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On the projections of the multifractal Hewitt-Stromberg dimensions

Bilel Selmi[a]

[a]*Laboratory of Analysis, Probability & Fractals, LR18ES17, Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir, Avenue of the Environment, 5019 Monastir, Tunisia.*

Abstract. This paper studies the behavior of the multifractal Hewitt-Stromberg measures and dimensions under projections onto almost all m-dimensional subspaces.

1. Introduction

Recently, the projection behavior of dimensions and multifractal spectra of sets and measures have generated large interest in the mathematical literature [5, 7, 9, 17, 18, 24–26, 34, 35, 37]. Hewitt-Stromberg measures were introduced in [22, Exercise (10.51)]. Since then, they have been investigated by several authors, highlighting their importance in the study of local properties of fractals and products of fractals. One can cite, for example [4, 10–13, 20, 21, 27, 31, 39, 40, 42]. In particular, Edgar's textbook [14, pp. 32-36] provides an excellent and systematic introduction to these measures. Such measures appear also explicitly, for example, in Pesin's monograph [33, 5.3] and implicitly in Mattila's text [29]. Motivated by the above papers, the authors in [2, 3, 38] introduced and studied a multifractal formalism based on the Hewitt-Stromberg measures. However, we point out that this formalism is completely parallel to Olsen's multifractal formalism introduced in [30] which is based on the Hausdorff and packing measures.

In the present paper we pursue those kinds of studies and consider the multifractal formalism developed in [32]. We will start by introducing the multifractal Hewitt-Stromberg measures and dimensions which slightly differ from those introduced in [2, 3, 38]. Our approach is to consider the behavior of multifractal Hewitt-Stromberg functions dimensions under projections onto (almost) all m-dimensional subspaces.

We will now give a brief description of the organization of the paper. In Section 2 we introduce the multifractal Hewitt-Stromberg measures and separator functions which slightly differ from those introduced in [2, 3], and study their properties. Section 3 contain our main results. The proofs are given in Sections 4-5.

2. Multifractal Hewitt-Stromberg measures and separator functions

Our main reason for modifying the definitions in [2, 3, 38] is to allow us to prove results for non necessary doubling measures. One main cause and motivation is the fact that such characteristics is not in fact preserved under projections. In the following, we will set up, for $q, t \in \mathbb{R}$ and a compactly supported

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Email addresses: bilel.selmi@fsm.rnu.tn (Bilel Selmi), bilel.selmi@isetgb.rnu.tn (Bilel Selmi)

probability measure μ on $\mathbb{R}^n$, the lower and upper multifractal Hewitt-Stromberg measures $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$ (see also [39]). For $E \subseteq \operatorname{supp}\mu$, the pre-measure of E is defined by

$$\mathsf{C}_\mu^{q,t}(E) = \limsup_{r\to 0} M_{\mu,r}^q(E)\, r^t,$$

where

$$M_{\mu,r}^q(E) = \sup\left\{\sum_i \mu\left(B\left(x_i, \frac{r}{3}\right)\right)^q \;\middle|\; \left(B(x_i, r)\right)_i \text{ is a packing of } E\right\}.$$

Observe that $\mathsf{C}_\mu^{q,t}$ is increasing and $\mathsf{C}_\mu^{q,t}(\emptyset) = 0$. However it is not σ-additive. For this, we introduce the $\mathsf{P}_\mu^{q,t}$-measure defined by

$$\mathsf{P}_\mu^{q,t}(E) = \inf\left\{\sum_i \mathsf{C}_\mu^{q,t}(E_i) \;\middle|\; E \subseteq \bigcup_i E_i \text{ and the } E_i'\text{s are bounded}\right\}.$$

In a similar way we define

$$\mathsf{L}_\mu^{q,t}(E) = \liminf_{r\to 0} M_{\mu,r}^q(E)\, r^t.$$

Since $\mathsf{L}_\mu^{q,t}$ is not countably subadditive, one needs a standard modification to get an outer measure. Hence, we modify the definition as follows

$$\mathsf{H}_\mu^{q,t}(E) = \inf\left\{\sum_i \mathsf{L}_\mu^{q,t}(E_i) \;\middle|\; E \subseteq \bigcup_i E_i \text{ and the } E_i'\text{s are bounded}\right\}$$

The measure $\mathsf{H}_\mu^{q,t}$ is of course a multifractal generalization of the lower t-dimensional Hewitt-Stromberg measure H^t, whereas $\mathsf{P}_\mu^{q,t}$ is a multifractal generalization of the upper t-dimensional Hewitt-Stromberg measures P^t (see [39]). In fact, it is easily seen that, for $t > 0$, one has

$$\mathsf{H}_\mu^{0,t} = \mathsf{H}^t \quad \text{and} \quad \mathsf{P}_\mu^{0,t} = \mathsf{P}^t.$$

The following result describes some of the basic properties of the multifractal Hewitt-Stromberg measures including the fact that $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$ are (metric) outer measures and summarises the basic inequalities satisfied by the multifractal Hewitt-Stromberg measures, the multifractal Hausdorff measure and the multifractal packing measure.

