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A study of triple sequence spaces in fuzzy anti-normed linear spaces

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Abstract. The proposed article aims to study some topological properties of triple real sequence spaces with respect to fuzzy-anti-normed linear space (**FANLS**). In this work the algebra of fuzzy $\mathcal{I}_\lambda$-limit and fuzzy $\mathcal{I}_\lambda$ -anti limit, where $0 < \lambda < 1$ and $\mathcal{I}$ is an ideal on $\mathcal{N}^3$, of triple real sequences along with an interesting example have been studied. Furthermore, the completeness of a special kind of sequence space with respect to fuzzy anti-norm has been examined.

1. Introduction

In 1965, Professor L. Zadeh [19] invented the notion of fuzzy set theory, which later proved widely practical in many areas of science and technology, playing a significant role in dealing with circumstances where vagueness, impreciseness, and inexactness exist in research. Following that, many breakthroughs in the study of the fuzzy theory were recorded, and many scholars published their work on the topic. In 1984, Katsaras [8] established the fuzzy norm and in 1992, Clementina [4] analyzed it further in the context of fuzzy numbers with regard to a linear space. In 1994, S. C. Cheng and J. M. Mordeson [3] expanded the concept of fuzzy norm on linear space in a slightly different method. Later, many authors investigated the topic in their own perspective and modified it. For example, in 2003, T. Bag and S. K. Samanta [1] improved the definition of fuzzy norm presented in [3] and in 2008, T. Bag and S. K. Samanta [2] investigated the notion given in [1, 4, 8] further and developed several important enhancements. In 2010, I. H. Jebril and T. K. Samanta [7] compared a new notion the fuzzy anti-norm, to the idea of the fuzzy norm, described in [2]. The fuzzy anti-norm combines the theory of the fuzzy set with a degree of non-belongingness of an item to some certain subject and produce a comprehensive strategy for dealing with ambiguity.

In 2000, P. Kostyrko [13] introduced the notion of ideal convergence ($\mathcal{I}-$ convergence) of real sequences, which generalizes the concept of statistical convergence of sequences. The idea of statistical convergence was first examined by Fast [6] and Schoenberg [17] individually in their own way, which elaborates the concepts of ordinary convergence to some higher context. The concept of statistical convergence is based on the technique of natural density of the set with respect to the set of natural numbers $\mathcal{N}$. Let F be the subset of $\mathcal{N}$. Then the natural density of F is defined as

$$\delta(F) = \lim_{p\to\infty} \frac{1}{p}\text{card}\left\{q \le p : q \in F \quad \text{and} \quad p \in \mathcal{N}\right\}.$$

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Later, various scholars examined the concept of ideal convergence in their own approach and generalized it (see [5]). Some authors have recently presented the concepts of statistical convergence and ideal convergence in the framework of fuzzy theory [9, 12, 14, 15].

The proposed article is inspired by [11] and [10] in which, Kočinac [11] investigated fuzzy anti-normed linear space with its topological properties, and V. A. Khan[10] generalized the study to some fuzzy anti $\lambda-$ ideal convergent double sequence spaces.

We now recall some preliminaries which will be used later on. An ideal $\mathcal{I}$ on $\mathcal{N}$ is a collection of subsets of $\mathcal{N}$ that meet the requirements, (i) $\emptyset \in \mathcal{I}$, (ii) $G_1, G_2 \in \mathcal{I} \Rightarrow G_1 \cup G_2 \in \mathcal{I}$, (iii) $G_1 \in \mathcal{I}$ and $G_2 \subseteq G_1 \Rightarrow G_2 \in \mathcal{I}$. An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{N}$ does not lie in $\mathcal{I}$, $\mathcal{I} \neq \emptyset$, and ideal $\mathcal{I}$ is called admissible if it contains all finite subsets of $\mathcal{N}$. There is another family of subsets of the set $\mathcal{N}$, we denote it by $\mathcal{I}^*$ and termed a filter on $\mathcal{N}$ that meets the requirements, (i)$\emptyset \notin \mathcal{I}^*$, (ii) $G_1, G_2 \in \mathcal{I}^* \Rightarrow G_1 \cap G_2 \in \mathcal{I}^*$, (iii)$G_1 \in \mathcal{I}^*$ and $G_2 \supseteq G_1 \Rightarrow G_2 \in \mathcal{I}^*$. Corresponding to each ideal $\mathcal{I}$ there exist a filter $\mathcal{F}(\mathcal{I})$ defined as

$$\mathcal{F}(\mathcal{I}) = \{M \subseteq \mathcal{N} : M = \mathcal{N} - F \text{ and } F \in \mathcal{I}\}$$

Throughout the work $\mathcal{I}$ denotes a triple ideal on $\mathcal{N}^3$ and $\mathcal{N}^3$ deals the triple product $\mathcal{N} \times \mathcal{N} \times \mathcal{N}$ and L stands for unit interval $[0, 1]$.

