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On the complex valued metric-like spaces

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Abstract. The main purpose of this paper is to study complex valued metric-like spaces as an extension of metric-like spaces, complex valued partial metric spaces, partial metric spaces, complex valued metric spaces and metric spaces. In this article, the concepts such as quasi-equal points, completely separate points, convergence of a sequence, Cauchy sequence, cluster points and complex diameter of a set are defined in a complex valued metric-like space. Moreover, this paper is an attempt to present compatibility definitions for the complex distance between a point and a subset of a complex valued metric-like space and also for the complex distance between two subsets of a complex valued metric-like space. In addition, the topological properties of this space are also investigated.

1. Introduction and preliminaries

Distance is an important and fundamental notion in mathematics and there exist many generalizations of this concept in the literature (see [6]). One of such generalizations is the partial metric which was introduced by Matthews (see [11]). It differs from a metric in that points are allowed to have non-zero "self-distances" (i.e., $d(x,x) \geq 0$), and the triangle inequality is modified to account for positive self-distances. O'Neill [12] extended Matthews definition to partial metrics with "negative distances". Before describing the material of this paper, let us recall some definitions and set the notations which we use in what follows.

Definition 1.1. *A mapping $\mathfrak{p} : X \times X \to \mathbb{R}^+$, where X is a non-empty set, is said to be a partial metric on X if for any $x, y, z \in X$, the following four conditions hold true:*
(i) $x = y$ if and only if $\mathfrak{p}(x,x) = \mathfrak{p}(y,y) = \mathfrak{p}(x,y)$;
(ii) $\mathfrak{p}(x,x) \leq \mathfrak{p}(x,y)$;
(iii) $\mathfrak{p}(x,y) = \mathfrak{p}(y,x)$;
(iv) $\mathfrak{p}(x,z) \leq \mathfrak{p}(x,y) + \mathfrak{p}(y,z) - \mathfrak{p}(y,y)$.

The pair $(X, \mathfrak{p})$ is then called a *partial metric space*. A sequence $\{x_n\}$ in a partial metric space $(X, \mathfrak{p})$ converges to a point $x_0 \in X$ if $\lim_{n\to\infty} \mathfrak{p}(x_n, x_0) = \mathfrak{p}(x_0, x_0)$. A sequence $\{x_n\}$ of elements of X is called *Cauchy* if the limit

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$\lim_{m,n\to\infty} \mathfrak{p}(x_n, x_m)$ exists and is finite. The partial metric space $(X, \mathfrak{p})$ is called complete if for each Cauchy sequence $\{x_n\}$, there is some $x \in X$ such that

$$\lim_{n\to\infty} \mathfrak{p}(x_n, x) = \mathfrak{p}(x, x) = \lim_{m,n\to\infty} \mathfrak{p}(x_n, x_m).$$

An example of a partial metric space is the pair $(\mathbb{R}^+, \mathfrak{p})$, where $\mathfrak{p}(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. For more material about the partial metric spaces, see, e.g. [3, 5, 9] and references therein.

In 2012, A. Amini-Harandi [1] introduced a new extension of the concept of partial metric space, called a *metric-like space*. After that, the concept of b-metric-like space which generalizes the notions of partial metric space, metric-like space, and b-metric space was introduced by Alghamdi et al. in [4]. Recently, Zidan and Mostefaoui [14] introduced the double controlled quasi metric-like spaces and studied some topological properties of this space. Here, we state the concept of a metric-like space.

Definition 1.2. *A mapping $\mathfrak{D} : X \times X \to \mathbb{R}^+$, where X is a non-empty set, is said to be a metric-like on X if for any $x, y, z \in X$, the following three conditions hold true:*
(i) $\mathfrak{D}(x, y) = 0 \Rightarrow x = y$;
(ii) $\mathfrak{D}(x, y) = \mathfrak{D}(y, x)$;
(iii) $\mathfrak{D}(x, y) \leq \mathfrak{D}(x, z) + \mathfrak{D}(z, y)$.

The pair $(X, \mathfrak{D})$ is called a metric-like space. A metric-like on X satisfies all of the conditions of a metric except that $\mathfrak{D}(x, x)$ may be positive for some $x \in X$. The study of partial metric spaces has wide area of application, especially in computer sciences, see, e.g. [5, 10, 13] and references therein. That is why working on this topic can be very useful in practice. Since metric-likes are generalizations of partial metrics, knowing them can therefore provide us more applicable fields. In fact, this is our motivation to study the metric-like spaces. Each metric-like $\mathfrak{D}$ on X generates a topology $\tau_{\mathfrak{D}}$ on X whose base is the family of open balls. An open ball in a metric-like space $(X, \mathfrak{D})$, with center x and radius $r > 0$, is the set

$$B(x, r) = \{y \in X : |\mathfrak{D}(x, y) - \mathfrak{D}(x, x)| < r\}.$$

It is clear that a sequence $\{x_n\}$ in the metric-like space $(X, \mathfrak{D})$ converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \mathfrak{D}(x_n, x) = \mathfrak{D}(x, x)$. A sequence $\{x_n\}$ of elements of a metric-like space $(X, \mathfrak{D})$ is called Cauchy if the limit $\lim_{n,m\to\infty} \mathfrak{D}(x_n, x_m)$ exists and is finite. The metric-like space $(X, \mathfrak{D})$ is called complete if for each Cauchy sequence $\{x_n\}$, there is some $x_0 \in X$ such that

$$\lim_{n\to\infty} \mathfrak{D}(x_n, x_0) = \mathfrak{D}(x_0, x_0) = \lim_{n,m\to\infty} \mathfrak{D}(x_n, x_m).$$

For more details on this topic, see, e.g. [1, 7]. Note that every partial metric space is a metric-like space. But, the converse is not true in general. For example, let $X = \mathbb{R}$, and let $\mathfrak{D}(x, y) = \max\{|x - 5|, |y - 5|\}$ for all $x, y \in \mathbb{R}$. Then $(X, \mathfrak{D})$ is a metric-like space, but since $\mathfrak{D}(0, 0) \nleq \mathfrak{D}(1, 2)$, then $(X, \mathfrak{D})$ is not a partial metric space. We now state another extension of the notion of distance that allows distance to be a complex value. Azam et al., [2] introduced the concept of a complex valued metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions. In that article, they consider a partial order $\precsim$ on the set of complex numbers $\mathbb{C}$ and then introduce a complex valued metric. The partial order $\precsim$ is as follows:

$$z_1 \precsim z_2 \Leftrightarrow Re(z_1) \leq Re(z_2),\ Im(z_1) \leq Im(z_2).$$

Let X be a non-empty set. Suppose that the mapping $\mathfrak{D} : X \times X \to \mathbb{C}$ satisfies the following conditions:
(i) $0 \precsim \mathfrak{D}(x, y)$ for all $x, y \in X$;
(ii) $\mathfrak{D}(x, y) = 0 \Leftrightarrow x = y$;
(iii) $\mathfrak{D}(x, y) = \mathfrak{D}(y, x)$ for all $x, y \in X$;
(iv) $\mathfrak{D}(x, y) \precsim \mathfrak{D}(x, z) + \mathfrak{D}(z, y)$ for all $x, y, z \in X$.
Then $\mathfrak{D}$ is called a complex valued metric on X , and $(X, \mathfrak{D})$ is called a complex valued metric space.