Theorem 2.1. *Let $q, t \in \mathbb{R}$. Then for every set $E \subseteq \mathbb{R}^n$ we have*

1. *the set functions $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$ are outer measures and thus they are measures on the Carathéodory-measurable algebra.*
2. *The function $\mathsf{H}_\mu^{q,t}$ is a metric outer measure and thus it is a measure on the Borel algebra.*
3. *The function $\mathsf{P}_\mu^{q,t}$ is not necessarily a metric outer measure.*
4. *There exists an integer $\xi \in \mathbb{N}$, such that*

$$\mathscr{H}_\mu^{q,t}(E) \le \xi \mathsf{H}_\mu^{q,t}(E)$$

and

$$\begin{array}{ccccc} \mathsf{L}_\mu^{q,t}(E) & \le & \mathsf{C}_\mu^{q,t}(E) & \le & \overline{\mathscr{P}}_\mu^{q,t}(E) \\ \vee\!\text{l} & & \vee\!\text{l} & & \vee\!\text{l} \\ \mathsf{H}_\mu^{q,t}(E) & \le & \mathsf{P}_\mu^{q,t}(E) & \le & \mathscr{P}_\mu^{q,t}(E), \end{array}$$

where $\mathscr{H}_\mu^{q,t}$, $\mathscr{P}_\mu^{q,t}$ and $\overline{\mathscr{P}}_\mu^{q,t}$ denote, respectively, the Hausdorff, packing and prepacking multifractal measures introduced in [32].

Proof. The proof of the first and second parts is straightforward and mimics that in [2, Theorem 2.1]. The proof of the third part is a straightforward application of Besicovitch's covering theorem and we omit it here (for more details we can see also [2, Theorem 2.1]). □

The measures $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$ and the pre-measures $\mathsf{L}_\mu^{q,t}$ and $\mathsf{C}_\mu^{q,t}$ assign in the usual way a multifractal dimension to each subset E of $\mathbb{R}^n$, they are respectively denoted by $\mathsf{b}_\mu^q(E)$, $\mathsf{B}_\mu^q(E)$, $\mathscr{L}_\mu^q(E)$ and $\Delta_\mu^q(E)$.

Proposition 2.2. *Let $q \in \mathbb{R}$ and $E \subseteq \mathbb{R}^n$. Then*

1. *there exists a unique number $\mathsf{b}_\mu^q(E) \in [-\infty, +\infty]$ such that*

$$\mathsf{H}_\mu^{q,t}(E) = \begin{cases} \infty & \text{if} \quad t < \mathsf{b}_\mu^q(E), \\ 0 & \text{if} \quad \mathsf{b}_\mu^q(E) < t, \end{cases}$$

2. *there exists a unique number $\mathsf{B}_\mu^q(E) \in [-\infty, +\infty]$ such that*

$$\mathsf{P}_\mu^{q,t}(E) = \begin{cases} \infty & \text{if} \quad t < \mathsf{B}_\mu^q(E), \\ 0 & \text{if} \quad \mathsf{B}_\mu^q(E) < t, \end{cases}$$

3. *there exists a unique number $\Delta_\mu^q(E) \in [-\infty, +\infty]$ such that*

$$\mathsf{C}_\mu^{q,t}(E) = \begin{cases} \infty & \text{if} \quad t < \Delta_\mu^q(E), \\ 0 & \text{if} \quad \Delta_\mu^q(E) < t, \end{cases}$$

4. *there exists a unique number $\mathscr{L}_\mu^q(E) \in [-\infty, +\infty]$ such that*

$$\mathsf{L}_\mu^{q,t}(E) = \begin{cases} \infty & \text{if} \quad t < \mathscr{L}_\mu^q(E), \\ 0 & \text{if} \quad \mathscr{L}_\mu^q(E) < t. \end{cases}$$

In addition, we have

$$\mathsf{b}_\mu^q(E) \le \mathsf{B}_\mu^q(E) \le \Delta_\mu^q(E).$$

The number $\mathsf{b}_\mu^q(E)$ is an obvious multifractal analogue of the lower Hewitt-Stromberg dimension $\underline{\dim}_{MB}(E)$ of E whereas $\mathsf{B}_\mu^q(E)$ is an obvious multifractal analogues of the upper Hewitt-Stromberg dimension $\overline{\dim}_{MB}(E)$ of E. Observe that the number $\mathscr{L}_\mu^q(E)$ is an obvious multifractal analogue of the lower box-dimension $\underline{\dim}_B(E)$ of E whereas $\Delta_\mu^q(E)$ is an obvious multifractal analogues of the upper box-dimension $\overline{\dim}_B(E)$ of E. In fact, it follows immediately from the definitions that

$$\mathscr{L}_\mu^0(E) = \underline{\dim}_B(E), \quad \Delta_\mu^0(E) = \overline{\dim}_B(E)$$

and

$$\mathsf{b}_\mu^0(E) = \underline{\dim}_{MB}(E), \quad \mathsf{B}_\mu^0(E) = \overline{\dim}_{MB}(E).$$

See [3, 12, 27, 31] for precise definitions of these dimensions.

Remark 2.3. *It follows from Theorem 2.1 that*

$$\begin{array}{ccccccc} & & \mathscr{L}^q_\mu(E) & \leq & \mathsf{\Delta}^q_\mu(E) & \leq & \Lambda^q_\mu(E) \\ & & \vee\!\mid & & \vee\!\mid & & \vee\!\mid \\ b^q_\mu(E) & \leq & \mathsf{b}^q_\mu(E) & \leq & \mathsf{B}^q_\mu(E) & \leq & B^q_\mu(E) \end{array}$$

where b^q_μ, B^q_μ and Λ^q_μ denote, respectively, the Hausdorff, packing and prepacking multifractal measures introduced in [32].

The definition of these dimension functions makes it clear that they are counterparts of the τ_μ-function which appears in the *multifractal formalism*. This being the case, it is important that they have the properties described by the physicists. The next theorem shows that these functions do indeed have some of these properties.

