Filomat 37:15 (2023), 5005–5015
https://doi.org/10.2298/FIL2315005A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by a modification of operators of exponential type associated with the Baskakov operators

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Dedicated to Prof. Vijay Gupta on the occasion of his 60-th birthday

Abstract. In the current article, we modify the operators of exponential type associated with the Baskakov operators so as to preserve the linear functions. Initially, we obtain the moments and central moments for the modified form. Further, we derive few convergence results including Voronovskaja type asymptotic formula and validate our results through graphical illustration. In the end, we obtain the difference estimate between the exponential type operators associated with the Baskakov operators and its modified form.

1. Introduction

The following sequence of linear positive operators was proposed in [11, (3.14)]:

$$(R_n f)(x) = \sum_{k=0}^{\infty} r_{n,k}(x) f\left(\frac{k}{n}\right), \quad (0 \leq x < \infty) \tag{1}$$

where

$$r_{n,k}(x) = e^{-(n+k)x/(x+1)} \frac{n(n+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k.$$

These operators are of exponential type and preserve both constant as well as linear functions. Numerous academics have explored a number of exponential operators during the past fifty years, including Bernstein, Post-Widder, Szász, Baskakov, Gauss Weierstrass, etc. The approximation properties of these operators and of their modified forms have been studied by several researchers in [1–4, 7–10]. Recently, the authors in [8] introduced the hybrid operators with Baskakov basis function $v_{n,k}$ as the weights in the following way:

$$(V_n f)(x) = \sum_{k=0}^{\infty} r_{n,k}(x) \frac{\langle v_{n,k}, f\rangle}{\langle v_{n,k}, 1\rangle}, \quad x \geq 0 \tag{2}$$

2020 *Mathematics Subject Classification*. 41A25.
Keywords. Exponential type approximation operators; Baskakov basis function; Modulus of continuity; Asymptotic formula; Difference estimate.
Received: 01 September 2022; Accepted: 26 December 2022
Communicated by Snezana Živković Zlatanović
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where $\langle h, f\rangle = \int_0^\infty h(t)f(t)dt$ and Baskakov basis functions is given as

$$v_{n,k}(t) = \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}.$$

It can be easily seen that $\langle v_{n,k}, 1\rangle = 1/(n-1)$. These operators preserve constant but fail to reproduce linear functions, which motivated us to define the modification of the operators (2) in the following way:

$$(\widetilde{V}_n f)(x) = \sum_{k=0}^{\infty} r_{n,k}(a_n(x))\frac{\langle v_{n,k}, f\rangle}{\langle v_{n,k}, 1\rangle}, \tag{3}$$

where

$$a_n(x) = \frac{(n-2)x-1}{n}.$$

Obviously, $(\widetilde{V}_n e_0)(x) = 1$ and it is observed that these operators reproduce linear functions. We may remark here that the operators are defined for $x \in [\frac{-1}{n}, \infty)$ such that $a_n(x) > 0$, which is a flexible interval.

2. Auxiliary Results

In the sequel, we require the following basic results.

Lemma 2.1. *[7] For m-th moment denoted by* $(R_n e_m)(x) = \sum_{k=0}^{\infty} r_{n,k}(x)\left(\frac{k}{n}\right)^m$, $m \in N\cup\{0\}$, $x \in [\frac{-1}{n}, \infty)$ *we have:*

$$(R_n e_{m+1})(x) = \frac{x(x+1)^2}{n}[(R_n e_m)(x)]' + x(R_n e_m)(x).$$

Alternatively, we may observe from the above recurrence relation that for $c_j \neq 0$ and $j \geq 0$, one has

$$\begin{aligned}\sum_{j\geq 0}(R_n c_j e_j)(x) &= c_0 + c_1 x + c_2\left(x^2 + \frac{x(x+1)^2}{n}\right) + c_3\left(x^3 + \frac{3x^2(x+1)^2}{n} + \frac{x(x+1)^3(1+3x)}{n^2}\right)\\ &\quad + c_4\left(x^4 + \frac{6x^3(x+1)^2}{n} + \frac{x^2(x+1)^3(7+15x)}{n^2} + \frac{x(x+1)^4(1+10x+15x^2)}{n^3}\right) + \ldots\end{aligned}$$

