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An L-fuzzy rough set model based on L-double fuzzy generalized neighborhood systems

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Abstract. In this paper, we consider a commutative quantale L as the truth value table to introduce the notion of L-double fuzzy generalized neighborhood (L-DFGN for short) systems. In addition, we specify and study a pair of L-double rough approximation operators based on L-DFGN systems. Moreover, we study and characterize the related L-double rough approximation (L-DRApprox for short) operators when the L-DFGN system satisfies the conditions of seriality, reflexivity, transitivity, and being unary, respectively. Furthermore, we define and study the measure of L-DRApprox, which characterizes the quality of the obtained approximation. Finally, we interpret the operators of double measures of L-double fuzzy lower and upper approximation as an L-double fuzzy topology and an L-double fuzzy co-topology on a set X, respectively.

1. Introduction

Pawlak **[33, 34]** established the rough set theory, which is an important technique that deals with inexact, ambiguous, or uncertain data. It's been used in a variety of fields like machine learning, knowledge discovery, data mining, expert systems, pattern recognition, granular computing, graph theory, algebraic systems, and partially ordered sets **[9, 16, 18, 25, 35, 43]**.

The majority of rough-set studies and their beginnings have focused on constructive techniques. Equivalence relation was a strict condition and primitive concept in Pawlak's rough set model **[32]**. Thus, the classical rough model has been extended to include binary relations **[8, 60, 61]** and coverings **[52, 54, 59]** and generalized neighborhood systems **[54, 57]**.

According to the development of fuzzy mathematics, the concept of Pawlak's rough set models has been generalized to a fuzzy environment. Dubois and Prade **[15]** firstly proposed fuzzy generalizations of rough sets. Several authors have studied the generalization of rough sets; for instance, Radzikowska and Kerre **[38]** examined fuzzy rough sets models based on L-fuzzy relations.

The notion of L-fuzzy generalized neighborhood (L-FGN for short) systems was offered in **[56]**. It was shown that the L-FGN systems based on approximation operators included the notions of generalized neighborhood system **[48, 54, 57]** (resp., L-fuzzy relation **[22, 44]** and L-fuzzy covering **[28, 29]**) based approximation operators as their special cases.

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Recently, there has been an increased interest in studying the link between fuzzy rough sets and L-topology **[23, 36, 49, 53]**. In 2014, Sostak **[46]** proposed an interpretation of measures of rough approximation based on transitive, and reflexive L-relation in terms of L-fuzzy topologies **[26, 47]**.

On the other hand, the notion of an intuitionistic fuzzy set **[2, 3]** appeared as a useful tool for dealing with imprecise, and imperfect data. One of the most important applications of intuitionistic fuzzy is the area of multi-attribute decision making (see **[30, 50, 51]**). Combining intuitionistic fuzzy set theory and rough set theory could be a fascinating field worth further investigation. Concerning this subject, some studies have already been done **[24, 37, 41]**. Çoker **[12]**, for example, was the first to establish a link between intuitionistic fuzzy set theory and the theory of rough set, demonstrating that a fuzzy rough set was actually an intuitionistic L-fuzzy set.

Using intuitionistic (which is named L-double **[19]**) fuzzy sets, Çoker and his colleagues **[11, 13]** established the notion of intuitionistic fuzzy topology. As a generalization of L-fuzzy topology **[47]** and intuitionistic fuzzy topology **[11]**, Samanta and Mondal **[31]** developed the notion of intuitionistic gradation of openness (which is called L-double fuzzy topology **[19]**).

In 2016, as L is a completely distributive lattice with an order reversing involution $' : L \longrightarrow L$, Abd el-Latif and A. Ramadan **[1]** used the notion of Goguen L-fuzzy sets **[21]** to define the concept of L-double relation, and they used it as a tool to define and study L-double fuzzy rough set models. Recently, there have been some other generalized fuzzy neighborhood system-base rough sets, for example **[17, 27]**.

In this paper, assuming that L is a commutative quantale, we propose the notion of L-DFGN systems as a generalization of L-FGN systems **[55, 56]** and then a pair of L-double rough approximation operators based on it and study some of the properties. Also, it is illustrated that L-double relation-based approximation operators **[1]** can be considered as special L-DFGN system-based approximation operators. Finally, we interpret the operators of double measures of L-double fuzzy lower and upper approximation as an L-double fuzzy topology and an L-double fuzzy co-topology on a set X, respectively.

The following is a description of the paper's structure. Some concepts and results from this study are reviewed in Section 2. In Section 3, we define the concept of L-DFGN systems and utilize it to introduce a pair of L-double rough approximation operators and study some of their properties. In Section 4, through the constructive approach, we study and characterize the related L-DRApprox operators when the L-DFGN system is seriality, reflexivity, transitivity, and unary, respectively. Also, we define the double measure of L-DRApprox, which characterizes the quality of the obtained approximation. Accordingly some properties of such double measures are established. In Section 5, from L-DRApprox operators, we generate the concepts of L-double fuzzy topology and L-double fuzzy co-topology, respectively.

2. Preliminaries

A complete lattice $(L, \leq, \bigvee, \bigwedge, \top_L, \perp_L)$ endowed with a binary operation $\otimes : L \times L \longrightarrow L$ and denoted by a semi-quantale $L = (L, \leq, \otimes)$ **[39]**. Also, we called

(1) L is a unital **[39]** when $\otimes$ has element $e \in L$, with $e \otimes u = u \otimes e = u, \ \forall u \in L$. If $e = \top_L$ is defined to be a strictly two-sided (st-s for short) semi-quantale.

(2) L is a commutative **[39]** when $u \otimes v = v \otimes u, \forall\, u, v \in L$.

(3) L is a quantale **[40]** when $\otimes$ is a associative and

$$u \otimes (\bigvee_{j \in J} v_j) = \bigvee_{j \in J} (u \otimes v_j) \text{ and } (\bigvee_{j \in J} v_j) \otimes u = \bigvee_{j \in J} (v_j \otimes u) \text{ for all } u \in L, \{v_j : j \in J\} \subseteq L.$$

In a commutative quantale $(L, \leq, \otimes)$ the function $u \otimes (-) : L \longrightarrow L$ has a right adjoint $u \to (-) : L \longrightarrow L$ specified by $u \to v = \bigvee \{c : u \otimes c \leq v\}$. The residual $\to: L \times L \to L$ fulfilling the next axiom

$$u \otimes v \leq c \Leftrightarrow u \leq v \to c.$$

Now, L is always taken to be a commutative quantale with the double negation law through this paper, unless otherwise stated.

Suppose that X is a non-empty set and L is a semi-quantale. The family of all L- subsets on X denoted by L^X. The smallest and largest elements in L^X are denoted by $\underline{\bot}$ and $\underline{\top}$, respectively. The operators $\otimes, \bigvee, \rightarrow$ on L can be interpreted onto L^X in a pointed wise as follows:

$$(A \otimes B)(x) = A(x) \otimes B(x), x \in X,$$

$$(\bigvee_{j\in J} A_j)(x) = \bigvee_{j\in J} A_j(x),$$

$$(A \rightarrow B)(x) = A(x) \rightarrow B(x),$$

where $A, B, A_j \in L^X$. One can see that $(L^X, \otimes, \bigvee)$ is a semi-quantale.

Lemma 2.1. **[5, 7, 20, 40, 45]** *For all $u, v, w \in L$ and $\{u_j, v_j : j \in J\} \subseteq L$, the next properties are achieved:*

(1) $u \otimes (u \rightarrow v) \leq v$, *and* $v \leq u \rightarrow (u \otimes v)$;

(2) *If* $(L, \leq, \otimes)$ *is st-s, then* $u \rightarrow v = \top_L$ *whenever* $u \leq v$;

(3) $(\bigvee_{j\in J} v_j) \rightarrow w = \bigwedge_{j\in J}(v_j \rightarrow w)$;

(4) $u \rightarrow (\bigwedge_{j\in J} v_j) = \bigwedge_{j\in J}(u \rightarrow v_j)$, *and* $u \otimes (\bigwedge_{j\in J} v_j) \leq \bigwedge_{j\in J}(u \otimes v_j)$;

(5) $u \otimes (v \rightarrow w) \leq v \rightarrow (u \otimes w)$;

(6) $\bigvee_{j\in J}(u \rightarrow v_j) \leq u \rightarrow (\bigvee_{j\in J} v_j)$;

(7) $(\bigvee_{j\in J} u_j) \otimes v = \bigvee_{j\in J}(u_j \otimes v)$;

(8) $\bigwedge_{j\in J}(u_j \rightarrow v_j) \leq (\bigvee_{j\in J} u_j) \rightarrow (\bigvee_{j\in J} v_j)$ *and* $\bigwedge_{j\in J}(u_j \rightarrow v_j) \leq (\bigwedge_{j\in J} u_j) \rightarrow (\bigwedge_{j\in J} v_j)$.

L is said to fulfill the double negation if

$$(u \rightarrow \bot) \rightarrow \bot = u, \forall u \in L.$$

Additionally, we denote $u \oplus v = \neg(\neg u \otimes \neg v)$ for every $u, v \in L$, where $\neg u$ is used to denote $u \rightarrow \bot$.

Proposition 2.2. **[14]** *For all $u, v \in L$ and $\{u_j : j \in J\} \subseteq L$, the next properties are achieved by satisfying the law of double negation:*

(1) $u \rightarrow v = \neg(u \otimes \neg v)$;

(2) $u \rightarrow (\neg v) = v \rightarrow (\neg u)$;

(3) $\neg(\bigvee_{j\in J} u_j) = \bigwedge_{j\in J} \neg(u_j)$;

(4) $u \leq v$ *implies* $\neg v \leq \neg u$.

The subsethood degree $S : L^X \times L^X \longrightarrow L$ **[6]** and the intersection degree $T : L^X \times L^X \longrightarrow L$ **[10]**, of any two L-subsets $P, Q \in L^X$, are given by

$$S(P, Q) = \bigwedge_{x\in X}(P(x) \rightarrow Q(x)) \text{ and } T(P, Q) = \bigvee_{x\in X}(P(x) \otimes Q(x)),$$

respectively.

Lemma 2.3. **[5, 6, 10]** *For all $P, Q, D, E \in L^X$, $\alpha \in L$ and $\{P_j, Q_j : j \in J\} \subseteq L^X$, the next properties are achieved:*

(1) $P \leq Q \Rightarrow S(D, P) \leq S(D, Q)$ *and* $S(Q, D) \leq S(P, D)$;

(2) $S(P, Q) \otimes S(Q, D) \leq S(P, D)$;

(3) $S(P,Q) \otimes S(D,E) \leq S(P \otimes D, Q \otimes E)$;

(4) $S(P, \bigwedge_{j\in J} Q_j) = \bigwedge_{j\in J} S(P, Q_j)$ and $S(\bigvee_{j\in J} P_j, Q) = \bigwedge_{j\in J} S(P_j, Q)$;

(5) $T(P, \bigvee_{j\in J} Q_j) = \bigvee_{j\in J} T(P, Q_j)$ and $T(P, \bigwedge_{j\in J} Q_j) \leq \bigwedge_{j\in J} T(P, Q_j)$;

(6) *If L satisfies the double negation law then* $S(P,Q) = S(\neg Q, \neg P)$.

Definition 2.4. **[38, 44]** *An L-relation* $\mathcal{R} \in L^{X\times X}$*, is called:*

(1) *serial when* $\bigvee_{y\in X} \mathcal{R}(x,y) = \top$, $\forall x \in X$,

(2) *reflexive when* $\mathcal{R}(x,x) = \top$, $\forall x \in X$,

(3) *transitive when* $\mathcal{R}(x,y) \otimes \mathcal{R}(y,z) \leq \mathcal{R}(x,z)$, $\forall x,y,z \in X$.

Definition 2.5. **[38, 44]** *For an L-relation* $\mathcal{R} \in L^{X\times X}$ *and* $A \in L^X$*, the upper and lower approximation operators are given as follows:*

$$\overline{\mathcal{R}}(A)(x) = T(\mathcal{R}(x,-), A) = \bigvee_{y\in X} (\mathcal{R}(x,y) \otimes A(y)).$$

$$\underline{\mathcal{R}}(A)(x) = S(\mathcal{R}(x,-), A) = \bigwedge_{y\in X} (\mathcal{R}(x,y) \to A(y)),$$

respectively.

Definition 2.6. **[55, 56, 58]** *By an L-FGN system operator on a universe of discourse X, we mean a function* $\mathcal{N} : X \longrightarrow L^{L^X}$*, if* $\mathcal{N}(x)$ *is non-empty, i.e.,* $\bigvee_{A\in L^X} \mathcal{N}(x)(A) = \top_L$, $\forall x \in X$.

