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# Some new characterizations of $(b, c)$-inverses and Bott-Duffin $(e, f)$-inverses

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**Abstract.** The $(b, c)$-inverse and the Bott-Duffin $(e, f)$-inverse are two classes of outer inverses, a few characterizations of which have been presented by certain researchers. In this paper, we give some new characterizations of $(b, c)$-inverses and Bott-Duffin $(e, f)$-inverses. First, we present a number of ring theoretic characterizations of $(b, c)$-inverses. Then we characterize $(b, c)$-inverses by equations. Finally, we present some characterizations of Bott-Duffin $(e, f)$-inverses. More specifically, we use Bott-Duffin $(e, f)$-inverses to characterize some classes of rings, such as directly finite rings, Abelian rings and left min-abel rings.

## 1. Introduction

Let $R$ be an associative ring with unity 1 and $b, c \in R$. An element $a \in R$ is said to be $(b, c)$-invertible if there exists $y \in R$ such that $y \in bRy \cap yRc$, $yab = b$, and $cay = c$. If such a $y$ exists, it is unique and is called the $(b, c)$-inverse of $a$, denoted by $a^{\|(b,c)}$.

As a new class of outer inverse, the concept of the $(b, c)$-inverse was for the first time introduced by Drazin in [2, Definition 1.3] in the setting of rings, which generalized the group inverse, the Drazin inverse, the Moore-Penrose inverse, the Chipman's weighted inverse and the Bott-Duffin inverse. Afterwards, certain researchers further studied and generalized it. Rakić et al. [9] connected the core and dual core inverses with the $(b, c)$-inverse. Wang et al. [11] gave some characterizations of the $(b, c)$-inverse, in terms of the direct sum decomposition, the annihilator and the invertible elements. Ke et al. [7] investigated the existence and the expression of the $(b, c)$-inverse in a ring with an involution. Boasso and Kantún-Montiel [1] presented some other conditions for the existence of the $(b, c)$-inverse in rings, proving that the conditions which ensure the existence of the $(b, c)$-inverse, of the annihilator $(b, c)$-inverse and of the hybrid $(b, c)$-inverse are equivalent. For more results on $(b, c)$-inverses, we refer to [3, 4, 6, 8, 10].

In [2], Drazin introduced another outer generalized inverse which intermediates between the Bott-Duffin inverse and the $(b, c)$-inverse. This class of generalized inverses is called Bott-Duffin $(e, f)$-inverses, where $e, f \in R$ are idempotents. Recall that the Bott-Duffin $(e, f)$-inverse of $a \in R$ is the element $y \in R$ which

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satisfies $y = ey = yf$, $yae = e$, and $fay = f$. If the Bott-Duffin $(e, f)$-inverse of $a$ exists, it is unique and denoted by $a^{BD(e,f)}$. The Bott-Duffin $(e, f)$-inverse and the $(b, c)$-inverse are formally very similar. It is not difficult to find that a $(b, c)$-inverse $y$ of $a$ is a Bott-Duffin $(e, f)$-inverse of $a$ if and only if $b$ and $c$ are both idempotents. Conversely, if $y$ is the $(b, c)$-inverse of $a$, then $y$ is also the Bott-Duffin $(ya, ay)$-inverse of $a$ [2, Proposition 3.3]. More properties and applications of the Bott-Duffin $(e, f)$-inverse are studied by Ke and Chen in [5].

In this paper, we present some new characterizations of $(b, c)$-inverses and Bott-Duffin $(e, f)$-inverses. First, we give certain ring theoretic characterizations of the $(b, c)$-inverse of an element $a \in R$. The following conditions are proved to be equivalent: (a) $a$ is $(b, c)$-invertible; (b) $c \in cabRc$ and $R = bR \oplus r(ab)$; (c) $r(ab) = r(b)$, $l(cab) = l(c)$, and $ab$ is right $c$-regular. Next, we characterize $(b, c)$-inverses by equations. It is showed that $a$ is $(b, c)$-invertible if and only if the equation $bxab = b$ has solution $x_0$ in $Rc$ and its every solution is similar to $x_0$. Finally, we give some characterizations of Bott-Duffin $(e, f)$-inverses. To be specific, we use Bott-Duffin $(e, f)$-inverses to characterize directly finite rings, Abelian rings and left min-abel rings.

## 2. Ring theoretic characterizations of $(b, c)$-inverses

In this section, we will characterize $(b, c)$-inverses in ring theory. First, we have the following proposition.