We now outline the work in the article as follows. Section 1, is the introduction part containing brief history and some concepts, Section 2, contains the basic definitions and results which are useful in the work, Section 3, contains some new definitions with examples, and Section 4, is devoted to main theorems and results that established the relationships between studied notions in Section 3.

2. Background

Definition 2.1. *[18] A binary operation $\diamond : L^2 \to L$ meeting the requirements*

(i) *$\diamond$ is commutative.*
(ii) *$\diamond$ is associative.*
(iii) *$s_1 \diamond s_2 \leq s_3 \diamond s_4$ if $s_1 \leq s_3$ and $s_2 \leq s_4$ for every $s_1, s_2, s_3, s_4 \in [0, 1]$.*
(iv) *$s \diamond 0 = s$ for all $s \in [0, 1]$.*

is defined as $t-$conorm.

Remark 2.2. *[16]*

(i) *For any $0 < \epsilon_1 < 1$, we can find $0 < \epsilon_2 < 1$ such that $\epsilon_2 \diamond \epsilon_2 \leq \epsilon_1$.*
(ii) *For any $0 < \epsilon, \epsilon_3 < 1$, and $\epsilon > \epsilon_3$, we can find $0 < \epsilon_4 < 1$ such that $\epsilon \geq \epsilon_3 \diamond \epsilon_4$.*

Definition 2.3. *[11] A 3-tuple object $(\mathcal{V}, \beta, \diamond)$, where $\mathcal{V}$ is a linear space, $\beta : \mathcal{V} \times \mathcal{R} \to \mathcal{R}$, gives fuzzy anti-norm and $\diamond$ is t-conorm, is defined as a fuzzy anti-norm linear space(**FANLS**) if for all $u, v \in \mathcal{V}$, the following requirements meets.*

(i) *$\beta(u, \tau) = 1$ for all $-\infty < \tau < 0$.*
(ii) *$\beta(u, \tau) = 0$ iff $u = \theta$, where θ is zero element of $\mathcal{V}$.*
(iii) *$\beta(\alpha u, \tau) = \beta\left(u, \frac{\tau}{|\alpha|}\right)$ if $\alpha \neq 0$.*
(iv) *$\beta(u + v, \tau + t) \leq \beta(u, \tau) \diamond \beta(v, t)$.*
(v) *$\lim_{\tau\to\infty} \beta(u, \tau) = 0$.*

Remark 2.4. *[10] $\beta(x, \tau) : \mathcal{V} \times \mathcal{R} \to L$ is a non increasing function of τ for each u.*

Remark 2.5. *[11] A fuzzy norm can be induced by a norm. As an example, let $(\mathcal{V}, ||.||)$ be a normed linear space and $\diamond$ be a t co-norm defined as $u \diamond v = max\ \{u, v\}$. Now define $\beta : \mathcal{V} \times \mathcal{R} \to L$ such that*

$$\beta(u, \tau) = \begin{cases} \frac{||u||}{\tau+||u||}; & if \ \ \tau > 0 \\ 1; & if \ \ \tau \leq 0 \end{cases} .$$

Definition 2.6. *[10] A sequence $u = (u_n)_{n\geq 1}$ of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ is said to be $\beta-$ convergent to a point ξ, if for all $\epsilon > 0$ and $\tau > 0$ there exists $n_\epsilon \in \mathcal{N}$ such that $\beta(u_n - \xi, \tau) < \epsilon \quad \forall \quad n \geq n_\epsilon$.*
Equivalently,

$$\lim_{n\to\infty} \beta(u_n - \xi, \tau) = 0. \tag{1}$$

We denote the case as $(u_n) \xrightarrow{\beta} \xi$.