Lemma 2.2. *For $e_m(t) = t^m$, the following relations hold:*

$$\begin{aligned}(\widetilde{V}_n e_0)(x) &= 1,\\ (\widetilde{V}_n e_1)(x) &= x,\\ (\widetilde{V}_n e_2)(x) &= \frac{1}{(n-3)(n-2)n^2}\Big\{\left(n^3-6n^2+12n-8\right)x^3 + \left(n^4-2n^3-7n^2+20n-12\right)x^2\\ &\quad +(2n^3-8n^2+11n-6)x - n^2 + +2n - 1\Big\},\end{aligned}$$

$$
\begin{aligned}
(\widetilde{V}_n e_3)(x) &= \frac{1}{(n-4)(n-3)(n-2)n^4}\Big\{\Big(3n^5-30n^4+120n^3-240n^2+240n-96\Big)x^5\\
&+(3n^6-14n^5-23n^4+264n^3-632n^2+640n-240t)x^4+(n^7-18n^5-12n^4\\
&+294n^3-708n^2+680n-240)x^3+(6n^6-24n^5-12n^4+204n^3-414n^2\\
&+360n-120)x^2+\Big(3n^5-24n^4+78n^3-124n^2+95n-30\Big)x\\
&-4n^4+12n^3-15n^2+10n-3\Big\},\\
(\widetilde{V}_n e_4)(x) &= \frac{1}{(n-5)(n-4)(n-3)(n-2)n^6}\Big\{(15n^7-210n^6+1260n^5-4200n^4+8400n^3\\
&-10080n^2+6720n-1920)x^7+(15n^8-110n^7-45n^6+3060n^5-13900n^4\\
&+30720n^3-37680n^2+24640n-6720)x^6+(6n^9-8n^8-209n^7+470n^6\\
&+3675n^5-21662n^4+50216n^3-61072n^2+38640n-10080)x^5+(n^{10}+4n^9\\
&-6n^8-308n^7+924n^6+2906n^5-20637n^4+47384n^3-55480n^2+33600n\\
&-8400)x^4+(12n^9-24n^8-252n^7+752n^6+1886n^5-12908n^4+27653n^3\\
&-30430n^2+17500n-4200)x^3+(30n^8-192n^7+276n^6+1064n^5-5209n^4\\
&+9886n^3-10055n^2+5460n-1260)x^2+(-4n^7-40n^6+390n^5-1224n^4\\
&+1987n^3-1850n^2+945n-210)x-15n^6+60n^5-126n^4+172n^3-146n^2+70n-15\Big\}.
\end{aligned}
$$

Lemma 2.3. *If we denote $\mu_{n,m}(x)=(\widetilde{V}_n(e_1-xe_0)^m)(x)$, then*

$$
\begin{aligned}
\mu_{n,0}(x) &= 1,\\
\mu_{n,1}(x) &= 0,\\
\mu_{n,2}(x) &= \frac{1}{(n-3)(n-2)n^2}\Big\{-n^2+\Big(n^3-6n^2+12n-8\Big)x^3+\Big(3n^3-13n^2+20n-12\Big)x^2\\
&+(2n^3-8n^2+11n-6)x+2n-1\Big\},\\
\mu_{n,3}(x) &= \frac{1}{(n-4)(n-3)(n-2)n^4}\Big\{\Big(3n^5-30n^4+120n^3-240n^2+240n-96\Big)x^5+(16n^5\\
&-131n^4+432n^3-728n^2+640n-240)x^4+(31n^5-204n^4+570n^3-852n^2\\
&+680n-240)x^3+\Big(24n^5-141n^4+354n^3-486n^2+360n-120\Big)x^2\\
&+\Big(6n^5-42n^4+105n^3-136n^2+95n-30\Big)x-4n^4+12n^3-15n^2+10n-3\Big\},\\
\mu_{n,4}(x) &= \frac{1}{(n-5)(n-4)(n-3)(n-2)n^6}\Big\{(15n^7-210n^6+1260n^5-4200n^4+8400n^3-10080n^2\\
&+6720n-1920)x^7+(3n^8+70n^7-1125n^6+6420n^5-19660n^4+35904n^3-39600n^2\\
&+24640n-6720)x^6+(18n^8+119n^7-2462n^6+13355n^5-37822n^4+63976n^3-65872n^2\\
&+38640n-10080)x^5+(39n^8+100n^7-2844n^6+14666n^5-38957n^4+61944n^3-60280n^2\\
&+33600n-8400)x^4+(36n^8+54n^7-1894n^6+9266n^5-23348n^4+35333n^3-32830n^2\\
&+17500n-4200)x^3+(12n^8+30n^7-750n^6+3414n^5-8189n^4+11906n^3-10655n^2\\
&+5460n-1260)x^2+(12n^7-168n^6+690n^5-1564n^4+2199n^3-1910n^2+945n-210)x\\
&-15n^6+60n^5-126n^4+172n^3-146n^2+70n-15\Big\}.
\end{aligned}
$$

Consequently, $\mu_{n,r}(x)=O\Big(n^{-[\frac{(r+1)}{2}]}\Big)$.

