Filomat 37:16 (2023), 5399–5412
https://doi.org/10.2298/FIL2316399B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On regular variations of fuzzy sequences via deferred statistical convergence

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Abstract. This paper introduces the idea of deferred statistical convergence of order β via the concept of regular variations of sequences of fuzzy numbers. The convergence of sequences of fuzzy numbers using ideas of variations such as regular, $O-$regular, translational regular and rapid, etc, are discussed in the context of deferred statistical convergence of order β. Results on generating relations among these different fuzzy deferred statistical variations are established.

1. Introduction

1.1. Statistical convergence

Due to its remarkable applications in various fields, the study of statistical convergence has attracted the recent research community, and subsequently been developed by several authors by various definitions. In 1935, Zygmund provided the idea of statistical convergence in general sense and it was reintroduced by Steinhaus [43] and Fast [23] independently in the context of sequence spaces. Schoenberg [42] developed the idea by applying it in operator theory and summability theory. The theory of statistical convergence was used in the convergence of trigonometric and Fourier series by Zygmund [46] and the theory of matrix characterization by Fridy and Miller [25]. Its applications were also found in various fields of mathematics such as in spaces of locally convex sets by Maddox [28], number theory by Erdős and Tenenbaum [21], integral summability theory by Connor and Swardson [11], measure theory by Miller [30], theory of lacunary summability by Fridy and Orhan [24], etc. Further, it was reintroduced and applied in approximation theory, single sequence spaces and different areas of functional analysis by Mursaleen et al. [32], Çakallı et al. [10], Di Maio and Kočinac [15], Et et al. [22], Salat [41], Baliarsingh et al. [6], Nuray and Aydın [39], Mohiuddine [31], Baliarsingh [7] and many others.

It is known that the idea of statistical convergence depends on the natural density of subsets of the set $\mathbb{N}$ (the set of all natural numbers). Before we define the statistical convergence, first we need to define the natural density of a set. The natural density of E, a subset of $\mathbb{N}$ is defined by

2020 *Mathematics Subject Classification*. Primary 40G15; Secomdary 41A36, 46A35, 46A45.
Keywords. Statistical convergence; Fuzzy deferred statistical convergence; Fuzzy sequence spaces; Regular variation.
Received: 01 October 2022; Revised: 23 December 2022; Accepted: 03 January 2023
Communicated by Ljubiša D.R. Kočinac
This research work is supported by NBHM, DAE, Mumbai, under the grant no. 02011/ 7/2020/NBHM(RP)/R&D/6289.
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$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k), \tag{1}$$

provided the limit in (1) exists and χ_E denotes the characteristic function of the set E. A sequence $x = (x_k)$ is said to be statistically convergent to ξ if, for every $\varepsilon > 0$, we have

$$\delta\left(\{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\}\right) = 0.$$

In this case, we say that $|x_k - \xi| \geq \varepsilon$ for almost all k and write $st - \lim x_k = \xi$. Note that if a sequence $x = (x_k)$ is convergent, then it is statistical convergent, but the converse is not true.

A sequence $x = (x_k)$ is said to be statistically Cauchy if, for every $\varepsilon > 0$, there exists a number N depending on ε, we have

$$\delta\left(\{k \in \mathbb{N} : |x_k - x_N| \geq \varepsilon\}\right) = 0.$$

In this case, we also say that $|x_k - x_N| \geq \varepsilon$ for almost all k. Note that every Cauchy sequence is statistical Cauchy, but the converse is not true.

Also, a sequence $x = (x_k)$ of real numbers is said to be statistically bounded if there exists a number K such that

$$\delta(\{k \in \mathbb{N} : |x_k| \geq K\}) = 0.$$

The main objective of this paper is to extend the idea of statistical convergence for fuzzy sequences via regular variations and deferred Cesàro mean.

1.2. *Fuzzy metric spaces*

Initially, in the year 1965, the idea of fuzzy number was introduced by Zadeh [45] via its arithmetic operations. Later on, it was applied in sequence space theory by Matloka [29] in 1986. Further, the idea was applied in various fields of summability theory including topological properties and other algebraic properties of sequence spaces of fuzzy numbers by Nanda [33]. The convergence and statistical convergence of fuzzy sequence was studied by Nuray and Savas [38], and then the idea was further generalized by many researchers like Altinok and Et [4], Altinok [3], Küçükaslan and Yilmaztürk [27], Nayak et al. [35, 36], etc. Some definitions related to fuzzy number are given below:

Fuzzy number is a function $\vartheta : \mathbb{R} \to [0, 1]$ which satisfies $\vartheta(r) = 1$ for some $r \in \mathbb{R}$ (normality), $\vartheta(r) \geq \min\{\vartheta(c), \vartheta(d)\}$ where $c < r < d$ (fuzzy convexity), ϑ is upper semi continuous, and the closure of $\vartheta^0 = \{\vartheta \in \mathbb{R} : \vartheta(r) > 0\}$ is compact.

