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Commutant hypercyclicity of Hilbert space operators

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Abstract. An operator T on a Hilbert space H is commutant hypercyclic if there is a vector x in H such that the set $\{Sx : TS = ST\}$ is dense in H. We prove that operators on finite dimensional Hilbert space, a rich class of weighted shift operators, isometries, exponentially isometries and idempotents are all commutant hypercyclic. Then we discuss on commutant hypercyclicity of 2×2 operator matrices. Moreover, for each integer number $n \geq 2$, we give a commutant hypercyclic nilpotent operator of order n on an infinite dimensional Hilbert space. Finally, we study commutant transitivity of operators and give necessary and sufficient conditions for a vector to be a commutant hypercyclic vector.

1. Introduction and Preliminaries

Throughout this paper, H is a separable complex Hilbert space and $B(H)$ denotes the Banach algebra of all bounded linear operators on H. A semigroup $\mathcal{S}$ of bounded linear operators on H is called hypercyclic if there is $x \in H$ such that the orbit

$$\mathrm{orb}(\mathcal{S}, x) = \{Tx : \ T \in \mathcal{S}\}$$

is dense in H. An element T in $B(H)$ is called commutant hypercyclic, cyclic and hypercyclic if $\mathcal{S} = \{T\}'$, the commutant of T, $\mathcal{S} = \{p(T) : p \text{ is a polynomial}\}$ and $\mathcal{S} = \{T^n : n \geq 0\}$, respectively. Every hypercylic (and even cyclic) operator is obviously commutant hypercyclic. Also the identity operator is always commutant hypercyclic (but not cyclic).

Dynamics of linear operators have become an active area of research over the last thirty years with two monographs [7] and [19]. Especially, cyclicity is a classical concept which it appears in many problems of functional analysis and applications to mathematical physics. Also there are many related concepts, some been around for many decades, and some in recent years. On the other hand, an important problem is to characterize the commutant of an operator, because if M is an invariant subspace of T then $\overline{SM}$, the closure of SM is also an invariant subspace of T for every operator S in the commutant of T. Finding the commutant of an operator is not an easy problem, however we prove commutant hypercyclicity of some operators without the description of their commutants. For the commutant of the multiplication and composition operators we refer the reader to recent papers [1, 23, 27]. The relation between the commutant of an operator and the existence of a cyclic vector for the operator is important and interesting. For example, Herrero and Salinas in [21] and Herrero in [22] analyzed the relationship between various statements concerning the commutant of a bounded linear operator on Hilbert space and the existence of cyclic vectors for the

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operator and its adjoint. Also, Gellar in [17, 18] investigated some behavior of elements of commutant of operators. See also [20], [24], [31] and the references there in.

Our motivation for the notion of commutant hypercyclicity is the famous invariant subspace problem: for a separable Hilbert space H, is there $T \in B(H)$ such that every non-zero vector $x \in H$ is cyclic for T? If this is valid, T lacks non-trivial invariant closed subspace. A weak version of the invariant subspace problem runs as follows:

Is there any operator $T \in B(H)$ (not multiple of the identity), for which every non-zero vector is a commutant hypercyclic vector?

Also, a more careful investigation of this concept raises some further surprising questions.

In this paper, we will show that some classes of operators like unilateral weighted shift operators, invertible bilateral weighted shift operators, direct sum of commutant hypercyclic operators, isometries, exponentially isometries and idempotents are all commutant hypercyclic. This rich class also contains all operators on finite dimensional spaces. Furthermore, we discuss the commutant transitivity.

Let $\mathbb{D}$ be the open unit disc $\{z : |z| < 1\}$, and $(\beta(n))_n$ be a sequence of positive numbers with $\beta(0) = 1$. The weighted Hardy space $H^2(\beta)$ consists of all formal power series $f(z) = \sum_{n=0}^{+\infty} \hat{f}(n)z^n$ for which

$$\|f\|_\beta = (\sum_{n=0}^{+\infty} |\hat{f}(n)|^2\beta(n)^2)^{\frac{1}{2}} < \infty.$$

The space $H^2(\beta)$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \sum_{n=0}^{+\infty} \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^2.$$

The classical Hardy space $H^2(\mathbb{D})$, the classical Bergman space $A^2(\mathbb{D})$, and the classical Dirichlet space $\mathcal{D}$ are weighted Hardy space with $\beta(n) = 1$, $\beta(n) = (n + 1)^{-\frac{1}{2}}$, and $\beta(n) = (n + 1)^{\frac{1}{2}}$ respectively.

Let $\{e_n : n \geq 0\}$ be an orthonormal basis of a Hilbert space H. A bounded linear operator T on H defined by $Te_n = \omega_n e_{n+1}$, n = 0, 1,... is called unilateral weighted shift, where $(\omega_n)_0^{+\infty}$ is a bounded sequence of complex numbers. Similarly, an operator $T \in B(H)$ defined in the same way is called bilateral weighted shift according to sequence $(\omega_n)_{-\infty}^{+\infty}$ and an orthonormal basis $\{e_n : n \in \mathbb{Z}\}$.

The multiplication operator M_z on $H^2(\beta)$ given by $(M_z(f))(s) = sf(s)$ is called the forward shift. Indeed, M_z is unitarily equivalent to an injective unilateral weighted shift operator, with the weight sequence $(\omega_n)_n$ defined by

$$0 < \omega_n = \frac{\beta(n + 1)}{\beta(n)}, \quad n \geq 0.$$

Furthermore, every injective unilateral weighted shift operator with positive weight sequence $(\omega_n)_0^{+\infty}$ can be represented as M_z on $H^2(\beta)$, for

$$\beta(n) = \begin{cases} \omega_0 \dots \omega_{n-1} & (n > 0), \\ 1 & (n = 0). \end{cases}$$

As in [29], $H^\infty(\beta)$ denotes the set of all formal power series $\varphi(z) = \sum_{n=0}^{+\infty} \hat{\varphi}(n)z^n$ such that $\varphi H^2(\beta) \subseteq H^2(\beta)$. For each $\varphi \in H^\infty(\beta)$ the multiplication operator M_φ on $H^2(\beta)$ is bounded. Moreover,

$$\{M_z\}' = \{M_\varphi : \varphi \in H^\infty(\beta)\}.$$

Also, when T is an injective bilateral weighted shift operator according to the sequence $(\omega_n)_{-\infty}^{+\infty}$,

$$\beta(n) = \begin{cases} \omega_0 \dots \omega_{n-1} & (n > 0), \\ 1 & (n = 0), \\ \frac{1}{\omega_{-1}\omega_{-2}\dots\omega_n} & (n < 0) \end{cases}$$

and

$$L^2(\beta) = \{f(z) = \sum_{-\infty}^{+\infty} \hat{f}(n)z^n : \sum_{-\infty}^{+\infty} |\hat{f}(n)|^2\beta(n)^2 < \infty\}$$

then T is a unitarily equivalent to M_z on $L^2(\beta)$. Conversely, M_z on $L^2(\beta)$ is unitarily equivalent to an injective bilateral weighted shift operator with weights $\omega_n = \frac{\beta(n+1)}{\beta(n)}$.