Lemma 2.4. *For the kernel $r_{n,k}(a_n(x))$ of the operators defined by (3), we have*

$$[r_{n,k}(a_n(x))]' = \frac{(k - na_n(x))}{a_n(x)[a_n(x)+1]^2} r_{n,k}(a_n(x))[a_n(x)]'$$

Proof. Consider

$$\begin{aligned}
[r_{n,k}(a_n(x))]' &= \frac{kn(k+n)^{k-1}}{k!}\left(\frac{n-2}{n\left(\frac{(n-2)x-1}{n}+1\right)} - \frac{(n-2)((n-2)x-1)}{n^2\left(\frac{(n-2)x-1}{n}+1\right)^2}\right)\left(\frac{(n-2)x-1}{n\left(\frac{(n-2)x-1}{n}+1\right)}\right)^{k-1} \\
&\exp\left(\frac{(-k-n)((n-2)x-1)}{n\left(\frac{(n-2)x-1}{n}+1\right)}\right) + \frac{n(k+n)^{k-1}}{k!}\left(\frac{(n-2)x-1}{n\left(\frac{(n-2)x-1}{n}+1\right)}\right)^{k} \\
&\left(\frac{(n-2)(-k-n)}{n\left(\frac{(n-2)x-1}{n}+1\right)} - \frac{(n-2)(-k-n)((n-2)x-1)}{n^2\left(\frac{(n-2)x-1}{n}+1\right)^2}\right)\exp\left(\frac{(-k-n)((n-2)x-1)}{n\left(\frac{(n-2)x-1}{n}+1\right)}\right) \\
&= \left(\frac{-(-2+n)n^2(-1-k+(-2+n)x)}{(-1+(-2+n)x)(-1+n-2x+nx)^2}\right) \\
&\left(\frac{n(k+n)^{k-1}\left(\frac{(n-2)x-1}{n\left(\frac{(n-2)x-1}{n}+1\right)}\right)^{k}\exp\left(\frac{(-k-n)((n-2)x-1)}{n\left(\frac{(n-2)x-1}{n}+1\right)}\right)}{k!}\right) \\
&= \frac{(k-na_n(x))}{a_n(x)[a_n(x)+1]^2} r_{n,k}(a_n(x))[a_n(x)]'.
\end{aligned}$$

Hence, the required relation holds. □

3. Convergence Estimates

Let us assume that $C_B[0,\infty)$ represent the collection of all bounded and continuous functions defined on $[0,\infty)$ equipped with the following norm:

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

For $f \in C_B[0,\infty)$, the K-functional is defined as

$$K_2(f,\xi) = \inf\{\| f - g \| + \xi \| g'' \|, g \in C_B^2[0,\infty)\},$$

where $\xi > 0$ and $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. It satisfies the inequality

$$K_2(f,\xi) \le M\omega_2(f,\xi^{1/2}), \tag{4}$$

for a positive constant M. Here, ω_2 is the modulus of continuity of second order.

Theorem 3.1. *For $f \in C_B[0,\infty)$, we have*

$$|(\widetilde{V}_n f)(x) - f(x)| \le M\omega_2\left(f, \sqrt{\mu_{n,2}(x)}\right)$$

where M is an absolute constant.

Proof. Let $g \in C_B^2[0,\infty)$. By Taylor's expansion, we may write

$$(\widetilde{V}_n g)(x) - g(x) = (\widetilde{V}_n(t-x))(x)g'(x) + (\widetilde{V}_n(\int_x^t (t-u)g''(u)du))(x).$$

Using Lemma 2.3 and the fact that

$$\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2\|g''\|,$$

we get

$$\left|(\widetilde{V}_n g)(x) - g(x)\right| \leq \mu_{n,2}(x)\|g''\|.$$

Using Lemma 2.3, the above inequality can be written as

$$\begin{aligned} |(\widetilde{V}_n g)(x) - g(x)| &\leq \frac{1}{(n-3)(n-2)n^2}\Big\{-n^2 + \Big(n^3 - 6n^2 + 12n - 8\Big)x^3 \\ &\quad + \Big(3n^3 - 13n^2 + 20n - 12\Big)x^2 + (2n^3 - 8n^2 + 11n - 6)x + 2n - 1\Big\}\|g''\|. \end{aligned}$$