**Proposition 2.1.** *Let $a, b, c \in R$. Then $a$ is $(b, c)$-invertible if and only if $b \in bRcab$ and $c \in cabR$.*

*Proof.* "⇐" Let $b = bucab$ and $c = cabv$, where $u, v \in R$. Take $y = buc$ and $x = bv$. Then $b = yab$ and $c = cax$. Moreover, $yay = yabuc = buc = y$. By

$$y = buc = bucabv = yabv = yax, \text{ and } x = bv = bucabv = yabv = yax,$$

we obtain $x = y$ and $c = cax = cay$. So

$$y = yabuc \in yRc \text{ and } y = bucay \in bRy.$$

Then $a^{\|(b,c)} = y = buc = x = bv$.

"⇒" Let $y = a^{\|(b,c)}$. Then $y \in bRy \cap yRc$, $yab = b$, and $cay = c$. Write $y = br_1y = yr_2c$, where $r_1, r_2 \in R$. Then we have

$$b = yab = br_1yab = br_1yr_2cab = b(r_1yr_2)cab \in bRcab,$$

and

$$c = cay = ca(br_1y) = (cab)r_1y \in cabR.$$

□

Note that $c = cay = ca(br_1y) = cabr_1(yr_2c) \in cabRc$. Hence, we get the following corollary from Proposition 2.1.

**Corollary 2.2.** *Let $a, b, c \in R$. Then $a$ is $(b, c)$-invertible if and only if $b \in bRcab$ and $c \in cabRc$.*

Similarly, we have the following proposition.

**Proposition 2.3.** *Suppose that $a, b, c \in R$. Then the following conditions are equivalent:*

*(1) $a$ is $(b, c)$-invertible;*
*(2) $b \in Rcab$ and $c \in cabRc$;*
*(3) $b \in bRcab$ and $c \in (cabR)^2$.*

*Proof.* (1) and (2) are equivalent by Proposition 2.1.
"(3)⇒(1)" Since $c \in (cabR)^2 = cabRcabR \subseteq cabR$, it is obvious from Proposition 2.1.
"(1)⇒(3)" It follows from Corollary 2.2 that $c \in cabRc$. Let $c = cabvc$, where $v \in R$. Then we have $c = cabvcabv \in cabRcabR = (cabR)^2$. □

For any $x \in R$, define $l(x) := \{y \in R \mid yx = 0\}$. Then we can characterize $(b,c)$-inverses using direct sum decomposition of rings.

**Proposition 2.4.** *Let $a, b, c \in R$. Then $a$ is $(b,c)$-invertible if and only if $b \in bRcab$ and $R = Rc \oplus l(ab)$.*

*Proof.* "⇒" From Proposition 2.1, we know that $b \in bRcab$. Let $y = a^{\|(b,c)}$. Then we have

$$y \in bRy \cap yRc,\ yab = b,\ cay = c, \text{ and } yay = y.$$

Notice that $y = br_1y = yr_2c$, where $r_1, r_2 \in R$. For every $x \in Rc \cap l(ab)$, one has that $xab = 0$. Let $x = uc$, where $u \in R$. Then

$$x = u(cay) = uca(br_1y) = (uc)abr_1y = xabr_1y = 0r_1y = 0.$$

Hence, $Rc \cap l(ab) = \{0\}$. Since

$$b = yab = (br_1y)ab = br_1(yr_2c)ab,$$

it follows that $ab = abr_1yr_2cab$. Moreover, $(1 - abr_1yr_2c)ab = 0$, i.e., $1 - abr_1yr_2c \in l(ab)$. Next, let

$$1 - abr_1yr_2c = t \in l(ab).$$

Then

$$1 = abr_1yr_2c + t \in Rc + l(ab).$$

Therefore, $R = Rc \oplus l(ab)$.

"⇐" Since $b \in bRcab$, there exists some $v \in R$ such that $b = bvcab$. Write $y = bvc$. Then $b = yab$ and $yay = y$. Thus, $y \in bRy \cap yRc$. Next we let $1 = wc + f$, where $w \in R$, $f \in l(ab)$, for $R = Rc \oplus l(ab)$. Then we get

$$ab = 1ab = wcab + fab = wcab,$$

$$b = yab = ywcab,$$

and

$$cab = ca(ywcab) = ca(yay)wcab = caya(ywcab) = cayab.$$

Moreover, $(c - cay)ab = 0$, i.e., $c - cay \in l(ab)$. Since $c - cay \in Rc$, it follows that $c - cay \in Rc \cap l(ab) = \{0\}$. Therefore, $c = cay$. Thus, $a$ is $(b,c)$-invertible. □

For any $x \in R$, define $r(x) := \{y \in R \mid xy = 0\}$, a right ideal of $R$. Using the same argument as in the proof of Proposition 2.4, we get the following proposition.