Combining the two concepts complex valued metric spaces and metric-like spaces, we get complex valued metric-like spaces. Also, we introduce the notion of a complex valued partial metric space. As will be seen, the notion of complex valued metric-like space is a generalization of the notions of metric-like space, complex valued metric space, partial metric space, complex valued partial metric space and metric space. Therefore, it is interesting to investigate this general notion.

In this article, we focus on the structure of complex valued metric-like spaces and study some topological properties of this space. For instance, we introduce some concepts such as quasi-equal points, completely separate points, convergence of a sequence, Cauchy sequence, cluster point, limit point, complex absolute value, complex distance between a point and a subset of a complex valued metric-like space, and complex distance between two subsets of a complex valued metric-like space.

Additionally, we present several results about complex valued metric-like spaces.

2. Results and proofs

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Following [2], we define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \Leftrightarrow Re(z_1) \leq Re(z_2),\ Im(z_1) \leq Im(z_2).$$

Hence, $z_1 \precsim z_2$ if one of the following conditions is satisfied:
(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(iii) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(iv) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied. Also, we write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$
$$z_1 \precsim z_2 \prec z_3 \Rightarrow z_1 \prec z_3$$
$$z_1 \precsim z_2 + z_3 \Rightarrow z_1 - z_2 \precsim z_3$$

for $z_1, z_2, z_3 \in \mathbb{C}$. But note that if $0 \prec z_1 \precsim z_2$, then the inequality $z_2^{-1} \precsim z_1^{-1}$ is not true in general. For example, if $z_1 = 1 + i$ and $z_2 = 4 + i$, then it is clear that $0 \prec z_1 \precsim z_2$. One can easily see that $z_1^{-1} \prec z_2^{-1}$. Also, $z_2 \succsim z_1$ (resp. $z_2 \succ z_1$) means that $z_1 \precsim z_2$ (resp. $z_1 \prec z_2$). Throughout the paper, the set $\{z \in \mathbb{C} \mid z \succsim 0\}$ is denoted by $\mathbb{C} \succsim 0$, i.e. $\mathbb{C} \succsim 0 = \{z \in \mathbb{C} \mid z \succsim 0\}$. A complex number z is called positive if $0 \prec z$.

Definition 2.1. *Let X be a non-empty set. A mapping $d : X \times X \to \mathbb{C} \succsim 0$ is called a complex valued metric-like on X if for any $x, y, z \in X$, the following conditions hold:*
(D1) $d(x, y) = 0 \Rightarrow x = y$;
(D2) $d(x, y) = d(y, x)$;
(D3)$d(x, y) \precsim d(x, z) + d(z, y)$.

The pair (X, d) is then called a complex valued metric-like space. Indeed, a complex valued metric-like on X satisfies all of the conditions of complex valued metric except that may be $0 \precnsim d(x, x)$ for some $x \in X$. For convenience, we write (CVML) for "complex valued metric-like".

Definition 2.2. *Let X be a non-empty set. A mapping $\Pi : X \times X \to \mathbb{C} \succsim 0$ is said to be a complex valued partial metric on X if for any $x, y, z \in X$, the following conditions hold:*

- *$x = y \Leftrightarrow \Pi(x, x) = \Pi(y, y) = \Pi(x, y)$;*
- *$\max\{\Pi(x, x), \Pi(y, y)\} \precsim \Pi(x, y)$;*

- $\Pi(x,y) = \Pi(y,x)$;
- $\Pi(x,z) \precsim \Pi(x,y) + \Pi(y,z) - \Pi(y,y)$.

The pair (X,Π) is called a complex valued partial metric space.

Definition 2.3. *A complex valued metric-like d is called* non-Archimedean *if instead of axiom (D3), it satisfies the following better inequality:*

$$d(x,y) \precsim \max\{d(x,z), d(z,y)\}, \text{ for all } x,y,z \in X.$$

Definition 2.4. *Let X be a non-empty set. A mapping $\mathfrak{d} : X \times X \to \mathbb{C} \succsim 0$ is called a complex valued pseudometric on X if for all $x,y,z \in X$, the following conditions hold:*

- $x = y \Rightarrow \mathfrak{d}(x,y) = 0$;
- $\mathfrak{d}(x,y) = \mathfrak{d}(y,x)$;
- $\mathfrak{d}(x,y) \precsim \mathfrak{d}(x,z) + \mathfrak{d}(z,y)$.

Below, we present some examples of complex valued metric-like spaces and complex valued partial metric spaces.

Example 2.5. *Let $X = \mathbb{C}$. A mapping $d : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ defined by $d(z_1,z_2) = e^{i\theta}(|z_1| + |z_2|)$, where $0 \leqslant \theta \leqslant \frac{\pi}{2}$ is a CVML on $\mathbb{C}$.*

Example 2.6. *Let $X = \mathbb{R}$. A mapping $d : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ defined by $d(x_1,x_2) = (1+i)(|x_1| + |x_2|)$ is a CVML on $\mathbb{R}$.*

Example 2.7. *Consider*

$$X_1 = \{z \in \mathbb{C} : Re(z) \geqslant 0, Im(z) = 0\},$$

$$X_2 = \{z \in \mathbb{C} : Im(z) \geqslant 0, Re(z) = 0\}$$

and $X = X_1 \cup X_2$. Define a mapping $d : X \times X \to \mathbb{C}$ as follows:

$$d(z_1,z_2) = \begin{cases} (x_1+x_2)(1+i); & z_1,z_2 \in X_1 \\ (y_1+y_2)(1+i); & z_1,z_2 \in X_2 \\ (x_1+y_1)(1+i); & z_1 \in X_1, z_2 \in X_2 \\ (x_2+y_1)(1+i); & z_1 \in X_2, z_2 \in X_1 \end{cases}$$

A straightforward verification shows the (X,d) is a CVML space.

Example 2.8. *Let $X = \mathbb{C}$ and $\mathfrak{D} : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ defined by $\mathfrak{D}(z_1,z_2) = (1+i)|z_1 - z_2|$ for all $z_1,z_2,z \in \mathbb{C}$. We define a mapping $\Pi : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ by*

$$\Pi(z_1,z_2) = \frac{\mathfrak{D}(z_1,z_2) + i(|z_1| + |z_2|)}{2}$$

for all $z_1,z_2,z \in \mathbb{C}$. We show that Π is a complex valued partial metric on $\mathbb{C}$. Clearly, $\Pi(z_1,z_2) \succsim 0$ for all $z_1,z_2 \in \mathbb{C}$. Suppose that $z_1 = z_2$. Then, $\mathfrak{D}(z_1,z_2) = 0$ and we have

$$\Pi(z_1,z_2) = \frac{0 + 2i|z_1|}{2} = i|z_1| = \Pi(z_1,z_1) = \Pi(z_2,z_2) = i|z_2|$$