Theorem 2.4. *Let $q \in \mathbb{R}$ and $E \subseteq \mathbb{R}^n$.*

1. *The functions $q \mapsto \mathsf{H}^{q,t}_\mu(E)$, $\mathsf{P}^{q,t}_\mu(E)$, $\mathsf{C}^{q,t}_\mu(E)$ are decreasing.*
2. *The functions $t \mapsto \mathsf{H}^{q,t}_\mu(E)$, $\mathsf{P}^{q,t}_\mu(E)$, $\mathsf{C}^{q,t}_\mu(E)$ are decreasing.*
3. *The functions $q \mapsto \mathsf{b}^q_\mu(E)$, $\mathsf{B}^q_\mu(E)$, $\mathsf{\Delta}^q_\mu(E)$ are decreasing.*
4. *The functions $q \mapsto \mathsf{B}^q_\mu(E)$, $\mathsf{\Delta}^q_\mu(E)$ are convex.*

Proof. The proof of this is straightforward and mimics that in [3, Theorem 3]. □

We note that for all $q \in \mathbb{R}$

$$\mathsf{b}^q_\mu(\emptyset) = \mathsf{B}^q_\mu(\emptyset) = \mathsf{\Delta}^q_\mu(\emptyset) = -\infty,$$

and if $\mu(E) = 0$, then

$$\mathsf{b}^q_\mu(E) = \mathsf{B}^q_\mu(E) = \mathsf{\Delta}^q_\mu(E) = -\infty \quad \text{for} \quad q > 0.$$

Next, we define the separator functions $\mathsf{\Delta}_\mu$, B_μ and $\mathsf{b}_\mu : \mathbb{R} \to [-\infty, +\infty]$ by,

$$\mathsf{\Delta}_\mu(q) = \mathsf{\Delta}^q_\mu(\operatorname{supp}\mu),\ \mathsf{B}_\mu(q) = \mathsf{B}^q_\mu(\operatorname{supp}\mu) \text{ and } \mathsf{b}_\mu(q) = \mathsf{b}^q_\mu(\operatorname{supp}\mu).$$

The multifractal formalism based on the Hewitt-Stromberg measures $\mathsf{H}^{q,t}_\mu$ and $\mathsf{P}^{q,t}_\mu$ and the Hewitt-Stromberg dimension functions b_μ, B_μ and $\mathsf{\Delta}_\mu$ provides a natural, unifying and very general multifractal theory which includes all the hitherto introduced multifractal parameters, i.e., the multifractal spectra functions $\alpha \mapsto \mathsf{f}_\mu(\alpha) =: \underline{\dim}_{MB} E_\mu(\alpha)$ and $\alpha \mapsto \mathsf{F}_\mu(\alpha) =: \overline{\dim}_{MB} E_\mu(\alpha)$, the multifractal box dimensions. The Hewitt-Stromberg dimension functions b_μ and B_μ are intimately related to the spectra functions f_μ and F_μ (see [3, 10–12, 38]), whereas the dimension function $\mathsf{\Delta}_\mu$ is closely related to the upper box spectrum (more precisely, to the upper multifractal box dimension function $\overline{\tau}_\mu$, see Proposition 5.2).

3. Main results

Let μ be a compactly supported probability measure on $\mathbb{R}^n$ and $q \in \mathbb{R}$. In the following, we require an alternative characterization of the upper and lower multifractal box-counting dimensions of μ in terms of a potential obtained by convolving μ with a certain kernel. For this purpose let us introduce some interesting notations. For $1 \leq s \leq n$ and $r > 0$ we define the function

$$\begin{array}{rccl} \phi^s_r : & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \min\Big\{1,\ r^s|x|^{-s}\Big\}, \end{array}$$

and

$$\mu * \phi_r^s(x) = \int \min\left\{1,\ r^s|x-y|^{-s}\right\} d\mu(y).$$

Let E be a compact subset of $\operatorname{supp}\mu$. For $1 \le s \le n$ and $q > 1$, write

$$N_{\mu,r}^{q,s}(E) = \int_E \left(\mu * \phi_{r/3}^s(x)\right)^{q-1} d\mu(x),$$

and

$$\overline{\tau}_\mu^{q,s}(E) = \limsup_{r\to 0} \frac{\log N_{\mu,r}^{q,s}(E)}{-\log r} \quad \text{and} \quad \underline{\tau}_\mu^{q,s}(E) = \liminf_{r\to 0} \frac{\log N_{\mu,r}^{q,s}(E)}{-\log r}.$$

These definitions are, frankly, messy, indirect and unappealing. In an attempt to make the concept more attractive, we present here an alternative approach to the dimensions $\underline{\tau}_\mu^{q,s}$ and $\overline{\tau}_\mu^{q,s}$, and their applications to projections in terms of a potential obtained by convolving μ with a certain kernel. For E a compact subset of $\operatorname{supp}\mu$ we can try to decompose E into a countable number of pieces $E_1, E_2, \dots$ in such a way that the largest piece has as small a dimension as possible. The present approach was first used by Falconer in [15, Section 3.3] and further developed by O'Neil and Selmi in [32, 36, 39]. This idea leads to the following modified dimensions in terms of the convolutions:

$$\underline{\mathfrak{T}}_\mu^{q,s}(E) = \inf\left\{ \sup_{1\le i<\infty} \underline{\tau}_\mu^{q,s}(E_i) \;\Big|\; E \subseteq \bigcup_i E_i \ \text{ with each } E_i \text{ compact} \right\},$$

$$\overline{\mathfrak{T}}_\mu^{q,s}(E) = \inf\left\{ \sup_{1\le i<\infty} \overline{\tau}_\mu^{q,s}(E_i) \;\Big|\; E \subseteq \bigcup_i E_i \ \text{ with each } E_i \text{ compact} \right\}$$

and

$$\underline{\mathfrak{T}}_\mu^{s}(q) = \underline{\mathfrak{T}}_\mu^{q,s}(\operatorname{supp}\mu) \quad \text{and} \quad \overline{\mathfrak{T}}_\mu^{s}(q) = \overline{\mathfrak{T}}_\mu^{q,s}(\operatorname{supp}\mu) \quad \text{for all} \quad s \ge 1.$$