3. Fuzzy $\mathcal{I}_\lambda-$anti-convergence of triple sequences

Definition 3.1. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. Then a triple sequence $u = (u_{kji})$ of the elements of **FANLS**$(\mathcal{V}, \beta, \diamond)$ is called fuzzy ideally $\beta-$ convergent ($\mathcal{I}_\beta-$ cgt) to ξ, if for all $\epsilon > 0$ and $\tau > 0$*

$$\left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon\right\} \in \mathcal{I}. \tag{2}$$

We denote the case as, $\mathcal{I}_\beta - \lim u = \xi$ and ξ is defined as fuzzy $\mathcal{I}_\beta$- limit of u.

Definition 3.2. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. Then for $0 < \lambda < 1$, a triple sequence $u = (u_{kji})$ of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ is said to be fuzzy $\mathcal{I}_\lambda-$ cgt to ξ, if for all $\tau > 0$*

$$\left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda\right\} \in \mathcal{I}. \tag{3}$$

We denote the case as, $\mathcal{I}_\lambda - \lim u = \xi$ and ξ is defined as fuzzy $\mathcal{I}_\lambda-$ limit of u.

Definition 3.3. *Let $\mathcal{I}$ defines an ideal on $\mathcal{N}$. Then a triple sequence $u = (u_{kji})$ of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ is said to be fuzzy ideally anti-$\beta-$cgt ($\mathcal{I}_\beta-$ anti-cgt) to ξ, if for all $\epsilon > 0$ and $\tau > 0$*

$$\left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon\right\} \in \mathcal{F}(\mathcal{I}). \tag{4}$$

We denote the case as, $u \xrightarrow{a-\mathcal{I}_\beta} \xi$, and ξ is known as fuzzy anti $\mathcal{I}_\beta$- limit of u.

Definition 3.4. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. Then for $0 < \lambda < 1$, a triple sequence $u = (u_{kji})$ of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ is said to be fuzzy $\mathcal{I}_\lambda-$anti convergent to ξ, if for all $\tau > 0$*

$$\left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - \xi, \tau\right) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}). \tag{5}$$

We denote the case, $u \xrightarrow{a-\mathcal{I}_\lambda} \xi$, and ξ is known as fuzzy $\mathcal{I}_\lambda-$anti limit of u.

Example 3.5. *Let $\mathcal{I} = \{E \subseteq \mathcal{N}^3 : \delta(E) = 0\}$ be an ideal on $\mathcal{N}^3$, and let $(\mathcal{R}^3, \beta, \diamond)$ be a **FANLS** with idempotent t-conorm $\diamond$, and $\beta(u, \tau) = \frac{\|u\|}{\tau + \|u\|}$. Let $u = (u_{kji})$ be a triple real sequence of the elements of $(\mathcal{R}^3, \beta, \diamond)$ defined as*

$$u = u_{kji} = \begin{cases} \begin{pmatrix} \frac{1}{i} \\ \frac{1}{j} \\ \frac{1}{k} \end{pmatrix}; & \text{if } k, j, i \text{ are perfect cube} \\ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}; & \text{Otherwise} \end{cases}$$

Then,

$$fuzzy - \mathcal{I}_\lambda - \lim(u) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \text{ and } fuzzy - \mathcal{I}_\lambda \text{ } anti - \lim(u) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Proof. Let

$$F = \left\{(r, q, p) \in \mathcal{N}^3 : r = k^3, q = j^3, p = i^3 \text{ and } k, j, i \in \mathcal{N}\right\}, \tag{6}$$

then natural density of $F = \delta(F) = 0$ since

$$0 \leq \delta(F) = \lim_{r,q,p\to\infty} \frac{1}{rqp} |F| \leq \lim_{r,q,p\to\infty} \frac{1}{rqp} \sqrt[3]{rqp} = 0. \tag{7}$$

Now for an arbitrary $\tau > 0$ and $0 < \lambda < 1$, there exists some $(r_0, q_0, p_0) \in F$ such that for all $(r, q, p) \in F$ where $r \geq r_0, \ q \geq q_0, \ p \geq p_0$, we get

$$\beta(u_{rqp} - (0, 0, 0)^t, \tau) = \frac{\sqrt{\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}}}{\tau + \sqrt{\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}}} = 0 < 1 - \lambda. \tag{8}$$

Furthermore the set,

$$F_1 = \left\{(r, q, p) \in \mathcal{N}^3 : \beta\left(u_{rqp} - (0, 0, 0)^t, \tau\right) < 1 - \lambda\right\} \subseteq F. \tag{9}$$