Definition 2.7. **[55, 56, 58]** *For an L-FGN system operator* $\mathcal{N} : X \longrightarrow L^{L^X}$ *and* $A \in L^X$*, the lower and upper approximation operators* $\underline{\mathcal{N}}(A)$ *and* $\overline{\mathcal{N}}(A)$ *are given by:*

$$\underline{\mathcal{N}}(A)(x) = \bigvee_{K\in L^X} (\mathcal{N}(x)(K) \otimes S(K,A)),$$

$$\overline{\mathcal{N}}(A)(x) = \bigwedge_{K\in L^X} (\mathcal{N}(x)(K) \to T(K,A)).$$

Definition 2.8. *Let X be an arbitrary sets. The pair* $(\mathcal{R}, \mathcal{R}^*)$ *of maps* $\mathcal{R}, \mathcal{R}^* : X \times X \longrightarrow L$ *is called an L-double relation (or L-double fuzzy relation) on X, if* $\mathcal{R}(x,y) \leq \neg(\mathcal{R}^*(x,y))$, $\forall\,(x,y) \in X \times X$. $\mathcal{R}(x,y)$ *(resp.,* $\mathcal{R}^*(x,y)$*), referred to as the degree of relation (resp., non-relation) between x and y.*

If $L = (L, \wedge, \vee, ', 0_L, 1_L)$ is taken to an order reversed completely distributive lattice then the above definition coincided with the definition of **[1]**.

3. A double rough approximation operators

Through this section, we will introduce the notion of L-DFGN systems, and use it to define a pair of L-DRApprox operators, and study some of their properties. Also, we show that L-double relation-based approximation operators **[1]** can be considered as special cases of the above L-DFGN system-based approximation operators.

Definition 3.1. *Assume that X is the universe of discourse. The pair* $(\mathcal{N}, \mathcal{N}^*)$ *of maps* $\mathcal{N}, \mathcal{N}^* : X \longrightarrow L^{L^X}$ *is said to be an L-DFGN system operator on X, if for any* $x \in X$, $\bigvee_{A\in L^X} \mathcal{N}(x)(D) = \top_L$ *and* $\mathcal{N}(x)(D) \leq \neg(\mathcal{N}^*(x)(D))$*. The triplet* $(X, \mathcal{N}, \mathcal{N}^*)$ *is said to be an L-DFGN space.*
Usually, the pair $(\mathcal{N}(x), \mathcal{N}^*(x))$ *is said to be an L-DFGN system of x and* $\mathcal{N}(x)(D)$ *(resp.,* $\mathcal{N}^*(x)(D)$*) is interpreted as the degree of neighborhood (resp., non-neighborhood) of x.*

In what follow, we shall establish an example of an L-DFGN system operator.

Example 3.2. *Assume that $X = \{x\}$ is a single point set, and $L = [0,1]$ is the usual unit interval. Define an L-DFGN system operator $\mathcal{N}, \mathcal{N}^* : X \longrightarrow L^{L^X}$ by*

$$\mathcal{N}(x)(D) = \begin{cases} 1 & for \;\; D = 1_X; \\ \frac{1}{2} & for \;\; D = x_{\frac{1}{2}}; \\ 0 & otherwise. \end{cases} \qquad \mathcal{N}^*(x)(D) = \begin{cases} 0 & for \;\; D = 1_X; \\ \frac{2}{5} & for \;\; D = x_{\frac{1}{2}}; \\ 1 & otherwise. \end{cases}$$

It is easy to have that $\mathcal{N}, \mathcal{N}^* : X \longrightarrow L^{L^X}$ is an L-DFGN system operator.

Remark 3.3. *Assume that X is the universe of discourse and $\mathcal{N} : X \longrightarrow L^{L^X}$ be an L-DFGN system operator on X. Define a map $\mathcal{N}^* : X \longrightarrow L^{L^X}$ by $\mathcal{N}^*(x) = \neg\mathcal{N}(x)\ \forall x \in X$, then the pair $(\mathcal{N}, \mathcal{N}^*)$ is an L-DFGN system. Therefore every L-FGN system operator* **[55, 56]** *corresponds to the following L-DFGN system operator $(\mathcal{N}, \neg\mathcal{N})$ and we can say that an L-DFGN system is a generalization of L-FGN system* **[55, 56]**.

Definition 3.4. *Let $(\mathcal{N}, \mathcal{N}^*)$ be an L-DFGN system operator. Define two mappings $\underline{\mathcal{N}}, \underline{\mathcal{N}^*} : L^X \longrightarrow L^X$ as follows:*

$\underline{\mathcal{N}}(x)(A) = \bigvee_{K \in L^X} (\mathcal{N}(x)(K) \otimes S(K, A))$,

$\underline{\mathcal{N}^*}(x)(A) = \bigwedge_{K \in L^X} (\neg\mathcal{N}^*(x)(K) \rightarrow T(K, \neg A))$,

where $x \in X$, and $A \in L^X$.

The pair $(\underline{\mathcal{N}}, \underline{\mathcal{N}^*})$ is said to be an L-double fuzzy lower approximation (L-DFLApprox for short) operator and the triplets $(X, \underline{\mathcal{N}}, \underline{\mathcal{N}^*})$ is called an L-DFLApprox space.

Remark 3.5. *Assume that $\mathcal{N} : X \longrightarrow L^{L^X}$ is an L-DFGN system operator on X and $\underline{\mathcal{N}} : L^X \longrightarrow L^X$ be a lower approximation operator* **[55, 56]**. *Define a map $\underline{\mathcal{N}^*} : L^X \longrightarrow L^X$ by*

$$\underline{\mathcal{N}^*}(x)(A) = \neg\underline{\mathcal{N}}(x)(A)\ \forall x \in X \text{ and } A \in L^X.$$

Then the pair $(\underline{\mathcal{N}}, \neg\underline{\mathcal{N}})$ is an L-DFLApprox operator. Therefore every lower approximation operator $\underline{\mathcal{N}} : L^X \longrightarrow L^X$ **[55, 56]** *corresponds to the following L-DFLApprox operators $(\underline{\mathcal{N}}, \neg\underline{\mathcal{N}})$.*

Definition 3.6. *Assume that $(\mathcal{N}, \mathcal{N}^*)$ is an L-DFGN system operator. Define two mappings $\overline{\mathcal{N}}, \overline{\mathcal{N}}^* : L^X \longrightarrow L^X$ as follows:*

$\overline{\mathcal{N}}(x)(A) = \bigwedge_{K \in L^X} (\mathcal{N}(x)(K) \rightarrow T(K, A))$,

$\overline{\mathcal{N}}^*(x)(A) = \bigvee_{K \in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg A))$,

where $x \in X$, and $A \in L^X$.

The pair $(\overline{\mathcal{N}}, \overline{\mathcal{N}}^*)$ is called an L-double fuzzy upper approximation (L-DFUApprox for short) operator and the triplets $(X, \overline{\mathcal{N}}, \overline{\mathcal{N}}^*)$ is said to be an L-DFUApprox space.

Similarly to what given in **Remark 3.5**, we can say that every upper approximation operator $\overline{\mathcal{N}} : L^X \longrightarrow L^X$ **[55, 56]** corresponds to the following L-DFUApprox operator having the form $(\overline{\mathcal{N}}, \neg\overline{\mathcal{N}})$.

Definition 3.7. *Let $(X, \mathcal{N}, \mathcal{N}^*)$ be an L-DFGN space. Then the quaternary $(\underline{\mathcal{N}}, \underline{\mathcal{N}^*}\overline{\mathcal{N}}, \overline{\mathcal{N}}^*)$ is said to be L-double fuzzy rough set.*

Example 3.8. *Suppose that $X = \{x\}$ is a single point set and $L = [0,1]$ with the adjoint pair $(*, \rightarrow)$ on $[0,1]$ defined as follows for all $\varepsilon, \theta \in L$,*

$$\varepsilon * \theta = \max\{0, \varepsilon + \theta - 1\}, \;\; \varepsilon \rightarrow \theta = \min\{1, 1 - \varepsilon + \theta\}.$$

For L-DFGN system operator $(\mathcal{N},\mathcal{N}^*)$, given in **Example 3.2**, *an L-DFLApprox operator $\underline{\mathcal{N}},\underline{\mathcal{N}}^*$ given by: For $A = x_{\frac{1}{3}}$*

$$\begin{aligned}
\underline{\mathcal{N}}(x)(x_{\frac{1}{3}}) &= \bigvee_{K\in L^X}(\mathcal{N}_x(K) * S(K, x_{\frac{1}{3}}))\\
&= (\mathcal{N}_x(1_X) * S(1_X, x_{\frac{1}{3}})) \vee (\mathcal{N}_x(x_{\frac{1}{2}}) * S(x_{\frac{1}{2}}, x_{\frac{1}{3}}))\\
&= (1 * (1 \to \tfrac{1}{3})) \vee (\tfrac{1}{2} * (\tfrac{1}{2} \to \tfrac{1}{3}))\\
&= (1 * \tfrac{1}{3}) \vee (\tfrac{1}{2} * \tfrac{5}{6})\\
&= \tfrac{1}{3} \vee (\tfrac{1}{2} - \tfrac{1}{6})\\
&= \tfrac{1}{3} \vee \tfrac{1}{3} = \tfrac{1}{3}.\\
\underline{\mathcal{N}}^*(x)(x_{\frac{1}{3}}) &= \bigwedge_{K\in L^X}(\neg\mathcal{N}^*(K)(x) \to T(K, \neg(x_{\frac{1}{3}}))\\
&= (\neg\mathcal{N}^*(x)(1_X) \to T(1_X, \neg(x_{\frac{1}{3}}))) \wedge (\neg\mathcal{N}^*(x)(x_{\frac{1}{2}}) \to T(x_{\frac{1}{2}}, \neg(x_{\frac{1}{3}})))\\
&= (1 \to (1 * \tfrac{2}{3})) \wedge (\tfrac{3}{5} \to (\tfrac{1}{2} * \tfrac{2}{3}))\\
&= (1 \to \tfrac{2}{3}) \wedge (\tfrac{3}{5} \to \tfrac{1}{6})\\
&= \tfrac{2}{3} \wedge \tfrac{17}{30} = \tfrac{17}{30}.
\end{aligned}$$

Also, an L-DFUApprox operator $\overline{\mathcal{N}},\overline{\mathcal{N}}^*$ given by: For $A = x_{\frac{2}{3}}$

$$\begin{aligned}
\overline{\mathcal{N}}(x)(x_{\frac{2}{3}}) &= \bigwedge_{K\in L^X}(\mathcal{N}(x)(K) \to T(K, x_{\frac{2}{3}}))\\
&= (\mathcal{N}(x)(1_X) \to T(1_X, x_{\frac{2}{3}})) \wedge (\mathcal{N}(x)(x_{\frac{1}{2}}) \to T(x_{\frac{1}{2}}, x_{\frac{2}{3}}))\\
&= (1 \to (1 * \tfrac{2}{3})) \wedge (\tfrac{1}{2} \to (\tfrac{1}{2} * \tfrac{2}{3}))\\
&= (1 \to \tfrac{2}{3}) \wedge (\tfrac{1}{2} \to \tfrac{1}{6})\\
&= \tfrac{2}{3} \wedge \tfrac{2}{3} = \tfrac{2}{3}.\\
\overline{\mathcal{N}}^*(x)(x_{\frac{2}{3}}) &= \bigvee_{K\in L^X}(\neg\mathcal{N}^*(x)(K) * S(K, \neg(x_{\frac{2}{3}})))\\
&= (\neg\mathcal{N}^*(x)(1_X) * S(1_X, \neg(x_{\frac{2}{3}}))) \vee (\neg\mathcal{N}^*(x)(x_{\frac{1}{2}}) * S(\tfrac{1}{2}, \neg(x_{\frac{2}{3}})))\\
&= (1 * (1 \to \tfrac{1}{3})) \vee (\tfrac{3}{5} * (\tfrac{1}{2} \to \tfrac{1}{3}))\\
&= (1 * \tfrac{1}{3}) \vee (\tfrac{3}{5} * \tfrac{5}{6})\\
&= \tfrac{1}{3} \vee \tfrac{13}{30} = \tfrac{13}{30}.
\end{aligned}$$

In the sequel, we will prove that the L-DFGN system has quantale-valued (or L-double) relation-based approximation operators **[1]** as a special case. Before going to the end, we give the following definition:

Definition 3.9. *Assume that $(\mathcal{R},\mathcal{R}^*)$ is an L-double relation on X. Define four mappings $\underline{\mathcal{R}},\underline{\mathcal{R}}^*,\overline{\mathcal{R}},\overline{\mathcal{R}}^* : L^X \to L^X$ as follows*

(*i*) $\underline{\mathcal{R}}(A)(x) = S(\mathcal{R}(x,-), A) = \bigwedge_{y\in X}(\mathcal{R}(x,y) \to A(y))$, *and*

$\underline{\mathcal{R}}^*(A)(x) = T(\neg\mathcal{R}^*(x,-), \neg A) = \bigvee_{y\in X}(\neg\mathcal{R}^*(x,y) \otimes \neg A(y))$,

(*ii*) $\overline{\mathcal{R}}(A)(x) = T(\mathcal{R}(x,-), A) = \bigvee_{y\in X}(\mathcal{R}(x,y) \otimes A(y))$, *and*

$\overline{\mathcal{R}}^*(A)(x) = S(\neg\mathcal{R}^*(x,-), \neg A) = \bigwedge_{y\in X}(\neg\mathcal{R}^*(x,y) \to \neg A(y))$,

where $x \in X$ and $A \in L^X$.
The pairs $(\underline{\mathcal{R}},\underline{\mathcal{R}}^*)$ and $(\overline{\mathcal{R}},\overline{\mathcal{R}}^*)$ are said to be L-DFLApprox and L-DFUApprox operators, respectively, and the triplets $(X,\underline{\mathcal{R}},\underline{\mathcal{R}}^*)$, $(X,\overline{\mathcal{R}},\overline{\mathcal{R}}^*)$ are said to be L-DFLApprox and L-DFUApprox spaces, respectively.