Let m be an integer with $0 < m \le n$ and $G_{n,m}$ the Grassmannian manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$. Denote by $\gamma_{n,m}$ the invariant Haar measure on $G_{n,m}$ such that $\gamma_{n,m}(G_{n,m}) = 1$. For $V \in G_{n,m}$, we define the projection map $\pi_V : \mathbb{R}^n \longrightarrow V$ as the usual orthogonal projection onto V. Then, the set $\{\pi_V,\ V \in G_{n,m}\}$ is compact in the space of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and the identification of V with π_V induces a compact topology for $G_{n,m}$. Also, for a Borel probability measure μ with compact support $\operatorname{supp}\mu \subset \mathbb{R}^n$ and for $V \in G_{n,m}$, we denote by μ_V, the projection of μ onto V, i.e.,

$$\mu_V(A) = \mu \circ \pi_V^{-1}(A) \quad \forall A \subseteq V.$$

Since μ is compactly supported and $\operatorname{supp}\mu_V = \pi_V(\operatorname{supp}\mu)$ for all $V \in G_{n,m}$, then, for any continuous function $f : V \longrightarrow \mathbb{R}$, we have

$$\int_V f\, d\mu_V = \int f(\pi_V(x)) d\mu(x),$$

whenever these integrals exist. Then for all $V \in G_{n,m}$, $x \in \mathbb{R}^n$ and $0 < r < 1$, we have

$$\mu * \phi_r^m(x) = \int \mu_V(B(x_V, r)) dV = \int \min\left\{1,\ r^m|x-y|^{-m}\right\} d\mu(y).$$

In the following, we compare the lower and upper multifractal Hewitt-Stromberg dimensions of a set E of $\mathbb{R}^n$ with respect to a measure μ with those of their projections onto m-dimensional subspaces.

Theorem 3.1. *Let μ be a compactly supported probability measure on $\mathbb{R}^n$ and $E \subseteq \operatorname{supp}\mu$. For $q \leq 1$ and all $V \in G_{n,m}$, we have*

$$\mathscr{L}^q_{\mu_V}(\pi_V(E)) \leq \mathscr{L}^q_\mu(E), \quad \Delta^q_{\mu_V}(\pi_V(E)) \leq \Delta^q_\mu(E)$$

and

$$\mathsf{b}^q_{\mu_V}(\pi_V(E)) \leq \mathsf{b}^q_\mu(E), \quad \mathsf{B}^q_{\mu_V}(\pi_V(E)) \leq \mathsf{B}^q_\mu(E).$$

The next result presents alternative expressions of the multifractal dimension functions $\mathscr{L}^q_\mu(E)$ and $\Delta^q_\mu(E)$ of a set E and that of its orthogonal projections.

Theorem 3.2. *Let E be a compact subset of* $\operatorname{supp}\mu$. *Then, we have*

1. *for all $q > 1$ and $V \in G_{n,m}$,*
$$\mathscr{L}^q_{\mu_V}(\pi_V(E)) \geq \underline{\tau}^{q,m}_\mu(E) \geq \mathscr{L}^q_\mu(E)$$
and
$$\Delta^q_{\mu_V}(\pi_V(E)) \geq \overline{\tau}^{q,m}_\mu(E) \geq \Delta^q_\mu(E).$$
2. *For all $1 < q \leq 2$ and $\gamma_{n,m}$-almost every $V \in G_{n,m}$,*
$$\Delta^q_{\mu_V}(\pi_V(E)) = \overline{\tau}^{q,m}_\mu(E) = \max\Big(m(1-q), \Delta^q_\mu(E)\Big)$$
and
$$\mathscr{L}^q_{\mu_V}(\pi_V(E)) = \underline{\tau}^{q,m}_\mu(E).$$
3. *For all $q > 2$ and $\gamma_{n,m}$-almost every $V \in G_{n,m}$,*
   (a) *If $\Delta^q_\mu(E) \geq -m$ then $\Delta^q_{\mu_V}(\pi_V(E)) = \overline{\tau}^{q,m}_\mu(E) = \Delta^q_\mu(E)$.*
   (b) $\mathscr{L}^q_{\mu_V}(\pi_V(E)) = \max\Big(m(1-q), \underline{\tau}^{q,m}_\mu(E)\Big)$.

In Theorem 3.3, we show that the upper multifractal Hewitt-Stromberg function $\mathsf{B}^q_\mu(E)$ is preserved under $\gamma_{n,m}$-almost every orthogonal projection for $q > 1$.

Theorem 3.3. *Let E be a compact subset of* $\operatorname{supp}\mu$ *and $q > 1$.*

1. *For all $V \in G_{n,m}$, we have*
$$\mathsf{B}^q_{\mu_V}(\pi_V(E)) \geq \mathsf{B}^q_\mu(E).$$
2. *If $1 < q \leq 2$, one has*
$$\mathsf{B}^q_{\mu_V}(\pi_V(E)) = \overline{\mathfrak{T}}^{q,m}_\mu(E) = \max\Big(m(1-q), \mathsf{B}^q_\mu(E)\Big), \text{ for } \gamma_{n,m}\text{-almost every } V \in G_{n,m}.$$
3. *If $q > 2$ and $(E_i)_i$ is a cover of E by a countable collection of compact sets is such that $\Delta^q_\mu(E_i) \geq -m$ for all i, then*
$$\mathsf{B}^q_{\mu_V}(\pi_V(E)) = \overline{\mathfrak{T}}^{q,m}_\mu(E) = \mathsf{B}^q_\mu(E), \text{ for } \gamma_{n,m}\text{-almost every } V \in G_{n,m}.$$

The following theorem enables us to study the lower multifractal Hewitt-Stromberg dimension of the projection of sets on m-dimensional linear subspaces for $q > 1$. In particular, we prove that $\mathsf{b}^q_\mu(E)$ is not preserved under $\gamma_{n,m}$-almost every orthogonal projection for $q > 1$.