The notation $\mathscr{L}(\mathbb{R})$ is used for the set of all fuzzy numbers on $\mathbb{R}$. The $\alpha-$level cut of the fuzzy number ϑ is defined by

$$[\vartheta]_\alpha = \{r \in \mathbb{R} : \vartheta(r) \geq \alpha\},$$

where $\alpha \in (0, 1]$. Suppose $[\vartheta]^+_\alpha$ and $[\vartheta]^-_\alpha$, denote the upper and lower bounds of the $\alpha-$level cut of the fuzzy number ϑ, respectively. Then the distance between two fuzzy numbers ϑ_1 and ϑ_2 is given by

$$\begin{aligned} d(\vartheta_1, \vartheta_2) &= \sup_{\alpha\in[0,1]} d_H([\vartheta_1]_\alpha, [\vartheta_2]_\alpha) \\ &= \sup_{\alpha\in[0,1]} \max\left\{\left|[\vartheta_1]^-_\alpha - [\vartheta_1]^-_\alpha\right|, \left|[\vartheta_1]^+_\alpha - [\vartheta_2]^+_\alpha\right|\right\}, \end{aligned}$$

where the function $d_H(.,.)$ is the Hausdorff metric. Let $\vartheta_1, \vartheta_2 \in \mathscr{L}(\mathbb{R})$. Then the fuzzy sum $\vartheta_1 \oplus \vartheta_2$ and fuzzy product $\vartheta_1 \odot \vartheta_2$ are, respectively defined by

$$[\vartheta_1 \oplus \vartheta_2]_\alpha = [\vartheta_1]_\alpha + [\vartheta_2]_\alpha,$$

and

$$[\vartheta_1 \odot \vartheta_2]_\alpha = [\vartheta_1]_\alpha [\vartheta_2]_\alpha.$$

The pair $(\mathscr{L}(\mathbb{R}), d)$ (see, [40]) is called a fuzzy metric space with the metric d and for $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathscr{L}(\mathbb{R})$, it satisfies

(i) $d(\gamma\vartheta_1, \gamma\vartheta_2) = |\gamma| d(\vartheta_1, \vartheta_2)$ for $\gamma \in \mathbb{C}$ (the set of all complex scalars),

(ii) $d(\vartheta_1 + \vartheta_2, \vartheta_3 + \vartheta_2) = d(\vartheta_1, \vartheta_3)$,

(iii) $d(\vartheta_1 + \vartheta_3, \vartheta_2 + \vartheta_4) \leq d(\vartheta_1, \vartheta_2) + d(\vartheta_3, \vartheta_4)$,

(iv) $|d(\vartheta_1, \overline{0}) - d(\vartheta_2, \overline{0})| \leq d(\vartheta_1, \vartheta_2) \leq d(\vartheta_1, \overline{0}) + d(\vartheta_2, \overline{0})$.

Note that $(\mathscr{L}(\mathbb{R}), d)$ is a complete metric space with metric d and $\overline{0}$ is the additive identity of $\mathscr{L}(\mathbb{R})$.

1.3. Deferred Cesàro mean

Suppose that $p = (p_n)$ and $q = (q_n)$ are the sequences of non-negative integers(see [1]) satisfying

(i) $p_n < q_n$ for all $n \in \mathbb{N}_0$,

(ii) $\lim_{n\to\infty} q_n = \infty$,

Then, the deferred Cesàro mean (see [34]) of the sequence $u = (u_n)$ is defined by

$$(D_{p,q}\xi)_n = \frac{u_{p_n+1} + u_{p_n+2} + \cdots + u_{q_n}}{q_n - p_n} = \sum_{k=0}^{\infty} d_{nk}\xi_k,$$

where

$$d_{nk} = \begin{cases} \frac{1}{q_n - p_n}, & (p_n < k \leq q_n) \\ 0, & (\text{ otherwise}). \end{cases}$$

The conditions (i) and (ii) are the regularity conditions for the deferred Cesàro mean $D_{p,q}$. In particular, $D_{p,q}$ mean generalizes various transforms such as I, the identity transform for $p_n = n - 1$ and $q_n = n$, $(C, 1)$, the Cesàro transform for $p_n = 0$ and $q_n = n$, and (V, λ^*), de la Vallée Poussin transform $p_n = n - \lambda_n^* + 1$ and $q_n = n$, etc. Note that $\lambda^* = (\lambda_n^*)$ is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_n^* \leq \lambda_n^* + 1$ and $\lambda_0^* = 1$. In this work, we extend the idea of convergence and divergence of fuzzy sequences in deferred statistical context via regular variations.

1.4. Regular variations

Generally, the theory of regular variations deals with the asymptotic analysis of convergent and divergent processes. The idea is being used to find the rate at which a given sequence or a function converges or diverges. In 1930, Karamata [26] initially developed the idea while working on the Tauberian theorems of Hardy and Littlewood, and then de Haan [12] applied it to the weak convergence theory. In 1973, Bojanić and Seneta [9] redefined the idea in order to use it in the theory of functions. Then the general idea of $O-$regular variations was given by Aljančić and Arandjelović [2], and was further extended by Djurčić and Božin [17], and Djurčić [18], etc. Recently, the idea of $O-$regular variations has been applied in the theory of uniform convergence by Arandjelović [5], function theory by Tasković [44], the theory of sequence spaces by Djurčić et al. [16, 19], and the theory of statistical convergence by Demirci et al. [13] Di Maio et al. [14], Dutta and Das [20], Nayak et al. [37] etc. Now, we provide some primary definitions of regular variations given in [8].

A positive real sequence $x = (x_k)$ is said to be regularly varying if it satisfies

$$\lim_{k\to\infty} \frac{x_{[\mu k]}}{x_k} = k(\mu) < \infty, \text{ for all } \mu > 0.$$

The sequence $x = (x_k)$ is called slowly varying if $k(\mu) = 1$, for each $\mu > 0$, and if the function $k(\alpha)$ is of the form μ^ρ for some $\rho \in \mathbb{R}$, then the number ρ is called the index of variability of the sequence x.