Example 3.10. *According to* **Remark 3.5**, *we have the following:*

(1) *Every lower L-fuzzy rough approximation operator $\underline{\mathcal{R}} : L^X \longrightarrow L^X$* **[38]** *can be recognized with an L-DFLApprox operator in the form $(\underline{\mathcal{R}},\neg\underline{\mathcal{R}})$.*

(2) *Every upper L-fuzzy rough approximation operator* $\overline{\mathcal{R}} : L^X \longrightarrow L^X$ **[38]** *can be recognized with an L-DFUApprox operator in the form* $(\overline{\mathcal{R}}, \neg\overline{\mathcal{R}})$.

Now, it is time to explain that an L-DRApprox operator based on an L-double relation **[1]** is a special case of an L-DRApprox operator based on L-DFGN systems.

Lemma 3.11. *Let* $(L, \leq, \otimes)$ *be a st-s, and let* $\mathcal{R}, \mathcal{R}^* : X \times X \longrightarrow L$ *be an L-double relation on a set X. We define an L-DFGN system operator* $\mathcal{N}_{\mathcal{R}}, \mathcal{N}^*_{\mathcal{R}^*}$ *as follows: For any* $x \in X$, $K \in L^X$,

$$\mathcal{N}_{\mathcal{R}}(x)(K) = \begin{cases} \top_L, & K = R(x,-); \\ \bot_L, & otherwise. \end{cases} \qquad \mathcal{N}^*_{\mathcal{R}^*}(x)(K) = \begin{cases} \bot_L, & K = \neg R^*(x,-); \\ \top_L, & otherwise. \end{cases}$$

Then, $\underline{\mathcal{N}}_{\mathcal{R}}(A) = \underline{\mathcal{R}}(A)$, $\underline{\mathcal{N}}^*_{\mathcal{R}^*}(A) = \underline{\mathcal{R}}^*(A)$ *and* $\overline{\mathcal{N}}_{\mathcal{R}}(A) = \overline{\mathcal{R}}(A)$, $\overline{\mathcal{N}}^*_{\mathcal{R}^*}(A) = \overline{\mathcal{R}}^*(A)$.

Proof. For any $x \in X$, we have

$$\bigvee_{K \in L^X} \mathcal{N}_{\mathcal{R}}(x)(K) \geq \mathcal{N}_{\mathcal{R}}(x)(\mathcal{R}(x,-)) = \top_L, \quad \bigwedge_{K \in L^X} \mathcal{N}^*_{\mathcal{R}^*}(x)(K) \leq \mathcal{N}^*_{\mathcal{R}^*}(x)(\neg\mathcal{R}^*(x,-)) = \bot_L.$$

Hence $\mathcal{N}_{\mathcal{R}}, \mathcal{N}^*_{\mathcal{R}^*}$ is an L-DFGN system operator. Then for any $A \in L^X$ and $x \in X$. By the definition of $\mathcal{N}_{\mathcal{R}}, \mathcal{N}^*_{\mathcal{R}^*}$, we get

$$\underline{\mathcal{N}}_{\mathcal{R}}(x)(A) = \bigvee_{K \in L^X}(\mathcal{N}_{\mathcal{R}}(x)(K) \otimes S(K, A)) = \top_L \otimes S(\mathcal{R}(x,-), A) = \underline{\mathcal{R}}(A)(x),$$

$$\underline{\mathcal{N}}^*_{\mathcal{R}^*}(x)(A) = \bigwedge_{K \in L^X}(\neg\mathcal{N}^*_{\mathcal{R}^*}(K) \to T(K, \neg A)) = \neg \bot_L \to T(\neg\mathcal{R}^*(x,-), \neg A) = \underline{\mathcal{R}}^*(x)(A),$$

$$\overline{\mathcal{N}}_{\mathcal{R}}(x)(A) = \bigwedge_{K \in L^X}(\mathcal{N}_{\mathcal{R}}(x)(K) \to T(K, A)) = \top_L \to T(\mathcal{R}(x,-), A) = \overline{\mathcal{R}}(x)(A),$$

$$\overline{\mathcal{N}}^*_{\mathcal{R}^*}(x)(A) = \bigvee_{K \in L^X}(\neg\mathcal{N}^*_{\mathcal{R}^*}(x)(K) \otimes S(K, \neg A)) = \neg \bot_L \otimes S(\neg\mathcal{R}^*(x,-), \neg A) = \overline{\mathcal{R}}^*(x)(A).$$

Hence, $\underline{\mathcal{N}}_{\mathcal{R}}(A) = \underline{\mathcal{R}}(A)$, $\underline{\mathcal{N}}^*_{\mathcal{R}^*}(A) = \underline{\mathcal{R}}^*(A)$, and $\overline{\mathcal{N}}_{\mathcal{R}}(A) = \overline{\mathcal{R}}(A)$, $\overline{\mathcal{N}}^*_{\mathcal{R}^*}(A) = \overline{\mathcal{R}}^*(A)$ for any $A \in L^X$. □

Theorem 3.12. *Let* $(\mathcal{N}, \mathcal{N}^*)$ *be an L-DFGN system operator on X. Then the L-DFLApprox operator* $(\underline{\mathcal{N}}, \underline{\mathcal{N}}^*)$ *satisfies the next properties: For all* $A, B \in L^X$, *and* $A_i \subseteq L^X$,

(1) $\underline{\mathcal{N}}(x)(A) \leq \neg\underline{\mathcal{N}}^*(x)(A)$;

(2) *(i)* $\underline{\mathcal{N}}(x)(\bigwedge_{i \in I} A_i) \leq \bigwedge_{i \in I} \underline{\mathcal{N}}(x)(A_i)$; *and* *(ii)* $\underline{\mathcal{N}}^*(x)(\bigwedge_{i \in I} A_i) \geq \bigvee_{i \in I} \underline{\mathcal{N}}^*(x)(A_i)$;

(3) *(i)* $\underline{\mathcal{N}}(x)(\bigvee_{i \in I} A_i) \geq \bigvee_{i \in I} \underline{\mathcal{N}}(x)(A_i)$; *and* *(ii)* $\underline{\mathcal{N}}^*(x)(\bigvee_{i \in I} A_i) \leq \bigwedge_{i \in I} \underline{\mathcal{N}}^*(x)(A_i)$;

(4) *If L is st-s (sometimes called integral), then*
(i) $\underline{\mathcal{N}}(\underline{\top}) = \underline{\top}$; *and* *(ii)* $\underline{\mathcal{N}}^*(\underline{\top}) = \underline{\bot}$;

(5) *If* $A \leq B$, *then*
(i) $\underline{\mathcal{N}}(x)(A) \leq \underline{\mathcal{N}}(x)(B)$; *and* *(ii)* $\underline{\mathcal{N}}^*(x)(A) \geq \underline{\mathcal{N}}^*(x)(B)$;

(6) *(i)* $\underline{\mathcal{N}}(x)(A) = \neg\overline{\mathcal{N}}(x)(\neg A)$; *and* *(ii)* $\underline{\mathcal{N}}^*(x)(A) = \neg\overline{\mathcal{N}}^*(x)(\neg A)$.

Proof. (1)
$$\begin{aligned}
\underline{\mathcal{N}}(x)(A) &= \bigvee_{K \in L^X} (\mathcal{N}(x)(K) \otimes S(K, A)) \\
&\leq \bigvee_{K \in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, A)) \\
&= \bigvee_{K \in L^X} (\neg\mathcal{N}^*(x)(K) \otimes \neg T(K, \neg A)) \text{ (by **Proposition 2.2** (1))} \\
&= \bigvee_{K \in L^X} \neg(\neg\mathcal{N}^*(x)(K) \to T(K, \neg A)) \\
&= \neg(\bigwedge_{K \in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \neg A))) \text{ (by **Proposition 2.2** (3))} \\
&= \neg\underline{\mathcal{N}}^*(x)(A).
\end{aligned}$$

(2) For all $\{A_i : i \in I\} \subseteq L^X$, we get

(*i*) $$\begin{aligned}\underline{\mathcal{N}}(x)(\bigwedge_{i\in I} A_i) &= \bigwedge_{K\in L^X} (\mathcal{N}(x)(K) \otimes S(K, \bigwedge_{i\in I} A_i))\\ &= \bigwedge_{K\in L^X} (\mathcal{N}(x)(K) \otimes \bigwedge_{i\in I} S(K, A_i)) \text{ (by \textbf{Lemma 2.3 (4)})}\\ &\le \bigwedge_{i\in I}(\bigwedge_{K\in L^X} (\mathcal{N}(x)(K) \otimes S(K, A_i))) \text{ (by \textbf{Lemma 2.3 (5)})}\\ &= \bigwedge_{i\in I} \underline{\mathcal{N}}(x)(A_i).\end{aligned}$$

(*ii*) $$\begin{aligned}\underline{\mathcal{N}}^*(x)(\bigwedge_{i\in I} A_i) &= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \neg \bigwedge_{i\in I} A_i))\\ &= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \bigvee_{i\in I} \neg A_i))\\ &= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to \bigvee_{i\in I} T(K, \neg A_i)) \text{ (by \textbf{Lemma 2.3 (5)})}\\ &\ge \bigvee_{i\in I}(\bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \neg A_i))) \text{ (by \textbf{Lemma 2.1 (6)})}\\ &= \bigvee_{i\in I} \underline{\mathcal{N}}^*(x)(A_i).\end{aligned}$$

(3) For all $\{A_i : i \in I\} \subseteq L^X$, we have

(*i*) $$\begin{aligned}\underline{\mathcal{N}}(x)(\bigvee_{i\in I} A_i) &= \bigvee_{K\in L^X} (\mathcal{N}(x)(K) \otimes S(K, \bigvee_{i\in I} A_i))\\ &\ge \bigvee_{K\in L^X} (\mathcal{N}(x)(K) \otimes \bigvee_{i\in I} S(K, A_i)) \text{ (by \textbf{Lemma 2.1 (6)})}\\ &= \bigvee_{i\in I}(\bigvee_{K\in L^X} (\mathcal{N}(x)(K) \otimes S(K, A_i))) \text{ (by \textbf{Lemma 2.3 (5)})}\\ &= \bigvee_{i\in I} \underline{\mathcal{N}}(x)(A_i).\end{aligned}$$

(*ii*) $$\begin{aligned}\underline{\mathcal{N}}^*(x)(\bigvee_{i\in I} A_i) &= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \neg \bigvee_{i\in I} A_i))\\ &= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \bigwedge_{i\in I} \neg A_i))\\ &\le \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to \bigwedge_{i\in I} T(K, \neg A_i)) \text{ (by \textbf{Lemma 2.3 (5)})}\\ &= \bigwedge_{i\in I}(\bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \neg A_i))) \text{ (by \textbf{Lemma 2.3 (4)})}\\ &= \bigwedge_{i\in I} \underline{\mathcal{N}}^*(x)(A_i).\end{aligned}$$

For the items (4) – (6), we prove only the second part (*ii*), since the proof of the first part (*i*) is the same as given in **[56]**.

(4) Suppose that L is st-s quantale, then

$$\begin{aligned}\underline{\mathcal{N}}^*(x)(\top) &= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \neg\underline{\top}))\\ &= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \underline{\bot}))\\ &= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to \bot)\\ &= \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K)) \to \bot \text{ (by \textbf{Lemma 2.1 (3)})}\\ &= \neg(\bigwedge_{K\in L^X} \mathcal{N}^*(x)(K)) \to \bot\\ &= \neg\bot \to \bot = \top \to \bot = \bot_L.\end{aligned}$$

(5) $\forall A, B \in L^X$, with $A \le B$, we find

$$\underline{\mathcal{N}}^*(x)(A) = \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \neg A))$$

$$\geq \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K,\neg B)) \text{ (by **Lemma 2.3** (1))}$$
$$= \underline{\mathcal{N}}^*(x)(B).$$

(6) $\neg\overline{\mathcal{N}}^*(x)(\neg A) = \neg \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K)\otimes S(K,A))$
$$= \bigwedge_{K\in L^X} \neg(\neg\mathcal{N}^*(x)(K)\otimes S(K,A))$$
$$= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to \neg S(K,A)) \text{ (by **Proposition 2.2** (1))}$$
$$= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K,\neg A))$$
$$= \underline{\mathcal{N}}^*(x)(A).$$

□

Theorem 3.13. *Let $(\mathcal{N},\mathcal{N}^*)$ be an L-DFGN system operator on X. Then the L-DFUApprox operator $(\overline{\mathcal{N}}, \overline{\mathcal{N}}^*)$ satisfies the next properties:*

(1) $\overline{\mathcal{N}}(x)(A) \geq \neg\overline{\mathcal{N}}^*(x)(A)$;

(2) (i) $\overline{\mathcal{N}}(x)(\bigwedge_{i\in I} A_i) \leq \bigwedge_{i\in I} \overline{\mathcal{N}}(x)(A_i)$; *and* (ii) $\overline{\mathcal{N}}^*(x)(\bigwedge_{i\in I} A_i) \geq \bigvee_{i\in I} \overline{\mathcal{N}}^*(x)(A_i)$;

(3) (i) $\overline{\mathcal{N}}(x)(\bigvee_{i\in I} A_i) \geq \bigvee_{i\in I} \overline{\mathcal{N}}(x)(A_i)$; *and* (ii) $\overline{\mathcal{N}}^*(x)(\bigvee_{i\in I} A_i) \leq \bigwedge_{i\in I} \overline{\mathcal{N}}^*(x)(A_i)$;

(4) *If L is st-s (sometimes called integral), then*

(i) $\overline{\mathcal{N}}(\underline{\perp}) = \underline{\perp}$; *and* (ii) $\overline{\mathcal{N}}^*(\underline{\perp}) = \underline{\top}$;

(5) *If $A \leq B$, then*

(i) $\overline{\mathcal{N}}(x)(A) \leq \overline{\mathcal{N}}(x)(B)$; *and* (ii) $\overline{\mathcal{N}}^*(x)(A) \geq \overline{\mathcal{N}}^*(x)(B)$;

(6) (i) $\overline{\mathcal{N}}(x)(A) = \neg\underline{\mathcal{N}}(x)(\neg A)$; *and* (ii) $\overline{\mathcal{N}}^*(x)(A) = \neg\underline{\mathcal{N}}^*(x)(\neg A)$,

where $A, B \in L^X$, and $A_i \subseteq L^X$.