**Proposition 2.5.** *Let $a, b, c \in R$. Then $a$ is $(b,c)$-invertible if and only if $c \in cabRc$ and $R = bR \oplus r(ca)$.*

**Definition 2.6.** *Let $d, c \in R$. Element $d$ is said to be right (left) $c$-regular, if there exists an element $x \in R$, such that $d = dxcd$ ($d = dcxd$).*

**Proposition 2.7.** *Let $a, b, c \in R$. Then $a$ is $(b,c)$-invertible if and only if $r(ab) = r(b)$, $l(cab) = l(c)$, and $ab$ is right $c$-regular.*

*Proof.* "⇐" Since $ab$ is a right $c$-regular, there exists an element $x \in R$, such that $ab = abxcab$. Thus,

$$ab(1 - xcab) = 0,\ 1 - xcab \in r(ab) = r(b),$$

and

$$b(1 - xcab) = 0, \text{ and } b = bxcab \in bRcab.$$

Notice that $cab = cabxcab$. We obtain that

$$(1 - cabx)cab = 0,\ 1 - cabx \in l(cab) = l(c), \text{ and } (1 - cabx)c = 0.$$

Therefore, $c = cabxc \in cabR$. From Proposition 2.1, we know that $a$ is $(b, c)$-invertible.

"⇒" Let $y = a^{\|(b,c)}$. Then

$$yab = b,\ cay = c,\ yay = y,\ y = br_1y, \text{ and } y = yr_2c, \text{ where } r_1, r_2 \in R.$$

Obviously, $r(b) \subseteq r(ab)$. Now, for any $x \in r(ab)$, one gets that

$$abx = 0,\ bx = (yab)x = y(abx) = 0, \text{ and } x \in r(b).$$

Therefore, $r(ab) \subseteq r(b)$. It is straightforward that $l(c) \subseteq l(cab)$. Conversely, let $x \in l(cab)$. Then

$$xcab = 0,\ xc = xcay = xca(br_1y) = xcab(r_1y) = 0,\ x \in l(c), \text{ and } l(cab) \subseteq l(c).$$

Therefore, $l(cab) = l(c)$. Since

$$ab = a(yab) = a(br_1y)ab = abr_1(yr_2c)ab = ab(r_1yr_2)cab,$$

we have that $ab$ is a right $c$-regular. □

Similarly, we get the following proposition.

**Proposition 2.8.** *Let $a, b, c \in R$. Then $a$ is $(b, c)$-invertible if and only if*

$$r(cab) = r(b),\ l(ca) = l(c),$$

*and $ca$ is left $b-$regular.*

**Corollary 2.9.** *Let $a, b, c \in R$. Then the following conditions are equivalent:*

*(1) $a$ is $(b, c)$-invertible;*
*(2) $r(b) = r(cab)$, $l(c) = l(cab)$, and $ab$ is right $c-$regular;*
*(3) $r(b) = r(cab)$, $l(c) = l(cab)$, and $ca$ is left $b-$regular.*

Recall that an element $a \in R$ is regular if there exists $x \in R$ satisfying $axa = a$. In this case, $x$ is a regular (or inner) inverse of $a$.

**Proposition 2.10.** *Let $a, b, c \in R$. Then the following conditions are equivalent:*

*(1) $a$ is $(b, c)$-invertible;*
*(2) $Rb = Rcab$, $cR = cabR$, and $ab$ is right $c-$regular;*
*(3) $Rb = Rcab$, $cR = cabR$, and $cab$ is regular;*
*(4) $Rb = Rcab$, $cR = cabR$, and $ca$ is left $b-$regular.*

*Proof.* "(1)⇒(2)" It follows from Propositions 2.1 and 2.7.

"(2)⇒(3)" and "(4)⇒(3)" are obvious.

"(1)⇒(4)" It follows from Propositions 2.1 and 2.8.