A positive real sequence $x = (x_k)$ is said to be $O-$regularly varying if for each $\mu > 0$,

$$\limsup_{k\to\infty} \frac{x_{[\mu k]}}{x_k} = o(\mu) < \infty.$$

Note that every regular varying sequence is $O-$regularly varying but the converse is not true in general.

A positive real sequence $x = (x_k)$ is said to be translationally regularly varying if for each $\mu > 0$,

$$\lim_{k\to\infty} \frac{x_{[k+\mu]}}{x_k} = r(\mu) < \infty,$$

where the function $r(\mu)$, for each $\mu > 0$ is of the form $e^{\rho[\mu]}$ for some $\rho \in \mathbb{R}$, and ρ is the index of variability of the sequence x.

A sequence $x = (x_k)$ (of positive real numbers) is said to be rapidly varying sequence(of index of variability ∞) if for each $\mu > 1$,

$$\lim_{k\to\infty} \frac{x_{[\mu k]}}{x_k} = \infty,$$

and for each $0 < \mu < 1$

$$\lim_{k\to\infty} \frac{x_{[\mu k]}}{x_k} = 0.$$

Similarly, it is said to be rapidly varying sequence of index of variability $-\infty$ if for each $\mu > 1$,

$$\lim_{k\to\infty} \frac{x_{[\mu k]}}{x_k} = 0.$$

Now, extend the above definitions to the sequence of positive fuzzy numbers in deferred statistical sense as follows:

Definition 1.1. A sequence of fuzzy numbers (u_k) (here and onwards as sequence of positive fuzzy numbers) is said to be deferred-statistically convergent of order β to ξ_0 if, for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)^\beta} |\{p_n < k \leq q_n : d(u_k, \xi_0) \geq \varepsilon\}| = 0.$$

In other words, the natural density with respect to deferred Cesàro mean $\delta^\beta_{pq}(k : d(u_k, \xi_0)) = 0$, and we say that $dst - \lim u = \xi_0$.

Definition 1.2. A sequence of fuzzy numbers (u_k) is said to be deferred-statistically regularly varying of order β if it satisfies

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)^\beta} |\{p_n < k \leq q_n : d\left(\frac{u_{[\mu k]}}{u_k}, k_{ds}(\mu)\right) \geq \varepsilon\}| = 0, \text{ for all } \mu > 0 \text{ and } k_{ds}(\mu) < \infty,$$

or, symbolically

$$\delta^\beta_{pq}\left(k : d\left(\frac{u_{[\mu k]}}{u_k}, k_{ds}(\mu)\right)\right) = 0, \text{ for all } \mu > 0 \text{ and } k_{ds}(\mu) < \infty.$$

By $DSRV^F, DSSV^F$, and $DSRV^F_\rho$, we denote the classes of all deferred-statistically regularly, slowly and regularly with index ρ varying of order β positive fuzzy sequences, respectively.

Definition 1.3. A sequence of fuzzy numbers (u_k) is said to be deferred-statistically $O-$regularly varying of order β if for each $\mu > 0$,

$$\delta^{\beta}_{pq}\left(k : d\left(\sup_{k} \frac{u_{[\mu k]}}{u_k}, u_{ds}(\mu)\right)\right) = 0, \quad \text{for all } u_{ds}(\mu) < \infty.$$

The set $DSORV^F$ denotes such class of all deferred-statistically $O-$regularly varying of order β positive fuzzy sequences.

Definition 1.4. A sequence of fuzzy numbers (u_k) is said to be deferred-statistically translationally regularly varying of order β if for each $\mu > 0$,

$$\delta^{\beta}_{pq}\left(k : d\left(\frac{u_{[k+\mu]}}{u_k}, r_{ds}(\mu)\right)\right) = 0, \quad \text{for all } r_{ds}(\mu) < \infty.$$

By $DSTRV^F$ and $DSTRV^F_{\rho}$, we denote the classes for all deferred-statistically translationally regularly and with index ρ varying of order β fuzzy sequences, respectively.

Definition 1.5. A sequence of fuzzy numbers (u_k) is said to be deferred-statistically rapidly varying of order β fuzzy sequence(of index of variability ∞) if for each $\mu > 1$,

$$\delta^{\beta}_{pq}\left(k : d\left(\frac{u_{[\mu k]}}{u_k}, \infty\right)\right) = 0,$$

and for each $0 < \mu < 1$

$$\delta^{\beta}_{pq}\left(k : d\left(\frac{u_{[\mu k]}}{u_k}, 0\right)\right) = 0.$$

For each $0 < \mu < 1$ we denote the set $DSRV^F_{\infty}$ for the class of all deferred-statistically rapidly varying of order β positive fuzzy sequences of index of variability ∞. The sequence (u_k) is said to be deferred-statistically rapidly varying of order β fuzzy sequence of index of variability $-\infty$ if for each $\mu > 1$,

$$\delta^{\beta}_{pq}\left(k : d\left(\frac{u_{[\mu k]}}{u_k}, 0\right)\right) = 0.$$

and we denote such class of sequences as the set $DSRV^F_{-\infty}$.