Proof. (1) $\neg\overline{\mathcal{N}}^*(x)(A) = \neg \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K)\otimes S(K,\neg A))$
$$= \bigwedge_{K\in L^X} \neg(\neg\mathcal{N}^*(x)(K)\otimes S(K,\neg A)) \text{ (by **Proposition 2.2** (3))}$$
$$= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to \neg S(K,\neg A)) \text{ (by **Proposition 2.2** (1))}$$
$$\leq \bigwedge_{K\in L^X} (\mathcal{N}(x)(K) \to T(K,A))$$
$$= \overline{\mathcal{N}}(x)(A).$$

(2) For all $\{A_i : i\in I\} \subseteq L^X$, we have

(i) $\overline{\mathcal{N}}(x)(\bigwedge_{i\in I} A_i) = \bigwedge_{K\in L^X} (\mathcal{N}(x)(K) \to T(K,\bigwedge_{i\in I} A_i))$
$$\leq \bigvee_{i\in I} \bigvee_{K\in L^X} (\mathcal{N}(x)(K) \to \bigwedge_{i\in I} T(K,A_i)) \text{ (by **Lemma 2.3** (5))}$$
$$= \bigwedge_{i\in I}(\bigvee_{K\in L^X} (\mathcal{N}(x)(K) \to T(K,A_i))) \text{ (by **Lemma 2.3** (4))}$$
$$= \bigwedge_{i\in I} \overline{\mathcal{N}}(x)(A_i).$$

$$(ii)\ \overline{\mathcal{N}}^{*}(x)(\textstyle\bigwedge_{i\in I} A_i) = \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg \bigwedge_{i\in I} A_i))$$
$$= \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \textstyle\bigvee_{i\in I} \neg A_i))$$
$$\geq \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes \textstyle\bigvee_{i\in I} S(K, \neg A_i)) \text{ (by **Lemma 2.1** (6))}$$
$$= \textstyle\bigvee_{i\in I}(\bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg A_i))) \text{ (by **Lemma 2.1** (7))}$$
$$= \textstyle\bigvee_{i\in I} \overline{\mathcal{N}}^{*}(x)(A_i).$$

(3) For all $\{A_i : i \in I\} \subseteq L^X$, we have

$$(i)\ \overline{\mathcal{N}}(x)(\textstyle\bigvee_{i\in I} A_i) = \bigwedge_{K\in L^X} (\mathcal{N}(x)(K) \to T(K, \bigvee_{i\in I} A_i))$$
$$= \bigwedge_{K\in L^X} (\mathcal{N}(x)(K) \to \textstyle\bigvee_{i\in I} T(K, A_i)) \text{ (by **Lemma 2.3** (5))}$$
$$\geq \textstyle\bigvee_{i\in I}(\bigwedge_{K\in L^X} (\mathcal{N}(x)(K) \to T(K, A_i))) \text{ (by **Lemma 2.1** (6))}$$
$$= \textstyle\bigvee_{i\in I} \overline{\mathcal{N}}(x)(A_i).$$

$$(ii)\ \overline{\mathcal{N}}^{*}(x)(\textstyle\bigvee_{i\in I} A_i) = \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg \bigvee_{i\in I} A_i))$$
$$= \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \textstyle\bigwedge_{i\in I} \neg A_i))$$
$$= \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes \textstyle\bigwedge_{i\in I} S(K, \neg A_i)) \text{ (by **Lemma 2.3** (4))}$$
$$\leq \textstyle\bigwedge_{i\in I}(\bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg A_i))) \text{ (by **Lemma 2.3** (5))}$$
$$= \textstyle\bigwedge_{i\in I} \overline{\mathcal{N}}^{*}(x)(A_i).$$

For the items (4) – (6), we prove only the second part (*ii*), since the proof of the first part (*i*) is the same as given in **[56]**.

(4) Suppose that L is st-s quantale, then

$$\overline{\mathcal{N}}^{*}(x)(\perp) = \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg\underline{\perp}))$$
$$= \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \underline{\top}))$$
$$= \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes \top) \text{ (by **Lemma 2.1** (2))}$$
$$= \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K)) \otimes \top \text{ (by **Lemma 2.1** (7))}$$
$$= \neg(\bigwedge_{K\in L^X} \mathcal{N}^*(x)(K)) \otimes \top$$
$$= \neg\perp \otimes \top = \top \otimes \top = \top_L.$$

(5) $\forall A, B \in L^X$, with $A \leq B$, we find

$$\overline{\mathcal{N}}^{*}(x)(A) = \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg A))$$
$$\geq \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg B)) \text{ (by **Lemma 2.3** (1))}$$
$$= \overline{\mathcal{N}}^{*}(x)(B).$$

(6) $\neg\underline{\mathcal{N}}^*(x)(\neg A) = \neg(\bigwedge_{K\in L^X}(\neg\mathcal{N}^*(x)(K)\rightarrow T(K,A)))$
$= \bigvee_{K\in L^X}\neg(\neg\mathcal{N}^*(x)(K)\rightarrow T(K,A)))$
$= \bigvee_{K\in L^X}(\neg\mathcal{N}^*(x)(K)\otimes\neg T(K,A)))$ (by **Proposition 2.2** (1))
$= \bigvee_{K\in L^X}(\neg\mathcal{N}^*(x)(K)\otimes S(K,\neg A))$
$= \overline{\mathcal{N}}^*(x)(A).$

□

Let $(\mathcal{N},\mathcal{N}^*)$ be an L-DFGN system operator on X. The L-double measure of roughness of L-DFUApprox $(\mathcal{U}_N,\mathcal{U}_{N^*})$, of an $A\in L^X$, given by:

$$\mathcal{U}_N(A)=S(\overline{\mathcal{N}}(A),A),\quad \mathcal{U}_{N^*}(A)=T(\neg\overline{\mathcal{N}}^*(A),\neg A),$$

and the L-double measure of roughness of L-DFUApprox $(\mathcal{L}_N,\mathcal{L}_{N^*})$ by

$$\mathcal{L}_N(A)=S(A,\underline{\mathcal{N}}(A)),\quad \mathcal{L}_{N^*}(A)=T(A,\underline{\mathcal{N}}^*(A)).$$

By the above definition, we can denote the double measures of roughness of L-DFLApprox and L-DFUApprox by the following mapping:

$$\mathcal{U}_N,\mathcal{U}_{N^*}:L^X\longrightarrow L \text{ and } \mathcal{L}_N,\mathcal{L}_{N^*}:L^X\longrightarrow L,$$

respectively.
In the following corollary, we give some properties of the L-double operator of L-DFLApprox and L-DFUApprox $\mathcal{L}_N,\mathcal{L}_{N^*}:L^X\longrightarrow L$, and $\mathcal{U}_N,\mathcal{U}_{N^*}:L^X\longrightarrow L$, respectively.

Corollary 3.14. *An L-double measure of roughness of L-DFLApprox* $\mathcal{L}_N,\mathcal{L}_{N^*}:L^X\longrightarrow L$ *satisfies the next properties:*

(1) *If L is st-s, then*
   (i) $\mathcal{L}_N(\underline{\top})=\top_L$, *and* ($ii$) $\mathcal{L}_{N^*}(\underline{\top})=\bot_L$,
(2) (i) $\mathcal{L}_N(\bigvee_{i\in I}A_i)\geq\bigwedge_{i\in I}\mathcal{L}_N(A_i)$, *and* ($ii$) $\mathcal{L}_{N^*}(\bigvee_{i\in I}A_i)\leq\bigvee_{i\in I}\mathcal{L}_{N^*}(A_i)$.

Proof. (1) If L is st-s, then

(i) $\mathcal{L}_N(\underline{\top})=S(\underline{\top},\underline{\mathcal{N}}(\underline{\top}))=S(\underline{\top},\underline{\top})=\top_L$.
(ii) $\mathcal{L}_{N^*}(\underline{\top})=T(\underline{\top},\underline{\mathcal{N}}^*(\underline{\top}))=T(\underline{\top},\underline{\bot})=\bot_L$.

(2) (i) $\mathcal{L}_N(\bigvee_{i\in I}A_i)=S(\bigvee_{i\in I}A_i,\underline{\mathcal{N}}(\bigvee_{i\in I}A_i))$
$\geq S(\bigvee_{i\in I}A_i,\bigvee_{i\in I}\underline{\mathcal{N}}(A_i))$ (by **Theorem 3.12(3)**)
$\geq\bigwedge_{i\in I}S(A_i,\underline{\mathcal{N}}(A_i))$(by **Lemma 2.1 (8)**)
$=\bigwedge_{i\in I}\mathcal{L}_{N^*}(A_i)$.

(ii) $\mathcal{L}_{N^*}(\bigvee_{i\in I}A_i)=T(\bigvee_{i\in I}A_i,\underline{\mathcal{N}}^*(\bigvee_{i\in I}A_i))$
$\leq T(\bigvee_{i\in I}A_i,\bigwedge_{i\in I}\underline{\mathcal{N}}^*(A_i))$ (by **Theorem 3.12(5)**)
$\leq T(\bigvee_{i\in I}A_i,\underline{\mathcal{N}}^*(A_i))$
$=\bigvee_{i\in I}T(A_i,\underline{\mathcal{N}}^*(A_i))$(by **Lemma 2.1 (7)**)
$=\bigvee_{i\in I}\mathcal{L}_{N^*}(A_i)$.

□

Corollary 3.15. *An L-double measure of roughness of L-DFUApprox* $\mathcal{U}_{N}, \mathcal{U}_{N^*} : L^X \longrightarrow L$ *satisfies the next properties:*

(1) $(i)\ \mathcal{U}_{N}(\underline{\bot}) = \top_L$, *and* $(ii)\ \mathcal{U}_{N^*}(\underline{\bot}) = \bot_L$,

(2) $(i)\ \mathcal{U}_{N}(\bigwedge_{i\in I} A_i) \geq \bigwedge_{i\in I} \mathcal{U}_{N}(A_i)$, *and* $(ii)\ \mathcal{U}_{N^*}(\bigwedge_{i\in I} A_i) \leq \bigvee_{i\in I} \mathcal{U}_{N^*}(A_i)$.

Proof. (1) $(i)\ \mathcal{U}_{N}(\underline{\bot}) = S(\overline{\mathcal{N}}(\underline{\bot}), \underline{\bot}) = S(\underline{\bot}, \underline{\bot}) = \top_L$.

$(ii)\ \mathcal{U}_{N^*}(\underline{\bot}) = T(\neg\overline{\mathcal{N}}^*(\underline{\bot}), \neg\underline{\bot}) = T(\neg\underline{\top}, \underline{\top}) = T(\underline{\bot}, \underline{\top}) = \bot_L$.