Recall that the operator T^*, the adjoint of T, is called the backward weighted shift operator. Moreover, every weighted shift operator T can be considered with non-negative weight sequence $(\omega_n)_n$ by unitary equivalence (see Corollary 1 of [29]).

2. Commutant hypercyclicity of some classical operators

In the first proposition, we observe that commutant hypercyclicity is invariant under similarity. We use $CH(T)$ to denote the set of all commutant hypercyclic vectors for T. In the following $\{T\}''$ denotes the double commutant of T, defined as

$$\{A \in B(H) : AS = SA \text{ for every } S \text{ in } \{T\}'\}.$$

Proposition 2.1. *Suppose that H_1 and H_2 are Hilbert spaces, $T_1 \in B(H_1)$, $T_2 \in B(H_2)$, $X_{21} : H_1 \to H_2$, $X_{12} : H_2 \to H_1$, $T_2X_{21} = X_{21}T_1$ and $T_1X_{12} = X_{12}T_2$ where X_{12} and X_{21} are dense range continuous maps. If T_1 is commutant hypercyclic and $X_{12}X_{21} \in \{T_1\}''$ then T_2 is also commutant hypercyclic. In particular, if T_1 and T_2 are similar and T_1 is commutant hypercyclic then so is T_2.*

Proof. The following commuting diagram illustrates the hypotheses.

$$\begin{array}{ccc} H_1 & \xrightarrow{T_1} & H_1 \\ \downarrow X_{21} & & \downarrow X_{21} \\ H_2 & \xrightarrow{T_2} & H_2 \\ \downarrow X_{12} & & \downarrow X_{12} \\ H_1 & \xrightarrow{T_1} & H_1 \end{array}$$

Note that if $B \in \{T_1\}'$, then $X_{21}BX_{12} \in \{T_2\}'$; indeed,

$$(X_{21}BX_{12})\, T_2 = X_{21}B(T_1X_{12}) = X_{21}T_1BX_{12} = T_2\,(X_{21}BX_{12}).$$

Suppose that $h_1 \in CH(T_1)$. Thus $\{(X_{21}BX_{12})(X_{21}h_1) : B \in \{T_1\}'\}^- = \{X_{21}X_{12}X_{21}Bh_1 : B \in \{T_1\}'\}^- = H_2$ because X_{12} and X_{21} have dense range and composition of dense range continuous maps has dense range. Since $X_{21}BX_{12} \in \{T_2\}'$, we conclude that $X_{21}h_1$ is a commutant hypercyclic vector for T_2. In particular, when T_1 and T_2 are similar we can take $X_{12} = X_{21}^{-1}$ and so $X_{12}X_{21} = I \in \{T_1\}''$. □

The next result is a powerful tool, in spite of its simple proof.

Theorem 2.2. *For every natural number i, let H_i be a Hilbert space and $T_i \in B(H_i)$. If each T_i is commutant hypercyclic, then so is $\oplus_{i=1}^{\infty} T_i$. Conversely, if $\oplus_{i=1}^{\infty} T_i$ is commutant hypercyclic and $\sigma(T_j) \cap \sigma(\oplus_{i\neq j} T_i) = \emptyset$ for some j then T_j is commutant hypercyclic. In particular, if the sets $\sigma(T_i)$ are pairwise disjoint for $1 \le i \le n$ and $\oplus_{i=1}^{n} T_i$ is commutant hypercyclic then so is each T_i.*

Proof. Suppose that $h_i \in H_i$ is a commutant hypercyclic vector for T_i. Put $h = \oplus_{i=1}^{\infty} \frac{h_i}{2^i\|h_i\|} \in \oplus_{i=1}^{\infty} H_i$. We will show that h is a commutant hypercyclic vector for $\oplus_{i=1}^{\infty} T_i$. To see this, let $g = \oplus_{i=1}^{\infty} g_i$ be an arbitrary element in $\oplus_{i=1}^{\infty} H_i$. For $\varepsilon > 0$, choose a natural number N such that $\sum_{i=N+1}^{\infty} \|g_i\|^2 < \varepsilon^2/2$.

Now, for $1 \le i \le N$ take $S_i \in \{T_i\}'$ such that

$$\|g_i - S_i(\frac{h_i}{2^i\|h_i\|})\|^2 < \frac{\varepsilon^2}{2N}$$

and $S_i = 0$, $i \ge N+1$. Hence $\oplus_{i=1}^{\infty} S_i \in \{\oplus_{i=1}^{\infty} T_i\}'$ and

$$\|(\oplus_{i=1}^{\infty} S_i)\left(\oplus_{i=1}^{\infty} \frac{h_i}{2^i\|h_i\|}\right) - \oplus_{i=1}^{\infty} g_i\|^2 = \sum_{i=1}^{N} \|g_i - S_i(\frac{h_i}{2^i\|h_i\|})\|^2 + \sum_{i=N+1}^{\infty} \|g_i\|^2 \le \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} \le \varepsilon^2.$$

The converse follows from the facts that $\{T_i \oplus T_j\}' = \{T_i\}' \oplus \{T_j\}'$ when $\sigma(T_i) \cap \sigma(T_j) = \emptyset$ and $\sigma(\oplus_{i=1}^{n} T_i) = \cup_{i=1}^{n} \sigma(T_i)$. □

Question 2.3. *If $T \oplus T$ is commutant hypercyclic, is so T?*

Corollary 2.4. *Every linear operator T on a finite dimensional space H is commutant hypercylic.*