As $|(\widetilde{V}_n f)(x)| \leq \|f\|$, this leads to

$$\begin{aligned} |(\widetilde{V}_n f)(x) - f(x)| &\leq |(\widetilde{V}_n(f-g))(x) - (f-g)(x)| + |(\widetilde{V}_n g)(x) - g(x)| \\ &\leq 2\|f-g\| + \frac{1}{(n-3)(n-2)n^2}\Big\{-n^2 + \Big(n^3 - 6n^2 + 12n - 8\Big)x^3 \\ &\quad + \Big(3n^3 - 13n^2 + 20n - 12\Big)x^2 + (2n^3 - 8n^2 + 11n - 6)x + 2n - 1\Big\}\|g''\|. \end{aligned}$$

So, considering infimum on the right hand side over $g \in C_B^2[0,\infty)$ and using the property of K-functional as defined in (4), we get the required result. □

Remark 3.2. *Under the assuptions of Theorem 3.1, for the original operators discussed in (2), we have*

$$|(V_n f)(x) - f(x)| \leq M\omega_2\left(f, \sqrt{\eta_2(x)}\right) + \omega\left(f, \eta_1(x)\right),$$

where

$$\eta_1(x) = \frac{1+2x}{n-2},$$

$$\eta_2(x) = \frac{nx^3 + 3nx^2 + 2nx + 6x^2 + 6x + 2}{(n-2)(n-3)}$$

and ω is usual modulus of continuity.
It is clear that the convergence estimate obtained in Theorem (3.1) for the modified operators (3) is better than that for the original operators (2).

Next, we prove Voronovskaja type asymptotic formula for $\left(\frac{d}{d\omega}\widetilde{V}_n(f,\omega)\right)_{\omega=x}$. Let

$$C_\gamma[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leq Ct^\gamma, \text{for some } \gamma > 0,\ t \in [0,\infty)\}.$$

Theorem 3.3. *Let $f \in C_\gamma[0,\infty)$ admitting the derivative of 3-rd order at a fixed point $x \in (0,\infty)$, we have*

$$\lim_{n\to\infty} n\left(\left(\frac{\partial}{\partial\omega}(\widetilde{V}_n f)(\omega) - f'(x)\right)_{\omega=x}\right) = \frac{f''(x)}{2}(3x^2 + 6x + 2) + f'''(x)\left\{\frac{x\left(x^2 + 3x + 2\right)}{2}\right\}.$$

Proof. From the Taylor's theorem, we may write

$$f(t)=\sum_{s=0}^{3}\frac{(t-x)^s}{s!}f^{(s)}(x)+\psi(t,x)(t-x)^3,\ \ t\in[0,\infty), \tag{5}$$

where $\lim\limits_{t\to x}\psi(t,x)=0$.
From equation (5), we obtain

$$\begin{aligned}\Big(\frac{\partial}{\partial\omega}(\widetilde{V}_n f(t))(\omega)\Big)_{\omega=x} &= f'(x)\Big(\frac{\partial}{\partial\omega}((\widetilde{V}_n t)(\omega)-x)\Big)_{\omega=x}\\ &+\frac{f''(x)}{2}\Big(\frac{\partial}{\partial\omega}((\widetilde{V}_n t^2)(\omega)-2x(\widetilde{V}_n t)(\omega)+x^2)\Big)_{\omega=x}\\ &+\frac{f'''(x)}{3!}\Big(\frac{\partial}{\partial\omega}((\widetilde{V}_n t^3)(\omega)-3x(\widetilde{V}_n t^2)(\omega)+3x^2(\widetilde{V}_n t)(\omega)-x^3)\Big)_{\omega=x}\\ &+\Big(\frac{\partial}{\partial\omega}((\widetilde{V}_n\psi(t,x))(t-x)^3)(\omega)\Big)_{\omega=x}.\end{aligned}$$

Using Lemma 2.2, we get

$$\begin{aligned}\Big(\frac{\partial}{\partial\omega}(\widetilde{V}_n f(t))(\omega)\Big)_{\omega=x} &= f'(x)+\frac{f''(x)}{2}\Big\{\frac{1}{(n-3)n^2}\Big\{\Big(3n^2-12n+12\Big)x^2+\Big(6n^2-14n+12\Big)x\\ &+2n^2-4n+3\Big\}\Big\}+\frac{f'''(x)}{3!}\Big\{\frac{1}{(n-4)(n-3)n^4}\Big\{(15n^4-120n^3+360n^2\\ &-480n+240)x^4+\Big(3n^5+40n^4-336n^3+888n^2-1040n+480\Big)x^3\\ &+(9n^5+36n^4-324n^3+786n^2-840n+360)x^2+(6n^5+12n^4-129n^3\\ &+300n^2-300n+120)x+3n^4-18n^3+42n^2+-40n+15\Big\}\Big\}\\ &+\Big(\frac{\partial}{\partial\omega}((\widetilde{V}_n\psi(t,x))(t-x)^3)(\omega)\Big)_{\omega=x}.\end{aligned}$$