Now, suppose that $\Pi(z_1,z_2) = \Pi(z_1,z_1) = \Pi(z_2,z_2)$. We shall to show that $z_1 = z_2$. Since we are assuming that $\Pi(z_1,z_2) = \Pi(z_1,z_1)$, we have the following expressions:

$$\frac{\mathfrak{D}(z_1,z_2) + i(|z_1| + |z_2|)}{2} = |z_1|\,i$$

Then,

$$\mathfrak{D}(z_1, z_2) = 2|z_1|i - i|z_1| - i\,|z_2| = |z_1|i - i|z_2| = i(|z_1| - |z_2|), \tag{1}$$

which means that

$$(1 + i)|z_1 - z_2| = i\,(|z_2| - |z_1|) \tag{2}$$

It follows from (2) *that* $z_1 = z_2$*. Now, we are going to show that*

$$\max\{\Pi(z_1, z_1), \Pi(z_2, z_2)\} \precsim \Pi(z_1, z_2).$$

It is evident that for all $z_1, z_2 \in \mathbb{C}$*, we have*

$$||z_1| - |z_2|| \leqslant |z_1 - z_2|.$$

So $i\,(|z_1| - |z_2|) \precsim (i + 1)\,|z_1 - z_2|$ *and also*

$$2i\,|z_1| - i|z_1| - i|z_2| \precsim (1 + i)\,|z_1 - z_2| = \mathfrak{D}(z_1, z_2)$$

Hence, we see that

$$2i\,|z_1| \precsim \mathfrak{D}(z_1, z_2) + i\,(|z_1| + |z_2|)$$

and consequently,

$$\Pi(z_1, z_1) = i\,|z_1| \precsim \frac{\mathfrak{D}(z_1, z_2) + i\,(|z_1| + |z_2|)}{2} = \Pi(z_1, z_2)$$

for all $z_1, z_2 \in \mathbb{C}$*. Similarly, we have* $\Pi(z_2, z_2) \precsim \Pi(z_1, z_2,)$ *for all* $z_1, z_2 \in \mathbb{C}$*. Therefore,*

$$\max\{\Pi(z_1, z_1), \Pi(z_2, z_2)\} \precsim \Pi(z_1, z_2)$$

for all $z_1, z_2 \in \mathbb{C}$*. Evidently,* $\Pi(z_1, z_2) = \Pi(z_2, z_1)$ *for all* $z_1, z_2 \in \mathbb{C}$*. Moreover, for any* $z_1, z_2, z_3 \in \mathbb{C}$*, we have*

$$\frac{\mathfrak{D}(z_1, z_2) + i\,(|z_1| + |z_2|)}{2} \precsim \frac{\mathfrak{D}(z_1, z_3) + i\,(|z_1| + |z_3|) + \mathfrak{D}(z_3, z_2) + i\,(|z_3| + |z_2|)}{2} - i\,|z_3|$$

and so

$$\Pi(z_1, z_2) \precsim \Pi(z_1, z_3) + \Pi(z_3, z_2) - \Pi(z_3, z_3).$$

Notice that every complex valued partial metric space is a complex valued metric-like space, but the converse is not true in general.

Example 2.9. *Let* $X = \mathbb{C}$ *and* $d_1, d_2 : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *defined by* $d_1(z_1, z_2) = \begin{cases} 2i & z_1 = z_2 = 0 \\ i & \text{otherwise} \end{cases}$ *and* $d_2(z_1, z_2) = i \max\{|z_1 - 5|, |z_2 - 5|\}$ *for all* $z_1, z_2 \in \mathbb{C}$*. Then, a straightforward verification shows that both* d_1 *and* d_2 *are CVML on* $\mathbb{C}$*. But note that* $d_1(0, 0) \precsim d_1(1, 2)$ *and* $d_2(0, 0) \precsim d_2(2, 1)$*. Hence, both* (X, d_1) *and* (X, d_2) *are not complex valued partial metric spaces.*

Example 2.10. *Suppose that* (X, d_1) *is a CVML space. It is easy to check that*

$$d(x, y) = \begin{cases} 0 & x = y\,, \\ d_1(x, y) & x \neq y \end{cases}$$

is a complex valued metric on X*. Also,* $\mathfrak{d}(x, y) = |d_1(x, c) - d_1(c, y)|$ *is a complex valued pseudometric on* X*, where* c *is an arbitrary fixed element of* X*.*

In the following, we introduce the complex absolute value of $z \in \mathbb{C}$ which is denoted by $|\cdot|_c$.

$$|z|_c = |\mathrm{Re(z)}| + i\,|Im(z)|$$

for any $z \in \mathbb{C}$. Clearly, $0 \precsim |z|_c$ for all $z \in \mathbb{C}$. In the next proposition, we present some properties of the complex absolute value.

Proposition 2.11. *The complex absolute value has the following properties:*
(i) $|z_1 + z_2|_c \precsim |z_1|_c + |z_2|_c$;
(ii) $||z|_c| = |z|$;
(iii) $|z|_c \precsim r \Leftrightarrow -r \precsim z \precsim r$ $(r \in \mathbb{C} \succsim 0)$;
(iv) $z \precsim |z|_c \precsim (1+i)|z|$
for all $z_1, z_2, z \in \mathbb{C}$.

Proof. We prove (iii) and leave the rest to the interested reader. Let r be an arbitrary element of $\mathbb{C} \succsim 0$. Suppose that $|z|_c \precsim r$ for some $z \in \mathbb{C}$. So $|Re(z)| + i|Im(z)| \precsim Re(r) + iIm(r)$ and this yields that $|Re(z)| \leq Re(r)$ and $|Im(z)| \leq Im(r)$. It follows from the previous equalities that

$$\begin{aligned} &-Re(r) \leq Re(z) \leq Re(r) \\ &-iIm(r) \precsim iIm(z) \precsim iIm(r) \end{aligned}$$

We now get that $-r \precsim z \precsim r$. Conversely, suppose that $-r \precsim z \precsim r$. This implies that

$$-Re(r) - iIm(r) \precsim Re(z) + iIm(z) \precsim Re(r) + iIm(r)$$

The previous equalities imply that $|Re(z)| \leq Re(r)$ and $|Im(z)| \leq Im(r)$. Therefore, $|Re(z)| + i|Im(z)| \precsim Re(r) + iIm(r)$, which means that $|z|_c \precsim r$, as desired. □

Definition 2.12. *Let (X, d) be a CVML space and $A \subseteq X$.*
(i) An open ball with center $x_0 \in X$ and radius $0 \prec r \in \mathbb{C}$ is the set

$$N(x_0; r) = \{y \in X : |d(x_0, y) - d(x_0, x_0)|_c \prec r\}.$$

(ii) $a \in A$ is called an interior point of A, whenever there is a complex number $0 \prec r$ such that $N(a; r) \subseteq A$. The set of all interior points of A is denoted by A°. Obviously, $A^\circ \subseteq A$.
(iii) A is called an open set, whenever each element of A is an interior point of A, i.e. $A \subseteq A^\circ$.

Note that the family $\mathfrak{F} = \{N(x; r) \,:\, x \in X,\ 0 \prec r \in \mathbb{C}\}$ is a sub-basis for a topology on X.

In the following, we provide an example of an open ball in a CVML space.