Proof. Let A be the matrix of T. We know that there is a decomposition $H = W_1 \oplus ... \oplus W_m$ such that the matrix A is a block diagonal matrix with elementary Jordan blocks $J_{n_i}(\lambda_i)$, $i = 1, ..., m$ where $\sum_{i=1}^{m} n_i = \dim H$. Therefore, $A = J_{n_1}(\lambda_1) \oplus J_{n_2}(\lambda_2) \oplus ... \oplus J_{n_m}(\lambda_m)$. But the characteristic polynomial of every elementary Jordan block coincides with its minimal polynomial; so they are cyclic and the proof is complete.
□

Remark 2.5. *Since an operator T on a finite dimensional space is cyclic if and only if $\{T\}' = \{p(T) : p(z)$ is a polynomial$\}$, we observe that the set $CH(T)$ equals to the set of all cyclic vectors for T.*

Let $(\lambda_n)_n$ be a bounded sequence of complex numbers and $(e_n)_n$ be an orthonormal basis for H. Then the diagonal operator $De_n = \lambda_n e_n$ is cyclic if and only if $\lambda_n \ne \lambda_m$ for all $n \ne m$ (see Theorem 4 of [15]). As a result of the above theorem we have:

Corollary 2.6. *Every diagonal operator is commutant hypercyclic.*

Next we prove that many weighted shift operators are commutant hypercyclic.

Theorem 2.7. *The unilateral forward and backward weighted shift operators are commutant hypercyclic. Also, every non-injective and every invertible forward and backward bilateral weighted shift operator is commutant hypercyclic.*

Proof. Let $H = \bigvee\{e_n : n \ge 0\}$ and T be a unilateral forward shift defined by $Te_n = \omega_n e_{n+1}, (n \ge 0)$. If $\omega_n > 0$, for all n, then e_0 is a cyclic vector for T. So suppose that $\omega_n = 0$ for some n. Without loss of generality, let $\omega_0 = 0$. Suppose that n_1 is the largest non-negative integer number such that $\omega_n = 0$ for all $n \le n_1$. In this case, put $M_1 = \bigvee\{e_0, e_1, \dots, e_{n_1}\}$. Then $\omega_{n_1+1} \ne 0$. Now suppose that $n_2 \ge n_1 + 1$ is the largest non-negative integer number such that $\omega_n \ne 0$ for all $n_1 + 1 \le n \le n_2$ and put $M_2 = \bigvee\{e_{n_1+1}, e_{n_1+2}, \dots, e_{n_2}\}$. By continuing this process we get $H = \oplus_{i=1}^{\infty} M_i$ where $TM_i \subseteq M_i$; so $T = \oplus_{i=1}^{\infty} T|_{M_i}$. Now, each $T|_{M_i}$ is an operator on a finite dimesional space or the zero operator or an injective unilateral forward shift operator. Hence the operator T is commutant hypercyclic. On the other hand, if $\omega_n > 0$, for all n then T^* is indeed, cyclic ([7], Example 1.15, Page 9) and otherwise T^* is a direct sum of operators on a finite dimensional space or the zero operator or a backward shift operator with positive weights. Hence again by Theorem 2.2, T^* is commutant hypercyclic. Next, suppose that $H = \bigvee\{e_n : -\infty < n < +\infty\}$ and $Te_n = \omega_n e_{n+1}$ is not injective. Thus, one can assume that $\omega_0 = 0$. If $\omega_n > 0$ for all $n < 0$, put $f_n = e_{-n}$, $n \ge 0$; therefore, $Tf_0 = 0$, $Tf_n = \omega_n f_{n-1}$ and consequently $T|_{\bigvee\{e_n:\, n \le 0\}}$ is a unilateral backward weighted shift operator. The above argument shows that T is a countable direct sum of finite dimensional operators or a unilateral forward or a unilateral backward weighted shift with positive weights (some of them may be absense). Hence T is commutant hypercyclic.
To prove the next part let H be an infinite dimesional Hilbert space and $T \in B(H)$ be an invertible bilateral

weighted shift. Denote T by M_z on $L^2(\beta)$ for suitible $\beta(n)$. Let $g(z) = \sum_{-\infty}^{+\infty} \hat{g}(n)z^n \in L^2(\beta)$. Since $M_z^{-1} = M_{\frac{1}{z}}$ exists on $L^2(\beta)$, we conclude that for every $N \geq 0$

$$\varphi_N(z) := \sum_{n=-N}^{N} \hat{g}(n)z^n \in L^\infty(\beta) = \{f \in L^2(\beta) : fL^2(\beta) \subseteq L^2(\beta)\}.$$

But $\| \varphi_N(z) - \sum_{-\infty}^{\infty} \hat{g}(n)z^n\|_{L^2(\beta)} \to 0$ as $N \to \infty$ and $M_{\varphi_N} \in \{M_z\}'$. Thus the constant 1 is a commutant hypercyclic vector for M_z. Finally, define the unitary operator U on H by $U(\sum_{-\infty}^{+\infty} \gamma_n e_n) = \sum_{-\infty}^{+\infty} \gamma_n e_{-n}$. It is easily seen that $SU = UT^*$, where $Se_n = \omega_{-n-1}e_{n+1}$. Hence the commutant hypercyclicity of S and T^* are equivalent. Therefore, T^* is also commutant hypercyclic. □

Now, we consider the multiplication operator M_z on the space $L^2(\mu)$ when μ is a compactly supported measure. We give necessary and sufficient condition for a vector to be in $CH(M_z)$.

Theorem 2.8. *Suppose that μ is a compactly supported measure on $\mathbb{C}$. Then $f \in L^2(\mu)$ is a commutant hypercyclic vector for M_z if and only if f is non-zero almost everywhere. Consequently, the multiplication operator M_ψ is commutant hypercyclic on $L^2(\mu)$ for all $\psi \in L^\infty(\mu)$.*

Proof. Note that $\{M_z\}' = \{M_\varphi : \varphi \in L^\infty(\mu)\}$ (see Page 279 of [11]). Suppose that f is a commutant hypercyclic vector for M_z. If f vanishes on a set E with non-zero measure then so does φf for $\varphi \in L^\infty(\mu)$. Let $g = \chi_E$ and $(\varphi_n)_n$ be a sequence in $L^\infty(\mu)$ so that $\varphi_n f \to g$ in $L^2(\mu)$. Therefore, by the Riesz-Fischer theorem, there is a subsequence $(n_k)_k$ such that $\varphi_{n_k} f \to g$ almost everywhere, which is contradiction.