"(3)⇒(1)" Let $cab = cabwcab$, $b = vcab$, and $c = cabs$. Then

$$b = vcab = vcabwcab = bwcab \in bRcab,$$

and

$$c = cabs = cabwcabs = cabwc \in cabRc.$$

By Proposition 2.1, the assertion holds. □

**Corollary 2.11.** *Let $a, b, c \in R$. Then the following conditions are equivalent:*

*(1) $a$ is $(b, c)$-invertible;*
*(2) there exists some $x \in R$ such that $xax = x$, $xR = bR$ and $Rx = Rc$.*

*Proof.* "(1)⇒(2)" In view of Proposition 2.10, we know that $b = vcab$, $c = cabs$, and $cab = cabwcab$. Then

$$b = vcab = vcabwcab = bwcab,$$

and

$$c = cabs = cabwcabs = cabwc.$$

Take $x = bwc$. Then

$$b = xab,\ c = cax, \text{ and } xax = xabwc = bwc = x.$$

Thus

$$xR = bR \text{ and } Rx = Rc.$$

"(2)⇒(1)" Since $1 - xa \in l(x) = l(b)$ and $1 - ax \in r(x) = r(c)$, one has that $b = xab$ and $c = cax$. Denote $x = bs = tc$. Then $x = xax = bsax \in bRx$, and

$$x = xax = xatc \in xRc.$$

Thus $a^{\|(b,c)} = x$. □

## 3. Characterizing (*b*, *c*)-inverses by equations

In this section, we characterize $(b, c)$-inverses by equations. Let $a, b, c \in R$. If there exists an element $u \in Rc$, such that $buab = b$, then $x = u$ is said to be a solution of the equation $bxab = b$ in $Rc$.

**Definition 3.1.** *Suppose that $x_1$ and $x_2$ are two solutions of the equation $bxab = b$. If $x_2 = x_2abx_1$ and $x_1 = x_1abx_2$, then $x_2$ is said to be similar to $x_1$.*

**Proposition 3.2.** *Let $a, b, c \in R$. Then $a$ is $(b, c)$-invertible if and only if the equation $bxab = b$ has solution $x_0$ in $Rc$ and its every solution is similar to $x_0$.*

*Proof.* "⇒" Let $a^{\|(b,c)} = y$. Then we have

$$y = br_1y = yr_2c,\ yab = b,\ cay = c, \text{ and } yay = y, \text{ where } r_1, r_2 \in R.$$

Moreover,

$$b(r_1yr_2c)ab = yr_2cab = yab = b.$$

Thus, $x_0 = r_1yr_2c$ is a solution of the equation $bxab = b$ in $Rc$. Next we suppose that $x = uc$ is a solution of the equation $bxab = b$ in $Rc$. Then $b(uc)ab = b$. Since

$$r_1yr_2 = r_1(br_1y)r_2 = r_1(bucab)r_1yr_2 = r_1bucayr_2,$$

we have

$$r_1yr_2c = r_1bucayr_2c = r_1bucay = r_1buc,$$

$$b = b(r_1yr_2c)ab = b(r_1buc)ab = br_1b,$$

$$y = br_1y = br_1yr_2c = b(r_1yr_2c) = br_1buc = buc,$$

$$uc = ucay = ucabr_1y = ucabr_1yr_2c = uc(ab)(r_1yr_2c) = ucabx_0,$$

and

$$\begin{aligned} x_0 &= r_1yr_2c = r_1yr_2cay = r_1yr_2cabr_1y = r_1yr_2ca(yab)r_1y \\ &= r_1yr_2ca(buc)abr_1y = x_0abucay = x_0ab(uc). \end{aligned}$$

Thus, $x = uc$ is similar to $x_0$.

"⇐" Assume that $x_0 = uc$ is a solution of the equation $bxab = b$ in $Rc$. Then we have $bucab = b$. Let $y = buc$. Then $b = yab$ and $yay = yabuc = buc = y$. Take

$$v = uc + (1 - cabu)c = uc + c - cabuc = uc + c - cay \in Rc.$$

Then

$$bvab = b(uc + c - cay)ab = bucab + bcab - bcayab = b + bcab - bcab = b.$$

Thus, $x = v$ is also a solution of the equation $bxab = b$ in $Rc$. From the assumption, we know that $v$ is similar to $x_0 = uc$. Moreover,

$$\begin{aligned} v &= vabuc = vay = (uc + c - cay)ay = ucay + cay - cayay \\ &= ucay + cay - cay = ucay, \end{aligned}$$