Example 2.13. *Let $X = \mathbb{R}$. A straightforward verification shows that the mapping $d : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by $d(x, y) = (1 + i)max\{|x|, |y|\}$ is a CVML on $\mathbb{R}$. For $x_0 = 2$ and $r_0 = 1 + i$, we have*

$$\begin{aligned} N(2; 1+i) &= \{x \in \mathbb{R} : |d(x, 2) - d(2, 2)|_c \prec 1 + i\} \\ &= \Big\{x \in \mathbb{R} : \Big|(1+i)max\{|x|, |2|\} - (1+i)max\{|2|, |2|\}\Big|_c \prec 1 + i\Big\} \\ &= \Big\{x \in \mathbb{R} : \Big|(1+i)\Big(max\{|x|, |2|\} - 2\Big)\Big|_c \prec 1 + i\Big\} \\ &= \Big\{x \in \mathbb{R} : \Big|max\{|x|, |2|\} - 2\Big| + i\Big|max\{|x|, |2|\} - 2\Big| \prec 1 + i\Big\} \\ &= \Big\{x \in \mathbb{R} : \Big|max\{|x|, |2|\} - 2\Big| < 1\Big\} \\ &= \{x \in \mathbb{R} : 1 < max\{|x|, |2|\} < 3\} \\ &= (-3, 3) \end{aligned}$$

Definition 2.14. *Let (X, d) be a CVML space, let $\{x_n\}_{n\geq 1}$ be a sequence of X and let $x_0 \in X$. We say that*

- *The sequence $\{x_n\}_{n\geqslant 1}$ converges to x_0 if for every $0 \prec r \in \mathbb{C}$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in N(x_0, r)$ for all $n > n_0$, i.e. $|d(x_n, x_0) - d(x_0, x_0)|_c \prec r$ for all $n > n_0$. We denote this by $\lim_{n\to+\infty} x_n = x_0$, or $x_n \to x_0$ as $n \to +\infty$.*
- *The sequence $\{x_n\}_{n\geqslant 1}$ is called Cauchy if $\lim_{m,n\to+\infty} d(x_n, x_m)$ exists and is finite. It means that sequence $\{x_n\}_{n\geqslant 1}$ is Cauchy if and only if $\lim_{m,n\to+\infty} d(x_n, x_m) = z_0$ for some $z_0 \in \mathbb{C}$.*
- *The complex valued metric-like space (X, d) is complete if every Cauchy sequence of X is convergent. On the other hand, the complex valued metric-like space (X, d) is complete if for each Cauchy sequence $\{x_n\}$, there is some $x \in X$ such that*

$$\lim_{n\to+\infty} d(x_n, x) = d(x, x) = \lim_{m,n\to+\infty} d(x_n, x_m)$$

In the following, we give an example of a convergent sequence in a CVML space which is not Cauchy!.

Example 2.15. *Let $X = \{\bigcirc, \triangle, \square\}$. We define a mapping $d : X \times X \to \mathbb{C}$ as follows:*

$$d(\triangle,\triangle) = d(\triangle,\square) = d(\square,\triangle) = d(\square,\square) = i$$
$$d(\bigcirc,\square) = d(\square,\bigcirc) = d(\bigcirc,\triangle) = d(\triangle,\bigcirc) = d(\bigcirc,\bigcirc) = 1 + 3i$$

It is easy to check that d is a CVML on X. Considering the sequence

$$x_n = \begin{cases} \triangle & ; n = 3k \\ \bigcirc & ; n = 3k + 1 \\ \square & ; n = 3k + 2 \end{cases}$$

we have

$$d(x_n, x_m) = \begin{cases} i & ; n = 3k, m = 3q \\ 1 + 3i & ; n = 3k, n = 3q + 1 \\ i & ; n = 3k, m = 3q + 2 \\ 1 + 3i & ; n = 3k+1, m = 3q + 1 \\ 1 + 3i & ; n = 3k+1, m = 3q + 2 \end{cases}$$

It means that $\{x_n\}$ is not a Cauchy sequence of (X, d). But note that

$$d(x_n, \bigcirc) - d(\bigcirc, \bigcirc) = \begin{cases} 0 & ; n = 3k \\ 0 & ; n = 3k + 1 \\ 0 & ; n = 3k+2 \end{cases}$$

This means that $x_n \to \bigcirc$, as $n \to \infty$.

Definition 2.16. *Let X be a vector space over a complex or real field $\mathbb{F}$. A complex valued norm on X is a map $\|\cdot\|_c : X \longrightarrow \mathbb{C} \gtrsim 0$ satisfying the following three conditions:*
(i) $\|x\|_c = 0 \Leftrightarrow x = 0$;
(ii) $\|\lambda x\|_c = |\lambda|\|x\|_c$ for all $x \in X, \lambda \in \mathbb{F}$;
(iii) $\|x + y\|_c \precsim \|x\|_c + \|y\|_c$ for all $x, y \in X$.
A complex valued normed space is a pair $(X, \|.\|_c)$, where X is a complex or real vector space and $\|.\|_c$ is a complex valued norm on X. For example, let $(X, \|.\|)$ be a normed space. Define $\|.\|_c : X \to \mathbb{C}^+$ by $\|x\|_c = i\|x\|$ for all $x \in X$. Clearly, $\|\cdot\|_c$ is a complex valued norm on X.
Now, suppose that $\|.\|_c$ is a complex valued norm an X. If we define $d : X \times X \to \mathbb{C} \gtrsim 0$ by $d(x, y) = \|x - y\|_c$, then d is a complex valued metric on X. Moreover, if $\|\cdot\|_c$ is a complex valued norm an X, then the mapping $d : X\times X \longrightarrow \mathbb{C}$ defined by $d(x, y) = \|x\|_c + \|y\|_c$ is a CVML on X.

Proposition 2.17. *Let (X, d) be a complex valued metric-like space, and let x_0 be an arbitrary element of X. Then $\frac{d(x_0,x_0)}{2} \precsim d(x, x_0)$ for all $x \in X$.*

Proof. Suppose that there exists an element $\mathfrak{x} \in X$ such that $d(\mathfrak{x}, x_0) \prec \frac{d(x_0,x_0)}{2}$. Therefore, we have

$$d(x_0, x_0) \precsim d(x_0, \mathfrak{x}) + d(\mathfrak{x}, x_0) \prec \frac{d(x_0, x_0)}{2} + \frac{d(x_0, x_0)}{2} = d(x_0, x_0),$$

which is a contradiction. This contradiction shows that $\frac{d(x_0,x_0)}{2} \precsim d(x, x_0)$ for all $x \in X$. □

It follows immediately from the above proposition that if (X, d) is a CVML space and x_0 is an arbitrary element of X, then $\{x \in X \mid d(x, x_0) \prec \frac{d(x_0,x_0)}{2}\} = \emptyset$. Hence, we deduce that

$$max\Big\{\frac{d(x, x)}{2}, \frac{d(y, y)}{2}\Big\} \precsim d(x, y)$$

for all $x, y \in X$.

Remark 2.18. *Note that in CVML spaces the limit of a convergent sequence is not necessarily unique and this means that topology of these spaces is not necessarily a Hausdorff topology. For instance, suppose that $X = \mathbb{C}$ and $d(x, y) = i \max\{|x|, |y|\}$ for each $x, y \in X$. Putting $x_n = \frac{i}{n}$, we have $\lim_{n\to\infty} d(\frac{i}{n}, i) = \lim_{n\to\infty} i \max\{|\frac{i}{n}|, |i|\} = i = d(i, i)$. It means that the sequence $\{\frac{i}{n}\}$ converses to i, i.e. $\frac{i}{n} \to i$. Moreover, we have $\lim_{n\to\infty} d(\frac{i}{n}, 2i) = \lim_{n\to\infty} i \max\{|\frac{i}{n}|, |2i|\} = 2i = d(2i, 2i)$, and consequently, $\frac{i}{n} \to 2i$ as well. This demonstrates that the sequence $\{\frac{i}{n}\}$ converges to two different points.*

The above example leads us to the next definition.