Taking limit as $n\to\infty$ on both sides of the above equation, we have

$$\begin{aligned}\lim_{n\to\infty}n\Big(\Big(\frac{\partial}{\partial\omega}(\widetilde{V}_n f)(\omega)-f'(x)\Big)_{\omega=x}\Big) &= \frac{f''(x)}{2}(3x^2+6x+2)+f'''(x)\Big\{\frac{x\Big(x^2+3x+2\Big)}{2}\Big\}\\ &+\lim_{n\to\infty}n\Big(\frac{\partial}{\partial\omega}((\widetilde{V}_n\psi(t,x))(t-x)^3)(\omega)\Big)_{\omega=x}.\end{aligned}$$

Now, consider

$$I:=\Big(\frac{\partial}{\partial\omega}((\widetilde{V}_n\psi(t,x))(t-x)^3)(\omega)\Big)_{\omega=x}.$$

From Lemma 2.4, we obtain

$$I \ \leq\ [a_n(x)]'\sum_{k=0}^{\infty}\frac{|k-na_n(x)|}{a_n(x)(1+a_n(x))^2}r_{n,k}(a_n(x))\int_0^{\infty}v_{n,k}(t)|\psi(t,x)||t-x|^3dt$$

Since $\psi(t,x) \to 0$ as $t \to x$, for a given $\epsilon > 0$ there exists $\delta > 0$ such that $|\psi(t,x)| < \epsilon$ whenever $|t-x| < \delta$. For $|t-x| \geq \delta$, we have $|\psi(t,x)(t-x)^3| \leq M_1 t^\gamma$, for some $M_1 > 0$. Thus,

$$\begin{aligned} I &\leq [a_n(x)]' \sum_{k=0}^{\infty} \frac{|k-na_n(x)|}{a_n(x)(1+a_n(x))^2} r_{n,k}(a_n(x)) \Big(\epsilon \int_{|t-x|<\delta} v_{n,k}(t)|t-x|^3 dt \\ &\quad + M_1 \int_{|t-x|\geq\delta} v_{n,k}(t) t^\gamma dt \Big) \\ &:= I_1 + I_2. \end{aligned} \tag{6}$$

Using Schwarz inequality for integration and then for summation, we can write

$$\begin{aligned} I_1 &= \epsilon[a_n(x)]' \sum_{k=0}^{\infty} \frac{|k-na_n(x)|}{a_n(x)(1+a_n(x))^2} r_{n,k}(a_n(x)) \int_{|t-x|<\delta} v_{n,k}(t)|t-x|^3 dt \\ &\leq \frac{\epsilon[a_n(x)]'}{a_n(x)(1+a_n(x))^2} \sum_{k=0}^{\infty} |k-na_n(x)| r_{n,k}(a_n(x)) \Big(\int_0^\infty v_{n,k}(t)(t-x)^6 dt \Big)^{1/2} \\ &\leq \frac{\epsilon[a_n(x)]'}{a_n(x)(1+a_n(x))^2} \Big(\sum_{k=0}^{\infty} r_{n,k}(a_n(x))(k-na_n(x))^2 \Big)^{1/2} \Big(\sum_{k=0}^{\infty} r_{n,k}(a_n(x)) \int_0^\infty v_{n,k}(t)(t-x)^6 dt \Big)^{1/2} \\ &= \frac{\epsilon[a_n(x)]'}{a_n(x)(1+a_n(x))^2} \left(n^2 \left(\sum_{k=0}^{\infty} r_{n,k}(a_n(x)) \Big(\frac{k}{n} - a_n(x) \Big)^2 \right) \right)^{1/2} \Big(\sum_{k=0}^{\infty} r_{n,k}(a_n(x)) \int_0^\infty v_{n,k}(t)(t-x)^6 dt \Big)^{1/2} \\ &= \epsilon.O(1), n \to \infty \text{ in view of Lemmas 2.3} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $I_1 \to 0$ as $n \to \infty$. Next