(2) (i)

$$\begin{aligned}\mathcal{U}_{N}(\bigwedge_{i\in I} A_i) &= S(\overline{\mathcal{N}}(\bigwedge_{i\in I} A_i), \bigwedge_{i\in I} A_i)\\ &\geq S(\bigwedge_{i\in I} \overline{\mathcal{N}}(A_i), \bigwedge_{i\in I} A_i) \text{ (by \textbf{Theorem 3.13(2)})}\\ &\geq \bigwedge_{i\in I} S(\overline{\mathcal{N}}(A_i), A_i) \text{ (by \textbf{Lemma 2.1 (8)})}\\ &= \bigwedge_{i\in I} \mathcal{U}_{N}(A_i).\end{aligned}$$

(ii)

$$\begin{aligned}\mathcal{U}_{N^*}(\bigwedge_{i\in I} A_i) &= T(\neg\overline{\mathcal{N}}^*(\bigwedge_{i\in I} A_i), \neg\bigwedge_{i\in I} A_i)\\ &\leq T(\neg\bigvee_{i\in I} \overline{\mathcal{N}}^*(A_i), \bigvee_{i\in I} \neg A_i) \text{ (by \textbf{Theorem 3.13(4)})}\\ &= T(\bigwedge_{i\in I} \neg\overline{\mathcal{N}}^*(A_i), \bigvee_{i\in I} \neg A_i) \text{ (by \textbf{Proposition 2.2(3)})}\\ &\leq T(\neg\overline{\mathcal{N}}^*(A_i), \bigvee_{i\in I} \neg A_i)\\ &= \bigvee_{i\in I} T(\neg\overline{\mathcal{N}}^*(A_i), \neg A_i) \text{ (by \textbf{Lemma 2.3(5)})}\\ &= \bigvee_{i\in I} \mathcal{U}_{N^*}(A_i).\end{aligned}$$

□

Corollary 3.16. *For an L-double measures of roughness of L-DFLApprox and L-DFUApprox* $\mathcal{L}_{N}, \mathcal{L}_{N^*}, \mathcal{U}_{N}, \mathcal{U}_{N^*} : L^X \longrightarrow L$, *we have*

$(i)\ \mathcal{L}_{N}(\neg D) = \mathcal{U}_{N}(D)$ *and* $\mathcal{U}_{N}(\neg D) = \mathcal{L}_{N}(D)$,

$(ii)\ \mathcal{L}_{N^*}(\neg D) = \mathcal{U}_{N^*}(D)$ *and* $\mathcal{U}_{N^*}(\neg D) = \mathcal{L}_{N^*}(D)$, *for all* $D \in L^X$.

Proof. (i)

$$\begin{aligned}\mathcal{L}_{N}(\neg D) &= S(\neg D, \underline{\mathcal{N}}(\neg D))\\ &= S(\neg D, \neg\overline{\mathcal{N}}(D))\\ &= S(\overline{\mathcal{N}}(D), D) \text{ (by \textbf{Proposition 2.2(2)})}\\ &= \mathcal{U}_{N}(D).\end{aligned}$$

Now, we prove the second part,

$$\begin{aligned}\mathcal{U}_{N}(\neg D) &= S(\overline{\mathcal{N}}(\neg D), \neg D)\\ &= S(\neg\underline{\mathcal{N}}(D), \neg D)\\ &= S(D, \underline{\mathcal{N}}(D)) \text{ (by \textbf{Proposition 2.2(2)})}\\ &= \mathcal{L}_{N}(D).\end{aligned}$$

(ii) $\mathcal{L}_{N^*}(\neg D) = T(\neg D, \underline{\mathcal{N}}^*(\neg D))$
$= T(\neg D, \neg\overline{\mathcal{N}}^*(D))$
$= T(\neg\overline{\mathcal{N}}^*(D), \neg D)$
$= \mathcal{U}_{N^*}(D).$

Now, we prove the second part, $\mathcal{U}_{N^*}(\neg D) = T(\neg\overline{\mathcal{N}}^*(\neg D), \neg(\neg D))$
$= T(\underline{\mathcal{N}}^*(D), D)$
$= T(D, \underline{\mathcal{N}}^*(D))$
$= \mathcal{L}_{N^*}(D).$

□

4. Special *L*-double fuzzy generalized neighborhood systems and related *L*-double rough approximation operators

Some special *L*-DFGN systems and related *L*-DRApprox operators will be proposed in this section. Also, we shall show that different *L*-DRApprox operators correspond to different modal logic systems, respectively.

4.1. *Serial L-DFGN systems*

The concept of serial *L*-DFGN system operators will be introduced and we will discuss their related *L*-DRApprox operators

Definition 4.1. *An L-DFGN system operator* $(\mathcal{N}, \mathcal{N}^*)$ *is called a serial, if*

(SE) $\mathcal{N}(x)(A) \leq \bigvee_{y\in X} A(y)$, *and* (SE*) $\mathcal{N}^*(x)(A) \geq \bigwedge_{y\in X}(\neg A(y))$,

where $x \in X$, $A \in L^X$.

Remark 4.2. *Every serial L-FGN system operator* $\mathcal{N} : X \longrightarrow L^{L^X}$ **[56]**, *can be identified with a serial L-DFGN system operator of the form* $(\mathcal{N}, \neg\mathcal{N})$. *Thus, the serial condition in L-DFGN system operator is an extension of the corresponding condition in L-FGN system operator. Moreover, it is easily observed that: for an L-double relation* $(\mathcal{R}, \mathcal{R}^*)$ **[1]**, $(\mathcal{N}_{\mathcal{R}}, \mathcal{N}^*_{\mathcal{R}^*})$ *is serial iff* $(\mathcal{R}, \mathcal{R}^*)$ *is serial. Where* $(\mathcal{N}_{\mathcal{R}}, \mathcal{N}^*_{\mathcal{R}^*})$ *is defined in* **Lemma 3.11**.

Proposition 4.3. *Let* $(L, \leq, \otimes)$ *be st-s. Then* $(\mathcal{N}, \mathcal{N}^*)$ *is serial iff*

(i) $\underline{\mathcal{N}}(\underline{\perp}) = \underline{\perp}$, *and* $\overline{\mathcal{N}}(\underline{\top}) = \underline{\top}$,

(ii) $\underline{\mathcal{N}}^*(\underline{\perp}) = \underline{\top}$, *and* $\overline{\mathcal{N}}^*(\underline{\top}) = \underline{\perp}$.

Proof. Let $(\mathcal{N}, \mathcal{N}^*)$ be a serial *L*-DFGN system operator , then:

(i) By Propositions **4.2** and **4.3** of **[56]**, we have:

$$\mathcal{N}(x)(A) \leq \bigvee_{y\in X} A(y) \Leftrightarrow \underline{\mathcal{N}}(\underline{\perp}) = \underline{\perp}, \text{ and } \overline{\mathcal{N}}(\underline{\top}) = \underline{\top}.$$

(ii) Let $\mathcal{N}^*(x)(A) \geq \bigwedge_{y\in X}(\neg A(y))$, then for any $x \in X$, we have that

$$\underline{\mathcal{N}}^*(x)(\perp) = \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \neg\underline{\perp}))$$
$$= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \underline{\top}))$$

$$= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to \bigvee_{y\in X} (K(y) \otimes \top_L))$$

$$= \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to ([\bigvee_{y\in X} K(y)] \otimes \top_L)) \text{ (by **Lemma 2.1 (7)**)}$$

$$\overset{SE^*}{\geq} \bigwedge_{K\in L^X} ([\bigvee_{y\in X} K(y)] \to ([\bigvee_{y\in X} K(y)] \otimes \top_L))$$

$$\geq \top_L \text{ (by **Lemma 2.1 (1)**)}.$$

Hence, $\underline{\mathcal{N}}^*(\underline{\perp}) = \underline{\top}$.

We prove the second part,

$$\overline{\mathcal{N}}^*(x)(\top) = \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg\underline{\top}))$$

$$= \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \underline{\perp}))$$

$$= \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes \bigwedge_{y\in X} (K(y) \to \perp_L))$$

$$\overset{SE^*}{\leq} \bigvee_{K\in L^X} [(\bigvee_{y\in X} K(y)) \otimes ((\bigvee_{y\in X} K(y)) \to \perp_L)] \text{(by **Lemma 2.1 (4)**)}$$

$$\leq \perp_L \text{ (by **Lemma 2.1 (1)**)}.$$

So, $\overline{\mathcal{N}}^*(\underline{\top}) = \underline{\perp}$.

Conversely, suppose that $\underline{\mathcal{N}}^*(x)(\perp) = \top$. Then, for any $x \in X$, we get

$$\underline{\mathcal{N}}^*(x)(\perp) = \bigwedge_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \neg\underline{\perp})) = \top.$$

It follows that, for any $K \in L^X$,

$$\neg\mathcal{N}^*(x)(K) \to T(K, \neg\underline{\perp}) \geq \underline{\top} \Rightarrow \underline{\top} \otimes \neg\mathcal{N}^*(x)(K) \leq T(K, \neg\underline{\perp})$$

$$\Rightarrow \neg\mathcal{N}^*(x)(K) \leq T(K, \underline{\top})$$

$$\Rightarrow \neg\mathcal{N}^*(x)(K) \leq \bigvee_{y\in X} (K(y) \otimes \top_L)$$

$$\Rightarrow \neg\mathcal{N}^*(x)(K) \leq (\bigvee_{y\in X} K(y)) \otimes \top_L \text{ (by **Lemma 2.1 (7)**)}$$

$$\Rightarrow \neg\mathcal{N}^*(x)(K) \leq \bigvee_{y\in X} K(y)$$

$$\Rightarrow \mathcal{N}^*(x)(K) \geq \neg \bigvee_{y\in X} K(y) \text{ (by **Proposition 2.2(4)**)}$$

$$\Rightarrow \mathcal{N}^*(x)(K) \geq \bigwedge_{y\in X} \neg K(y) \text{ (by **Proposition 2.2(3)**)}.$$

And, suppose that $\overline{\mathcal{N}}(x)(\top) = \perp$. Then, for any $x \in X$, we get

$\overline{\mathcal{N}}^*(x)(\top) = \bigvee_{K\in L^X} (\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg\underline{\top})) = \perp$. It follows that for any $K \in L^X$,

$$\neg\mathcal{N}^*(x)(K) \otimes S(K, \neg\underline{\top}) \leq \perp \Rightarrow \neg\mathcal{N}^*(x)(K) \leq S(K, \underline{\perp}) \to \perp$$

$$\Rightarrow \neg\mathcal{N}^*(x)(K) \leq \bigwedge_{y\in X} (K(y) \to \perp) \to \perp$$

$$\Rightarrow \neg\mathcal{N}^*(x)(K) \leq ((\bigvee_{y\in X} K(y)) \to \perp) \to \perp \; (by\, \textbf{Lemma2.1(3)})$$

$$\Rightarrow \neg\mathcal{N}^*(x)(K) \leq \bigvee_{y\in X} K(y)$$

$$\Rightarrow \mathcal{N}^*(x)(K) \geq \neg \bigvee_{y \in X} K(y) \text{ (by **Proposition 2.2(4)**)}$$

$$\Rightarrow \mathcal{N}^*(x)(K) \geq \bigwedge_{y \in X} \neg K(y) \text{ (by **Proposition 2.2(3)**)}.$$

□

4.2. *Reflexive L-DFGN systems*

The concept of reflexive L-DFGN system operators will be introduced and we will discuss their related L-DRApprox operators.

Definition 4.4. *An L-DFGN system operator* $(\mathcal{N}, \mathcal{N}^*)$ *is called a reflexive, if*
(*RE*) $\mathcal{N}(x)(A) \leq A(x)$, *and* (*RE**) $\mathcal{N}^*(x)(A) \geq \neg A(x)$,

where $x \in X$, $A \in L^X$.

Remark 4.5. *Every reflexive L-FGN system operator* $\mathcal{N} : X \longrightarrow L^{L^X}$ **[56]**, *can be identified with a reflexive L-DFGN system operator of the form* $(\mathcal{N}, \neg\mathcal{N})$. *Thus, the reflexive condition in L-DFGN system operator is an extension of the corresponding condition in L-FGN system operator. Moreover, it is easily observed that: for an L-double relation* $(\mathcal{R}, \mathcal{R}^*)$ **[1]**, $(\mathcal{N}_{\mathcal{R}}, \mathcal{N}^*_{\mathcal{R}^*})$ *is reflexive iff* $(\mathcal{R}, \mathcal{R}^*)$ *is reflexive. Where* $(\mathcal{N}_{\mathcal{R}}, \mathcal{N}^*_{\mathcal{R}^*})$ *is defined in* **Lemma 3.11**.

Proposition 4.6. *Let* $(\mathcal{N}, \mathcal{N}^*)$ *be an L-DFGN system operator on X. If* $(\mathcal{N}, \mathcal{N}^*)$ *is reflexive, then for each* $A \in L^X$.