Definition 2.19. *(Quasi-equal points) Let (X, d) be a CVML space. The points $x, y \in X$ are called* quasi-equal points *if there exists a sequence $\{x_n\}$ of X converging to both x and y, i.e. $x_n \to x$ and $x_n \to y$.*

From the previous remark, one can easily deduce that if $X = \mathbb{C}$ and $d(x, y) = i \max\{|x|, |y|\}$ for each $x, y \in \mathbb{C}$, then i and $2i$ are two quasi-equal points.

Definition 2.20. *(Completely separate points) Let (X, d) be a CVML space. The points x, y of X are called* completely separate points *if the following condition holds true:*

$$d(x, x) + d(y, y) \prec d(x, y)$$

Example 2.21. *Let X be a non-empty set. Define the mapping $d_1 : X \times X \to \mathbb{C}$ by $d_1(x, y) = \begin{cases} 0 & x = y \\ 1 + 4i & otherwise \end{cases}$. A straightforward verification shows that d_1 is a complex valued metric on X. Now we define $d : X \times X \to \mathbb{C}$ by$d(x, y) = i + d_1(x, y)$. Clearly, d is a CVML on X. Note that $d(x, x) = i = d(y, y)$ for all $x, y \in X$. If x, y are distinct, i.e. $x \neq y$, then $1 + 4i = d_1(x, y) \succ i$ and this yields that*

$$d(x, x) + d(y, y) \prec d(x, y)$$

This shows that every two distinct points of X are completely separate points. Obviously, in this space, a sequence $\{x_n\}$ of X converges to a point $x_0 \in X$ if and only if there exists a positive integer N such that $x_n = x_0$ for all $n > N$.

Theorem 2.22. *Let (X, d) be a CVML space. Then there are no convergent sequences to two completely separate points.*

Proof. Suppose that x, y are two completely separate points. To obtain a contradiction, let $\{x_n\}$ be a sequence of X converging to both x, y. Put $r_1 = \frac{1}{3}r$, where $r = d(x, y) - d(x, x) - d(y, y) \succ 0$. Since $x_n \to x$ and $x_n \to y$, for $\varepsilon = r_1$, there exist two positive integers N_1 and N_2 such that

$$\max\{|d(x_N, x) - d(x, x)|_c, |d(x_N, y) - d(y, y)|_c\} \prec r_1$$

for all $n \geq N = \max\{N_1, N_2\}$. Therefore,

$$\begin{aligned}
d(x, y) = |d(x, y)|_c &\precsim |d(x, x_N) + d(x_N, y)|_c \\
&= |d(x, x_N) + d(x_N, y) - d(x, x) + d(x, x) - d(y, y) + d(y, y)|_c \\
&\precsim |d(x, x_N) - d(x, x)|_c + |d(x_N, y) - d(y, y)|_c + d(x, x) + d(y, y) \\
&\prec r_1 + r_1 + d(x, x) + d(y, y) \\
&= 2r_1 + d(x, x) + d(y, y) \\
&= \frac{2}{3}\Big[d(x, y) - d(x, x) - d(y, y)\Big] + d(x, x) + d(y, y) \\
&= \frac{2}{3}d(x, y) + \frac{1}{3}d(x, x) + \frac{1}{3}d(y, y).
\end{aligned}$$

So we obtain that $d(x, y) \prec d(x, x) + d(y, y)$, a contradiction. This contradiction proves our claim. □

Immediate conclusion from the above theorem demonstrates that completely separate points are not quasi-equal.

Theorem 2.23. *Let (X, d) be a CVML space. Then the points x_1, x_0 of X are quasi-equal if and only if $N(x_0; r) \cap N(x_1; r) \neq \phi$ for all $0 \prec r \in \mathbb{C}$.*

Proof. Suppose that x_0, x_1 are two quasi-equal points of X. Hence, there exists a sequence $\{x_n\}$ of X such that $x_n \to x_0$ and $x_n \to x_1$. Assume that there exists a positive complex number r such that $N(x_0; r) \cap N(x_1; r) = \phi$. Since $x_n \to x_0$ and $x_n \to x_1$, for $\varepsilon = r$, there exist two positive integers N_1 and N_2 such that $x_n \in N(x_0; r)$ and also $x_n \in N(x_1; r)$ for all $n \geq M = \max\{N_1, N_2\}$. We see that $x_M \in N(x_0; r) \cap N(x_1; r) = \phi$, a contradiction. Conversely, assume that $N(x_0; r) \cap N(x_1; r) \neq \phi$ for all $0 \prec r \in \mathbb{C}$. Our next task is to show that there is a sequence $\{x_n\} \subseteq X$ converging to both x_0 and x_1. Putting $r_n = \frac{i}{n}$ and using our assumption, we deduce that for any positive integer n, there exists an element $x_n \in X$ such that $x_n \in N(x_0; \frac{i}{n}) \cap N(x_1; \frac{i}{n})$. It is clear that, for a given positive complex number r, there is a natural number $N \in \mathbb{N}$ such that $\frac{i}{N} \prec r$. Thus, for each $n \geq N$, we have

$$x_n \in N(x_0; \frac{i}{n}) \cap N(x_1; \frac{i}{n}) \subseteq N(x_0; \frac{i}{N}) \cap N(x_1; \frac{i}{N}) \subseteq N(x_0; r) \cap N(x_1; r).$$

It means that for any positive complex number r, there exists a positive integer N such that $x_n \in N(x_0; r)$ and $x_n \in N(x_1; r)$ for all $n \geq N$. This means that the sequence $\{x_n\}$ converges to both x_0 and x_1, i.e. $x_n \to x_0$, $x_n \to x_1$. Consequently, x_0 and x_1 are quasi-equal points, as desired. □

There is a consequence of the above theorem as follows:

Corollary 2.24. *Let (X, d) be a CVML space. The points x_0, x_1 of X are not quasi-equal if and only if there exists a positive complex number r such that $N(x_0; r) \cap N(x_1; r) = \phi$.*

It is clear that if x, y are two completely separate points of a CVML space (X, d), then there is a positive complex number r such that $N(x; r) \cap N(y; r) = \phi$. It is enough to assume that $r = \frac{1}{2}\Big[d(x, y) - d(x, x) - d(y, y)\Big]$. Below, we show that the converse of this statement is not true in general. To see this, let $X = \mathbb{C}$, $d(x, y) =$

$(1+i)(|x|+|y|)$, $x_0 = i$ and $x_1 = 4i$. Obviously, $d(x_0, x_1) \prec d(x_0, x_0) + d(x_1, x_1)$ and it means that the points x_0, x_1 are not completely separate points. We claim that $N(x_0; 1+i) \cap N(x_1; 1+i) = \phi$. To show the claim, we have

$$\begin{aligned} N(x_0; 1+i) = N(i; 1+i) &= \{z \in \mathbb{C} : |d(z,i) - d(i,i)|_c \prec 1+i\} \\ &= \{z \in \mathbb{C} : \left|(1+i)(|z|+1) - 2(1+i)\right|_c \prec 1+i\} \\ &= \{z \in \mathbb{C} : \left|(1+i)(|z|-1)\right|_c \prec 1+i\} \\ &= \{z \in \mathbb{C} : ||z|-1| + i||z|-1| \prec 1+i\} \\ &= \{z \in \mathbb{C} : ||z|-1| < 1\} \\ &= \{z \in \mathbb{C} : -1 < |z|-1 < 1\} \\ &= \{z \in \mathbb{C} : 0 < |z| < 2\} \\ &= \{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 4\} \end{aligned}$$