$$\begin{aligned} I_2 &\leq \frac{M_1[a_n(x)]'}{a_n(x)(1+a_n(x))^2} \Big(\sum_{k=0}^{\infty} (k-na_n(x))^2 r_{n,k}(a_n(x)) \Big)^{1/2} \Big(\int_0^\infty v_{n,k}(t) dt \Big)^{1/2} \\ &\quad \times \Big(\sum_{k=0}^{\infty} r_{n,k}(a_n(x)) \int_{|t-x|\geq\delta} v_{n,k}(t) t^{2\gamma} dt \Big)^{1/2} \\ &\leq \frac{M_1[a_n(x)]'}{a_n(x)(1+ca_n(x))} \Big(n^2 \sum_{k=0}^{\infty} r_{n,k}(a_n(x)) \Big(\frac{k}{n} - a_n(x) \Big)^2 \Big)^{1/2} \Big(\sum_{k=0}^{\infty} r_{n,k}(a_n(x)) \int_{|t-x|\geq\delta} v_{n,k}(t) t^{2\gamma} dt \Big)^{1/2} \\ &= \Big\{ n.O\Big(\frac{1}{n^{1/2}} \Big) \Big\} M_1 n^{-m} = O\Big(\frac{1}{n^{\frac{2m-1}{2}}} \Big), \text{ for any } m > 0. \end{aligned}$$

Combining the estimates of I_1 and I_2, we get $nI \to 0$ as $n \to \infty$. Hence, the required result is obtained. □

4. Difference of Operators

Inspired by the work of Gupta et al. [5, 6] related to the difference of operators with distinct basis functions, we present here the difference estimates of exponential type operators associated with the Baskakov operators (2) and its modified form (3).

Here, we consider $F_{n,k}(f) := \frac{\langle v_{n,k}, f \rangle}{\langle v_{n,k}, 1 \rangle}$, then the operators represented by (2) and (3) takes the form

$$(V_n f)(x) = \sum_{k=0}^{\infty} r_{n,k}(x) F_{n,k}(f)$$

and

$$(\widetilde{V}_n f)(x) = \sum_{k=0}^{\infty} r_{n,k}(a_n(x)) F_{n,k}(f)$$

respectively.

Lemma 4.1. *If we denote* $\nu_i^{F_{n,k}} = F_{n,k}(e_1 - \phi^{F_{n,k}} e_0)^i$, *where* $\phi^{F_{n,k}} = F_{n,k}(e_1)$, *then*

$$F_{n,k}(e_r) = \frac{\langle v_{n,k}, e_r \rangle}{\langle v_{n,k}, 1 \rangle} = \frac{(k+r)!(n-r-2)!}{k!(n-2)!}$$

and

$$\phi^{F_{n,k}} = F_{n,k}(e_1) = \frac{k+1}{n-2}.$$

Thus,

$$\begin{aligned}
\nu_2^{F_{n,k}} &= F_{n,k}(e_1 - \phi^{F_{n,k}} e_0)^2 \\
&= \frac{(k+1)(k+n-1)}{(n-3)(n-2)^2}, \\
\nu_3^{F_{n,k}} &= F_{n,k}(e_1 - \phi^{F_{n,k}} e_0)^3 \\
&= F_{n,k}(e_3) - 3F_{n,j}(e_2)\left(\frac{k+1}{n-2}\right) + 3F_{n,j}(e_1)\left(\frac{k+1}{n-2}\right)^2 \\
&\quad -F_{n,k}(e_0)\left(\frac{k+1}{n-2}\right)^3 \\
&= \frac{(k+1)\left(k^2\left(3n^2-21n+40\right) + k\left(6n^2-36n+68\right) + 5n^2-23n+36\right)}{(n-4)(n-3)(n-2)^3}
\end{aligned}$$

and

$$\begin{aligned}
\nu_4^{F_{n,k}} &= F_{n,k}(e_1 - \psi^{F_{n,k}} e_0)^4 \\
&= F_{n,j}(e_4) - 4F_{n,j}(e_3)\left(\frac{k+1}{n-2}\right) + 6F_{n,j}(e_2)\left(\frac{k+1}{n-2}\right)^2 \\
&\quad -4F_{n,j}(e_1)\left(\frac{k+1}{n-2}\right)^3 + F_{n,k}(e_0)\left(\frac{k+1}{n-2}\right)^4 \\
&= \frac{1}{(n-5)(n-4)(n-3)(n-2)^4}\Big\{k^4[(n-2)^3 - 4(n-5)(n-2)^2 + 6(n-5)(n-4) \\
&\quad -3(n-5)(n-4)(n-3)] + k^3[10(n-2)^3 - 28(n-5)(n-2)^2 + 12(n-5)(n-4) \\
&\quad -12(n-5)(n-4)(n-3) + 6(n-5)(n-4)n] + k^2[35(n-2)^3 - 68(n-5)(n-2)^2 \\
&\quad -18(n-5)(n-4)(n-3) + 18(n-5)(n-4)n) + k(50(n-2)^3 - 68(n-5)(n-2)^2 \\
&\quad -12(n-5)(n-4) - 12(n-5)(n-4)(n-3) + 18(n-5)(n-4)n] + 24(n-2)^3 \\
&\quad -24(n-5)(n-2)^2 - 6(n-5)(n-4) - 3(n-5)(n-4)(n-3) + 6(n-5)(n-4)n\Big\}.
\end{aligned}$$