Conversely, suppose that $f \in L^2(\mu)$ is non-zero almost everywhere and h is an arbitrary function in $L^\infty(\mu)$. Put

$$h_n(x) = \begin{cases} 0 & (|f(x)| < \frac{1}{n}), \\ \frac{h(x)}{f(x)} & (|f(x)| \geq \frac{1}{n}). \end{cases}$$

Then h_n and $h_n f$ are in $L^\infty(\mu)$. Moreover, $\|h_n f\|_\infty \leq \|h\|_\infty$ and $h_n f$ converges to h almost everywhere. Now, an application of the dominated convergence theorem shows that $h_n f$ converges to h in $L^2(\mu)$. On the other hand, $L^\infty(\mu)$ is a dense subset of $L^2(\mu)$, so f is a commutant hypercyclic vector for M_z. The last part follows from the fact that $\{M_z\}' = \{M_\varphi : \varphi \in L^\infty(\mu)\} \subseteq \{M_\psi\}'$. □

It is known that normal operators are commutant hypercyclic [24]. Another proof of this fact can be deduced from Theorems 1 and 3.

Corollary 2.9. *Every normal operator is commutant hypercyclic.*

Proof. Let N be a normal operator on a separable Hilbert space H. It is unitary equivalent to a countable direct sum of multiplication operators. Indeed, for $x \in H$ let $H(x) = \{p(N, N^*)x : p \textit{ is a polynomial in } z \textit{ and } \bar{z}\}$. Then Zorn's Lemma shows that there is a maximal sequence $(x_n)_n$ in H such that $H(x_n) \perp H(x_m)$, $n \neq m$. Since $\{x_n\}$ is maximal $H = \oplus_n H(x_n)$. Moreover, if E is the spectral measure for N and $\mu_n(\Omega) = \|E(\Omega)x_n\|^2$ then N is unitary equivalent to $\oplus_n M(z, n)$ where $M(z, n)$ is the operator of multiplication by z on the space $L^2(\mu_n)$, where μ_n has compact support (see Page 269 of [11]). Now, the result follows from Theorems 2.2 and 2.8. □

Remark 2.10. *Observe that for $\varphi \in H^\infty$ the multiplication operator M_φ is commutant hypercyclic on the Hardy space H^2, because $M_z \in \{M_\varphi\}'$ is cyclic on H^2.*

Another important class of bounded operators consists of isometries. For an isometry T, by the von Neumann-Wold decomposition, $T = S \oplus U$ where S is a unilateral shift and U is a unitary operator. Hence by combining the preceding corollary with Theorems 2.2 and 2.7 the following is obtained.

Corollary 2.11. *Every isometry on a Hilbert space is commutant hypercyclic.*

A generalization of the class of isometries is the class of m-isometries. For a positive integer m, a bounded linear operator T on H is said to be m-isometry if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k}=0.$$

Note that each 1-isometry is an isometry. Such operators are introduced in [2] and they have applications to Brownian motion, differential operators and disconjugacy (see [3–5]). Recently, the dynamics of m-isometric operators have been considered by several authors in [8, 10, 16, 25]. For an m-isometry T the covariance operator Δ_T is defined as $\frac{1}{(m-1)!}\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}T^{*k}T^{k}$. It is known that Δ_T is a positive operator ([3], Proposition 1.5) and $\langle \Delta_T x, x\rangle = \lim_{n\to\infty}\frac{\|T^n x\|^2}{n^{m-1}}$ for all $x \in H$ ([10], Proposition 2.3). Bermúdez et al. [10] have shown that if Δ_T is injective then the orbit of any N-dimesional subspace under T is not dense in H for all $N \geq 1$. However, we will show that a subclass of m-isometric operators is commutant hypercyclic.

Theorem 2.12. *Every m-isometric operator T which its covariance operator Δ_T is injective and has closed range is commutant hypecyclic.*

Proof. Since Δ_T is positive and injective, we observe that $\langle\langle x, y\rangle\rangle := \langle \Delta_T x, y\rangle$, $(x, y \in H)$, is an inner product on H; moreover,

$$|||Tx|||^2 = \langle\langle Tx, Tx\rangle\rangle = \langle \Delta_T Tx, Tx\rangle = \lim_{n\to\infty}\frac{\|T^{n+1}x\|^2}{n^{m-1}} = \lim_{n\to\infty}\frac{\|T^n x\|^2}{n^{m-1}} = |||x|||^2.$$

Let H_2 be the completion of H with respect to the norm $|||.|||$ and note that the extension T_2 of T on H_2 remains isometry. On the other hand, $(0) = \ker\,\Delta_T = (\mathrm{ran}\ \ \Delta_T)^{\perp}$; thus, $H = \overline{ran\ \Delta_T} = \mathrm{ran}\ \Delta_T$ which implies that Δ_T is invertible. Therefore, $|||x|||^2 = \langle \Delta_T x, x\rangle = \|(\Delta_T)^{\frac{1}{2}}x\|^2$ yields $|||x||| = \|(\Delta_T)^{\frac{1}{2}}x\| \geq \|(\Delta_T)^{-\frac{1}{2}}\|^{-1}\|x\|$. Consequently, $c_1\|x\| \leq |||x||| = \langle \Delta_T x, x\rangle^{\frac{1}{2}} \leq c_2\|x\|$ for all $x \in H$ where $c_1 = \|(\Delta_T)^{-\frac{1}{2}}\|^{-1}$ and $c_2 = \|\Delta_T\|^{\frac{1}{2}}$. This, in turn, implies that the commutant of T with respect to the norm $|||.|||$ is equal to the commutant of T with respect to the original norm $\|.\|$. Hence by the preceding corollary T is commutant hypercyclic. □

The positive operator Δ_T is surjective if and only if it is bounded from below. Therefore, the following result holds.

Corollary 2.13. *Every m-isometric operator with surjective covariance operator is commutant hypercyclic.*

It is natural to ask the following question.

Question 2.14. *Is every m-isometric operator with $m \geq 2$, commutant hypercyclic?*

Remark 2.15. *Any m-isometric operator is injective and has a closed range ([3], Lemma 1.21). Since the range of an injective forward and backward bilateral weighted shift operator is dense, by Theorem 2.7, any m-isometric bilateral weighted shift and unilateral forward weighted shift is commutant hypercyclic. Note that unilateral backward weighted shift operators are not m-isometry, because they are not injective.*

An operator $T \in B(H)$ is called an exponentially isometry if $exp(T) = \sum_0^{+\infty}\frac{T^n}{n!}$ is an isometry. By the spectral mapping theorem, $\sigma(T)$, is a subset of the imaginary axis. Observe that the operators of the form $iA + 2\pi iE$ are examples of such operators when A is selfadjoint, E is idempotent and $AE = EA$. This class of operators are also commutant hypercyclic.