Theorem 3.4. *Let E be a compact subset of* $\operatorname{supp}\mu$ *and $q > 1$.*

1. *For all $V \in G_{n,m}$, we have*
$$\mathsf{b}^q_{\mu_V}(\pi_V(E)) \geq \mathsf{b}^q_\mu(E).$$

2. *If* $1 < q \leq 2$, *one has*

$$\mathsf{b}^q_{\mu_V}(\pi_V(E)) = \underline{\mathfrak{T}}^{q,m}_{\mu}(E),\ \textit{for } \gamma_{n,m}\textit{-almost every } V \in G_{n,m}.$$

3. *If* $q > 2$, *then*

$$\mathsf{b}^q_{\mu_V}(\pi_V(E)) = \max\Big(m(1-q), \underline{\mathfrak{T}}^{q,m}_{\mu}(E)\Big),\ \textit{for } \gamma_{n,m}\textit{-almost every } V \in G_{n,m}.$$

For an integer s with $1 \leq m \leq s < n$, we define the s-energy of a measure μ by

$$I_s(\mu) = \int\int |x-y|^{-s} d\mu(x) d\mu(y).$$

Frostman [19] showed that the Hausdorff dimension of a Borel subset E of $\mathbb{R}^n$ is the supremum of the positive reals s for which there exists a Borel probability measure μ charging E and for which the s-energy of μ is finite. This characterization is used by Kaufmann [28] and Mattila [29] to prove their results on the preservation of the Hausdorff dimension. The condition $I_s(\mu) < \infty$ implies that $\dim_H(\mu) \geq s$. On the other hand, if $\mu(B(x,r)) \leq r^s$, for all x and all sufficiently small r then μ has a finite s-energy. Notice that Mattila [29] proved that if $I_m(\mu)$ is finite, then for almost every m-dimensional subspace V, the measure μ_V is absolutely continuous with respect to Lebesgue measure $\mathcal{L}^m_V$ on V identified with $\mathbb{R}^m$ and $\mu_V \in L^2(V)$, where $\mathcal{L}^m_V(E) = \mathcal{L}^m(E \cap V)$ for $E \subset \mathbb{R}^m$. Theorem 3.5 enables us to describe the behavior of large measures under projection.

Theorem 3.5. *Suppose that* μ *is a compactly supported Radon measure on* $\mathbb{R}^n$ *and* $0 < m \leq s < n$ *are such that* $I_s(\mu) < \infty$. *Then*

1. *if* $2m < s < n$, *then for* $\gamma_{n,m}$*-almost every* $V \in G_{n,m}$ *and* $q \geq 0$

$$\mathsf{b}_{\mu_V}(q) = \mathsf{B}_{\mu_V}(q) = m(1-q),$$

2. *if* $m \leq s \leq 2m$, *then for* $\gamma_{n,m}$*-almost every* $V \in G_{n,m}$ *and* $q \geq 0$

$$m(1-q) \leq \mathsf{b}_{\mu_V}(q) \leq \mathsf{B}_{\mu_V}(q) \leq \max\Big(m(1-q), -\frac{sq}{2}\Big).$$

Remark 3.6. *Fix* $0 < m \leq n$ *and let* μ *be a self-similar measure on* $\mathbb{R}^n$ *with support equal to* K *such that* $\dim_P(K) = s \leq m$. *Let* $q \geq 0$ *and* $(E_i)_i$ *be a cover of* E *by a countable collection of compact sets is such that* $\Delta^q_\mu(E_i) \geq -m$ *for all* i. *By using Theorems 3.3 and 3.4 and [32, Corollary 5.12], we have for* $\gamma_{n,m}$*-almost every* $V \in G_{n,m}$

$$B_{\mu_V}(q) = \mathsf{b}_{\mu_V}(q) = \mathsf{B}_{\mu_V}(q) = b_{\mu_V}(q) = b_\mu(q) = \mathsf{b}_\mu(q) = \mathsf{B}_\mu(q) = B_\mu(q) = \underline{\mathfrak{T}}^m_\mu(q) = \overline{\mathfrak{T}}^m_\mu(q).$$

4. Proof of Theorem 3.1

When $q < 0$ it suffice to observe that if $\Big(B\left(x_i, r\right)\Big)_{i\in\mathbb{N}}$ is a centered packing of $\pi_V(E)$ then $\Big(B\left(y_i, r\right)\Big)_{i\in\mathbb{N}}$ is a centered packing of E where $y_i \in E$ is such that $x_i = \pi_V\left(y_i\right)$ which implies that $\mu_V(B\left(x_i \frac{r}{3}\right))^q \leq \mu(B\left(y_i, \frac{r}{3}\right))^q$. This easily gives the desired result.