Theorem 4.2. *Let* $f \in C_B[0,\infty]$. *Then for the operators* $(V_n f)(x)$ *and for the modified operators* $(\widetilde{V}_n f)(x)$, *we have*

$$|((V_n - \widetilde{V}_n)f)(x)| \leq \frac{A(x)}{2}\|f''\| + 2\omega_1(f,\alpha_1) + 2\omega_1(f,\alpha_2),$$

with $\|.\| = \sup_{x\in[0,\infty)} |f(x)| < \infty$.

The values of $A(x)$, α_1 *and* α_2 *are indicated in the proof of the theorem.*

Proof. Using Lemmas 2.1 and 4.1 and with the help of software Mathematica, we obtain

$$\begin{aligned} A(x) &= \sum_{k=0}^{\infty} r_{n,k}(x)\nu_2^{F_{n,k}} + \sum_{k=0}^{\infty} r_{n,k}(a_n(x))\nu_2^{F_{n,k}} \\ &= \frac{1}{(n-3)(n-2)^2n^2}\Big\{2n^4x(x+1) + n^3\left(2x^3-2x+1\right) - n^2\left(6x^3+7x^2+2x+2\right) \\ &+ n(2x+1)^2(3x+2) - (2x+1)^3\Big\}. \end{aligned}$$

Also,

$$\begin{aligned} \alpha_1^2 &= \sum_{k=0}^{\infty} r_{n,k}(x)\left[F_{n,k}(e_1)-x\right]^2 \\ &= \frac{nx^3+2(n+2)x^2+(n+4)x+1}{(n-2)^2} \end{aligned}$$

and

$$\begin{aligned} \alpha_2^2 &= \sum_{k=0}^{\infty} r_{n,k}(a_n(x))\left[F_{n,k}(e_1)-x\right]^2 \\ &= \frac{((n-2)x-1)(nx+n-2x-1)^2}{(n-2)^2n^2}. \end{aligned}$$

This completes the proof. □

Theorem 4.3. *Let $f \in C_B[0,\infty)$. Then for the operators $(V_nf)(x)$ and for the modified operators $(\widetilde{V}_nf)(x)$, we have*

$$\begin{aligned} |((V_n-\widetilde{V}_n)f)(x)| &\le \zeta_1(x)\|f^{iv}\| + \zeta_2(x)\|f'''\| + \zeta_3(x)\|f''\| \\ &+ 2\omega_1(f,\alpha_1) + 2\omega_1(f,\alpha_2), \end{aligned}$$

where the etimates of $\zeta_1(x)$, $\zeta_2(x)$, $\zeta_3(x)$, α_1 and α_2 are indicated in the proof.

Proof. Using Lemmas 2.1 and 4.1 and with the help of software Mathematica, we obtain

$$\begin{aligned} \zeta_1(x) &= \frac{1}{4!}\sum_{k=0}^{\infty}\left(r_{n,k}(x)\nu_4^{F_{n,k}} + r_{n,k}(a_n(x))\nu_4^{F_{n,k}}\right) \\ &= \frac{-1}{8(n-5)(n-4)(n-3)(n-2)^4n^6}\Big\{4n^{13}x^4 + 8n^{12}x^3\left(3x^2-2x+4\right) \\ &+ 2n^{11}x^2\left(30x^4-100x^3-45x^2-158x+25\right) + 2n^{10}x(30x^6-400x^5-54x^4-94x^3 \\ &+ 279x^2-335x+2) + n^9(-1140x^7+3660x^6+5908x^5+6586x^4+3478x^3+3327x^2 \\ &- 28x+4) + n^8\left(10350x^7-240x^6-22922x^5-34962x^4-24718x^3-11460x^2-651x-58\right) \\ &+ n^7\left(-61890x^7-100465x^6-40522x^5+40288x^4+52258x^3+23569x^2+2732x+106\right) \\ &+ n^6\left(260820x^7+685200x^6+776615x^5+482466x^4+179352x^3+44848x^2+9413x+1128\right) \\ &- n^5(2x+1)^2\left(192150x^5+429045x^4+406988x^3+205295x^2+56660x+7135\right) \\ &+ n^4(2x+1)^3\left(194670x^4+399765x^3+325384x^2+124059x+18758\right) - n^3(2x+1)^4 \\ &\left(134010x^3+226875x^2+133838x+27330\right) + n^2(2x+1)^5\left(60030x^2+72585x+22694\right) \\ &- 155n(2x+1)^6(102x+65) + 1860(2x+1)^7\Big\}, \end{aligned}$$