Corollary 2.16. *Every exponentially isometric operator is commutant hypercyclic. Specially, every idempotent operator is commutant hypercyclic.*

Proof. Let T be an exponentially isometric operator. Since e^T is unitary, the spectral theorem yields $\{T\}' = \{e^T\}'$. By Corollary 2.11, e^T is commutant hypercyclic, and so is T. □

For an operator T in $B(H)$, let $W(T)$ be the closure of polynomials in T in the weak operator topology. If T_1 and T_2 are two cyclic operators then $T_1 \oplus T_2$ is not necessary cyclic even if $\{T_i\}' = W(T_i)$ $i = 1, 2$. For example the multiplication by the independent variable z, M_z on the Hilbert Hardy space is cyclic but $M_z \oplus M_z$ is not. In the following result we give sufficient conditions under which the direct sum of two operators is cyclic.

Proposition 2.17. *Suppose that $T_i \in B(H_i)$, $\{T_i\}' = W(T_i)$ $i = 1, 2$, and $W(T_1 \oplus T_2) = W(T_1) \oplus W(T_2)$. If $T_1 \oplus T_2$ is commutant hypercyclic then it is cyclic. Consequently, T_1 and T_2 are cyclic.*

Proof. Let $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ be in the commutant of $T_1 \oplus T_2$. Therefore, $S_{12}T_2 = T_1S_{12}$, $S_{21}T_1 = T_2S_{21}$. Since $S_{12}T_2 = T_1S_{12}$, we conclude that closed subspace $M = \{S_{12}x \oplus x : x \in H_2\}$ is invariant under $T_1 \oplus T_2$; so it is an invariant subspace of every operator in $W(T_1 \oplus T_2) = W(T_1) \oplus W(T_2)$ especially under $I \oplus 0$. Hence $S_{12} = 0$. Similarly $S_{21} = 0$. Consequently,

$$W(T_1 \oplus T_2) = W(T_1) \oplus W(T_2) = \{T_1\}' \oplus \{T_2\}' = \{T_1 \oplus T_2\}'.$$

Now, suppose that $x_1 \oplus x_2$ is a commutant hypercyclic vector for $T_1 \oplus T_2$ and $y_1 \oplus y_2 \in H_1 \oplus H_2$ is such that

$$\langle p(T_1 \oplus T_2)(x_1 \oplus x_2), y_1 \oplus y_2 \rangle = 0$$

for every polynomial p. For $A_1 \oplus A_2 \in \{T_1 \oplus T_2\}'$, there is a net of polynomials $(p_i)_i$ such that $p_i(T_1 \oplus T_2) \to A_1 \oplus A_2$ in the weak operator topology. Thus

$$\langle (A_1 \oplus A_2)(x_1 \oplus x_2), y_1 \oplus y_2 \rangle = 0$$

which, in turn, implies that $y_1 \oplus y_2 = 0$. Hence $x_1 \oplus x_2$ is a cyclic vector for $T_1 \oplus T_2$. □

Recall that an operator T in $B(H)$ is algebraic if there exists a non-zero polynomial $p(z)$ such that $p(T) = 0$.

Corollary 2.18. *Suppose that $T_i \in B(H_i)$, $i = 1, 2$ are algebraic such that $\sigma(T_1) \cap \sigma(T_2) = \emptyset$. Moreover, suppose that $\{T_i\}' = W(T_i)$, $i = 1, 2$. If $T_1 \oplus T_2$ is commutant hypercyclic then it is cyclic. In particular, if A_1 and A_2 are two cyclic matrices whose spectrums are disjoint, then $A_1 \oplus A_2$ is cyclic.*

Proof. Since $\sigma(T_1) \cap \sigma(T_2) = \emptyset$, it is known that $\{T_1 \oplus T_2\}' = \{T_1\}' \oplus \{T_2\}'$. Therefore, $\{T_1 \oplus T_2\}'' = \{T_1\}'' \oplus \{T_2\}''$. On the other hand, by a result of Turner [30], $\{T\}'' = W(T)$ for every algebraic operator T. Thus $W(T_1 \oplus T_2) = W(T_1) \oplus W(T_2)$. Now, the result follows from the preceding proposition. Moreover, since every finite matrix is commutant hypercyclic and $\{A_i\}' = W(A_i)$, $i = 1, 2$ ([28], Theorem 3) the next part is obvious. □

To discuss commutant hypercyclicity of a 2×2 operator matrix we need some preliminaries. The Hilbert-Schmidt class, $B_2(H)$, is the class of all $T \in B(H)$ such that $\|T\|_2^2 = \sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$ where $\{e_n : n \geq 1\}$ is an orthonormal basis for H. This space is a Hilbert space with the inner product $\langle T, S \rangle = tr(S^*T)$, the trace of the operator S^*T. For simplicity of notation we denote the space $B_2(H)$ by B_2. Recall that B_2 is an ideal in $B(H)$ [13] . In the following $\sigma_{ap}(T)$ is the approximate point spectrumof T and $\sigma_p(T)$ is the point spectrum of T.

By the Berberian extension theorem [9] there exists a Hilbert space $B_2^\circ \supseteq B_2$ and a unital linear map $\Gamma : B(B_2) \to B(B_2^\circ)$ such that $\sigma(\Gamma(T)) = \sigma(T)$, $\sigma_{ap}(\Gamma(T)) = \sigma_{ap}(T) = \sigma_p(\Gamma(T))$ and $\Gamma(TS) = \Gamma(T)\,\Gamma(S)$ for all T and S in $B(B_2)$.