then $ucay$ is also a solution of the equation $bxab = b$ in $Rc$. From the definition of the similarity of solutions, we have

$$uc = x_0 = x_0abucay = ucabucay = ucayay = ucay = v = uc + c - cay.$$

That is, $c = cay$, $y = yay = bucay \in bRy$, and

$$y = yay = yabuc \in yRc.$$

Hence, $y$ is the $(b, c)$-inverse of $a$, i.e., $a^{\|(b,c)} = y$. □

Let $a \in R$. It is well known that the regular inverse of $a$, if there is one, is not always unique. We denote $a^-$ the set of all regular inverse of $a$. For convenience, $a^-$ also indicates an arbitrary regular inverse of $a$ when no confusion can arise.

**Proposition 3.3.** *Let $a, b, c \in R$, $e, f \in E(R)$, $bR = eR$, and $Rc = Rf$. Then the following are equivalent:*

*(1) $a$ is $(b, c)$-invertible;*
*(2) The system of equations*

$$\begin{cases} bxcae = e \\ fabxc = f \end{cases} \tag{1}$$

*is solvable;*
*(3) $cae$ and $fab$ are regular, $e = bb^-e(cae)^-(cae)$, and $f = fab(fab)^-fc^-c$.*

*Proof.* "(1)⇒(2)" Let $a^{\|(b,c)} = y$, $e = bd$ and $f = tc$, where $d, t \in R$. Then

$$yab = b,\ cay = c, \text{ and } y = br_1y = yr_2c, \text{ where } r_1, r_2 \in R.$$

Since $b = eb$ and $c = cf$, one has that

$$y = br_1y = ebr_1y = ey,\ y = yr_2c = yr_2cf = yf,\ e = bd = yabd = yae,$$

and

$$f = tc = tcay = fay.$$

Thus

$$e = yae = br_1yae = br_1yr_2cae = b(r_1yr_2)cae,$$

and

$$f = fay = fabr_1y = fabr_1yr_2c = fab(r_1yr_2)c.$$

Hence the system of equations (1) admits a solution $x = r_1yr_2$.

"(2)⇒(3)" Let $x = u$ be a solution of the system of equations (1). Then

$$bucae = e \text{ and } fabuc = f.$$

Hence

$$cae = caee = cae(bucae) = caebucae.$$

Thus $cae$ is regular. Then $(cae)^-$ exists. Similarly, we can prove that $(fab)^-$ exists. Denote

$$e = bd \text{ and } f = tc, \text{ where } d, t \in R.$$

Then $b = eb = bdb$, and $c = cf = ctc$. Thus both $b^-$ and $c^-$ exist. Moreover,

$$\begin{aligned} bb^-e(cae)^-cae &= bb^-bd(cae)^-cae = bd(cae)^-cae = e(cae)^-cae \\ &= bucae(cae)^-cae = bucae = e. \end{aligned}$$

Similarly, one can prove that $fab(fab)^-fc^-c = f$.

"(3)⇒(1)" We know that

$$b = eb = bb^-e(cae)^-caeb = b(b^-e(cae)^-)cab \in bRcab,$$

and

$$c = cf = cfab(fab)^-fc^-c = cab((fab)^-fc^-)c \in cabRc \subseteq cabR.$$

By Proposition 2.1, one obtains that $a$ is $(b, c)$-invertible. □

## 4. Characterizations of Bott-Duffin $(e, f)$-inverses

As we know Bott-Duffin $(e, f)$-inverses are particular $(b, c)$-inverses. They, however, has its own research significance. Some results and approaches of $(b, c)$-inverses can be borrowed from to study Bott-Duffin $(e, f)$-inverses. In this section, we give some characterizations of Bott-Duffin $(e, f)$-inverses. Mainly, we use Bott-Duffin $(e, f)$-inverses to characterize some classes of rings. First, we have the following proposition similar to Proposition 3.3. It is the basis of some propositions in this section.

**Proposition 4.1.** *Let $a \in R$ and $e, f \in E(R)$. Then the following are equivalent:*

*(1) $a$ is Bott-Duffin $(e, f)$-invertible;*
*(2) The system of equations*

$$\begin{cases} exfae = e \\ faexf = f \end{cases} \tag{2}$$

*is solvable;*
*(3) $fae$ is regular, $e = e(fae)^-fae$, and $f = fae(fae)^-f$.*

*Proof.* It follows from Proposition 3.3 by taking $b = e$ and $c = f$. □

Recall that a ring $R$ is said to be Abelian if $E(R) \subseteq C(R)$.