Example 1.6. Let us define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} t, & \text{if } k \text{ is a square,} \\ 1-t, & \text{otherwise,} \end{cases} \quad t \in [1, 0].$$

For fixed $\mu > 0$, the sequence $\frac{u_{[\mu k]}}{u_k}$ can be given by

$$\frac{u_{[\mu k]}}{u_k} = \begin{cases} \dfrac{t}{1-t} \ (= u^{(1)}), & \text{if } [\mu k] \text{ is a square and } k \text{ is a non-square (CASE I),} \\ \dfrac{1-t}{t} \ (= u^{(2)}), & \text{if } [\mu k] \text{ is a non-square and } k \text{ is a square (CASE II),} \\ 1, & \text{otherwise.} \end{cases}$$

Now, we can calculate

$$d\left(\frac{u_{[\mu k]}}{u_k}, u^{(1)}\right) = \begin{cases} 0, & \text{for CASE I,} \\ \sup\limits_{\alpha\in[0,1]} \max\{|[u^{(2)}]^-_{\alpha} - [u^{(1)}]^-_{\alpha}|, |[u^{(2)}]^+_{\alpha} - [u^{(1)}]^+_{\alpha}|\}, & \text{for CASE II,} \\ \sup\limits_{\alpha\in[0,1]} \max\{1 - |[u^{(1)}]^-_{\alpha}|, |1 - [u^{(1)}]^+_{\alpha}|\}, & \text{otherwise.} \end{cases}$$

$$d\left(\frac{u_{[\mu k]}}{u_k}, u^{(2)}\right) = \begin{cases} \sup\limits_{\alpha\in[0,1]} \max\{|[u^{(1)}]^-_\alpha - [u^{(2)}]^-_\alpha|, |[u^{(1)}]^+_\alpha - [u^{(2)}]^+_\alpha|\}, & \text{for CASE I,} \\ 0, & \text{for CASE II,} \\ \sup\limits_{\alpha\in[0,1]} \max\{1 - |[u^{(2)}]^-_\alpha|, |1 - [u^{(2)}]^+_\alpha|\}, & \text{otherwise.} \end{cases}$$

$$d\left(\frac{u_{[\mu k]}}{u_k}, 1\right) = \begin{cases} \sup\limits_{\alpha\in[0,1]} \max\{|[u^{(1)}]^-_\alpha - 1|, |[u^{(1)}]^+_\alpha - 1|\}, & \text{for CASE I,} \\ \sup\limits_{\alpha\in[0,1]} \max\{|[u^{(2)}]^-_\alpha - 1|, |[u^{(2)}]^+_\alpha - 1|\}, & \text{for CASE II,} \\ 0, & \text{otherwise.} \end{cases}$$

In order to maintain the uniqueness and existence of the deferred-statistically convergence of order β, we presume that $1/2 < \beta \leq 1$ and $p_n = p$, being a fixed positive integer. Then for given $\varepsilon > 0$, we have

$$\delta^\beta_{pq}\left(k : d\left(\frac{u_{[k+\mu]}}{u_k}, u^{(1)}\right)\right)$$
$$\leq \lim_{n\to\infty} \frac{(q_n - p) - \min\left\{\sqrt{(q_n - p)}, (q_n - p) - \frac{\sqrt{(q_n-p)}}{\mu}\right\}}{(q_n - p)^\beta} = \lim_{n\to\infty} \frac{(q_n - p) - C_1\sqrt{(q_n - p)}}{(q_n - p)^\beta} \neq 0,$$

$$\delta^\beta_{pq}\left(k : d\left(\frac{u_{[k+\mu]}}{u_k}, u^{(2)}\right)\right)$$
$$\leq \lim_{n\to\infty} \frac{(q_n - p) - \min\left\{(q_n - p) - \sqrt{(q_n - p)}, \frac{\sqrt{(q_n-p)}}{\mu}\right\}}{(q_n - p)^\beta} = \lim_{n\to\infty} \frac{(q_n - p) - C_2\sqrt{(q_n - p)}}{(q_n - p)^\beta} \neq 0,$$

and

$$\delta^\beta_{pq}\left(k : d\left(\frac{u_{[k+\mu]}}{u_k}, 1\right)\right) \leq \lim_{n\to\infty} \frac{(C_1 + C_2)\sqrt{(q_n - p)}}{(q_n - p)^\beta} = 0.$$

Note that $C_1 = \frac{\min\left\{\sqrt{(q_n-p)}, (q_n-p) - \frac{\sqrt{(q_n-p)}}{\mu}\right\}}{\sqrt{(q_n-p)}}$ and $C_2 = \frac{\min\left\{(q_n-p) - \sqrt{(q_n-p)}, \frac{\sqrt{(q_n-p)}}{\mu}\right\}}{\sqrt{(q_n-p)}}$. This example illustrates that the deferred-statistically regularly varying of order β limit of fuzzy sequence is 1.

Example 1.7. Consider the sequence (u_k) of fuzzy numbers, defined by

$$u_k(t) = \begin{cases} 1, & \text{if } k \text{ is a cube,} \\ t, & \text{otherwise,} \end{cases} \quad t \in [1, 0].$$

For fixed $\mu > 0$, the sequence $\frac{u_{[\mu+k]}}{u_k}$ can be given by

$$\frac{u_{[\mu+k]}}{u_k} = \begin{cases} \frac{1}{t}\ (= u^{(3)}), & \text{if } [\mu + k] \text{ is a cube and } k \text{ is a non-cube (CASE III),} \\ t\ (= u^{(4)}), & \text{if } [\mu + k] \text{ is a non-cube and } k \text{ is a cube (CASE IV),} \\ 1, & \text{otherwise.} \end{cases}$$