Thus, $F_1 \in \mathcal{I}$ (since $F \in \mathcal{I}$).
Hence,

$$\text{fuzzy} - \mathcal{I}_\lambda - \lim(u) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10}$$

on the other hand, if $(k, j, i) \notin F$ then for an arbitrary $\tau > 0$ and $0 < \lambda < 1$,

$$\beta(u_{kji} - (0, 1, -1)^t, \tau) = \frac{0}{\tau + 0} = 0 < 1 - \lambda. \tag{11}$$

Here, we observe that, the set

$$\left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - (0, 1, -1)^t, \tau\right) < 1 - \lambda\right\} = \mathcal{N}^3 - F \in \mathcal{F}(\mathcal{I}) \tag{12}$$

hence,

$$\text{fuzzy} - \mathcal{I}_\lambda\text{−anti} - \lim(u) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{13}$$

□

Definition 3.6. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. Then for $0 < \lambda < 1$, a triple sequence $u = (u_{kji})$ of the elements of* ***FANLS*** *$(\mathcal{V}, \beta, \diamond)$ is called fuzzy $\mathcal{I}_\lambda$− Cauchy sequence if for all $\tau > 0$, there exist $r, q, p \in \mathcal{N}$ such that for all $k \geq r$, $j \geq q$ and $i \geq p$ the set*

$$\left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - u_{rqp}, \tau\right) < 1 - \lambda\right\} \in \mathcal{I}. \tag{14}$$

Definition 3.7. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. Then for $0 < \lambda < 1$, a triple sequence $u = (u_{kji})$ of the elements* ***FANLS*** *$(\mathcal{V}, \beta, \diamond)$ is called fuzzy $\mathcal{I}_\lambda$−anti Cauchy sequence, if for all $\tau > 0$ there exist $r, q, p \in \mathcal{N}$ such that for all $k \geq r$, $j \geq q$ and $i \geq p$ the set*

$$\left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - u_{rqp}, \tau\right) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}). \tag{15}$$

Definition 3.8. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. Then for $0 < \lambda < 1$, a triple sequence $u = (u_{kji})$ of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ is called fuzzy $\mathcal{I}_\lambda-$ complete, if every fuzzy $\mathcal{I}_\lambda-$ Cauchy sequence in $\mathcal{V}$ is fuzzy $\mathcal{I}_\lambda-$ cgt in $\mathcal{V}$.*

Definition 3.9. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. Then for $0 < \lambda < 1$, a triple sequence $u = (u_{kji})$ of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ is called fuzzy $\mathcal{I}_\lambda-$anti complete, if every fuzzy $\mathcal{I}_\lambda-$ anti Cauchy sequence in $\mathcal{V}$ is fuzzy anti $\mathcal{I}_\lambda-$ cgt in $\mathcal{V}$.*

Remark 3.10. *Let $(\mathcal{V}, \beta, \diamond)$ be a fuzzy anti norm linear space then an open ball $3^{B_u(r)}$, centred at u with radius r with respect to t-conorm $\diamond$ is the collection of triple sequences $(w_{kji}) \in l^3_\infty$ such that $\beta(u_{kji} - w_{kji}, \tau) < r$ or,*

$$3^{B_u(r)} = \left\{(w_{kji}) \in l^3_\infty : \beta(u_{kji} - w_{kji}, \tau) < r\right\}, \tag{16}$$

where l^3_∞ stands for the space of triple bounded sequences.

Definition 3.11. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. Then a triple sequence $u = (u_{kji})$ of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ is called $\mathcal{I}$ -convergent to a point ξ with fuzzy anti-norm β, if for each $\epsilon > 0$ and $\tau > 0$, we have*

$$\left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) \geq \epsilon\right\} \in \mathcal{I}. \tag{17}$$

We denote the case as $(\beta) - \mathcal{I} - \lim u \to \xi$.