**Lemma 4.2.** *A ring $R$ is an Abelian ring if and only if $(1-e)Re = 0$ for all $e \in E(R)$.*

*Proof.* "⇒" Since $e \in E(R)$, one has that $(1-e)Re = (1-e)eR = 0$.

"⇐" Suppose that $(1-e)Re = 0$ for any $e \in E(R)$. Since $1-e \in E(R)$, we have that $[1-(1-e)]R(1-e) = 0$, that is $eR(1-e) = 0$. Thus for any $a \in R$, it follows that $ea(1-e) = 0 = (1-e)ae$. This gives $ea = eae = ae$. Hence $R$ is an Abelian ring. □

**Proposition 4.3.** *The following conditions are equivalent:*

*(1) $R$ is an Abelian ring;*
*(2) for any $a \in R$ and any $e, f \in E(R)$, if $a$ is Bott-Duffin $(e, f)$-invertible, then $e = f$.*

*Proof.* "(1)⇒(2)" Let $R$ be an Abelian ring, and $a$ be Bott-Duffin $(e, f)$-invertible. By Proposition 4.1, we have that

$$e = e(fae)^{-}fae, \text{ and } f = fae(fae)^{-}f.$$

Since $R$ is an Abelian ring, one has that $f, e \in C(R)$. Hence

$$e = e(fae)^{-}faef = ef, \text{ and } f = efae(fae)^{-}f = ef.$$

Thus $e = f$.

"(2)⇒(1)" Suppose that $R$ is not Abelian. Then $(1-e)Re \neq 0$ for some $e \in E(R)$. By Lemma 4.2, there exists some $a \in R$ such that $(1-e)ae \neq 0$. Write $g = e + (1-e)ae$. Then

$$eg = e, \; ge = e + (1-e)ae = g, \text{ and } g^2 = gg = (ge)g = g(eg) = ge = g.$$

Hence $g \in E(R)$. It can easily be verified that $e^{BD(g,e)} = g$. Hence $g = e$ by hypothesis, it follows that $(1-e)ae = 0$, a contradiction. Then $(1-e)Re = 0$, and $R$ is an Abelian ring. □

Recall that a ring $R$ is directly finite, if for any $a, b \in R$, $ab = 1$ implies $ba = 1$. Clearly, Abelian rings are directly finite.

**Proposition 4.4.** *The following conditions are equivalent:*

*(1) $R$ is a directly finite ring;*
*(2) for any right invertible element $a \in R$ and any $e \in E(R)$, if $a$ is Bott-Duffin $(e, 1)$-invertible, then $e = 1$.*

*Proof.* "(1)⇒(2)" Let $a$ be a right invertible element in $R$, and $e \in E(R)$, such that $a$ is Bott-Duffin $(e, 1)$-invertible. By Proposition 4.1, we have that

$$1 = ae(ae)^{-}, \text{ and } e = e(ae)^{-}ae.$$

Since $R$ is a directly finite ring and $a$ a right invertible element, one has that $a$ is invertible. Hence, there exists some $b \in R$ such that $ba = 1 = ab$. Thus

$$e = 1e = bae, \text{ and } b = b1 = bae(ae)^{-} = e(ae)^{-}.$$

Therefore

$$(1-e)b = (1-e)e(ae)^{-} = 0,$$

so

$$(1-e) = (1-e)1 = (1-e)ba = 0.$$

Then $e = 1$.

"(2)⇒(1)" Let $a, b \in R$ such that $ab = 1$. Denote $e = ba$. Then

$$ae = a(ba) = (ab)a = 1a = a, \; eb = (ba)b = b(ab) = b1 = b,$$

and

$$e^2 = ee = eba = ba = e.$$

It is obvious that $a^{BD(e,1)} = b$. Then by hypothesis, we obtain that $e = 1$, namely $ba = 1$. Hence $R$ is a directly finite ring. □

Recall that an idempotent $e$ of a ring $R$ is called left minimal idempotent, if $Re$ is a minimal left ideal of $R$. Denote by $ME_l(R)$ the set of all left minimal idempotent elements of $R$.

Let $e \in E(R)$. If $(1-e)Re = 0$, we call $e$ a left semi-central idempotent element of $R$.