For $0 \leq q \leq 1$, fix $V \in G_{n,m}$ and let $\Big(B(x_i, r)\Big)_{i\in\mathbb{N}}$ be a packing of $\pi_V(E)$. For each i we consider the collection of balls as follows

$$\left\{B\left(y, \frac{r}{9}\right) \;\Big|\; y \in E \cap \pi_V^{-1}\left(V \cap B\left(x_i, \frac{r}{3}\right)\right)\right\},$$

and let $E_i = E \cap \pi_V^{-1}\Big(V \cap B(x_i, \frac{r}{3})\Big)$. So Besicovitch's covering theorem (see [32, Theorem 2.2]) provides a positive integer $\xi = \xi(n)$ and index sets, $I_{i1}, I_{i2}, \ldots, I_{i\xi}$ such that $E_i \subset \bigcup_{j=1}^{\xi} \bigcup_{i\in I_j} B\left(y, \frac{r}{9}\right)$ and the subset

$\left\{B(y,\frac{r}{9}) \mid y\in I_{ij}\right\}$ is a disjoint family for each j. Also, we observe that for a fixed i and j, we may make a simple volume estimate to further subdivide J_{ij} into at most 7^n disjoint subfamilies, $J_{ij1}, J_{ij2}, ..., J_{ij7^n}$ such that for each j and k, $\left\{B\left(y,\frac{r}{3}\right) \mid y\in \cup_i I_{ijk}\right\}$ is a centered packing of E. Since $0<q\le 1$, we have

$$\sum_i \mu_V\left(B\left(x_i,\frac{r}{3}\right)\right)^q \le \sum_i \mu\left(\bigcup_{j=1}^{\xi}\bigcup_{k=1}^{7^n}\bigcup_{y\in I_{ijk}} B\left(y,\frac{r}{9}\right)\right)^q \le \sum_i\sum_{j=1}^{\xi}\sum_{k=1}^{7^n}\sum_{y\in I_{ijk}} \mu\left(B\left(y,\frac{r}{9}\right)\right)^q \le \xi 7^n M^q_{\mu,\frac{r}{3}}(E).$$

This implies that

$$M^q_{\mu_V,r}(\pi_V(E)) \le \xi 7^n M^q_{\mu,\frac{r}{3}}(E).$$

Letting $r\downarrow 0$, now yields

$$\mathsf{L}^{q,t}_{\mu_V}(\pi_V(E)) \le 3^t\xi 7^n \mathsf{L}^{q,t}_{\mu}(E) \quad \text{and} \quad \mathsf{C}^{q,t}_{\mu_V}(\pi_V(E)) \le 3^t\xi 7^n \mathsf{C}^{q,t}_{\mu}(E).$$

We deduce from the previous inequalities that

$$\mathscr{L}^q_{\mu_V}(\pi_V(E)) \le \mathscr{L}^q_{\mu}(E) \quad \text{and} \quad \Delta^q_{\mu_V}(\pi_V(E)) \le \Delta^q_{\mu}(E).$$

Now, let $t > \mathsf{B}^q_\mu(E)$ which implies that $\mathsf{P}^{q,t}_\mu(E)<\infty$, then we can choose $(E_i)_i$ a covering of E such that $\sum_i \mathsf{C}^{q,t}_\mu(E_i) < 1$. We therefore conclude that $\pi_V(E)\subseteq \bigcup_i \pi_V(E_i)$ and

$$\mathsf{P}^{q,t}_{\mu_V}(\pi_V(E)) \le \sum_i \mathsf{C}^{q,t}_{\mu_V}(\pi_V(E_i)) \le 3^t\xi 7^n \sum_i \mathsf{C}^{q,t}_{\mu}(E_i) \le 3^t\xi 7^n < \infty.$$

This implies that

$$\mathsf{B}^q_{\mu_V}(\pi_V(E)) \le t \quad \text{for all} \quad t > \mathsf{B}^q_\mu(E).$$

We now infer that

$$\mathsf{B}^q_{\mu_V}(\pi_V(E)) \le \mathsf{B}^q_\mu(E). \tag{4.1}$$

The proof of the statement $\mathsf{b}^q_{\mu_V}(\pi_V(E)) \le \mathsf{b}^q_\mu(E)$ is identical to the proof of the statement (4.1) and is therefore omitted.

5. Proof of Theorems 3.2, 3.3, 3.4 and 3.5

We present the tools, as well as the intermediate results, which will be used in the proof of our main theorems.

5.1. Preliminary results

Let μ be a compactly supported probability measure on $\mathbb{R}^n$ and $q\in\mathbb{R}$. Recall that the upper and lower multifractal box-counting dimensions $\overline{\tau}^q_\mu$ and $\underline{\tau}^q_\mu$ of E are defined respectively by

$$\overline{\tau}^q_\mu(E) = \limsup_{r\to 0} \frac{\log M^q_{\mu,r}(E)}{-\log r} \quad \text{and} \quad \underline{\tau}^q_\mu(E) = \liminf_{r\to 0} \frac{\log M^q_{\mu,r}(E)}{-\log r}.$$

For technical convenience we shall assume that, if $r\in(0,1)$: $\frac{\log 0}{-\log r} = -\infty$.

The next result is essentially a restatement of [6, Proposition 4.2] and [8, Proposition 5.1] (see also [16, Lemma 2.6 (a)] and [41]), and has recently been obtained in [36, Proposition 4.2].

Proposition 5.1. *Let E be a compact subset of* supp μ. *For $q > 1$, we have*

$$\underline{\tau}_\mu^q(E) = \liminf_{r\to 0} \frac{1}{-\log r} \log \int_E \mu\left(B\left(x, \frac{r}{3}\right)\right)^{q-1} d\mu(x)$$

and

$$\overline{\tau}_\mu^q(E) = \limsup_{r\to 0} \frac{1}{-\log r} \log \int_E \mu\left(B\left(x, \frac{r}{3}\right)\right)^{q-1} d\mu(x).$$

In the next we investigate the relation between the lower and upper multifractal Hewitt-Stromberg functions b_μ and B_μ and the multifractal box dimension, the multifractal packing dimension and the multifractal pre-packing dimension.