Moreover,

$$\begin{aligned} N(x_1; 1+i) = N(4i; 1+i) &= \{z \in \mathbb{C} : |d(z,4i) - d(4i,4i)|_c \prec 1+i\} \\ &= \{z \in \mathbb{C} : \left|(1+i)(|z|-4)\right|_c \prec 1+i\} \\ &= \{z \in \mathbb{C} : ||z|-4| + i||z|-4| \prec 1+i\} \\ &= \{z \in \mathbb{C} : ||z|-4| < 1\} \\ &= \{z \in \mathbb{C} : 3 < |z| < 5\} \\ &= \{(x,y) \in \mathbb{R}^2 : 9 < x^2 + y^2 < 25\}, \end{aligned}$$

It is observed that $N(i; 1+i) \cap N(4i; 1+i) = \phi$.

Checking the following properties for CVML spaces is straightforward and we leave it to the interested reader. Let (X, d) be a CVML space.

1. If x_0, x_1 are completely separate points, then there exists a positive complex number r such that $N(x_0; r) \cap N(x_1; r) = \phi$.
2. The points x_0, x_1 are not quasi-equal if and only if there exists a positive complex number r such that $N(x_0; r) \cap N(x_1; r) = \phi$.
3. If x_0, x_1 are completely separate points, then they are not quasi-equal points.

Remark 2.25. *We know that topology of CVML space is not necessarily a Hausdorff topology, since the limit of a convergent sequence in these spaces is not always unique. In a CVML space, completely separate points help us to obtain a Hausdorff subspace of that space. Indeed, let (X, d) be a CVML space and let $\mathfrak{Z}$ be the set of all completely separate points of X. Then the subspace $(\mathfrak{Z}, d)$ of (X, d) is a Hausdorff space.*

Definition 2.26. *(Cluster points) Let (X, d) be a CVML space and let A be a subset of X. A point $x_0 \in X$ is said to be a cluster point of A if for every positive complex number r there exists an element $a \in A$ such that $|d(a, x_0) - d(x_0, x_0)|_c \prec r$.*

As usual, the set of all *cluster points* of A is called the closure of A and is denoted by $\overline{A}$. Note that $x_0 \in \overline{A}$ if and only if $N(x_0; r) \cap A \neq \phi$ for all $0 \prec r \in \mathbb{C}$. Indeed, we have

$$\overline{A} = \left\{x_0 \in X : N(x_0; r) \cap A \neq \phi \ \textit{ for all } 0 \prec r \in \mathbb{C}\right\}$$

It is clear that if (X, d) is a CVML space and A is a subset of X, then $A \subseteq \overline{A}$. In the following, we establish a theorem to present a necessary and sufficient condition for cluster points in the CVML spaces. First, we prove a theorem about the convergence of sequences in CVML spaces.

Theorem 2.27. *Let (X, d) be a CVML space and let $\{x_n\}$ be a sequence of X. Then $\{x_n\}$ converges to x if and only if* $\lim_{n\to+\infty} \left| d(x_n, x) - d(x, x) \right| = 0$.

Proof. Suppose that $\{x_n\}$ converges to x. Let $\varepsilon = \frac{\alpha}{\sqrt{2}} + i\frac{\alpha}{\sqrt{2}} = \frac{\alpha}{\sqrt{2}}(1+i)$, where α is a given positive real number. Since $\{x_n\}$ converges to x, there exists a positive integer N such that $|d(x_n, x) - d(x, x)|_c \prec \varepsilon$ for all $n > N$. Therefore,

$$\left| |d(x_n, x) - d(x, x)|_c \right| < |\varepsilon| = \alpha$$

for all $n > N$, which means that $|d(x_n, x) - d(x, x)| < \alpha$ for all $n > N$. So

$$\lim_{n\to+\infty} |d(x_n, x) - d(x, x)| = 0.$$

Conversely, suppose that $\lim_{n\to+\infty} |d(x_n, x) - d(x, x)| = 0$. Therefore, for each positive real number α, there exists a positive integer N such that $|d(x_n, x) - d(x, x)| < \alpha$ for all $n > N$. It is clear that for any $0 \prec \varepsilon \in \mathbb{C}$, there exists a positive real number α such that $\alpha + i\alpha = (1+i)\alpha \prec \varepsilon$. Since $\lim_{n\to+\infty} |d(x_n, x) - d(x, x)| = 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that $|d(x_n, x) - d(x, x)| < \alpha$ for all $n \geq n_0$. Hence, for all $n \geq n_0$, we have

$$|d(x_n, x) - d(x, x)|_c \precsim (1+i) |d(x_n, x) - d(x, x)| \prec (1+i)\alpha \prec \varepsilon$$

Indeed, we get that for all $0 \prec \varepsilon \in \mathbb{C}$ there exists a positive integer n_0 such that $|d(x_n, x) - d(x, x)|_c \prec \varepsilon$ for all $n \geq n_0$, and this means that $\lim_{n\to+\infty} x_n = x$, as desired. □

Theorem 2.28. *Let (X, d) be a CVML space and let A be a subset of X. Then $x_0 \in \overline{\mathcal{A}}$ if and only if there is a sequence $\{a_n\} \subseteq \mathcal{A}$ converging to x_0.*

Proof. Suppose that $x_0 \in \overline{A}$. So for each $r_n = \frac{1+i}{n}$ $(n \in \mathbb{N})$, there is an element $a_n \in \mathcal{A}$ such that $|d(a_n, x_0) - d(x_0, x_0)|_c \prec \frac{1+i}{n}$. It implies that $\lim_{n\to+\infty} |d(a_n, x_0) - d(x_0, x_0)| = 0$, and it follows from Theorem 2.27 that the sequence $\{a_n\} \subseteq A$ converges to x_0. Conversely, assume that $\{a_n\}$ is a sequence of A converging to x_0. We must to show that $x_0 \in \overline{A}$. Let r be an arbitrary positive complex number. Therefore, there exists a positive number N such that $a_n \in N(x_0, r)$ for all $n \geq N$. It means that $A \cap N(x_0; r) \neq \phi$. Since we are assuming that r is an arbitrary positive complex number, it is deduced that $x_0 \in \overline{A}$, as desired. □

Example 2.29. *Let $X = \mathbb{C}$, $A = \{z \in \mathbb{C} : |z| < 1\}$, and let z_0 be an arbitrary complex number. Considering $d(z_1, z_2) = i \max\{|z_1|, |z_2|\}$ for each $z_1, z_2 \in \mathbb{C}$, we have*

$$\lim_{n\to+\infty} d\left(\frac{i}{n}, z_0\right) = \lim_{n\to\infty} i \max\left\{\frac{1}{n}, |z_0|\right\} = i|z_0| = d(z_0, z_0),$$

which means that $\frac{i}{n} \to z_0$. Since $\{\frac{i}{n}\} \subseteq A$ and also z_0 is an arbitrary element of $\mathbb{C}$, $\overline{A} = \mathbb{C}$.