(*i*) $\underline{\mathcal{N}}(x)(A) \leq A(x)$, *and* (*ii*) $\underline{\mathcal{N}^*}(x)(A) \geq \neg A(x)$,
and the opposite is true if $(L, \leq, \otimes)$ *is st-s.*

Proof. Let $(\mathcal{N}, \mathcal{N}^*)$ be reflexive, then:

(*i*) By [**[56]**, Proposition **4.5**], we have:

$$\mathcal{N}(x)(A) \leq A(x) \Leftrightarrow \underline{\mathcal{N}}(x)(A) \leq A(x).$$

(*ii*) Let $\mathcal{N}^*(x)(A) \geq \neg A(x)$, then

$$\begin{aligned}
\underline{\mathcal{N}^*}(x)(A) &= \bigwedge_{K \in L^X} [\neg \mathcal{N}^*(x)(K) \rightarrow T(K, \neg A)] \\
&= \bigwedge_{K \in L^X} [\neg \mathcal{N}^*(x)(K) \rightarrow \bigvee_{x \in X} (K(x) \otimes \neg A(x))] \\
&\geq \bigwedge_{K \in L^X} [\neg \mathcal{N}^*(x)(K) \rightarrow (K(x) \otimes \neg A(x))] \\
&\overset{RE^*}{\geq} \bigwedge_{K \in L^X} [K(x) \rightarrow (K(x) \otimes \neg A(x))] \\
&\geq \neg A(x) \text{ (by **Lemma 2.1 (1)**)}.
\end{aligned}$$

Conversely, suppose that $(L, \leq, \otimes)$ is st-s and $\underline{\mathcal{N}^*}(x)(A) \geq \neg A(x)$ for each $A \in L^X$. For any $x \in X$, we get

$$\begin{aligned}
&\bigwedge_{K \in L^X} [\neg \mathcal{N}^*(x)(K) \rightarrow T(K, \neg A)] \geq \neg A(x) \\
\Rightarrow &\bigwedge_{K \in L^X} [\neg T(K, \neg A) \rightarrow \mathcal{N}^*(x)(K)] \geq \neg A(x) \\
\Rightarrow &\bigwedge_{K \in L^X} [S(K, A) \rightarrow \mathcal{N}^*(x)(K)] \geq \neg A(x) \\
\Rightarrow &S(K, A) \rightarrow \mathcal{N}^*(x)(K) \geq \neg A(x).
\end{aligned}$$

Taking $K = A$, we get

$$S(A,A) \to \mathcal{N}^*(x)(A) \geq \neg A(x)$$
$$\Rightarrow \top_L \to \mathcal{N}^*(x)(K) \geq \neg A(x)$$
$$\Rightarrow \mathcal{N}^*(x)(K) \geq \neg A(x).$$

□

Proposition 4.7. *Let $(\mathcal{N},\mathcal{N}^*)$ be an L-DFGN system operator on X. Then $(\mathcal{N},\mathcal{N}^*)$ is reflexive iff*

(i) $\overline{\mathcal{N}}(x)(A) \geq A(x)$, and *(ii) $\overline{\mathcal{N}}^*(x)(A) \leq \neg A(x)$ for each $A \in L^X$.*

Proof. Let $(\mathcal{N},\mathcal{N}^*)$ is reflexive, then:

(*i*) By [[**56**], Proposition **4.6**], we have that

$$\mathcal{N}(x)(A) \leq A(x) \Leftrightarrow \overline{\mathcal{N}}(x)(A) \geq A(x).$$

(*ii*) Let $\mathcal{N}^*(x)(A) \geq \neg A(x)$, then

$$\begin{aligned}\overline{\mathcal{N}}^*(x)(A) &= \bigvee_{K\in L^X} [\neg\mathcal{N}^*(x)(K) \otimes S(K,\neg A)] \\ &= \bigvee_{K\in L^X} [\neg\mathcal{N}^*(x)(K) \otimes \textstyle\bigwedge_{x\in X}(K(x) \to \neg A(x))] \\ &\leq \bigvee_{K\in L^X} [\neg\mathcal{N}^*(x)(K) \otimes (K(x) \to \neg A(x))] \\ &\overset{RE^*}{\leq} \bigvee_{K\in L^X} [K(x) \otimes (K(x) \to \neg A(x))] \leq \neg A(x) \text{ (by \textbf{Lemma 2.1 (1)})}.\end{aligned}$$

Conversely, suppose that $\overline{\mathcal{N}}^*(x)(A) \leq \neg A(x)$, for $A \in L^X$. Then for any $x \in X$, we get

$$\overline{\mathcal{N}}^*(x)(\neg A) \leq A(x), \text{ i.e., } \neg\overline{\mathcal{N}}^*(x)(\neg A) \geq \neg A(x)$$
$$\Rightarrow \underline{\mathcal{N}}^*(x)(A) \geq \neg A(x)$$

From **Theorems 3.13** and **4.6**, we get $\mathcal{N}^*(x)(A) \geq \neg A(x)$.

□

4.3. *Transitive L-DFGN systems*

The concept of transitive L-DFGN system operators will be introduced and we will establish their related L-DRApprox operators.

Definition 4.8. *An L-DFGN system operator $(\mathcal{N},\mathcal{N}^*)$ is called a transitive, if*

$$(TR)\quad \mathcal{N}(x)(A) \leq \bigvee_{B\in L^X} \{\mathcal{N}_x(B) \otimes \bigwedge_{y\in X} (B(y) \to \bigvee_{B_y\in L^X} (\mathcal{N}_y(B_y) \otimes S(B_y,A)))\}, \text{ and}$$

$$(TR^*)\quad \mathcal{N}^*(x)(A) \geq \bigwedge_{B\in L^X} \{\neg\mathcal{N}^*(x)(B) \to \bigvee_{y\in X} (B(y) \otimes \bigwedge_{B_y\in L^X} (\neg\mathcal{N}^*(y)(B_y) \to T(B_y,\neg A)))\},$$

where $x \in X$, $A \in L^X$.

Remark 4.9. *Every transitive L-FGN system operator $\mathcal{N} : X \longrightarrow L^{L^X}$* [**56**]*, can be identified with a reflexive L-DFGN system operator of the form $(\mathcal{N},\neg\mathcal{N})$. Thus, the transitive condition in L-DFGN system operator is an extension of the corresponding condition in L-FGN system operator. Moreover, it is easily observed that: for an L-double relation $(\mathcal{R},\mathcal{R}^*)$* [**1**]*, $(\mathcal{N}_{\mathcal{R}},\mathcal{N}^*_{\mathcal{R}^*})$ is transitive iff $(\mathcal{R},\mathcal{R}^*)$ is transitive Where $(\mathcal{N}_{\mathcal{R}},\mathcal{N}^*_{\mathcal{R}^*})$ is defined in* **Lemma 3.11**.

Proposition 4.10. *Let $(\mathcal{N},\mathcal{N}^*)$ be an L-DFGN system operator on X. If $(\mathcal{N},\mathcal{N}^*)$ is transitive, then*

(i) $\underline{\mathcal{N}}(x)(A) \leq \underline{\mathcal{N}}(x)(\underline{\mathcal{N}}(A))$, *and* *(ii)* $\underline{\mathcal{N}}^*(x)(A) \geq \underline{\mathcal{N}}^*(x)(\neg\underline{\mathcal{N}}^*(A))$ *for each* $A \in L^X$,

and the opposite is true if $(L, \leq, \otimes)$ *is st-s.*

Proof. Let $(\mathcal{N},\mathcal{N}^*)$ is transitive, then:

(*i*) By [**[56]**, Proposition **4.5**], we have:
$\mathcal{N}(x)(A) \leq \bigvee_{B\in L^X}\{\mathcal{N}_x(B) \otimes \bigwedge_{y\in X}(B(y) \to \bigvee_{B_y\in L^X}(\mathcal{N}_y(B_y) \otimes S(B_y, A)))\} \Leftrightarrow \underline{\mathcal{N}}(x)(A) \leq \underline{\mathcal{N}}(x)(\underline{\mathcal{N}}(A))$.

(*ii*) Let $\mathcal{N}^*(x)(A) \geq \bigwedge_{B\in L^X}\{\neg\mathcal{N}^*(x)(B) \to \bigvee_{y\in X}(B(y) \otimes \bigwedge_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \to T(B_y, \neg A)))\}$, then

$$\underline{\mathcal{N}}^*(x)(A) = \bigwedge_{K\in L^X}\{\neg\mathcal{N}^*(x)(K) \to T(K, \neg A)\}$$
$$\overset{TR^*}{\geq} \bigwedge_{K\in L^X}\{\neg(\bigwedge_{B\in L^X}[\neg\mathcal{N}^*(x)(B) \to \bigvee_{y\in X}(B(y) \otimes \bigwedge_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \to T(B_y, \neg K)))]) \to T(K, \neg A)\}$$
$$= \bigwedge_{K\in L^X}\{\bigvee_{B\in L^X}\neg[\neg\mathcal{N}^*(x)(B) \to \bigvee_{y\in X}(B(y) \otimes \bigwedge_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \to T(B_y, \neg K)))] \to T(K, \neg A)\} \text{(by } \textbf{Proposition 2.2(3)}\text{)}$$
$$= \bigwedge_{K,B\in L^X}\{\neg[\neg\mathcal{N}^*(x)(B) \to \bigvee_{y\in X}(B(y) \otimes \bigwedge_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \to T(B_y, \neg K)))] \to T(K, \neg A)\} \text{ (by } \textbf{Lemma 2.1 (3)}\text{)}$$
$$= \bigwedge_{K,B\in L^X}[(\neg\mathcal{N}^*(x)(B) \otimes \neg(\bigvee_{y\in X}(B(y) \otimes \bigwedge_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \to T(B_y, \neg K))))) \to T(K, \neg A)] \text{(by } \textbf{Proposition 2.2(1)}\text{)}$$
$$= \bigwedge_{K,B\in L^X}[(\neg\mathcal{N}^*(x)(B) \otimes \bigwedge_{y\in X}(B(y) \to \neg(\bigwedge_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \to T(B_y, \neg K))))) \to T(K, \neg A)] \text{(by } \textbf{Proposition 2.2(1)}\text{)}$$
$$= \bigwedge_{K,B\in L^X}[(\neg\mathcal{N}^*(x)(B) \otimes \bigwedge_{y\in X}(B(y) \to \bigvee_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \otimes \neg T(B_y, \neg K)))) \to T(K, \neg A)]$$
$$= \bigwedge_{K,B\in L^X}[(\neg\mathcal{N}^*(x)(B) \otimes \bigwedge_{y\in X}(B(y) \to \bigvee_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \otimes S(B_y, K)))) \to T(K, \neg A)]$$
$$= \bigwedge_{K,B\in L^X}\neg[(\neg\mathcal{N}^*(x)(B) \otimes \bigwedge_{y\in X}(B(y) \to \bigvee_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \otimes S(B_y, K)))) \otimes \neg T(K, \neg A)] \text{ (by } \textbf{Proposition 2.2 (1)}\text{)}$$
$$= \bigwedge_{K,B\in L^X}\neg[\neg\mathcal{N}^*(x)(B) \otimes \bigwedge_{y\in X}(B(y) \to \bigvee_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \otimes S(B_y, K))) \otimes S(K, A)]$$
$$\geq \bigwedge_{K,B\in L^X}\neg(\neg\mathcal{N}^*(x)(B) \otimes \bigwedge_{y\in X}(S(K, A) \otimes (B(y) \to \bigvee_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \otimes S(B_y, K))))) \text{ (by } \textbf{Lemma 2.1 (4)}\text{)}$$
$$\geq \bigwedge_{K,B\in L^X}\neg(\neg\mathcal{N}^*(x)(B) \otimes \bigwedge_{y\in X}(B(y) \to (S(K, A) \otimes \bigvee_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \otimes S(B_y, K))))) \text{ (by } \textbf{Lemma 2.1 (5)}\text{)}$$
$$= \bigwedge_{K,B\in L^X}\neg(\neg\mathcal{N}^*(x)(B) \otimes \bigwedge_{y\in X}(B(y) \to \bigvee_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \otimes S(B_y, K) \otimes S(K, A)))) \text{ (by } \textbf{Lemma 2.1 (7)}\text{)}$$
$$\geq \bigwedge_{B\in L^X}\neg(\neg\mathcal{N}^*(x)(B) \otimes \bigwedge_{y\in X}(B(y) \to \bigvee_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \otimes S(B_y, A)))) \text{ (}by\text{ } \textbf{Lemma2.3(2)}\text{)}$$
$$= \bigwedge_{B\in L^X}(\neg\mathcal{N}^*(x)(B) \to \bigvee_{y\in X}(B(y) \otimes \bigwedge_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \to \neg S(B_y, A)))) \text{ (by } \textbf{Proposition 2.2 (1)}\text{)}$$
$$= \bigwedge_{B\in L^X}(\neg\mathcal{N}^*(x)(B) \to \bigvee_{y\in X}(B(y) \otimes \bigwedge_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y) \to T(B_y, \neg A))))$$
$$= \bigwedge_{B\in L^X}(\neg\mathcal{N}^*(x)(B) \to T(B, \underline{\mathcal{N}}^*(A)))$$
$$= \underline{\mathcal{N}}^*(x)(\neg\underline{\mathcal{N}}^*(A)).$$

Conversely, Let $\underline{\mathcal{N}}^*(A) \geq \underline{\mathcal{N}}^*(\neg\underline{\mathcal{N}}^*(A))$ for each $A \in L^X$. Then for any $x \in X$, we get $\underline{\mathcal{N}}^*(x)(A) \geq \underline{\mathcal{N}}^*(x)(\neg\underline{\mathcal{N}}^*(A))$ and this lead to