4. Main Theorems

Theorem 4.1. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. If a triple sequence $u = (u_{kji})$ of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ is $\mathcal{I}$-convergent to ξ, then $u = (u_{kji})$ fuzzy $\mathcal{I}_\beta-$ anti convergent to ξ.*

Proof. Let $(\beta) - \mathcal{I} - \lim u = \xi$, then for every $\epsilon > 0$ and $\tau > 0$ we obtain

$$\mathbb{M} = \left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) \geq \epsilon\right\} \in \mathcal{I},$$

which implies

$$\mathbb{M}^c = \left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon\right\} = \mathcal{N}^3 - \mathbb{M} \in \mathcal{F}(\mathcal{I})$$

Hence, $u \overset{a-\mathcal{I}_\beta}{\longrightarrow} \xi$. □

Theorem 4.2. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}$, triple sequence $u = (u_{kji})$ of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ is fuzzy anti-$\mathcal{I}_\beta$ convergent to ξ if and only if it is fuzzy $\mathcal{I}_\lambda-$anti cgt to ξ.*

Proof. Let $u = (u_{jki})$ be fuzzy- $\mathcal{I}_\beta-$anti cgt to ξ. Then for every $\epsilon > 0$ and $\tau > 0$, we have

$$\mathbb{A} = \left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon\right\} \in \mathcal{F}(\mathcal{I}). \tag{18}$$

For every $\epsilon > 0$, we can choose $0 < \lambda < 1$ such that $\epsilon < 1 - \lambda$ and

$$\left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon\right\} \subseteq \left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda\right\}.$$

$$\Rightarrow \left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}) \tag{19}$$

Conversely
let $u = (u_{jki})$ be fuzzy- $\mathcal{I}_\lambda-$anti cgt to ξ, then for every $0 < \lambda < 1$ and $\tau > 0$, we have

$$\mathbb{A} = \left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}). \tag{20}$$

For every $0 < \lambda < 1$ we can find $\epsilon > 0$ such that $1 - \lambda < \epsilon$ and

$$\left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda\right\} \subseteq \left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon\right\}. \tag{21}$$

$$\Rightarrow \left\{(k, j, i) \in \mathcal{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon\right\} \in \mathcal{F}(\mathcal{I}). \tag{22}$$

□

Theorem 4.3. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. If a triple sequence $u = (u_{kji})$ of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ with idempotent t−conorm $\diamond$, is fuzzy -$\mathcal{I}_\lambda$−anti cgt to ξ, then fuzzy $\mathcal{I}_\lambda$− anti limit ξ is unique.*

Proof. Let the fuzzy-$\mathcal{I}_\lambda$− anti $\lim(u) = \xi$. If possible, we suppose there is another limit η such that fuzzy -$\mathcal{I}_\lambda$−anti $\lim(u) = \eta$. Now for every $0 < \lambda < 1$ and $\tau > 0$ we have,

$$A = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - \xi, \tau\right) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I})$$

and

$$A_1 = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - \eta, \tau\right) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}).$$

Let $(l, m, n) \in A \cap A_1$. Now since $\diamond$ is idempotent t− conorm then for every $0 < \lambda < 1$, we have $(1-\lambda)\diamond(1-\lambda) < 1 - \lambda$. Hence,

$$\begin{aligned}\beta(\xi - \eta, \tau) = \beta(\xi - u_{lmn} + u_{lmn} - \eta, \tau) &\le \beta\left(\xi - u_{lmn}, \frac{\tau}{2}\right) \diamond \beta\left(u_{lmn} - \eta, \frac{\tau}{2}\right) \\ &< (1 - \lambda) \diamond (1 - \lambda) < 1 - \lambda, \quad \forall \ \lambda.\end{aligned} \tag{23}$$

Now we can find $\epsilon > 0$ with respect to every $0 < \lambda < 1$ such that $1 - \lambda < \epsilon$. Thus we would have

$$\beta(\xi - \eta, \tau) < \epsilon \Rightarrow \xi - \eta = \theta \Rightarrow \xi = \eta. \tag{24}$$

□

Theorem 4.4. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. If the two triple sequences $u = (u_{kji})$ and $v = (v_{kji})$ of the elements **FANLS** $(\mathcal{V}, \beta, \diamond)$ with idempotent t-conorm $\diamond$, are such that fuzzy-$\mathcal{I}_\lambda$− anti* $\lim(u) = \xi$ *and fuzzy anti-$\mathcal{I}_\lambda$− anti* $\lim(v) = \eta$, *then fuzzy- $\mathcal{I}_\lambda$−anti-*$\lim(u + v) = \xi + \eta$.