Definition 2.30. *(Limit points) Let (X, d) be a CVML space and let A be a subset of X. A point $x_0 \in X$ is said to be a limit point of A if $N(x_0; r) \cap (A - \{x_0\}) \neq \phi$ for all positive complex numbers r.*

As usual, the set of all *limit points* of A is denoted by A'. A subset A of X is called a closed set, whenever each limit point of A belongs to itself, i.e. $A' \subseteq A$. One can easily prove the following theorem.

Theorem 2.31. *Let (X, d) be a CVML space and let A be a subset of X. Then $\overline{A} = A \cup A'$.*

In the following, we define the "*complex distance*" between a point and a subset of a CVML space.

Definition 2.32. *Let (X, d) be a CVML space and let A be a non-empty subset of X. The complex distance between a point $x_0 \in X$ and A is defined as follows:*

$$\begin{aligned} d(x_0, A) &= \inf\Big\{|d(x_0, a) - d(x_0, x_0)|_c : a \in A\Big\} \\ &= \inf\Big\{|Re(d(x_0, a) - d(x_0, x_0))| + i|Im(d(x_0, a) - d(x_0, x_0))| : a \in A\Big\} \\ &= \inf\Big\{|Re(d(x_0, a) - d(x_0, x_0))| : a \in A\Big\} + \\ &\quad i \inf\Big\{|Im(d(x_0, a) - d(x_0, x_0))| : a \in A\Big\}. \end{aligned}$$

Example 2.33. *(i) Let $X = \mathbb{C}$, $d(z_1, z_2) = i\max\{|z_1|, |z_2|\}$ for each $z_1, z_2 \in \mathbb{C}$ and let $A = \{n + \frac{i}{n} \mid n \in \mathbb{N}\}$. Then we have*

$$\begin{aligned} d(1, A) &= \inf\Big\{\Big|i\max\{|n + \frac{i}{n}|, |1|\} - i\max\{|1|, |1|\}\Big|_c : n \in \mathbb{N}\Big\} \\ &= \inf\Big\{\Big|i\max\{\sqrt{n^2 + \frac{1}{n^2}}, 1\} - i\Big|_c : n \in \mathbb{N}\Big\} \\ &= \inf\Big\{\Big|i\sqrt{n^2 + \frac{1}{n^2}} - i\Big|_c : n \in \mathbb{N}\Big\} \\ &= \inf\Big\{i\Big|\sqrt{n^2 + \frac{1}{n^2}} - 1\Big| : n \in \mathbb{N}\Big\} \\ &= \sqrt{2} - 1. \end{aligned}$$

(ii) As another example in this regard, let $X = \mathbb{R}$, let $d(x, y) = e^{i\theta}(|x| + |y|)$, where $0 \leqslant \theta \leqslant \frac{\pi}{2}$, let $A = (-1, 1)$ and let $x_0 = 4$. Then we have

$$\begin{aligned} d(x_0, A) = d(4, A) &= \inf\Big\{\Big|d(4, 4) - d(4, a)\Big|_c : a \in A\Big\} \\ &= \inf\Big\{|e^{i\theta}(|a| - 4)|_c : a \in A\Big\} \\ &= \inf\Big\{||a| - 4||e^{i\theta}|_c : a \in A\Big\} \\ &= 3|e^{i\theta}|_c. \end{aligned}$$

The next theorem demonstrates a relationship between the complex distance of a set and its closure.

Theorem 2.34. *Let (X, d) be a CVML space and let A be a non-empty subset of X. Then $\overline{A} = \{x \in X : d(x, A) = 0\}$.*

Proof. We first show that $\overline{A} \subseteq \{x \in X : d(x, A) = 0\}$. We have the following expressions:

$$\begin{aligned} x_0 \in \overline{A} &\Rightarrow \forall r \succ 0, N(x_0, r) \cap A \neq \phi, \\ &\Rightarrow \forall r \succ 0, \exists a_0 \in A : |d(x_0, x_0) - d(x_0, a_0)|_c \prec r, \\ &\Rightarrow \forall r \succ 0, d(x_0, A) \prec r, \\ &\Rightarrow d(x_0, A) = 0. \end{aligned}$$

Hence, $\overline{A} \subseteq \{x \in X : d(x, A) = 0\}$. Conversely, we show that if $d(x_0, A) = 0$, then $x_0 \in \overline{A}$. Since $d(x_0, A) = 0$, for each $r_n = \frac{1+i}{n}$ ($n \in \mathbb{N}$), there is an element $a_n \in A$ such that $|d(x_0, x_0) - d(x_0, a_n)|_c \prec \frac{1+i}{n}$. It follows from Theorem 2.27 that the sequence $\{a_n\}$ of A converges to x_0, and according to Theorem 2.28, $x_0 \in \overline{A}$. So we get that $\{x \in X : d(x, A) = 0\} \subseteq \overline{A}$, and the theorem is now proved. □

Meanwhile, one can easily prove the following proposition:

$$d(x_0, A) = 0 \Leftrightarrow \forall r \succ 0, \exists a \in A : |d(x_0, x_0) - d(x_0, a)|_c \prec r.$$

Now we illustrate the previous theorem by the following example.

Example 2.35. *Let $X = \mathbb{C}$, $A = \{z \in \mathbb{C} : |z| < 1\}$, and let $d(z_1, z_2) = i \max\{|z_1|, |z_2|\}$ for each $z_1, z_2 \in \mathbb{C}$. In Example 2.29 it was shown that $\overline{A} = \mathbb{C}$. Below, we show that $\{z \in \mathbb{C} : d(z, A) = 0\} = \mathbb{C}$. Suppose $z_0 \in \mathbb{C}$, with $|z_0| \geq 1$. Then for each $z \in A$, $d(z, z_0) = i \max\{|z|, |z_0|\} = i|z_0|$. In this case, we have*

$$\begin{aligned} d(z_0, A) &= \inf\left\{|d(z_0, z_0) - d(z_0, z)|_c : z \in A\right\} \\ &= \inf\left\{|i|z_0| - i|z_0||_c : z \in A\right\} \\ &= 0. \end{aligned}$$

Obviously, for each arbitrary element z_0 of A, we have $d(z_0, A) = 0$. Therefore,

$$\{z \in \mathbb{C} : d(z, A) = 0\} = \mathbb{C} = \overline{A}.$$

Below, we define the distance between two non-empty subsets of a CVML space.

