0$, then there exists $y \in Q$ such that $\left\|x-y\right\|_\infty = d(P,Q)$. Suppose that $Sx \notin Q_0$, then for all $z \in P$ we have $\|Sx - z\|_\infty > \left|x_1 - y_1\right| = d(P,Q)$. For $z = Ty = Hy$ ($Ty \in P$ because T is a cyclic mapping), we have

$$\left|x_1-y_1\right| < \left\|Sx - Ty\right\|_\infty = \left\|Tx - Ty\right\|_\infty \leq \max(\sigma, \Psi(\sigma)) \left|x_1-y_1\right| < \left|x_1-y_1\right|.$$

It is a contradiction. Therefore, $S(P_0) \subset Q_0$. Similarly, we show that $H(Q_0) \subset P_0$. Also, it is clear that (P,Q) verifies the P-property.

Now, we want to explain why in case 1, we just consider $(u,v) \in P \times Q$ such that $\|u-v\|_\infty > \left|x_1-y_1\right|$. Indeed, if there are $u \in P$ and $v \in Q$ so that

$$\|u-v\|_\infty = \left|x_1-y_1\right| = d(P,Q),$$

then $u \in P_0$ and $v \in Q_0$, and so P_0 and Q_0 are nonempty. We shall show here that there exist a best proximity point of S and a best proximity point of H. For this, we need to consider the functions $g_1 : P_0 \longrightarrow [0,\infty)$ defined by $g_1(w) = \|w - Sw\|_\infty$ and $g_2 : Q_0 \longrightarrow [0,\infty)$ defined by $g_2(w) = \|w - Hw\|_\infty$. Clearly, P_0 and Q_0 are compact and so the real functions g_1, g_2 are continuous. Then there exist $u_0 \in P_0$ and $v_0 \in Q_0$ such that $\|u_0 - Su_0\|_\infty = \inf_{u\in P_0} \|u - Su\|_\infty$ and $\|v_0 - Hu_0\|_\infty = \inf_{u\in Q_0} \|u - Hu\|_\infty$.

We claim that $\|u_0 - Su_0\|_\infty = d(P,Q) = \left|x_1-y_1\right|$. We argue by contradiction, so we have $\|u_0 - Su_0\|_\infty > \left|x_1-y_1\right|$. Thus, in view of case 1, one writes

$$d(Su_0, S^2u_0) \leq \max(\sigma, \Psi(\sigma)) d(u_0, Su_0).$$

Since $S^2u_0 \in P_0$, one gets

$$\begin{aligned} 0 < \|u_0 - Su_0\|_\infty &= \inf_{u\in P_0} \|u - Su\|_\infty \\ &\leq \left\|S^2u_0 - S^3u_0\right\|_\infty \\ &\leq \max(\sigma, \Psi(\sigma)) \left\|Su_0 - S^2u_0\right\|_\infty \\ &\leq (\max(\sigma, \Psi(\sigma)))^2 \|u_0 - Su_0\|_\infty \\ &< \|u_0 - Su_0\|_\infty . \end{aligned}$$

It is a contradiction. Hence, $\|u_0 - Su_0\|_\infty = d(P,Q) = \left|x_1-y_1\right|$, and so u_0 is a best proximity point of S in P_0. Similarly, v_0 is a best proximity point of H in Q_0. We conclude that the pair (u_0, v_0) is an optimal solution of the system (6.15).

Finally, all the conditions of Theorem 3.6 are verified, and so S admits a best proximity point (Without loss of generality, say $\Phi \in P$). Similarly, H admits a best proximity point, say $\Psi \in Q$. One then writes

$$\|S\Phi - \Phi\|_\infty = \left|x_1-y_1\right|, \tag{6.18}$$

and

$$\|H\Psi - \Psi\|_\infty = \left|x_1 - y_1\right|. \tag{6.19}$$

From (6.18) and (6.19), we have for all $t \in I_\delta$,

$$|S\Phi(t) - \Phi(t)| \leq \left|x_1 - y_1\right|,$$

and

$$|H\Psi(t) - \Psi(t)| \leq \left|x_1 - y_1\right|.$$

That is, the pair (Φ, Ψ) is an optimal solution of the system (6.15). The proof is completed. □

7. Conclusion

This paper dealt with a best proximity result involving a non-self-mapping $T : P \to Q$ involving a pair $(\mathcal{F}, h)$ of upper class of type II and a C- function f. The $\mathcal{P}$-property of the pair (P, Q) and the $\alpha - \lambda - a - b$-proximal sub-admissibility have been used to establish our theorem. We studied a system of fractional differential equations. By applying our best proximity point result, we ensure the existence of an optimal solution of such a system. Particularly, if the boundary conditions are equal, such an optimal solution turned to be an exact solution of the system. To our knowledge, our idea is new when resolving those types of systems. By using a best proximity point technique, we succeeded to pass from optimal solutions to exact solutions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper.

Availability of data and materials

The data used to support the findings of this study are available from the corresponding author upon request.

Funding

This research received no external funding.

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