Proposition 5.2. *Let $q \in \mathbb{R}$ and μ be a compact supported Borel probability measure on $\mathbb{R}^n$. Then for every $E \subseteq$ supp μ we have*

$$\mathscr{L}_\mu^q(E) = \underline{\tau}_\mu^q(E) \qquad \text{and} \qquad \Delta_\mu^q(E) = \overline{\tau}_\mu^q(E) = \Lambda_\mu^q(E).$$

Proof. We will prove the first equality, the second one is similar. Suppose that

$$\underline{\tau}_\mu^q(E) > \mathscr{L}_\mu^q(E) + \epsilon \quad \text{for some} \quad \epsilon > 0.$$

Then we can find $\delta > 0$ such that for any $r \leq \delta$,

$$M_{\mu,r}^q(E)\, r^{\mathscr{L}_\mu^q(E)+\epsilon} > 1 \quad \text{and then} \quad \mathsf{L}_\mu^{q,\mathscr{L}_\mu^q(E)+\epsilon} \geq 1$$

which is a contradiction. We therefore infer

$$\underline{\tau}_\mu^q(E) \leq \mathscr{L}_\mu^q(E) + \epsilon \quad \text{for any} \quad \epsilon > 0.$$

The proof of the following statement

$$\underline{\tau}_\mu^q(E) \geq \mathscr{L}_\mu^q(E) - \epsilon \quad \text{for any} \quad \epsilon > 0$$

is identical to the proof of the above statement and is therefore omitted.

We have the following additional property.

Proposition 5.3. *Let $q \in \mathbb{R}$ and μ be a compact supported Borel probability measure on $\mathbb{R}^n$. Then for every $E \subseteq$ supp μ we have*

$$\mathsf{b}_\mu^q(E) = \inf\left\{\sup_i \mathscr{L}_\mu^q(E_i) \;\middle|\; E \subseteq \bigcup_i E_i,\ \ E_i \ \textit{are bounded in}\ \ \mathbb{R}^n\right\}$$

and

$$\mathsf{B}_\mu^q(E) = \inf\left\{\sup_i \Delta_\mu^q(E_i) \;\middle|\; E \subseteq \bigcup_i E_i,\ \ E_i \ \textit{are bounded in}\ \ \mathbb{R}^n\right\}.$$

Proof. Denote

$$\beta = \inf\left\{\sup_i \mathscr{L}_\mu^q(E_i) \;\middle|\; E \subseteq \bigcup_i E_i,\ \ E_i \ \text{are bounded in}\ \ \mathbb{R}^n\right\}.$$

Assume that $\beta < \mathsf{b}_\mu^q(E)$ and take $\alpha \in (\beta, \mathsf{b}_\mu^q(E))$. Then we can choose $\{E_i\}$ of bounded subset of E such that $E \subseteq \cup_i E_i$, and $\sup_i \mathscr{L}_\mu^q(E_i) < \alpha$. Observe that $\mathsf{L}_\mu^{q,\alpha}(E_i) = 0$ which implies that $\mathsf{H}_\mu^{q,\alpha}(E) = 0$. It is a contradiction. Now suppose that $\mathsf{b}_\mu^q(E) < \beta$, then, for any $\alpha \in (\mathsf{b}_\mu^q(E), \beta)$, we have $\mathsf{H}_\mu^{q,\alpha}(E) = 0$. Thus, there exists $\{E_i\}$ of bounded subset of E such that $E \subseteq \cup_i E_i$, and $\sup_i \mathsf{L}_\mu^{q,\alpha}(E_i) < \infty$. We conclude that, $\sup_i \mathscr{L}_\mu^q(E_i) \leq \alpha$. It is also a contradiction. The proof of the second statement is identical to the proof of the statement in the first part and is therefore omitted.

The following proposition is a consequence of Propositions 5.2 and 5.3.

Proposition 5.4. *Let E be a compact subset of* supp μ *and* $q \in \mathbb{R}$*. One has*

$$\mathsf{b}_\mu^q(E) = \inf\left\{ \sup_{1\leq i<\infty} \underline{\tau}_\mu^q(E_i) \;\middle|\; E \subseteq \bigcup_i E_i \;\text{ with each } E_i \text{ compact} \right\}$$

and

$$\begin{aligned} \mathsf{B}_\mu^q(E) &= \inf\left\{ \sup_{1\leq i<\infty} \overline{\tau}_\mu^q(E_i) \;\middle|\; E \subseteq \bigcup_i E_i \;\text{ with each } E_i \text{ compact} \right\} \\ &= \inf\left\{ \sup_{1\leq i<\infty} \Lambda_\mu^q(E_i) \;\middle|\; E \subseteq \bigcup_i E_i \;\text{ with each } E_i \text{ compact} \right\}. \end{aligned}$$

Proposition 5.5. *Let E be a subset of* supp μ *and* $q \in \mathbb{R}$*. Then we have*

$$B_\mu^q(E) = \mathsf{B}_\mu^q(E).$$

Proof. It follows immediately from Proposition 5.4 and [36, Proposition 4.1].

The following result presents alternative expressions of the upper and lower multifractal box-counting dimensions in terms of the convolutions as well as general relations between the upper and lower multifractal box-counting dimensions of a measure and that of its orthogonal projections. This result has recently been obtained in [36, Theorem 4.1].