Proof. For $0 < \lambda < 1$ and $\tau > 0$, we have

$$A_1 = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - \xi, \frac{\tau}{2}\right) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}) \tag{25}$$

and

$$A_2 = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(v_{kji} - \eta, \frac{\tau}{2}\right) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}). \tag{26}$$

Let $(l, m, n) \in A_1 \cap A_2$. Now for every $0 < \lambda < 1$, we have $(1 - \lambda) \diamond (1 - \lambda) < 1 - \lambda$ and then

$$\begin{aligned}\beta(u_{lmn} + v_{lmn} - (\xi + \eta), \tau) &\le \beta\left(u_{lmn} - \xi, \frac{\tau}{2}\right) \diamond \beta\left(v_{lmn} - \eta, \frac{\tau}{2}\right) \\ &< (1 - \lambda) \diamond (1 - \lambda) < 1 - \lambda, \quad \forall \ \lambda.\end{aligned} \tag{27}$$

Hence we can find

$$A_1 \cap A_2 = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} + v_{kji} - (\xi + \eta), \frac{\tau}{2}\right) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}) \tag{28}$$

which means, $fuzzy - \mathcal{I}$−anti $\lim(u + v) = \xi + \eta$. □

Theorem 4.5. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. If the triple sequences $u = (u_{kji})$ of the elements of* ***FANLS*** $(\mathcal{V}, \beta, \diamond)$ *with idempotent t-conorm $\diamond$, is fuzzy -$\mathcal{I}_\lambda$−anti converges to ξ, then fuzzy- $\mathcal{I}_\lambda$− anti $\lim(\alpha u) = \alpha\xi$, where α is a scalar.*

Proof. Case I. If $\alpha = 0$, then theorem is obvious.
Case II. If $\alpha \neq 0$ and for every $0 < \lambda < 1$ and $\tau > 0$, we have

$$A = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - \xi, \tau\right) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}). \tag{29}$$

Since $\tau > 0$ is arbitrary, therefore we can replace it by $\frac{t}{|\alpha|}$, in (29) where $t > 0$ and $\alpha \neq 0$, which follows

$$A = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - \xi, \frac{t}{|\alpha|}\right) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}). \tag{30}$$

Equivalently,

$$A = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(\alpha u_{kji} - \alpha\xi, t\right) < 1 - \lambda\right\} \in \mathcal{F}(\mathcal{I}). \tag{31}$$

Since $t > 0$ was arbitrary, hence the theorem is proved. □

Theorem 4.6. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. If the triple sequences $u = (u_{kji}) \in l^3_\infty$ of the elements of* ***FANLS*** $(\mathcal{V}, \beta, \diamond)$ *with idempotent t-conorm $\diamond$, is fuzzy anti-$\mathcal{I}_\beta$ convergent to ξ, then fuzzy - $\mathcal{I}_\beta$− anti $\lim(u^2) = \xi^2$.*

Proof. Let $M > 0$ such that $\sup(u_{kji}) = M$, For all $k, j, i \in \mathcal{N}$and for every $\epsilon > 0$ and $\tau > 0$, we have

$$A = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - \xi, \tau\right) < \epsilon\right\} \in \mathcal{F}(\mathcal{I}). \tag{32}$$

Now for $\tau > 0$, *let*

$$A_1 = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u^2_{kji} - \xi^2, \tau\right) < \epsilon\right\}. \tag{33}$$

We now wish to prove that $A_1 \in \mathcal{F}(\mathcal{I})$. For this it suffices to show $A \subseteq A_1$.
Let $(l, m, n) \in A$. Then $\beta(u_{lmn} - \xi, \tau) < \epsilon$ and

$$\begin{aligned}\beta\left(u^2_{lmn} - \xi^2, \tau\right) &= \beta\left((u_{lmn} - \xi)(u_{lmn} + \xi), \tau\right) = \beta\left((u_{lmn} - \xi), \frac{\tau}{|u_{lmn} + \xi|}\right)\\ &= \beta\left((u_{lmn} - \xi), \frac{\tau}{M + |\xi|}\right) = \beta\left(u_{lmn} - \xi, t\right) < \epsilon, \text{by (32)}\end{aligned} \tag{34}$$

where, $t = \frac{\tau}{M+|\xi|} > 0$. Therefore $(l, m, n) \in A_1$,which implies that

$$A_1 = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u^2_{kji} - \xi^2, \tau\right) < \epsilon\right\} \in \mathcal{F}(\mathcal{I}). \tag{35}$$