Theorem 2.19. *Let H be a Hilbert space and T and S in $B(H)$ be commutant hypercyclic. If $\sigma_{ap}(T^*) \cap \sigma_{ap}(S) = \emptyset$, then the 2×2 upper triangular matrix $\begin{bmatrix} T & V \\ 0 & S \end{bmatrix}$ is commutant hypercyclic.*

Proof. Suppose that there is an operator $W \in B_2$ such that $TW - WS = -V$. Then the matrices $\begin{bmatrix} T & V \\ 0 & S \end{bmatrix}$ and $\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}$ are similar. Indeed,

$$\begin{bmatrix} T & V \\ 0 & S \end{bmatrix}\begin{bmatrix} I & W \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & W \\ 0 & I \end{bmatrix}\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}$$

and the inverse of $\begin{bmatrix} I & W \\ 0 & I \end{bmatrix}$ *is* $\begin{bmatrix} I & -W \\ 0 & I \end{bmatrix}$. So the result follows from Proposition 2.1 and Theorem 2.2. Define the operators L_T and R_S on the space B_2 by $L_T X = TX$ and $R_S X = XS$. To finish the proof it is sufficient to show that the operator $L_T - R_S$ is onto. On the contrary, assume that $L_T - R_S$ is not onto on B_2. Therefore,

$$0 \in \sigma_{ap}(L_T - R_S)^* = \sigma_{ap}(L_{T^*} - R_{S^*}) = \sigma_{ap}(\Gamma(L_{T^*}) - \Gamma(R_{S^*})).$$

Furthermore, since L_{T^*} and R_{S^*} commute, Berberian theorem implies that $\Gamma(L_{T^*})$ and $\Gamma(R_{S^*})$ also commute; thus, the non-zero subspace $N = \ker\,(\Gamma(L_{T^*}) - \Gamma(R_{S^*}))$ is invariant under $\Gamma(L_{T^*})$ and $\Gamma(R_{S^*})$. Moreover, $\Gamma(L_{T^*})|_N = \Gamma(R_{S^*})|_N$. In the next step we show that $\sigma_{ap}(R_{S^*}) \subseteq \sigma_{ap}(S)$ and $\sigma_{ap}(L_{T^*}) \subseteq \sigma_{ap}(T^*)$. Observe that it is sufficient to show that
(*i*) if $0 \in \sigma_{ap}(R_{S^*})$, then $0 \in \sigma_{ap}(S)$
and
(*ii*) if $0 \in \sigma_{ap}(L_{T^*})$, then $0 \in \sigma_{ap}(T^*)$.

To prove (*i*) assume that $0 \notin \sigma_{ap}(S)$. Then the operator S is bounded below and so $S : H \to \operatorname{ran} S$ is invertible. Define the operator $R : \operatorname{ran} S \oplus (\operatorname{ran} S)^\perp \to H$ by $R(Sx \oplus y) = x$. Since $RS = I$, we observe that $XRS = X$ for all $X \in B(H)$. Therefore, the operator R_S is onto which yields that R_{S^*} is one-to-one and has closed range. Thus, $0 \notin \sigma_{ap}(R_{S^*})$ which is a contradiction. The proof of (*ii*) follows from the fact that 0 is not in the approximate point spectrum of an operator if and only if it is left invertible.

Now, since the approximate point spectrum of an operator is nonempty, suppose that $\lambda \in \sigma_{ap}(\Gamma(L_{T^*})|_N)$. In the following for two subsets E and F of the complex plane by $E - F$ we mean the set $\{\lambda - \alpha : \lambda \in E,\ \alpha \in F\}$. Thus

$$\begin{aligned} 0 = \lambda - \lambda &\in \sigma_{ap}(\Gamma(L_{T^*})|_N) - \sigma_{ap}(\Gamma(R_{S^*})|_N) \\ &\subseteq \sigma_{ap}(\Gamma(L_{T^*})) - \sigma_{ap}(\Gamma(R_{S^*})) \\ &= \sigma_{ap}(L_{T^*}) - \sigma_{ap}(R_{S^*}) \\ &\subseteq \sigma_{ap}(T^*) - \sigma_{ap}(S) \end{aligned}$$

which contradicts the hypothesis. □

Corollary 2.20. *If $T \in B(H)$ is commutant hypercyclic then there is $\lambda \in \mathbb{C}$ such that 2×2 upper triangular matrices $\begin{bmatrix} T-\lambda & S \\ 0 & T-\lambda \end{bmatrix}$ are commutant hypercyclic for every $S \in B(H)$.*

Proof. Choose $\lambda \in \mathbb{C}$ such that $\sigma(T - \lambda I)$ lies on the upper half plane and apply the preceding theorem. □

Remark 2.21. *Suppose that $\sigma(T) \cap \sigma(S)$ is empty. Then by the Rosenblum theorem [26] for every operator V there is an operator W such that $TW - WS = -V$ hence if T and S are commutant hypercyclic then so is $\begin{bmatrix} T & V \\ 0 & S \end{bmatrix}$.*

In the next step, we present a result on the commutant hypercyclicity of nilpotent operators. Recall that $A \in B(H)$ is a nilpotent operator of order $n \geq 2$, if $A^n = 0$ but $A^{n-1} \neq 0$.

Proposition 2.22. *Suppose that $A \in B(H)$ is a commutant hypercyclic nilpotent operator of order n. If $B \in \{A\}'$ is invertible then the operator $T = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is a commutant hypercyclic nilpotent operator of order $n+1$.*

Proof. Observe that $T^k = \begin{bmatrix} A^k & A^{k-1}B \\ 0 & 0 \end{bmatrix}$ for all $k \geq 1$; therefore, T is a nilpotent operator of order $n+1$. Let $x \in CH(A)$ and $y \oplus z \in H \oplus H$. So there are two sequences $(A_n)_n$ and $(C_n)_n$ in $\{A\}'$ such that $A_n x \to y$ and $C_n x \to z$. Since

$$\begin{bmatrix} AA_nB^{-1} + BC_nB^{-1} & A_n \\ 0 & C_n \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} A_n x \\ C_n x \end{bmatrix} \to \begin{bmatrix} y \\ z \end{bmatrix}$$

and

$$\begin{bmatrix} AA_nB^{-1} + BC_nB^{-1} & A_n \\ 0 & 0 \end{bmatrix}$$

commutes with T, we conclude that $0 \oplus x \in CH(T)$. □

Corollary 2.23. *For every $n \geq 2$, there is a commutant hypercyclic nilpotent operator of order n on some infinite dimensional Hilbert space.*

Proof. Let H be an infinite dimensional Hilbert space. The operator T_2 defined on $H \oplus H$ by $T_2(x \oplus y) = 0 \oplus x$, is a nilpotent operator of order 2 with matrix representation $\begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$. Therefore, its commutant is

$$\{\begin{bmatrix} V & 0 \\ W & V \end{bmatrix} : V\,,\ W \text{ in } B(H)\}.$$

Thus $x \oplus y$ is a commutant hypercyclic vector for T_2 if and only if $x \neq 0$. In the next step put $A = T_2$ and $B = I$ in the preceding proposition. So we observe that $T_3 = \begin{bmatrix} T_2 & I \\ 0 & 0 \end{bmatrix}$ is a commutant hypercyclic operator of order 3. By continuing this process, the result follows for each $n \geq 2$. □

We propose the following question.