Theorem 5.6. *Let E be a compact subset of* supp μ*. Then, we have*

1. *for all* $q > 1$ *and* $V \in G_{n,m}$*,*

$$\underline{\tau}_{\mu_V}^q(\pi_V(E)) \geq \underline{\tau}_\mu^{q,m}(E) \quad \text{and} \quad \overline{\tau}_{\mu_V}^q(\pi_V(E)) \geq \overline{\tau}_\mu^{q,m}(E).$$

2. *For all* $1 < q \leq 2$ *and* $\gamma_{n,m}$*-almost every* $V \in G_{n,m}$*,*

$$\overline{\tau}_{\mu_V}^q(\pi_V(E)) = \overline{\tau}_\mu^{q,m}(E) = \max\Big(m(1-q), \overline{\tau}_\mu^q(E)\Big)$$

and

$$\underline{\tau}_{\mu_V}^q(\pi_V(E)) = \underline{\tau}_\mu^{q,m}(E).$$

3. *For all* $q > 2$ *and* $\gamma_{n,m}$*-almost every* $V \in G_{n,m}$*,*

   (a) *If* $-m \leq \overline{\tau}_\mu^q(E)$ *then* $\overline{\tau}_{\mu_V}^q(\pi_V(E)) = \overline{\tau}_\mu^{q,m}(E) = \overline{\tau}_\mu^q(E)$.

   (b) $\underline{\tau}_{\mu_V}^q(\pi_V(E)) = \max\Big(m(1-q), \underline{\tau}_\mu^{q,m}(E)\Big)$.

The assertion (2) is essentially a restatement of the main result of Hunt et al. in [23] and Falconer et al. in [16, Theorem 3.9]. The assertion (3) extends the result of Hunt and Kaloshin (of Falconer and O'Neil) to the case $q > 2$ untreated in their work.

5.2. *Proof of Theorem 3.2*

Follows directly from Proposition 5.2 and Theorem 5.6.

5.3. Proof of Theorem 3.3

The proof of the first part of Theorem 3.3 follows from Theorem 5.6 (1.), Proposition 5.3 and since $\overline{\tau}_\mu^{q,m}(E) \geq \overline{\tau}_\mu^q(E)$. By using Proposition 5.5, then the proof of the second and the third part of Theorem 3.3 has recently been obtained in [36, Theorem 3.2].

5.4. Proof of Theorem 3.4

1. The proof of the first part of Theorem 3.4 follows from Theorem 5.6 (1.), Proposition 5.3 and since $\underline{\tau}_\mu^{q,m}(E) \geq \underline{\tau}_\mu^q(E)$.
2. If $s > \underline{\mathfrak{T}}_\mu^{q,m}(E)$ we may cover E by a countable collection of sets E_i, which we may take to be compact, such that $\underline{\tau}_\mu^{q,m}(E_i) < s$. By using Theorem 5.6 (2.), we have $\underline{\tau}_{\mu_V}^q(\pi_V(E_i)) \leq s$ for $\gamma_{n,m}$-almost every $V \in G_{n,m}$. Proposition 5.4 implies that $\mathsf{b}_{\mu_V}^q(\pi_V(E)) \leq s$ for $\gamma_{n,m}$-almost every $V \in G_{n,m}$ and so, $\mathsf{b}_{\mu_V}^q(\pi_V(E)) \leq \underline{\mathfrak{T}}_\mu^{q,m}(E)$ for $\gamma_{n,m}$-almost every $V \in G_{n,m}$.
   Now, if $s < \underline{\mathfrak{T}}_\mu^{q,m}(E)$. Fix $V \in G_{n,m}$ and let $(\widetilde{E_i})_i$ be a cover of the compact set $\pi_V(E)$ by a countable collection of compact sets. Put for each i, $E_i = E \cap \pi_V^{-1}(\widetilde{E_i})$, then $\sup_i \underline{\tau}_\mu^{q,m}(E_i) > s$. By using Theorem 5.6 (1.), we have $\sup_i \underline{\tau}_{\mu_V}^q(\pi_V(E_i)) \geq s$ and $\sup_i \underline{\tau}_{\mu_V}^q(\widetilde{E_i}) \geq s$, this implies that $\mathsf{b}_{\mu_V}^q(\pi_V(E)) \geq s$. Therefore, we obtain $\mathsf{b}_{\mu_V}^q(\pi_V(E)) \geq \underline{\mathfrak{T}}_\mu^{q,m}(E)$.
3. First we suppose that $\underline{\mathfrak{T}}_\mu^{q,m}(E) \geq m(1-q)$. If $s > \underline{\mathfrak{T}}_\mu^{q,m}(E)$ we may cover E by a countable collection of sets E_i, which we may take to be compact, such that $\underline{\tau}_\mu^{q,m}(E_i) < s$. By using Theorem 5.6 (3.) we have $\underline{\tau}_{\mu_V}^q(\pi_V(E_i)) \leq s$ for $\gamma_{n,m}$-almost every $V \in G_{n,m}$. Proposition 5.4 implies that $\mathsf{b}_{\mu_V}^q(\pi_V(E)) \leq s$ for $\gamma_{n,m}$-almost every $V \in G_{n,m}$ and so, $\mathsf{b}_{\mu_V}^q(\pi_V(E)) \leq \underline{\mathfrak{T}}_\mu^{q,m}(E)$ for $\gamma_{n,m}$-almost every $V \in G_{n,m}$. The proof of $\mathsf{b}_{\mu_V}^q(\pi_V(E)) \geq \underline{\mathfrak{T}}_\mu^{q,m}(E)$ for all $V \in G_{n,m}$, is identical to the proof of the above statement and is therefore omitted.
   Now, we suppose that $\underline{\mathfrak{T}}_\mu^{q,m}(E) \leq m(1-q)$. It follows from Proposition 5.4 and Theorem 5.6 (3.) that $\mathsf{b}_{\mu_V}^q(\pi_V(E)) = m(1-q)$ for $\gamma_{n,m}$-almost every $V \in G_{n,m}$.

5.5. Proof of Theorem 3.5

The proof of Theorem 3.5 is straightforward from Remark 2.3 and [32, Corollary 4.4].

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