□

Theorem 4.7. *Let $\mathcal{I}$ be an ideal on $\mathcal{N}^3$. If the triple sequences $u = (u_{kji}) \in l^3_\infty$, $u_{jki} > 0$ of the elements of* ***FANLS*** $(\mathcal{V}, \beta, \diamond)$ *with idempotent t-conorm $\diamond$, is fuzzy -$\mathcal{I}_\beta$−anti convergent to ξ, then*

$$\textit{fuzzy-}\mathcal{I}_\beta - \textit{anti} \lim\left(\frac{1}{u}\right) = \frac{1}{\xi}.$$

Proof. Let fuzzy-$\mathcal{I}_\beta$−anti $\lim(u) = \xi$, and $u = (u_{kji}) \in l^3_\infty$. Then there exists some $M > 0$ such that $\sup(u_{kji}) = M$ for all $k, j, i \in \mathcal{N}$. Then for every $\epsilon > 0$ and $\tau > 0$, we have

$$A = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - \xi, \tau\right) < \epsilon\right\} \in \mathcal{F}(\mathcal{I}). \tag{36}$$

Let

$$A_1 = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(\frac{1}{u_{kji}} - \frac{1}{\xi}, \tau\right) < \epsilon\right\}. \tag{37}$$

We now wish to prove that $A_1 \in \mathcal{F}(\mathcal{I})$. For the requirement it is suffices to prove $A \subseteq A_1$.
Let $(l, m, n) \in A$. Then $\beta\,(u_{lmn} - \xi, \tau) < \epsilon$.*Thus*

$$\begin{aligned}\beta\left(\frac{1}{u_{lmn}} - \frac{1}{\xi}, t\right) &= \beta\left(\frac{u_{lmn} - \xi}{u_{lmn}\xi}, t\right) = \beta\,(u_{lmn} - \xi, t|u_{lmn}\xi|) \\ &= \beta\,(u_{lmn} - \xi, tM\xi) = \beta\,(u_{lmn} - \xi, \tau) < \epsilon, \text{by (36)}\end{aligned} \tag{38}$$

where $\tau = tM\xi > 0$. Hence, we obtain $(l, m, n) \in A_1$,which implies that

$$A_1 = \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(\frac{1}{u_{kji}} - \frac{1}{\xi}, \tau\right) < \epsilon\right\} \in \mathcal{F}(\mathcal{I}). \tag{39}$$

□

Remark 4.8. *Let $(\mathcal{V}, \beta, \diamond)$ be a **FANLS** with idempotent t-conorm $\diamond$. Then the triple sequence $u = (u_{kji})$ of the terms of $(\mathcal{X}, \beta, \diamond)$ is said to be a Cauchy sequence if for all $\tau > 0$ and $\epsilon > 0$ there exist $k_0, j_0, i_0 \in \mathcal{N}$, such that for all $r, k \geq k_0$, $q, j \geq j_0$ and $p, i \geq i_0$, we have*

$$\beta\left(u_{rqp} - u_{kji}, \tau\right) < \epsilon. \tag{40}$$

Remark 4.9. *By $\Omega_\lambda^{\mathcal{I}^3}$, we denote the space of all triple real bounded fuzzy $\mathcal{I}_\lambda-$ convergent sequences, where $0 < \lambda < 1$ and $\mathcal{I}$ is an ideal on $\mathcal{N}$,of the elements of **FANLS** $(\mathcal{V}, \beta, \diamond)$ with idempotent t-conorm $\diamond$.*
Moreover,

$$\Omega_\lambda^{\mathcal{I}^3} = \left\{(u_{kji}) \in l_\infty^3 : \exists\ \ u \in \mathcal{R} : \left\{(k, j, i) \in \mathcal{N}^3 : \beta\left(u_{kji} - u, \tau\right) < 1 - \lambda\right\} \in \mathcal{I}\right\}. \tag{41}$$

Theorem 4.10. *Let $(\mathcal{X}, \beta, \diamond)$ be a **FANLS** with idempotent t-conorm $\diamond$. Then the space $\Omega_\lambda^{\mathcal{I}^3}$ defined in (Remark 4.9), is complete with respect to fuzzy anti norm β, where β is given by (Remark 2.5).*