Now, we have

$$d\left(\frac{u_{[\mu+k]}}{u_k}, u^{(3)}\right) = \begin{cases} 0, & \text{for CASE III,} \\ \sup\limits_{\alpha\in[0,1]} \max\{|[u^{(4)}]^-_\alpha - [u^{(3)}]^-_\alpha|, |[u^{(4)}]^+_\alpha - [u^{(3)}]^+_\alpha|\}, & \text{for CASE IV,} \\ \sup\limits_{\alpha\in[0,1]} \max\{1 - |[u^{(3)}]^-_\alpha|, |1 - [u^{(3)}]^+_\alpha|\}, & \text{otherwise.} \end{cases}$$

$$d\left(\frac{u_{[\mu+k]}}{u_k}, u^{(4)}\right) = \begin{cases} \sup\limits_{\alpha\in[0,1]} \max\{|[u^{(3)}]^-_\alpha - [u^{(4)}]^-_\alpha|, |[u^{(3)}]^+_\alpha - [u^{(4)}]^+_\alpha|\}, & \text{for CASE III,} \\ 0, & \text{for CASE IV,} \\ \sup\limits_{\alpha\in[0,1]} \max\{1 - |[u^{(4)}]^-_\alpha|, |1 - [u^{(4)}]^+_\alpha|\}, & \text{otherwise.} \end{cases}$$

$$d\left(\frac{u_{[\mu+k]}}{u_k}, 1\right) = \begin{cases} \sup\limits_{\alpha\in[0,1]} \max\{|[u^{(3)}]^-_\alpha - 1|, |[u^{(3)}]^+_\alpha - 1|\}, & \text{for CASE III,} \\ \sup\limits_{\alpha\in[0,1]} \max\{|[u^{(4)}]^-_\alpha - 1|, |[u^{(4)}]^+_\alpha - 1|\}, & \text{for CASE IV,} \\ 0, & \text{otherwise.} \end{cases}$$

Using the uniqueness and existence of the deferred-statistically convergence of order β, we presume that $1/3 < \beta \leq 1$ and $p_n = 0, q_=n$. Then for given $\varepsilon > 0$, we have

$$\delta^\beta_{pq}\left(k : d\left(\frac{u_{[k+\mu]}}{u_k}, u^{(3)}\right)\right) \leq \lim_{n\to\infty} \frac{n - \min\left\{\sqrt[3]{n}, n - \sqrt[3]{n} - \mu\right\}}{n^\beta} = \lim_{n\to\infty} \frac{n - C_3\sqrt[3]{n}}{n^\beta} \neq 0,$$

$$\delta^\beta_{pq}\left(k : d\left(\frac{u_{[k+\mu]}}{u_k}, u^{(4)}\right)\right) \leq \lim_{n\to\infty} \frac{n - \min\left\{n - \sqrt[3]{n}, \sqrt[3]{n} - \mu\right\}}{n^\beta} = \lim_{n\to\infty} \frac{n - C_4\sqrt{n}}{n^\beta} \neq 0,$$

and

$$\delta^\beta_{pq}\left(k : d\left(\frac{u_{[k+\mu]}}{u_k}, 1\right)\right) \leq \lim_{n\to\infty} \frac{(C_3 + C_4)\sqrt[3]{n}}{n^\beta} = 0,$$

where $C_3 = \frac{\min\{\sqrt{n}, n - \sqrt{n} - \mu\}}{\sqrt{n}}$ and $C_4 = \frac{\min\{n - \sqrt{n}, \sqrt{n} - \mu\}}{\sqrt{n}}$.

This shows that the fuzzy sequence (u_k) is deferred-statistically translationally regularly varying of order β to the limit 1.

2. Main results

In this section, we establish some relationships on different types of deferred-statistically varying positive fuzzy sequences of order β.

Theorem 2.1. *Let $u = (u_k)$ be a sequence of positive fuzzy numbers and $0 < \beta \leq 1$. If*

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)^\beta} \sum_{k=p_n+1}^{q_n} d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right) = 0,$$

then $u \in DSORV^F$ or $u \in DSTRV^F$ and

$$\delta^\beta_{p,q}\left(k : d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\right) = 0.$$

Proof. We prove the theorem for $DSORV^F$ only and for others we use the similar techniques. As the hypothesis, $u = (u_k)$ is a sequence of positive fuzzy numbers which satisfies

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)^\beta} \sum_{k=p_n+1}^{q_n} d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right) = 0, \text{ for } \mu > 0.$$

This leads to

$$\frac{1}{(q_n-p_n)^\beta}\sum_{k=p_n+1}^{q_n} d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right) \geq \frac{1}{(q_n-p_n)^\beta}\sum_{\substack{k=p_n+1,\\ d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\geq\varepsilon}}^{q_n} d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)$$

For the limit as $n \to \infty$, we have

$$\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}\sum_{\substack{k=p_n+1,\\ d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\geq\varepsilon}}^{q_n} d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right) = 0,$$

as a consequence, we have $u \in DSORV^F$ and

$$\delta_{p,q}^{\beta}\left(k : d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\right) = 0.$$

□

For the converse of Theorem 2.1, we present the following theorem.

Theorem 2.2. *If $u = (u_k)$ is a bounded sequence of positive fuzzy numbers, $\beta = 1$, and $u \in DSORV^F$ or $u \in DSTRV^F$, then*

$$\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}\sum_{k=p_n+1}^{q_n} d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right) = 0.$$

Proof. Suppose (u_k) is a bounded sequence. Then for $\mu > 0$, there exists a number $\mathcal{K} < \infty$ such that

$$\sup_k d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right) < \mathcal{K}. \tag{2}$$