$$\begin{aligned}
\zeta_2(x) &= \frac{1}{3!}\left|\sum_{k=0}^{\infty} r_{n,k}(x)\nu_3^{F_{n,k}} - \sum_{k=0}^{\infty} r_{n,k}(a_n(x))\nu_3^{F_{n,k}}\right| \\
&= \frac{1}{3!(n-4)(n-3)(n-2)^3 n^4}\Big\{9n^8x^2(2x+1) + 9n^7x\left(8x^3-2x^2+3x+3\right) + n^6(90x^5 \\
&\quad -435x^4 - 534x^3 - 513x^2 - 167x + 8) + n^5\left(-990x^5 - 315x^4 + 828x^3 + 951x^2 + 179x - 56\right) \\
&\quad +n^4\left(4440x^5 + 8360x^4 + 7086x^3 + 3594x^2 + 1221x + 214\right) - 3n^3(2x+1)^2(880x^3 + 1266x^2 \\
&\quad +725x + 164) + n^2(2x+1)^3\left(1866x^2 + 2131x + 687\right) - n(2x+1)^4(726x + 463) + 120(2x+1)^5\Big\}, \\
\zeta_3(x) &= \frac{1}{2!}\left|\sum_{k=0}^{\infty} r_{n,k}(x)\nu_2^{F_{n,k}} - \sum_{k=0}^{\infty} r_{n,k}(a_n(x))\nu_2^{F_{n,k}}\right| \\
&= \frac{1}{2(n-3)(n-2)^2n^2}\left\{(2x+1)\left(n^3(2x+1) + n^2x(3x+2) - n\left(6x^2+7x+2\right) + (2x+1)^2\right)\right\}, \\
\alpha_1^2 &= \sum_{k=0}^{\infty} r_{n,k}(x)\left[F_{n,k}(e_1) - x\right]^2 \\
&= \frac{nx^3 + 2(n+2)x^2 + (n+4)x + 1}{(n-2)^2}
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2^2 &= \sum_{k=0}^{\infty} r_{n,k}(a_n(x))\left[F_{n,k}(e_1) - x\right]^2 \\
&= \frac{((n-2)x-1)(nx+n-2x-1)^2}{(n-2)^2n^2}.
\end{aligned}$$

□

Now, we consider an illustration to validate our convergence results.

Example 4.4. *For $f(x) = x^4 - 5x^3 + 5x^2 + 5x$, we predict the convergence of $\widetilde{V}_n$ to $f(x)$ for $n = 100, n = 200$ and $n = 500$ in Fig. 1.*

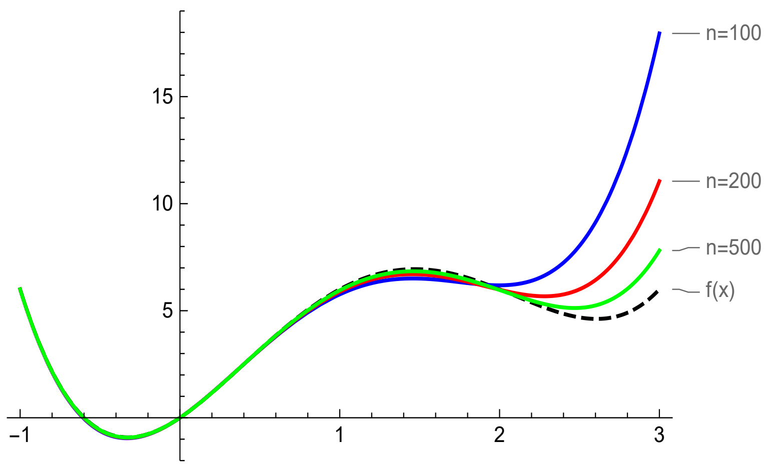

Figure 1: Graphs of $(\widetilde{V}_{100}f)(x)$ (blue), $(\widetilde{V}_{200}f)(x)$ (red), $(\widetilde{V}_{500}f)(x)$ (green) and $f(x) = x^4 - 5x^3 + 5x^2 + 5x$ (black).

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