Proof. Let (u_{kji}^n) be a Cauchy sequence in $\Omega_\lambda^{\mathcal{I}^3}$. We now wish to prove that there exists $(u_{kji}) \in \Omega_\lambda^{\mathcal{I}^3}$ such that for all $\epsilon > 0$ and $\tau > 0$, there exists some $n_\epsilon \in \mathcal{N}$ and for $n \geq n_\epsilon$, we have

$$\beta\left(\left(u_{kji}^n\right) - u_{kji}, \tau\right) < \epsilon. \tag{42}$$

Since (u_{kji}^n) is a Cauchy sequence, then there exists some $n_\epsilon \in \mathcal{N}$ such that for all $n, m \geq n_\epsilon$ we have

$$||u_{kji}^n - u_{kji}^m|| < \epsilon, \tag{43}$$

which implies that

$$\beta\left(u_{kji}^n - u_{kji}^m, \tau\right) < \epsilon \ \text{ for all } \ \ \epsilon > 0. \tag{44}$$

Since (u_{jki}^n) bounded real triple sequence and $\mathcal{R}$ is complete with respect to max$||.||$, therefore there exists a (u_{kji}) such that

$$\lim_{n\to\infty} u_{kji}^n = u_{kji}. \tag{45}$$

and hence

$$\beta\left(\left(u_{kji}^{n}\right)-u_{kji},\tau\right)<\epsilon \tag{46}$$

We now prove that $(u_{kji}) \in \Omega_{\lambda}^{\mathcal{I}^3}$. For this, it is sufficient to show that there exist $u \in \mathcal{R}$ such that fuzzy-$\mathcal{I}_{\lambda} - \lim\left(u_{kji}\right) = u$.
Let

$$\left\{(k,j,i)\in\mathcal{N}^3:\beta\left(u_{kji}^{n}-u_n,\tau\right)<1-\lambda\right\}\in\mathcal{I}. \tag{47}$$

Take $p > n_{\epsilon}$. And then

$$\beta\left(u_{kji}^{p}-u_{kji},\tau\right)<\epsilon<1-\lambda. \tag{48}$$

So, for every $\epsilon > 0$, we can choose $0 < \lambda < 1$ such that $\epsilon < 1 - \lambda$. Let

$$E=\left\{(k,j,i)\in\mathcal{N}^3:\beta\left(u_{kji}^{p}-u_{kji},\tau\right)<1-\lambda\right\} \tag{49}$$

and

$$F=\left\{(k,j,i)\in\mathcal{N}^3:\beta\left(u_{kji}^{p}-u_p,\tau\right)<1-\lambda\right\}\in\mathcal{I}. \tag{50}$$

The sequence (u_n) is convergent to u with respect to β. Then there exists $n_{\epsilon} \in \mathcal{N}$ such that for all $p \geq n_{\epsilon}$, we have

$$\beta\left(u_p-u,\tau\right)<\epsilon \quad \text{for all} \quad \epsilon>0. \tag{51}$$

Now we prove that $E \subseteq F$. Let $(l,m,n) \in E$. Then

$$\beta\left(u_{lmn}-u,\tau\right)<1-\lambda, \tag{52}$$

and

$$\begin{aligned}\beta\left(u_{lmn}^{p}-u_p,3\tau\right)&=\beta\left(u_{lmn}^{p}-u_{lmn}+u_{lmn}-u+u-u_p,3\tau\right)\\&\leq\beta\left(u_{lmn}^{p}-u_{lmn},\tau\right)\diamond\beta\left(u_{lmn}-u,\tau\right)\diamond\beta\left(u_p-u,\tau\right)\\&<(1-\lambda)\diamond(1-\lambda)\diamond(1-\lambda)\\&=(1-\lambda)\diamond(1-\lambda)=(1-\lambda).\end{aligned} \tag{53}$$

Since $\tau > 0$ was arbitrary, for all $t > 0$ and $t = 3\tau$, therefore

$$\beta\left(u_{lmn}^{p}-u_p,t\right)<1-\lambda \tag{54}$$

thus $(l,m,n) \in F$, and hence $E \subseteq F$. □

Conclusion

Our proposed work establishes more general results on the algebra of limits of ideally convergent sequences spaces with respect to fuzzy anti-norm linear space and discusses some examples to elaborate the notion of the spaces. The study also includes the completeness property of a special sequence space that is original.

Data Availability

No data is used in this study.

Conflict of Interest

The authors declare that there is no conflict of interest in the publication of this article.

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