Now, for given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{(q_n-p_n)^\beta}\sum_{k=p_n+1}^{q_n} d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right) &= \frac{1}{(q_n-p_n)^\beta}\sum_{\substack{k=p_n+1,\\ d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)<\varepsilon}}^{q_n} d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\\ &\quad + \frac{1}{(q_n-p_n)^\beta}\sum_{\substack{k=p_n+1,\\ d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\geq\varepsilon}}^{q_n} d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\\ &\leq \frac{\varepsilon}{(q_n-p_n)^\beta}\sup_k d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\sum_{\substack{k=p_n+1,\\ d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)<\varepsilon}}^{q_n} (1)\\ &\quad + \frac{1}{(q_n-p_n)^\beta}\sum_{\substack{k=p_n+1,\\ d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\geq\varepsilon}}^{q_n} d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\end{aligned}$$

By putting $\beta = 1$ and taking the limit as $n \to \infty$ along with the condition (2) in the above inequality, we have the result as desired. □

A sequence $u = (u_k)$ is said to be deferred-statistically regular varying Cauchy sequence if, for every $\varepsilon > 0$, there exists a number N depending on ε, we have

$$\delta^{\beta}_{p,q}\left(k : d\left(\frac{u_{[\mu k]}}{u_k}, \frac{u_{[\mu N]}}{u_N}\right)\right) = 0.$$

We denote such fuzzy sequences by the class $DSRVC^F$.

Theorem 2.3. *Suppose $u = (u_k)$ is a sequence of positive fuzzy numbers and $0 < \beta \leq 1$. Then the following statements are equivalent:*

(i) *$u \in DSRV^F$.*

(ii) *$u \in DSRVC^F$.*

(iii) *For some $v \in RV^F$(the space of all regular varying positive fuzzy sequence) such that*

$$\frac{u_{[\mu k]}}{u_k} = \frac{v_{[\mu k]}}{v_k}, \text{ for almost all } k.$$

Proof. Assume that **(i)** holds and $u = (u_k) \in DSRV^F$. Then

$$\delta^{\beta}_{p,q}\left(k : d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right)\right) = 0, \text{ for all } \mu > 0.$$

This implies that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)^{\beta}} \left|\left\{p_n < k \leq q_n : d\left(\frac{u_{[\mu k]}}{u_k}, r(\mu)\right) \geq \frac{\varepsilon}{2}\right\}\right| = 0.$$

Choose a number N such that for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)^{\beta}} \left|\left\{p_n < N \leq q_n : d\left(\frac{u_{[\mu N]}}{u_N}, r(\mu)\right) \geq \frac{\varepsilon}{2}\right\}\right| = 0.$$

By using triangle inequality to the above equations, we have

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)^{\beta}} \left|\left\{p_n < k \leq q_n : d\left(\frac{u_{[\mu k]}}{u_k}, \frac{u_{[\mu N]}}{u_N}\right) \geq \varepsilon\right\}\right| = 0.$$

This implies **(ii)** and completes the first part.

For second part, assume that **(ii)** holds and choose a number N such that the closed interval $I_0 = \left[\left[\frac{u_{[\mu N]}}{u_N}\right]_{\alpha} - 1, \left[\frac{u_{[\mu N]}}{u_N}\right]_{\alpha} + 1\right]$ contains $\frac{u_{[\mu k]}}{u_k}$ for almost all k. Similarly, choose another number N_1 such that $I' = \left[\left[\frac{u_{[\mu N_1]}}{u_{N_1}}\right]_{\alpha} - 1/2, \left[\frac{u_{[\mu N_1]}}{u_{N_1}}\right]_{\alpha} + 1/2\right]$ contains $\frac{u_{[\mu k]}}{u_k}$ for almost all k. Therefore, the interval $I_1 = I_0 \cap I'$ is of length less than or equal to 1, contains $\frac{u_{[\mu]}}{u_k}$ for almost all k. By choosing the number N_2, construct a closed interval $I'_1 = \left[\left[\frac{u_{[\mu N_2]}}{u_{N_2}}\right]_{\alpha} - 1/4, \left[\frac{u_{[\mu N_2]}}{u_{N_2}}\right]_{\alpha} + 1/4\right]$ containing $\frac{u_{[\mu k]}}{u_k}$ for almost all k. Proceeding this way, we claim that the interval $I_2 = I_1 \cap I'_1$ is of the length not greater than $1/2$ contains $\frac{u_{[\mu k]}}{u_k}$ for almost all k. By principle of induction, for a natural number m, we can construct an interval I_m of length not greater than $\frac{1}{2^{m-1}}$ contains $\frac{u_{[\mu k]}}{u_k}$ for almost all k. Using nested interval theorem, we can able to find a number σ, such that

$$I = \bigcap_{j=1}^{\infty} I_j.$$

Since I_m contains $\frac{u_{[\mu k]}}{u_k}$ for almost all k, we choose an increasing sequence of positive integers $\sigma = (\sigma_m)$ such that

$$\delta^{\beta}_{p,q}\left(\left\{p_n < k \leq q_n : \frac{u_{[\mu k]}}{u_k} \notin I_m\right\}\right) < \frac{1}{m} \text{ for each } n > \sigma_m. \tag{3}$$

Consider a subsequence $\varrho = (\varrho_k)$ of $u = (u_k)$ consisting of all the terms u_k such that $k > \sigma_1$ and if $\sigma_m < k \leq \sigma_{m+1}$, then $\frac{u_{[\mu k]}}{u_k} \notin I_m$.