$\bigwedge_{K\in L^X}\{\neg\mathcal{N}^*(x)(K)\to T(K,\neg A)\}\geq\bigwedge_{B\in L^X}\{\neg\mathcal{N}^*(x)(B)\to T(B,\underline{\mathcal{N}}^*(A))\}$. So, we find:

$$\neg\mathcal{N}^*(x)(K)\to T(K,\neg A)\geq\bigwedge_{B\in L^X}\{\neg\mathcal{N}^*(x)(B)\to T(B,\underline{\mathcal{N}}^*(A))\}$$
$$i.e.,\ \neg(\neg\mathcal{N}^*(x)(K)\otimes S(K,A))\geq\bigwedge_{B\in L^X}\{\neg\mathcal{N}^*(x)(B)\to T(B,\underline{\mathcal{N}}^*(A))\}.$$

taking $K=A$, we get

$$\neg(\neg\mathcal{N}^*(x)(A)\otimes\underline{\top})\geq\bigwedge_{B\in L^X}\{\neg\mathcal{N}^*(x)(B)\to T(B,\underline{\mathcal{N}}^*(A))\}$$
$$\Rightarrow\neg(\neg\mathcal{N}^*(x)(A))\geq\bigwedge_{B\in L^X}\{\neg\mathcal{N}^*(x)(B)\to T(B,\underline{\mathcal{N}}^*(A))\}$$
$$\Rightarrow\mathcal{N}^*(x)(A)\geq\bigwedge_{B\in L^X}\{\neg\mathcal{N}^*(x)(B)\to\bigvee_{y\in X}(B(y)\otimes\underline{\mathcal{N}}^*(A))\}$$
$$\geq\bigwedge_{B\in L^X}\{\neg\mathcal{N}^*(x)(B)\to\bigvee_{y\in X}(B(y)\otimes\bigwedge_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y)\to T(B_y,\neg A))))\}.$$

From (i) and (ii), we find $(\mathcal{N},\mathcal{N}^*)$ is a transitive.
□

Proposition 4.11. *Let $(\mathcal{N},\mathcal{N}^*)$ be an L-DFGN system operator. Then $(\mathcal{N},\mathcal{N}^*)$ is a transitive iff*

(i) $\overline{\mathcal{N}}(A)\geq\overline{\mathcal{N}}(\overline{\mathcal{N}}(A))$, *and* ($ii$) $\overline{\mathcal{N}}^*(A)\leq\overline{\mathcal{N}}^*(\neg\overline{\mathcal{N}}^*(A))$ *for each $A\in L^X$.*

Proof. Suppose that $(\mathcal{N},\mathcal{N}^*)$ is a transitive, then for any $A\in L^X$ and for any $x\in X$,

(i) By [**[56]**, Proposition **4.12**], we have that for any $A\in L^X$,

$\mathcal{N}(x)(A)\leq\bigvee_{B\in L^X}\{\mathcal{N}_x(B)\otimes\bigwedge_{y\in X}(B(y)\to\bigvee_{B_y\in L^X}(\mathcal{N}_y(B_y)\otimes S(B_y,A)))\}\Leftrightarrow\overline{\mathcal{N}}(x)(A)\geq\overline{\mathcal{N}}(x)(\overline{\mathcal{N}}(A))$.

(ii) Let $\mathcal{N}^*(x)(A)\geq\bigwedge_{B\in L^X}\{\neg\mathcal{N}^*(x)(B)\to\bigvee_{y\in X}(B(y)\otimes\bigwedge_{B_y\in L^X}(\neg\mathcal{N}^*(y)(B_y)\to T(B_y,\neg A)))\}$, then

$\overline{\mathcal{N}}^*(A)=\neg(\underline{\mathcal{N}}^*(\neg A))$ (by **Theorem 3.13**)
$\leq\neg\underline{\mathcal{N}}^*(\neg\underline{\mathcal{N}}^*(\neg A))$ (by **Proposition 4.10**)
$=\neg\underline{\mathcal{N}}^*(\neg(\neg\overline{\mathcal{N}}^*(A)))$
$=\overline{\mathcal{N}}^*(\neg\overline{\mathcal{N}}^*(A))$.

Conversely, it follows by **Theorem 3.13** and **Propositions 4.10**.

From (i) and (ii), we find $(\mathcal{N},\mathcal{N}^*)$ is a transitive.
□

4.4. *Unary L-DFGN systems*

The concept of unary L-DFGN system operators will be introduced and we will establish their related L-DRApprox operators.

Definition 4.12. *An L-DFGN system operator $(\mathcal{N},\mathcal{N}^*)$ is called unary, if*

(UN) $\mathcal{N}(x)(A)\otimes\mathcal{N}(x)(B)\leq\bigvee_{K\in L^X}(\mathcal{N}(x)(K)\otimes S(K,A\otimes B))$,

(UN^*) $\mathcal{N}^*(x)(A)\oplus\mathcal{N}^*(x)(B)\geq\bigwedge_{K\in L^X}(\neg\mathcal{N}^*(x)(K)\to T(K,\neg(A\otimes B))$,

where $x\in X$ and $A,B\in L^X$.

Remark 4.13. *Every unary L-FGN system operator* $\mathcal{N} : X \longrightarrow L^{L^X}$ **[56]**, *can be identified with a unary L-DFGN system operator of the form* $(\mathcal{N}, \neg\mathcal{N})$.

Proposition 4.14. *Let* $(\mathcal{N}, \mathcal{N}^*)$ *be an L-DFGN system operator on X. Then if* $(\mathcal{N}, \mathcal{N}^*)$ *is unary, then*

(*i*) $\underline{\mathcal{N}}(x)(A) \otimes \underline{\mathcal{N}}(x)(B) \leq \underline{\mathcal{N}}(x)(A \otimes B)$, *and*

(*ii*) $\underline{\mathcal{N}}^*(x)(A) \oplus \underline{\mathcal{N}}^*(x)(B) \geq \underline{\mathcal{N}}^*(x)(A \otimes B)$ *for each* $A, B \in L^X$.

The opposite is true if L is st-s.

Proof. Suppose that $(\mathcal{N}, \mathcal{N}^*)$ is a unary. Then for any $x \in X$ and $A, B \in L^X$,

(*i*) By [**[56]**, Proposition **4.8**], we get
$\mathcal{N}(x)(A) \otimes \mathcal{N}(x)(B) \leq \bigvee_{K \in L^X} (\mathcal{N}(x)(K) \otimes S(K, A \otimes B)) \Leftrightarrow \underline{\mathcal{N}}(x)(A) \otimes \underline{\mathcal{N}}(x)(B) \leq \underline{\mathcal{N}}(x)(A \otimes B)$ whenever L is st-s.

(*ii*) Suppose that $\mathcal{N}^*(x)(A) \oplus \mathcal{N}^*(x)(B) \geq \bigwedge_{K \in L^X} (\neg\mathcal{N}^*(x)(K) \to T(K, \neg(A \otimes B))$, then

$$\begin{aligned}
\underline{\mathcal{N}}^*(A) \oplus \underline{\mathcal{N}}^*(B) &= \neg(\neg\underline{\mathcal{N}}^*(A) \otimes \neg\underline{\mathcal{N}}^*(B)) \\
&= \neg(\neg \bigwedge_{K \in L^X} [\neg\mathcal{N}^*(x)(K) \to T(K, \neg A)] \otimes \neg \bigwedge_{V \in L^X} [\neg\mathcal{N}^*(x)(V) \to T(V, \neg B)] \\
&= \neg(\bigvee_{K \in L^X} [\neg\mathcal{N}^*(x)(K) \otimes S(K, A)] \otimes \bigvee_{V \in L^X} [\neg\mathcal{N}^*(x)(V) \otimes S(V, B)] \\
&= \neg(\bigvee_{K, V \in L^X} [\neg\mathcal{N}^*(x)(K) \otimes \neg\mathcal{N}^*(x)(V) \otimes S(K, A) \otimes S(V, B)]) \\
&\geq \neg(\bigvee_{K, V \in L^X} [\neg(\mathcal{N}^*(x)(K) \oplus \mathcal{N}^*(x)(V)) \otimes S(K \otimes V, A \otimes B)])(\text{ by } \textbf{Lemma 2.3(3)}) \\
&\overset{UN^*}{\geq} \neg(\bigvee_{K, V \in L^X} \neg(\bigwedge_{U \in L^X} \neg\mathcal{N}^*(x)(U) \to T(U, \neg(K \otimes V))) \otimes S(K \otimes V, A \otimes B)) \\
&= \neg(\bigvee_{K, V \in L^X} (\bigvee_{U \in L^X} \neg\mathcal{N}^*(x)(U) \otimes S(U, K \otimes V)) \otimes S(K \otimes V, A \otimes B)) \\
&= \neg(\bigvee_{K, V \in L^X} \bigvee_{U \in L^X} \neg\mathcal{N}^*(x)(U) \otimes S(U, K \otimes V) \otimes S(K \otimes V, A \otimes B)) \\
&\geq \neg(\bigvee_{U \in L^X} \neg\mathcal{N}^*(x)(U) \otimes S(U, A \otimes B)) \text{ (by } \textbf{Lemma 2.3 (2)}) \\
&= \bigwedge_{U \in L^X} (\neg\mathcal{N}^*(x)(U) \to T(U, \neg(A \otimes B))) \\
&= \underline{\mathcal{N}}^*(A \otimes B).
\end{aligned}$$

Conversely, suppose that L is a st-s (integral) quantale and $\underline{\mathcal{N}}^*(A \otimes B) \leq \underline{\mathcal{N}}^*(A) \oplus \underline{\mathcal{N}}^*(B)$. For any $x \in X$, we get $\underline{\mathcal{N}}^*(A \otimes B) \leq \underline{\mathcal{N}}^*(A) \oplus \underline{\mathcal{N}}^*(B)$. So, it follows:

$$\begin{aligned}
&\bigwedge_{U \in L^X} [\neg\mathcal{N}^*(x)(U) \to T(U, \neg(A \otimes B))] \\
&\leq [\neg\mathcal{N}^*(x)(K) \to T(K, \neg(A)] \oplus [\neg\mathcal{N}^*(x)(V) \to T(V, \neg(B)] \\
&= \neg(\neg[\neg\mathcal{N}^*(x)(K) \to T(K, \neg(A))] \otimes \neg[\neg\mathcal{N}^*(x)(V) \to T(V, \neg(B))]) \\
&= \neg([\neg\mathcal{N}^*(x)(K) \otimes S(K, A)] \otimes [\neg\mathcal{N}^*(x)(V) \otimes S(V, B)]).
\end{aligned}$$

By taking $K = A$ and $V = B$ in the above inequality we find

$$\begin{aligned}
&= \neg[(\neg\mathcal{N}^*(x)(A) \otimes \top_L) \otimes (\neg\mathcal{N}^*(x)(B) \otimes \top_L)] \\
&= \neg[\neg\mathcal{N}^*(x)(A) \otimes \neg\mathcal{N}^*(x)(B)]
\end{aligned}$$

$= \mathcal{N}^*(x)(A) \oplus \mathcal{N}^*(x)(B).$

From (i) and (ii), the proof completed.

□

The relationship between the double measures of roughness of their L-DFLApprox and an unary L-DFGN system operators is given in the next lemma:

Lemma 4.15. *Let $(\mathcal{N}, \mathcal{N}^*)$ be a unary L-DFGN system operator on X. Then the L-double measure of roughness of L-DFLApprox $\mathcal{L}_{N}, \mathcal{L}_{N^*} : L^X \longrightarrow L$ has the next properties:*

(i) $\mathcal{L}_{N}(A \otimes B) \geq \mathcal{L}_{N}(A) \otimes \mathcal{L}_{N}(B)$, *and*

(ii) $\mathcal{L}_{N^*}(A \otimes B) \leq \mathcal{L}_{N^*}(A) \oplus \mathcal{L}_{N^*}(B)$ *for all* $A, B \in L^X$.

Proof. (i)
$$\begin{aligned}\mathcal{L}_{N^*}(A \otimes B) &= S(A \otimes B, \underline{\mathcal{N}}(A \otimes B)) \\ &\geq S(A \otimes B, \underline{\mathcal{N}}(A) \otimes \underline{\mathcal{N}}(B)) \\ &\geq S(A, \underline{\mathcal{N}}(A)) \otimes S(B, \underline{\mathcal{N}}(B)) \\ &= \mathcal{L}_{N}(A) \otimes \mathcal{L}_{N}(B)\end{aligned}$$

(ii)
$$\begin{aligned}\mathcal{L}_{N^*}(A \otimes B) &= T(A \otimes B, \underline{\mathcal{N}}^*(A \otimes B)) \\ &\leq T(A \otimes B, \underline{\mathcal{N}}^*(A) \oplus \underline{\mathcal{N}}^*(B)) \\ &\leq T(A, \underline{\mathcal{N}}^*(A)) \oplus T(B, \underline{\mathcal{N}}^*(B)) \\ &= \mathcal{L}_{N^*}(A) \oplus \mathcal{L}_{N^*}(B).\end{aligned}$$

□

Proposition 4.16. *Let $(\mathcal{N}, \mathcal{N}^*)$ be an L-DFGN system operator on X. If $(\mathcal{N}, \mathcal{N}^*)$ is a unary, then*

(i) $\overline{\mathcal{N}}(A \oplus B) \leq \overline{\mathcal{N}}(A) \oplus \overline{\mathcal{N}}(B)$, *and*

(ii) $\overline{\mathcal{N}}^*(A \oplus B) \geq \overline{\mathcal{N}}^*(A) \otimes \overline{\mathcal{N}}^*(B)$ *for each* $A, B \in L^X$.

The opposite is true if L is st-s.

Proof. Assume that $(\mathcal{N}, \mathcal{N}^*)$ is a unary. For any $x \in X$ and $A, B \in L^X$, then

(i) By [[**56**], Proposition **4.9**], we have:

$\mathcal{N}(x)(A) \otimes \mathcal{N}(x)(B) \leq \bigvee_{K \in L^X} (\mathcal{N}(x)(K) \otimes S(K, A \otimes B)) \Leftrightarrow \overline{\mathcal{N}}(x)(A) \oplus \overline{\mathcal{N}}(x)(B) \geq \overline{\mathcal{N}}(x)(A \oplus B)$ whenever L is st-s.