Recall that a ring $R$ is said to be left min-abel [12] if either $ME_l(R) = \emptyset$, or every element of $ME_l(R)$ is a left semi-central idempotent element.

**Proposition 4.5.** *The following conditions are equivalent:*

*(1)R is a left min-abel ring;*
*(2) for any $a \in R$, $e \in ME_l(R)$ and $g \in E(R)$, if $a$ is Bott-Duffin $(g, e)$-invertible, then $e = ge$.*

*Proof.* "(1)⇒(2)" Suppose $a$ is Bott-Duffin $(g, e)$-invertible. Then by Proposition 4.1, we have that

$$g = g(eag)^{-}eag, \text{ and } e = eag(eag)^{-}g.$$

Since $R$ is a left min-abel ring, and $e$ is a left semi-central element, one has that

$$g(eag)^{-}e = e(g(eag)^{-})e.$$

Note that $g = eg(eag)^{-}eag = eg$. Then $l(e) \subseteq l(g)$. Define

$$f : Re \to Reg = Rg, xe \mapsto xeg.$$

It is easy to verify that $f$ is a left $R-$module map. Since $rg = reg$ for any $rg \in Rg$, one has that $f(re) = reg = rg$. Thus $f$ is surjective. Hence $Rg \cong Re/Kerf$, so $Kerf$ is a submodule of left $R-$module $Re$, i.e., $Kerf$ is a left ideal of $R$ contained in the minimal left ideal $Re$. If $Kerf \neq 0$, we have $Kerf = Re$. This gives $Rg \cong Re/Kerf = 0$, so $g = 0$ and therefore $e = 0$, which contradicts that $Re$ is a minimal left ideal of $R$. Then $Kerf = 0$, and $Rg \cong Re$. Hence $Rg$ is also a minimal left ideal of $R$, so $g \in ME_l(R)$. Since $R$ is a left min-abel ring, one has that $g$ is a left semi-central element. Then

$$eag(eag)^{-}g = geag(eag)^{-}g = ge, \text{ and } e = eag(eag)^{-}g = geag(eag)^{-}g = ge.$$

"(2)⇒(1)" If $ME_l(R) = \emptyset$, we know that $R$ is a left min-abel ring. We suppose $ME_l(R) \neq \emptyset$ below. Assume that there exist some $e \in ME_l(R)$ and some $a \in R$ such that $(1 - e)ae \neq 0$. Then $0 \neq R(1 - e)ae \subseteq Re$. Since $Re$ is a minimal left ideal, one has that $R(1 - e)ae = Re$. Write $h = (1 - e)ae$. One obtains that $Rh = Re$. Denote $e = ch$, where $c \in R$. We get that

$$h = (1 - e)ae = (1 - e)aee = he = hch.$$

Put $g = hc$. Then $h = gh$, and $g^2 = hchc = hc = g$, so $g \in E(R)$. It is easy to check that $c^{BD(g,e)} = h$. By hypothesis, one has that

$$e = ge = hce, \text{ and } e = ee = ehce = e(1 - e)aece = 0,$$

a contradiction. Thus $(1 - e)Re = 0$, so $R$ is a left min-abel ring. □

Recall that a ring $R$ is a strongly left min-abel ring [13] if either $ME_l(R) = \emptyset$, or every element of $ME_l(R)$ is a right semi-central element.

**Proposition 4.6.** *The following conditions are equivalent:*

*(1) R is a strongly left min-abel ring;*
*(2) for any $a \in R$, $e \in ME_l(R)$ and $g \in E(R)$, if $a$ is Bott-Duffin $(e, g)$-invertible, then $ge = g$.*

*Proof.* "(1)⇒(2)" Let $a$ be Bott-Duffin $(e, g)$-invertible. Then

$$e = e(gae)^{-}gae, \text{ and } g = gae(gae)^{-}g.$$

Since $R$ is a strongly left min-abel ring, one gets that $e$ is a right semi-central element. Thus $g = ge$.

"(2)⇒(1)" Suppose there exist some $e \in ME_l(R)$ and $a \in R$ such that $ea(1 - e) \neq 0$. Denote $g = e + ea(1 - e)$. Then

$$eg = g, ge = e, \text{ and } g^2 = g.$$

It can easily be verified that $e^{BD(e,g)} = g$. By hypothesis, we obtain that $g = ge = e$, so $ea(1 - e) = 0$, a contradiction. Thus, we have $eR(1 - e) = 0$, and therefore, $R$ is a strongly left min-abel ring. □

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