Question 2.24. *If H is an infinite dimensional Hilbert space and T is a nilpotent operator on H, is T commutant hypercyclic?*

Note that it follows from Theorem 2.7 that every nilpotent weighted shift operator is commutant hypercyclic.

In the rest of this section we discuss the non-commutant hypercyclicity. It is proved in [24] that every non-algebraic normal operator has an extension which is not commutant hypercyclic. Also, an example of bilateral weighted shift which is not commutant hypercyclic is given in [14].

Proposition 2.25. *Suppose that $Te_n = \omega_n e_{n+1}$ is an injective non-invertible bilateral weighted shift operator with $\sigma_p(T^*) \neq \emptyset$. Then T is not commutant hypercyclic. In particular, if $(\frac{1}{\omega_n})_n$ is an unbounded sequence and $\limsup_n [\omega_{-1} \ldots \omega_{-n}]^{\frac{1}{n}} < \liminf_n [\omega_0 \ldots \omega_{n-1}]^{\frac{1}{n}}$, then T is not commutant hypercyclic.*

Proof. By Page 91 of [29], $\{T\}' = S(T)$ where $S(T)$ is the strong limit of polynomials in T. Hence commutant hypercyclicity of T is equivalent with cyclicity of T. But by Theorem 4 of [22], T is not cyclic. For the second part, note that by Theorem 9 of [29], $\sigma_p(T^*)$ is non-empty. □

Example 2.26. *Let $r > 1$ and $c > 0$. Then the bilateral weighted shift T with weight sequence $(\omega_n)_n$ defined by*

$$\omega_n = \begin{cases} c & (n \geq 0), \\ \frac{1}{r^n} & (n < 0) \end{cases}$$

is not commutant hypercyclic.

Now, Theorem 2.2 helps us to construct a collection of operators which are not commutant hypercyclic. Indeed, if T is as in Example 2.26 and S is any operator such that $\sigma(T) \cap \sigma(S) = \emptyset$, then by Theorem 2.2, $T \oplus S$ is not commutant hypercyclic.

In our example of non-commutant hypercyclic operator T, we have $W(T) = \{T\}'$. We give an operator T such that $W(T) \neq \{T\}'$ and T is not commutant hypercyclic. It is known that a finite dimensional Hilbert space operator S is cyclic if and only if $W(S) = \{S\}'$. Let, in the above collection, S be a non-cyclic finite dimensional operator. Then $T \oplus S$ is not a commutant hypercyclic operator and moreover,

$$W(T \oplus S) = W(T) \oplus W(S) \subsetneqq \{T\}' \oplus \{S\}' = \{T \oplus S\}'.$$

We have found two ways to get new non-commutant hypercyclic operators from the old one. One is to take the direct sum and the other is to restrict to M or $M^{\perp}$ where M is a reducing subspace of T. Both of them follow from Theorem 2.2. Note that when $c \geq 1$ the operator T^*, in Example 2.26, is commutant hypercyclic (in fact, it is cyclic by Page 9 of [7]). Hence an operator may be commutant hypercyclic but its adjoint is not.

Remark 2.27. *Since the commutant hypercyclicity of an operator T is equivalent to commutant hypercyclicity of $T - \alpha I$ for every scalar α, we can conclude that commutant hypercyclicity is independent of invertibility.*

3. Commutant Transitivity

Definition 3.1. *An operator T in $\mathbf{B}(H)$ is commutant transitive if for every pair of non-empty open subsets U and V of H, there exists an operator $S \in \{T\}'$ with $S(U) \bigcap V \neq \emptyset$.*

Proposition 3.2. *An operator $T \in B(H)$ is commutant transitive if and only if the set of commutant hypercyclic vectors for T is a dense G_δ set.*

Proof. Let $\{V_k : k = 1, 2, \ldots\}$ be a countable basis of open sets in H. Observe that the set of commutant hypercyclic vectors of T can be written as $\bigcap_{k=1}^{+\infty} \bigcup_{S \in \{T\}'} S^{-1}(V_k)$ which is a G_δ set. Suppose that the operator T is commutant transitive, U is a non-empty open subset of H and $S_k \in \{T\}'$ such that $S_k(U) \bigcap V_k \neq \emptyset$. Thus $U \bigcap S_k^{-1}(V_k) \neq \emptyset$ which in turn implies that the set $\bigcup_{S \in \{T\}'} S^{-1}(V_k)$ is dense in H. Now, by the Baire category theorem, the set of commutant hypercyclic vectors of T is a dense subset of H.

Conversely, let U and V be two non-empty open subsets of H. If $x \in U$ is a commutant hypercyclic vector for T then there is an operator $S \in \{T\}'$ so that $Sx \in V$; hence $SU \cap V$ is non-empty. □

The following example shows that completeness is a necessary condition in the above proposition.

Example 3.3. *Let $C_C((0, \infty))$ be the vector space of all continuous complex functions with compact support on the interval $(0, \infty)$. Its completion is $C_0((0, \infty))$, the space of continuous functions on $(0, \infty)$ that vanishe at infinity, relative to the metric defined by the supremum norm. Define*

$$S : C_C((0, \infty)) \to C_C((0, \infty))$$

by $(Sf)(x) = f(x + 1)$ and let $T = \alpha S$ where $\alpha > 1$. Now, the operator T is commutant transitive but not commutant hypercyclic.