(ii) $(\Rightarrow)$ Let $\mathcal{N}^*(x)(A) \oplus \mathcal{N}^*(x)(B) \geq \bigwedge_{K \in L^X} (\neg \mathcal{N}^*(x)(K) \to T(K, \neg(A \otimes B))$, then

$$\begin{aligned}\overline{\mathcal{N}}^*(A) \otimes \overline{\mathcal{N}}^*(B) &= [\neg \underline{\mathcal{N}}^*(\neg A) \otimes \neg \underline{\mathcal{N}}^*(\neg B)] \text{ (by \textbf{Theorem 3.13})} \\ &= \neg[\underline{\mathcal{N}}^*(\neg A) \oplus \underline{\mathcal{N}}^*(\neg B)] \\ &\leq \neg(\underline{\mathcal{N}}^*(\neg A \otimes \neg B)) \text{ (by \textbf{Proposition 4.14})} \\ &= \neg(\underline{\mathcal{N}}^*(\neg(A \oplus B))) \\ &= \overline{\mathcal{N}}^*(A \oplus B).\end{aligned}$$

$(\Leftarrow)$ It follows by **Theorem 3.13** and **Proposition 4.14**.

From (i) and (ii), the proof completed.

□

The relationship between unary L-DFGN system operators and the double measures of roughness of their L-DFUApprox is given in the next lemma:

Lemma 4.17. *Let $(\mathcal{N},\mathcal{N}^*)$ be a unary L-DFGN system operator on X. Then the double measure of roughness of L-DFUApprox $\mathcal{U}_{N},\mathcal{U}_{N^*} : L^X \longrightarrow L$ has the next properties:*

(i) $\mathcal{U}_{N}(A \oplus B) \geq \mathcal{U}_{N}(A) \otimes \mathcal{U}_{N}(B)$, *and*

(ii) $\mathcal{U}_{N^*}(A \oplus B) \leq \mathcal{U}_{N^*}(A) \oplus \mathcal{U}_{N^*}(B)$, $\forall A, B \in L^X$.

Proof. (*i*)

$$\begin{aligned}\mathcal{U}_{N}(A \oplus B) &= S(\overline{\mathcal{N}}(A \oplus B), A \oplus B)\\ &\geq S((\overline{\mathcal{N}}(A) \oplus \overline{\mathcal{N}}(B)), A \oplus B)\\ &= S(\neg(\neg\overline{\mathcal{N}}(A) \otimes \neg\overline{\mathcal{N}}(B)), \neg(\neg A \otimes \neg B))\\ &= S(\neg A \otimes \neg B, \neg\overline{\mathcal{N}}(A) \otimes \neg\overline{\mathcal{N}}(B)) \text{ (by **Lemma 2.3 (6)**)}\\ &\geq S(\neg A, \neg\overline{\mathcal{N}}(A)) \otimes S(\neg B, \neg\overline{\mathcal{N}}(B)) \text{ (by **Lemma 2.3 (3)**)}\\ &= S(\overline{\mathcal{N}}(A), A) \otimes S(\overline{\mathcal{N}}(B), B) \text{ (by **Lemma 2.3 (6)**)}\\ &= \mathcal{U}_{N}(A) \otimes \mathcal{U}_{N}(B).\end{aligned}$$

(*ii*)

$$\begin{aligned}\mathcal{U}_{N^*}(A \oplus B) &= T(\neg\overline{\mathcal{N}}^*(A \oplus B), \neg(A \oplus B))\\ &\leq T(\neg(\overline{\mathcal{N}}^*(A) \otimes \overline{\mathcal{N}}^*(B)), \neg A \otimes \neg B)\\ &= T(\neg A \otimes \neg B, \neg\overline{\mathcal{N}}^*(A) \oplus \neg\overline{\mathcal{N}}^*(B))\\ &\leq T(\neg A, \neg\overline{\mathcal{N}}^*(A)) \oplus T(\neg B, \neg\overline{\mathcal{N}}^*(B))\\ &= T(\neg\overline{\mathcal{N}}^*(A), \neg A) \oplus T(\neg\overline{\mathcal{N}}^*(B), \neg B)\\ &= \mathcal{U}_{N^*}(A) \oplus \mathcal{U}_{N^*}(B).\end{aligned}$$

□

5. Relationships between *L*-double fuzzy topologies and *L*-double rough approximation operators

In this section, we shall study the relationship between L-DFUApprox operators based on L-DFGN system operator and L-double fuzzy topologies . In **[31, 42]**, we offered the notion of L-double fuzzy topology. For $(L, \leq, \otimes)$ is semi-quantales and X a non-empty set. The pair $(\mathcal{T},\mathcal{T}^*)$ of maps $\mathcal{T},\mathcal{T}^* : L^X \longrightarrow L$ is said to be an L-double fuzzy topology on X **[4]** if it satisfies the next conditions: For all $A, B \in L^X$ and for every family $\{A_j : j \in J\} \subseteq L^X$,

(T_1) $\mathcal{T}(A) \leq \neg(\mathcal{T}^*(A))$,

(T_2) $\mathcal{T}(\underline{\perp}) = \mathcal{T}(\underline{\top}) = \top_L$, and (T_2^*) $\mathcal{T}^*(\underline{\perp}) = \mathcal{T}^*(\underline{\top}) = \perp_L$,

(T_3) $\mathcal{T}(A) \otimes \mathcal{T}(B) \leq \mathcal{T}(A \otimes B)$, and (T_3^*) $\mathcal{T}^*(A) \oplus \mathcal{T}^*(B) \geq \mathcal{T}^*(A \otimes B)$,

(T_4) $\bigwedge_{j\in J} \mathcal{T}(A_j) \leq \mathcal{T}(\bigvee_{j\in J} A_j)$, and (T_4^*) $\bigvee_{j\in J} \mathcal{T}^*(A_j) \geq \mathcal{T}^*(\bigvee_{j\in J} A_j)$,

The triple $(X,\mathcal{T},\mathcal{T}^*)$ is called an L-double fuzzy topological space.

Example 5.1. **[4]** *Suppose that $X = \{c, d\}$ is a set, $L = M = [0, 1]$ and $c \otimes d = \max\{0, c + d - 1\}, c \oplus d = \min\{1, c + d\}$. Then $([0, 1], \leq, \otimes)$ is a left-continuous t-norm with an order-reversing involution defined by $c' = \min\{1 - c, 1\}$. Let*

$\delta, \gamma \in [0,1]^X$ be defined as follows: $\delta(c) = 0.6, \delta(d) = 0.3, \gamma(c) = 0.5, \gamma(d) = 0.7$. Define $\tau, \tau^* : [0,1]^X \to [0,1]$ as follows:

$$\tau(\eta) = \begin{cases} 1, & if \quad \eta = \underline{0}, \underline{1}; \\ 0.8, & if \quad \eta = \delta; \\ 0.3, & if \quad \eta = \gamma; \\ 0.7, & if \quad \eta = \delta \vee \gamma; \\ 0.2, & if \quad \eta = \delta \wedge \gamma; \\ 0, & otherwise. \end{cases} \qquad \tau^*(\eta) = \begin{cases} 0, & if \quad \eta = \underline{0}, \underline{1} \\ 0.2, & if \quad \eta = \delta; \\ 0.7, & if \quad \eta = \gamma; \\ 0.3, & if \quad \eta = \delta \vee \gamma; \\ 0.8, & if \quad \eta = \delta \wedge \gamma; \\ 1, & otherwise. \end{cases}$$

Then, the pair (τ, τ^*) is an (L, M)-double fuzzy topology on X.

Definition 5.2. *For $(L, \leq, \otimes)$ is semi-quantales and X a non-empty set. The pair $(\mathcal{K}, \mathcal{K}^*)$ of maps $\mathcal{K}, \mathcal{K}^* : L^X \longrightarrow L$ is called an L-double fuzzy co-topology on X* **[31, 42]** *if it satisfies the next conditions: For all $A, B \in L^X$ and for every family $\{A_j : j \in J\} \subseteq L^X$,*

(COT_1) $\mathcal{K}(A) \leq \neg(\mathcal{K}^*(A))$

(COT_2) $\mathcal{K}(\underline{\perp}) = \mathcal{K}(\underline{\top}) = \top_L$, *and* $\qquad (COT_2^*)$ $\mathcal{K}^*(\underline{\perp}) = \mathcal{K}^*(\underline{\top}) = \perp_L$,

(COT_3) $\mathcal{K}(A) \otimes \mathcal{K}(B) \leq \mathcal{K}(A \oplus B)$, *and* $\qquad (COT_3^*)$ $\mathcal{K}^*(A) \oplus \mathcal{T}^*(B) \geq \mathcal{K}^*(A \oplus B)$,

(COT_4) $\bigwedge_{j \in J} \mathcal{K}(A_j) \leq \mathcal{K}(\bigwedge_{j \in J} A_j)$, *and* $\qquad (COT_4^*)$ $\bigvee_{j \in J} \mathcal{K}^*(A_j) \geq \mathcal{K}^*(\bigwedge_{j \in J} A_j)$,

The triple $(X, \mathcal{K}, \mathcal{K}^*)$ is said to be an L-double fuzzy co-topological space, $\mathcal{K}$ and $\mathcal{K}^*$ may be interpreted as gradation of closedness and gradation of non closedness, respectively.

According to **Lemma 4.15** and **Corollary 3.14**, we get the next result:

Theorem 5.3. *An L-double measure of roughness of L-DFLApprox $\mathcal{L}_N, \mathcal{L}_{N^*} : L^X \longrightarrow L$ has the next properties: For all $A, B \in L^X$ and for every family $\{A_i : i \in I\} \subseteq L^X$;*

(1) *If L is st-s, then*
   (i) $\mathcal{L}_N(\underline{\top}) = \top_L$, *and* $\qquad (ii)$ $\mathcal{L}_{N^*}(\underline{\top}) = \perp_L$,

(2) (i) $\mathcal{L}_N(\bigvee_{i \in I} A_i) \geq \bigwedge_{i \in I} \mathcal{L}_N(A_i)$, *and* $\qquad (ii)$ $\mathcal{L}_{N^*}(\bigvee_{i \in I} A_i) \leq \bigvee_{i \in I} \mathcal{L}_{N^*}(A_i)$,

(3) (i) $\mathcal{L}_N(A \otimes B) \geq \mathcal{L}_N(A) \otimes \mathcal{L}_N(B)$, *and* $\qquad (ii)$ $\mathcal{L}_{N^*}(A \otimes B) \leq \mathcal{L}_{N^*}(A) \oplus \mathcal{L}_{N^*}(B)$.

The statements of such theorem means that the operators $\mathcal{L}_N, \mathcal{L}_{N^*} : L^X \longrightarrow L$ constitute an L-double fuzzy topology on X.

According to **Corollary 3.15**, and **Lemma 4.17**, we can conclude that:

Theorem 5.4. *An L-double measure of roughness of L-DFUApprox $\mathcal{U}_N, \mathcal{U}_{N^*} : L^X \longrightarrow L$ has the next properties: For all $A, B \in L^X$ and for every family $\{A_i : i \in I\} \subseteq L^X$;*

(1) (i) $\mathcal{U}_N(\underline{\perp}) = \top_L$, *and* $\qquad (ii)$ $\mathcal{U}_{N^*}(\underline{\perp}) = \perp_L$,

(2) (i) $\mathcal{U}_N(\bigwedge_{i \in I} A_i) \geq \bigwedge_{i \in I} \mathcal{U}_N(A_i)$, *and* $\qquad (ii)$ $\mathcal{U}_{N^*}(\bigwedge_{i \in I} A_i) \leq \bigvee_{i \in I} \mathcal{U}_{N^*}(A_i)$,

(3) (i) $\mathcal{U}_N(A \oplus B) \geq \mathcal{U}_N(A) \otimes \mathcal{U}_N(B)$, *and* $\qquad (ii)$ $\mathcal{U}_{N^*}(A \oplus B) \leq \mathcal{U}_{N^*}(A) \oplus \mathcal{U}_{N^*}(B)$.

What was stated in the previous theorem means that the operators $\mathcal{U}_N, \mathcal{U}_{N^*} : L^X \longrightarrow L$ constitute an L-double fuzzy co-topology on X.

6. Conclusions

In this paper, we have defined and studied the notion of L-DFGN systems as a generalization of L-FGN systems **[55, 56]**. Additionally, a pair of L-DFLApprox and$L$-DFUApprox operators based on L-DFGN systems have been proposed. Their respective double measure of roughness has been given. As L is a quantale, we have redefined the L-double relation **[1]** and used it to define the quantale-valued double fuzzy rough set. In addition, it has been proved that L-DFGN system-based approximation operators has L-double relation as a special case. Furthermore, different kinds of L-DRApprox operators corresponding to the different special L-DFGN system have been presented and studied. Finally, we have interpreted the operators of double measures of L-DFLApprox and L-DFUApprox as an L-double fuzzy topology and an L-double fuzzy co-topology on a set X, respectively. In the future, we will attempt to consider some potential applications of the L-double fuzzy rough set theory of multi-attribute decision making.

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