Now, using the subsequence $\varrho = (\varrho_k)$, define the sequence $v = (v_k)$ such that

$$\frac{v_{[\mu k]}}{v_k} = \begin{cases} \sigma, & \text{if } u_k \text{ is a term of } \varrho, \\ \frac{u_{[\mu k]}}{u_k}, & \text{otherwise.} \end{cases}$$

It is observed that $\lim_{k\to\infty} \frac{v_{[\mu k]}}{v_k} = I$. For $k > \sigma_m$ and $0 < \frac{1}{m} < \varepsilon$, we have u_k is either of the form ϱ_k or $\frac{u_{[\mu k]}}{u_k} = \frac{v_{[\mu k]}}{v_k} \in I_m$ and $|[\frac{v_{[\mu k]}}{v_k}]_\alpha - I|$ is not greater than the length of I_m. Using $\varrho_m < n < \varrho_{m+1}$, in (3) we obtain that

$$\begin{aligned}\delta^{\beta}_{p,q}\left(\left\{p_n < k \leq q_n : \frac{u_{[\mu k]}}{u_k} \neq \frac{v_{[\mu k]}}{u_k}\right\}\right) &< \delta^{\beta}_{p,q}\left(\left\{p_n < k \leq q_n : \frac{u_{[\mu k]}}{u_k} \notin I_m\right\}\right) \\ &< \frac{1}{m}.\end{aligned}$$

Thus it is concluded that

$$\delta^{\beta}_{p,q}\left(k : d\left(\frac{u_{[\mu k]}}{u_k}, \frac{v_{[\mu k]}}{v_k}\right) \geq \varepsilon\right) = 0.$$

This leads to **(iii)**. Finally, we consider **(iii)** and show that **(i)** is true i.e., $u \in DSRV^F$ or

$$\delta^{\beta}_{p,q}\left(\left\{p_n < k \leq q_n : \frac{u_{[\mu k]}}{u_k} \neq \frac{v_{[\mu k]}}{v_k}\right\}\right).$$

Since $v \in RV^F$, for every $\varepsilon > 0$ and $\mu > 0$,

$$\delta^{\beta}_{p,q}\left(\left\{p_n < k \leq q_n : d\left(\frac{v_{[\mu k]}}{v_k}, k_{ds}(\mu)\right) \geq \varepsilon\right\}\right) = 0.$$

Now, for every $\varepsilon > 0$ and $\mu > 0$, we have

$$\begin{aligned}\delta^{\beta}_{p,q}\left(\left\{p_n < k \leq q_n : d\left(\frac{u_{[\mu k]}}{u_k}, k_{ds}(\mu)\right) \geq \varepsilon\right\}\right) <& \delta^{\beta}_{p,q}\left(k : d\left(\frac{u_{[\mu k]}}{u_k}, \frac{v_{[\mu k]}}{v_k}\right) \geq \varepsilon\right) \\ &+ \delta^{\beta}_{p,q}\left(\left\{p_n < k \leq q_n : d\left(\frac{v_{[\mu k]}}{v_k}, k_{ds}(\mu)\right) \geq \varepsilon\right\}\right).\end{aligned}$$

Therefore, we have $\delta^{\beta}_{p,q}\left(\left\{k : d\left(\frac{u_{[\mu k]}}{u_k}, k_{ds}(\mu)\right) \geq \varepsilon\right\}\right) = 0$, i.e., $u \in DSRV^F$. □

Theorem 2.4. *Let sequence $u = (u_k)$ is a sequence of positive fuzzy numbers and $u \in STRV^F$ or $u \in SORV^F$. Then $u \in DSTRV^F$ or $u \in DSORV^F$, respectively provided the sequence $\left(\frac{p_n}{(q_n - p_n)^\beta}\right)_{n\in\mathbb{N}}$ is bounded.*

Proof. The proof is straightforward, hence omitted. □

Theorem 2.5. *Suppose $u = (u_k)$ is a sequence of positive fuzzy numbers and $0 < \beta \leq 1$. Then the following statements are equivalent:*

(i) $u \in DSTRV^F$.

(ii) $u \in DSTRVC^F$.

(iii) *For some $v \in TRV^F$(the space of all translationally varying fuzzy sequence) such that*

$$\frac{u_{[\mu k]}}{u_k} = \frac{v_{[\mu k]}}{v_k}, \text{ for almost all } k.$$

where the class $DSRVC^F$ is the set of all deferred-statistical translationally regular varying Cauchy positive fuzzy sequence.

Proof. The proof follows the similar lines as described in Theorem 2.3, hence omitted. □

Theorem 2.6. *Let $u = (u_k)$ be a sequence of positive fuzzy numbers. Then $u \in DSRV^F$ i.e.,*

$$dst - \lim_{k\to\infty} \frac{u_{[\mu k]}}{u_k} = k_{ds}(\mu), \text{ for all } \mu > 0,$$

if and only if there exists a subsequence $v = (v_k)$ of u such that

$$\lim_{k\to\infty} \frac{v_{[\mu k]}}{v_k} = k_{ds}(\mu),$$

and $k_{ds}(\mu) = \mu^\rho$ for some $\rho \in \mathbb{R}$.

Proof. This is an immediate consequence of Theorem 2.3. □

Theorem 2.7. *Let $u = (u_k)$ be a sequence of positive fuzzy numbers. Then $u \in DSTRV^F$ i.e.,*

$$dst - \lim_{k\to\infty} \frac{u_{[\mu k]}}{u_k} = r_{ds}(\mu), \text{ for all } \mu > 0,$$

If and only if there exists a subsequence $v = (v_k)$ of u such that

$$\lim_{k\to\infty} \frac{v_{[\mu k]}}{v_k} = r_{ds}(\mu),$$

and $r_{ds}(\mu) = e^{\rho[\mu]}$ for some $\rho \in \mathbb{R}$.