Definition 2.36. *Let (X, d) be a CVML space and let A, B be two non-empty subsets of X. The complex distance between A and B is defined as follows:*

$$d(A, B) := \min\left\{ \inf\{d(a, B) : a \in A\}, \inf\{d(b, A) : b \in B\}\right\}.$$

Example 2.37. *Suppose that $X = \mathbb{R}$ and $d(x, y) = (1 + i)(|x| + |y|)$. We want to calculate the complex distance between the sets $A = (-1, 1)$ and $B = (3, 4)$ in the CVML space (X, d). For an arbitrary element $a \in A$, we have*

$$\begin{aligned} d(a, B) &= \inf\left\{|d(a, a) - d(a, b)|_c : b \in B\right\} \\ &= \inf\left\{\Big|(1 + i)(|a| - |b|\Big|_c : b \in B\right\} \\ &= \inf\left\{||a| - |b|| + i||a| - |b|| : b \in B\right\} \\ &= (3 - |a|) + i(3 - |a|). \end{aligned}$$

So $\inf\{d(a, B) : a \in A\} = \inf\{(1 + i)(3 - |a|) : a \in A\} = 2 + 2i$. Moreover, if b is an arbitrary element of B, then

$$\begin{aligned} d(b, A) &= \inf\left\{|d(a, b) - d(b, b)|_c : a \in A\right\} \\ &= \inf\{|b - |a|| + i|b - |a|| : a \in A\} \\ &= (b - 1) + i(b - 1). \end{aligned}$$

Hence, $\inf\{d(b, A) : b \in B\} = \inf\{(b - 1)(1 + i) : b \in B\} = 2 + 2i$. Thus, the complex distance between A and B is $d(A, B) = \min\{2 + 2i, 2 + 2i\} = 2 + 2i$.

Definition 2.38. *Let (X, d) be a CVML space and let A be a subset of X. The complex diameter of A is defined as follows:*

$$diam(A) := \sup\left\{|d(x, y) - d(x, x)|_c, |d(x, y) - d(y, y)|_c : x, y \in A\right\}.$$

If $|diam(A)| < \infty$, then the subset $A \subseteq X$ is said to be bounded.

Example 2.39. *Let* $X = \mathbb{C}$, $d(x,y) = (1+i)(|z_1| + |z_2|)$, *and let* $A = \{z \in \mathbb{C} : 3 < |z| < 5\}$. *In this case, we have*

$$\begin{aligned} diam(A) &= \sup\Big\{|d(z_1,z_2) - d(z_1,z_1)|_c, |d(z_1,z_2) - d(z_2,z_2)|_c : z_1, z_2 \in A\Big\} \\ &= \sup\Big\{\big||z_1| - |z_2|\big| + i\big||z_1| - |z_2|\big| : z_1, z_2 \in A\Big\} \\ &= \sup\Big\{\big||z_1| - |z_2|\big|(1+i) : z_1, z_2 \in A\Big\} \\ &= 2(1+i). \end{aligned}$$

Example 2.40. *Let Let* $0 \precsim z_0 \in \mathbb{C}$, $X = \mathbb{C}$, $d(x,y) = z_0 \max\{|x|, |y|\}$ *and let* $A = \{\frac{in}{n+1} : n \in \mathbb{N}\}$. *Hence, we have*

$$\begin{aligned} diam(A) &= \sup\Big\{\Big|d(\frac{mi}{m+1}, \frac{ni}{n+1}) - d(\frac{mi}{m+1}, \frac{mi}{m+1})\Big|_c, \Big|d(\frac{mi}{m+1}, \frac{ni}{n+1}) - \\ & d(\frac{ni}{n+1}, \frac{ni}{n+1})\Big|_c : m, n \in \mathbb{N}, m \geq n\Big\} \\ &= \sup\Big\{\Big|z_0 \max\{\frac{m}{m+1}, \frac{n}{n+1}\} - \frac{z_0 m}{m+1}\Big|_c, \Big|z_0 \max\{\frac{m}{m+1}, \frac{n}{n+1}\} - \frac{z_0 n}{n+1}\Big|_c : \\ & m, n \in \mathbb{N}, m \geq n\Big\} \\ &= \sup\Big\{\Big|\frac{z_0 m}{m+1} - \frac{z_0 m}{m+1}\Big|_c, \Big|\frac{z_0 m}{m+1} - \frac{z_0 n}{n+1}\Big|_c : m, n \in \mathbb{N}, m \geq n\Big\} \\ &= \sup\Big\{\Big|z_0(\frac{m}{m+1} - \frac{n}{n+1})\Big|_c : m, n \in \mathbb{N}, m \geq n\Big\} \\ &= \sup\Big\{(\frac{m}{m+1} - \frac{n}{n+1})|z_0|_c : m, n \in \mathbb{N}, m \geq n\Big\} \\ &= \frac{1}{2}|z_0|_c. \end{aligned}$$

Convergence of sequences plays a fundamental role in CVML spaces. The following theorem shows that in a CVML space every convergent sequence is bounded.

Theorem 2.41. *Suppose that* (X, d) *is a CVML space and sequence* $\{x_n\}$ *converges to* $x \in X$. *Then the set* $A = \{x_n : n \in \mathbb{N}\}$ *is a bounded subset of* X.

Proof. Let r_0 be a positive complex number. Since the sequence $\{x_n\}$ converges to the point x, there exists a positive integer N such that $|d(x_n, x) - d(x, x)|_c \prec r_0$ for all $n \geq N$. Using Proposition 2.11(iii), we deduce that $d(x_n, x) \prec r_0 + d(x, x)$ for all $n \geq N$. Hence, for all $m, n \geq N$, we have

$$\begin{aligned} |d(x_n, x_m) - d(x_n, x_n)|_c &\precsim |d(x_m, x_n)|_c + |d(x_n, x_n)|_c \\ &\precsim |d(x_m, x) + d(x, x_n)|_c + |d(x_n, x) + d(x, x_n)|_c \\ &\precsim |d(x_m, x)|_c + |d(x, x_n)|_c + |d(x_n, x)|_c + |d(x, x_n)|_c \\ &\prec 4(r_0 + d(x, x)). \end{aligned}$$

Reasoning like the above-mentioned argument, we can achieve that $|d(x_n, x_m) - d(x_m, x_m)|_c \prec 4(r_0 + d(x, x))$, for all $m, n \geq N$. Hence, $diam(A) \precsim 4(r_0 + d(x, x))$, and it implies that A is a bounded set in the CVML space (X, d). □

We conclude the article with the following remark.

Remark 2.42. *Let* (X, d) *be a CVML space. If we define* $D(x, y) = |d(x, y)|$ *for all* $x, y \in X$, *then it is observed that* (X, D) *is a metric-like space. We can provide a sequence* $\{z_n\}$ *in* $\mathbb{C} \succsim 0$ *such that* $lim_{n\to\infty}||z_n| - |z_0|| = 0$ *for some* $z_0 \in \mathbb{C} \succsim 0$, *but* $lim_{n\to\infty}|z_n - z_0| \neq 0$. *Let* $z_n = 2 + \frac{i}{n}$ *and* $z_0 = \sqrt{3} + i$. *It is clear that* $|z_n| = \sqrt{4 + \frac{1}{n^2}}$, $|z_0| = 2$ *and* $z_n, z_0 \in \mathbb{C} \succsim 0$ *for all* $n \in \mathbb{N}$. *We have* $lim_{n\to\infty}|\sqrt{4 + \frac{1}{n^2}} - 2| = 0$, *but it is easy to see that* $lim_{n\to\infty}|2 + \frac{i}{n} - \sqrt{3} - i| \neq 0$. *Hence, one can present a CVML space* (X, d) *in which* $lim_{n\to\infty}||d(x_n, x_0)| - |d(x_0, x_0)|| = |D(x_n, x_0) - D(x_0, x_0)| = 0$, *but* $lim_{n\to\infty}|d(x_n, x_0) - d(x_0, x_0)| \neq 0$. *Therefore, the topologies of the spaces* (X, d) *and* (X, D) *are not equivalent.*

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