To see this let U and V be two non-empty open sets in $C_C((0, \infty))$ with $f \in U$ and $g \in V$. Choose $r > 0$ so that the neighborhood with radius r and center at f is in U. Moreover, there are two natural numbers m and k such that $f(x) = 0$ on $[m, \infty)$ and $g(x) = 0$ on $[k, \infty)$. On the other hand, let k be greater than m so that $\frac{1}{\alpha^k}\|g\| < r$. Put

$$h(x) = \begin{cases} f(x) & (x \leq m), \\ 0 & (m < x \leq k), \\ \frac{1}{\alpha^k} g(x - k) & (k < x). \end{cases}$$

Then $h \in C_{\mathbb{C}}((0,\infty))$, $\|h-f\| < r$, $T^k h = g$ *and* $T^k U \bigcap V \neq \emptyset$ *which implies that* T *is commutant transitive. Now, assume on the contrary that there is a function* $f \in C_{\mathbb{C}}((0,\infty))$ *that is a commutant hypercyclic vector of* T*. Let* k *be a natural number with* $T^k f = 0$ *and* $g \in C_{\mathbb{C}}((0,\infty))$ *such that* $g(x) = 1$ *for* $x \in [k, k+1]$*. Then* $T^k g$ *is non-zero. Moreover, there is a net* $(T_i)_i$ *in* $\{T\}'$ *with* $T_i f \to g$*; hence*

$$0 = T_i T^k f = T^k T_i f \to T^k g$$

which is a contradiction.

Note that by Proposition 3.2 and Birkhoff's transivity theorem [7], hypercyclicity implies commutant transitivity. We show that an operator may be commutant hypercyclic but not commutant transitive. In the following, for an open set Ω of the complex plane, $H(\Omega)$ denotes the space of all analytic functions on Ω.

Theorem 3.4. *If* $\sigma_p(M_z^*) \neq \{0\}$*, then the multiplication operator* M_z *is commutant hypercyclic but not commutant transitive on* $H^2(\beta)$*.*

Proof. Since M_z is cyclic, it is commutant hypercyclic. To prove the other part, first note that it follows from Theorem 8 and Theorem 10 of [29], that $r_2(M_z) = \liminf \beta(n)^{\frac{1}{n}} > 0$ and $H^2(\beta) \subseteq H(\Omega)$, where $\Omega = \{z \in \mathbf{C} : |z| < r_2(M_z)\}$. If $f \in H^2(\beta)$, then for all $z \in \Omega$

$$|f(z)| = |\Sigma_{n=0}^{\infty} \hat{f}(n) z^n| \leq (\Sigma_{n=0}^{\infty} |\hat{f}(n)|^2 \beta(n)^2)^{\frac{1}{2}} (\Sigma_{n=0}^{\infty} \frac{|z|^{2n}}{\beta(n)^2})^{\frac{1}{2}}$$

$$= \| f \|_2 \, (\Sigma_{n=0}^{\infty} \frac{|z|^{2n}}{\beta(n)^2})^{\frac{1}{2}}.$$

So the convergence of the series $\Sigma_{n=0}^{\infty} \frac{|z|^{2n}}{\beta(n)^2}$ guarantees that convegence in $H^2(\beta)$ implies the uniform convergence on compact subsets of Ω.

On the contrary, assume that M_z is commutant transitive and f is a commutant hypercyclic vector for M_z. Since $\{M_z\}' = \{M_\varphi : \varphi \in H^\infty(\beta)\}$ and $H^\infty(\beta) \subseteq H(\Omega)$, observe that $f(z) \neq 0$, *for all* $z \in \Omega$. By Proposition 3.2 and the above argument, for $g \in H^2(\beta)$ there is a sequence $(f_n)_n$ of commutant hypercyclic vectors that converges to g in $H(\Omega)$. Now, an application of the Hurwitz's theorem [12] shows that $g \equiv 0$ or g never vanishes on Ω. This is a contradiction. □

In the next result, we give some characterization for a vector to be a commutant hypercyclic vector. Let $\mathcal{S}_1 = \{x \in H : \|x\| = 1\}$ the unit sphere of H.

Proposition 3.5. *Suppose that* $T \in B(H)$*. Then*
(a) *The vector* $x \in H$ *is a commutant hypercyclic vector for* T *if and only if the set* $\{\frac{Sx}{\|Sx\|} : Sx \neq 0, S \in \{T\}'\}$ *is dense in* $\mathcal{S}_1$*.*
(b) *Suppose that* T *is commutant transitive. If*

$$\rho =: \inf\{\frac{\|Sx\|}{\|S\|} : S \in \{T\}'\} > 0,$$

then x *is a commutant hypercyclic vector for* T*.*

Proof. (a) Suppose that x is a commutant hypercyclic vector for T. Since the map $x \mapsto \frac{x}{\|x\|}$ is onto from $H \setminus \{0\}$ to $\mathcal{S}_1$, we conclude that the set $\{\frac{Sx}{\|S\|} : S \in \{T\}'\}$ is dense in $\mathcal{S}_1$. For the converse, let $y \in H$ be non-zero and $\epsilon > 0$. Then there is $S \in \{T\}'$ such that $\|\frac{Sx}{\|Sx\|} - \frac{y}{\|y\|}\| < \frac{\epsilon}{\|y\|}$. Therefore, $\|\frac{\|y\|}{\|Sx\|} Sx - y\| < \epsilon$. But $\frac{\|y\|}{\|Sx\|} S \in \{T\}'$ so the result follows.

(b) On the contrary assume that x is not a commutant hypercyclic vector for T. Thus there is a vector $y \in H$ so that $\langle Sx, y \rangle = 0$ for all $S \in \{T\}'$ and $\|y\| > \rho$. Therefore,

$$\frac{2\|Sx-y\|^2}{(1+\|S\|)^2} \geq \frac{(\|Sx\|+\|y\|)^2}{(1+\|S\|)^2} \geq (\frac{\rho(\|S\|+1)}{1+\|S\|})^2 = \rho^2.$$

So in light of Lemma 1 of [6], the proof is finished. In fact, the above inequality shows that $\|Sx - y\| > \frac{\rho}{2}(1 + \|S\|)$ for all $S \in \{T\}'$. On the other hand, commutant transivity of T implies the existence of $S \in \{T\}'$ and $z \in H$ such that $\|x - z\|$ and $\|Sz - y\|$ are less than $\rho/2$. This yields $\|Sx - y\| \leq \rho(1 + \|S\|)/2$ which is a contradiction. □

Remark 3.6. *The converse of part* (*b*) *of the above proposition is not correct. Indeed, if B is the backward shift operator then $T = 2B$ is hypercyclic, and so is commutant transitive. But for every $x \in H$*

$$0 = \inf\{\frac{\|T^n x\|}{\|T^n\|} : n \geq 0\} \geq \inf\{\frac{\|Sx\|}{\|S\|} : S \in \{T\}'\}.$$

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