Proof. This follows from Theorem 2.5. □

Theorem 2.8. *Let $u = (u_k)$ be a sequence of positive fuzzy numbers and $u \in DSRV^F$ i.e., $dst - \lim_{k\to\infty} \frac{u_{[\mu k]}}{u_k} = k_{ds}(\mu)$. Then $u \in RV^F$ with the same limit if*

$$d\left(\frac{u_{[\mu k]}}{u_k}, \frac{u_{[\mu(k+1)]}}{u_{k+1}}\right) = O((q_k - p_k)^{-1}).$$

Proof. Let $u \in DSRV^F$. Then by Theorem 2.3 there exists a sequence $v = (v_k)$ such that

$$\lim_{k\to\infty} \frac{v_{[\mu k]}}{v_k} = k_{ds}(\mu),$$

such that

$$\delta^{\beta}_{p,q}\left(k : \frac{u_{[\mu k]}}{u_k} \neq \frac{v_{[\mu k]}}{v_k}\right) = 0.$$

Set $q_k - p_k = A_1(k) + A_2(k)$, where

$$A_1(k) = \max\left\{p_k < n \leq q_k : \frac{u_{[\mu n]}}{u_n} = \frac{v_{[\mu n]}}{v_n}\right\}$$

and

$$A_2(k) = \max\left\{p_k < n \le q_k : \frac{u_{[\mu n]}}{u_n} \ne \frac{v_{[\mu n]}}{v_n}\right\}.$$

Now, we show that

$$\lim_{k\to\infty} \frac{A_2(k)}{A_1(k)} = 0.$$

This can be proved by induction method. In the contrary, suppose $\frac{A_2(k)}{A_1(k)} \ge \varepsilon > 0$, then

$$\begin{aligned}\frac{1}{(q_k-p_k)^\beta}\left(\left\{p_k < n \le q_k : \frac{u_{[\mu n]}}{u_n} = \frac{v_{[\mu n]}}{v_n}\right\}\right) &\le \frac{A_1(k)}{A_1(k)+A_2(k)}\\ &\le \frac{A_1(k)}{A_1(k)+\varepsilon A_1(k)}\\ &= \frac{\varepsilon}{1+\varepsilon}.\end{aligned}$$

This leads to the contradiction that

$$\delta^\beta_{p,q}\left(k : \frac{u_{[\mu k]}}{u_k} = \frac{v_{[\mu k]}}{v_k}\right) = 0.$$

As a result, this concludes that $\lim_{k\to\infty} \frac{A_2(k)}{A_1(k)} = 0$.

As per the hypothesis $d\left(\frac{u_{[\mu k]}}{u_k}, \frac{u_{[\mu(k+1)]}}{u_{k+1}}\right) = O((q_k-p_k)^{-1})$. Thus, for all k, there exist a constant C such that

$$d\left(\frac{u_{[\mu k]}}{u_k}, \frac{u_{[\mu(k+1)]}}{u_{k+1}}\right) = \frac{C}{q_k-p_k}.$$

By using properties (ii) and (iii) of fuzzy metric space $(\mathscr{L}(\mathbb{R}), d)$, we have

$$\begin{aligned}d\left(\frac{v_{[\mu A_1(k)]}}{v_{A_1(k)}}, \frac{u_{[\mu(q_k-p_k)]}}{u_{q_k-p_k}}\right) &= d\left(\frac{u_{[\mu A_1(k)]}}{u_{A_1(k)}}, \frac{u_{[\mu(A_1(k)+A_2(k))]}}{u_{A_1(k)+A_2(k)}}\right)\\ &\le \sum_{i=A_1(k)}^{A_1(k)+A_2(k)-1} d\left(\frac{u_{[\mu(i)]}}{u_i}, \frac{u_{[\mu(i+1)]}}{u_{i+1}}\right)\\ &\le \frac{C}{q_k-p_k}\sum_{i=A_1(k)}^{A_1(k)+A_2(k)-1}(1)\\ &= \frac{C(A_1(k)+A_2(k)-A_1(k))}{A_1(k)+A_2(k)}\\ &= \frac{C\left(\frac{A_2(k)}{A_1(k)}\right)}{1+\frac{A_2(k)}{A_1(k)}}.\end{aligned}$$

Using $\lim_{k\to\infty}\frac{v_{[\mu k]}}{v_k} = k_{ds}(\mu)$ in the above inequality and taking limit as $k\to\infty$ on both the sides, $\lim_{k\to\infty}\frac{u_{[\mu A_1(k)]}}{A_1(k)} = \lim_{k\to\infty}\frac{u_{[\mu(q_k-p_k)]}}{u_{q_k-p_k}} = k_{ds}(\mu)$. □

Theorem 2.9. *Let $u=(u_k)$ be a sequence of positive fuzzy numbers and $u \in DSTRV^F$ i.e., $dst-\lim_{\to\infty}\frac{u_{[k+\mu]}}{u_k} = r_{ds}(\mu)$. Then $u \in TRV^F$ with the same limit if*

$$d\left(\frac{u_{[\mu k]}}{u_k}, \frac{u_{[\mu(k+1)]}}{u_{k+1}}\right) = O((q_k-p_k)^{-1}).$$

Proof. Proof is similar to that of Theorem 2.8, hence omitted. □

3. Conclusion

In this paper, we have studied the idea of convergence of fuzzy sequence of positive real numbers via regular variations in deferred statistical context. Also, some relations among these newly defined convergence of fuzzy sequence were established. This work provides a new development in the direction of the convergence of fuzzy sequences and generalizes notion of usual regular variations. As a future scope, this idea may be studied to the case of different statistical convergence using various types of means for single and double sequences.

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